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A REGULARITY CRITERION OF STRONG SOLUTIONS TO THE 2D CAUCHY PROBLEM OF THE KINETIC-FLUID MODEL FOR FLOCKING*

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Abstract. In this paper, we consider the blow-up criterion for the two dimensional kinetic-fluid model in the whole space. For particle and fluid dynamics, we employ the Cucker-Smale-Fokker-Planck model for the flocking particle part, and the isentropic compressible Navier-Stokes equations for the fluid part, and the separate systems are coupled through the drag force. We show that the strong solution exists globally if the $L^\infty(0,T;L^\infty)$ norm of the fluid density $\rho(t,x)$ is bounded.

Keywords. Compressible Navier-Stokes equations; Cucker-Smale-Fokker-Planck equation; vacuum; blow-up criterion.

AMS subject classifications. 35A20; 35B45; 76N10; 76T99.

1. Introduction

In this paper, we consider a coupled kinetic-fluid model for the interactions between Cucker-Smale particles and compressible viscous fluid via a friction force in a random environment, which can be modeled by the coupled system of kinetic CS-FP type equation with a degenerate diffusion coefficient and compressible isentropic Navier-Stokes equations. More precisely, let $f=f(t,x,v)$ be the one-particle distribution function of a Cucker-Smale ensemble with velocity $v=(v_1,v_2)\in\mathbb{R}^2$ at position $x=(x_1,x_2)\in\mathbb{R}^2$ at time $t>0$ for the particle side, and let $\rho(t,x)$ be the density and $u=u(t,x)=(u_1,u_2)(t,x)$ be the bulk velocity of the compressible fluid. Then the coupled dynamics of $[f,\rho,u]$ is governed by the following kinetic-fluid system:

$$\begin{cases}\partial_t f+v\cdot\nabla_x f+\nabla_v\cdot(fL[f]+(u-v)f)=\Delta_v(|v-v_c|^2 f),\\ \partial_t\rho+\nabla_x\cdot(\rho u)=0,\\ \partial_t(\rho u)+\nabla_x\cdot(\rho u\otimes u)+\nabla_x P(\rho)=\mu\Delta_x u+(\mu+\lambda)\nabla_x \mathrm{div} u+\displaystyle\int_{\mathbb{R}^2}(v-u)f dv,\\ v_c:=\dfrac{\int_{\mathbb{R}^4} vf dvdx}{\int_{\mathbb{R}^4} f dvdx},\quad L[f](t,x,v):=\displaystyle\int_{\mathbb{R}^4}\psi(x-y)(v_*-v)f(t,y,v_*)dv_*dy,\end{cases}\tag{1.1}$$

subject to the initial condition:

$$(f(0,x,v),\rho(0,x),u(0,x))=(f_0(x,v),\rho_0(x),u_0(x)),\quad (x,v)\in\mathbb{R}^2\times\mathbb{R}^2,\tag{1.2}$$

and the far field behavior

$$(f(t,x,v),\rho(t,x),u(t,x))\to(0,0,0),\quad as(|x|,|v|)\to+\infty,\quad t>0,\tag{1.3}$$

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where $P(\rho)$ is the pressure given by

$$P(\rho)=\rho^{\gamma}, \quad \gamma>1,$$

$\psi(x-y)=\psi(|x-y|)$ in $L[f]$ is a communication weight representing a degree of communication between particles located at x and y. For definiteness, we assumed that ψ is uniformly bounded and away from zero, and is sufficiently regular: there exist positive constants ψ_m and ψ_M such that the communication function ψ satisfies the following property:

$$0<\psi(s)\leq\psi_M<+\infty, \quad s\geq 0, \qquad \sum_{1\leq i\leq 3}\left\|\frac{d^i\psi}{ds^i}\right\|_{L^\infty(\mathbb{R}_+)}<\infty.$$

Here, we only consider viscosity coefficients μ and λ are constant, and satisfy the following physical restrictions

$$\mu>0, \qquad \mu+\lambda\geq 0.$$

Moreover, ρ_f and u_f denote the local mass density, and average local velocity of particle ensemble, respectively:

$$\rho_f:=\int_{\mathbb{R}^2} f\mathrm{d}v \quad \text{and} \quad u_f:=\begin{cases} \frac{\int_{\mathbb{R}^2} vf\mathrm{d}v}{\int_{\mathbb{R}^2} f\mathrm{d}v} & if \qquad \rho_f\neq 0, \\ 0 & if \qquad \rho_f=0. \end{cases}$$

There is a huge amount of literature on the studies about the mechanism of blow-up and structure of possible singularities of strong and classical solutions to the compressible Navier-Stokes equations. The pioneering work can be traced to Serrin's criterion [25] on the Leray-Hopf weak solutions to the three-dimensional incompressible Navier-Stokes equations, which can be stated that if a weak solution u satisfies

$$u\in L^s(0,T;L^r), \quad \frac{2}{s}+\frac{3}{r}=1, \quad 3<r\leq+\infty \tag{1.4}$$

then u is regular one. Recently, Serrin-type blow-up criterion was extended to the compressible Navier-Stokes system in [15,17–19], especially the initial density is allowed to vanish. For more information about the blow-up criteria for compressible flows, we can refer to [26], in which the authors proved a blow-up criterion in terms of the upper bound of the density for the strong solution to the 3-D compressible Navier-Stokes equations. In [28], a blow-up criterion for the strong solution for 3D viscous liquid-gas two-phase flow model in a smooth bounded domain is obtained.

In [16, 22, 24, 29], the global solvability of the two-dimensional barotropic compressible Navier-Stokes system with vacuum and large initial data was considered. The main technical point in these proofs is to apply the spatial weight function on the energy estimates. These tools help us to control the fluid part in the coupled system (1.1).

In [12] and [13], the authors considered the global classical solutions to kinetic-fluid coupled system with large initial data and may contain vacuum. The main difficulty comes from the kinetic and fluid interaction term which has high nonlinearity. We can refer to [1, 8, 14] for existence of strong solution for the Cucker-Smale model and refer to [3–5, 11] for local/global existence of strong solution for the coupled system. Moreover, under suitable assumptions on the initial configurations, the finite-time blow-up phenomena of classical solutions for Vlasov/Navier-Stokes equations was considered in [7].

These previous results lead to a natural question: *is it possible to find a blow-up criterion for strong solution to the system (1.1) with fluid density which may contain vacuum ?* In this coupled system, the terms need be controlled not only from the fluid and kinetic parts but also from the interactions of the kinetic and fluid parts which have high nonlinearity.

The rest of paper is organized as follows. In Section 1 and 2, we briefly discuss a framework and present our main results. In Section 3, we provide several lemmas to be used later. In Section 4, we derive *a priori* estimates in the whole space and provide a proof of the main result.

Notation. Throughout the paper, C denotes a generic positive constant which may change line by line. The small constants to be chosen are denoted by ε and δ. For function spaces, $W^{k,p}(\mathbb{R}^2)$ and $W^{k,p}(\mathbb{R}^4)$ denote the standard Sobolev spaces with standard norm $\|\cdot\|_{W^{k,p}}$, and $H^k := W^{k,2}$. $\|\cdot\|_p := (\int_{\mathbb{R}^2}|\cdot|^p \mathrm{d}x)^{\frac{1}{p}}$ or $(\int_{\mathbb{R}^4}|\cdot|^p \mathrm{d}v\mathrm{d}x)^{\frac{1}{p}}$ with $1\le p\le +\infty$. For notational simplicity, we denote

$$\begin{aligned}
&\partial^{\alpha_*}_{\beta_*} f := \partial_x^{\alpha_*}\partial_v^{\beta_*} f, \quad \alpha_* = [\alpha_{*1},\alpha_{*2}], \quad \beta_* = [\beta_{*1},\beta_{*2}],\\
&\|f\|_{W_k^{N,p}} := \sum_{|\alpha_*|+|\beta_*|\le N} \|\langle v\rangle^k \partial^{\alpha_*}_{\beta_*} f\|_{L^p(\mathbb{R}^4)}, \quad k\ge 0.
\end{aligned}$$

Homogeneous Sobolev space $D^{\ell,p}(\ell\ge 1)$ is defined by $D^{\ell,p}(\mathbb{R}^2) = \{u\in L^1_{loc}(\mathbb{R}^2) | \|\nabla^\ell u\|_p < +\infty\}$ with $\|u\|_{D^{\ell,p}} := \|\nabla^\ell u\|_p$. For the special case $p=2$, we denote D^ℓ as $D^{\ell,2}$.

2. Main result

In this section, we present our main results. Before presenting our main result, let us recall the precise definition of strong solutions.

Definition 2.1. *If all derivatives involved in* (1.1) *for* $[f,\rho,u]$ *are regular distributions, equations* (1.1) *hold almost everywhere in* $(0,T)\times\mathbb{R}^2$*, and for* $q>2$*,* $p>4$*,*

$$\begin{aligned}
&\rho(t,x)\ge 0, \quad \forall (t,x)\in[0,T]\times\mathbb{R}^2,\\
&\rho(t,x)\in C([0,T];L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\cap W^{1,q}(\mathbb{R}^2)),\\
&\rho_t(t,x)\in C([0,T];L^q),\\
&u(t,x)\in C([0,T;D^{1,2}\cap D^{2,2})\cap L^2(0,T;D^{2,q}),\\
&u_t(t,x)\in L^2(0,T;D^{1,2}), \quad (\sqrt{\rho}u_t)(t,x)\in L^\infty(0,T;L^2),\\
&f(t,x,v)>0, \quad f(t,x,v)\in C(0,T;W_k^{2,p}).
\end{aligned} \tag{2.1}$$

then $[f,\rho,u]$ *is called a strong solution of the system* (1.1).

Theorem 2.1. *Suppose the initial value* $[f_0(x,v),\rho_0(x),u_0(x)]$ *satisfies that*

$$\begin{aligned}
&\rho_0(x)\ge 0, \bar{x}^{\frac{5a}{4}}\rho_0(x)\in L^1(\mathbb{R}^2), \bar{x}^a\rho_0(x)\in D^{1,2}(\mathbb{R}^2)\cap D^{1,q}(\mathbb{R}^2),\\
&\bar{x} = (\mathrm{e}+|x|^2)^{\frac{1}{2}}\log^{1+\sigma}(\mathrm{e}+|x|^2)\\
&\rho_0(x)\in L^1(\mathbb{R}^2)\cap H^1(\mathbb{R}^2)\cap W^{1,q}(\mathbb{R}^2), \nabla u_0(x)\in L^2(\mathbb{R}^2), \sqrt{\rho_0}u_0\in L^2(\mathbb{R}^2),\\
&f_0(x,v)>0, x^2 f_0(x,v)\in L^2(\mathbb{R}^2\times\mathbb{R}^2), f_0(x,v)\in L^\infty(0,T;W_k^{2,p}(\mathbb{R}^2)),
\end{aligned} \tag{2.2}$$

with $\sigma>0$*,* $q>2$*,* $p>4$*,* $1<a<\frac{8}{5}$*,* k *is sufficiently large, and the compatibility condition*

$$-\mu\Delta u_0 - (\mu+\lambda)\nabla\mathrm{div} u_0 + u_0\rho_{f_0} - u_{f_0}\rho_{f_0} + \nabla P(\rho_0) = \sqrt{\rho_0}g \tag{2.3}$$

for some $g(x)\in L^2(\mathbb{R}^2)$. Let $[f(t,x,v),\rho(t,x),u(t,x)]$ be a strong solution to the Cauchy problem (1.1)-(1.2). If $T^*<\infty$ is the maximal time for that solution, then we have

$$\lim_{T\to T^*}\|\rho\|_{L^\infty(0,T;L^\infty)}=\infty. \tag{2.4}$$

Some remarks are listed as below:

Remark 2.1.

- Our method can also be applied to the case that μ and λ are smooth functions dependent on $\rho(t,x)$ similar to [17, 22].
- We fail to quantify the index k, and it is insignificant. The reason results from that k is dependent on p in Lemma 4.4 and Corollary 4.1, and several places have used it in this paper.

Now, we briefly sketch the main ideas to derive our result in this paper. Owing to the fact that the initial density $\rho_0(x)$ may contain vacuum, it is difficult to bound the L^p-norm of $u(t,x)$ from the zero-order energy estimate. To overcome this difficulty, we introduce the Hardy-type $\|u\bar{x}^{-\eta}\|_{\frac{2+\epsilon}{\tilde{\eta}}}$ estimate in (3.3) for zero-order $u(t,x)$. In the meantime, imposing the space weight on fluid density $\rho(t,x)$ in (4.11) is essential. Next, the coupled terms in $(1.1)_2$ which have high nonlinearity have the following difficulties that we need to deal with. Fortunately, as (i) in Lemma 4.2 holds, we can have nice bounds for kinetic-fluid coupled terms in (4.14) and (4.21). These zero-order coupled-type estimates are crucial in the proof. The *a priori* estimates on $L_t^\infty L_x^q$-norm of $\nabla\rho(t,x)$ and the $L_t^1L_x^\infty$-norm of the velocity $u(t,x)$ can be obtained by solving a logarithmic Gronwall's inequality in Lemma 4.7. Finally, the higher-order estimate on kinetic density distribution $f(t,x,v)$ will be considered in Lemma 4.9.

3. Preliminaries

3.1. Elementary inequalities. In this part, we recall several elementary inequalities. These inequalities play an important role in our proof. We first state the following Sobolev inequality which will be used frequently:

Lemma 3.1. *[27] There exists a positive constant C such that the following estimates hold for any function $o\in\{o\in L^p|\nabla o\in L^{\frac{2p}{p+2}}(\mathbb{R}^2)\}$, $h\in H^1(\mathbb{R}^2)$, $w\in L^r(\mathbb{R}^2)\cap D^{1,p}(\mathbb{R}^2)$, for any $p\in(2,\infty)$ and $r\in(1,\infty)$,*

$$\begin{aligned}&\|o\|_p\le Cp^{\frac12}\|\nabla o\|_{\frac{2p}{p+2}},\quad \|h\|_p\le Cp^{\frac12}\|h\|_2^{\frac2p}\|\nabla h\|_2^{1-\frac2p},\\ &\|w\|_\infty\le C(p,r)\|w\|_r^{\frac{r(p-2)}{2p+r(p-2)}}\|\nabla w\|_p^{\frac{2p}{2p+r(p-2)}}.\end{aligned} \tag{3.1}$$

Then one has the following weighted L^p bounds for elements of $D^{1,2}(\mathbb{R}^2)$ (cf. Theorem B.1 in [23]):

Lemma 3.2. *For $m\in[2,\infty)$, $\theta\in(1+m/2,\infty)$ and $w\in D^{1,2}(\mathbb{R}^2)$, it holds that*

$$\left(\int_{\mathbb{R}^2}\frac{|w|^m}{e+|x|^2}(\log(e+|x|^2))^{-\theta}dx\right)^{\frac1m}\lesssim\|w\|_{2(B_1)}+\|\nabla w\|_2, \tag{3.2}$$

where $B_1:=\{x\in\mathbb{R}^2\big||x|<1\}$.

The combination of Lemma 3.1 and Lemma 3.2 yields the following Hardy-type inequality which cites the Lemma 2.4 in [16].

LEMMA 3.3. *Let $\bar{x}$ be as in (2.2). Assume that $\rho(t,x)$ is a non-negative function such that*

$$\int_{B_{N_1}} \rho \mathrm{d}x \ge M_1, \quad \|\rho\|_{L^1\cap L^\infty} \le M_2$$

for positive constants $M_i (i=1,2)$, a, and $N_1 \ge 1$. Then for $\epsilon,\eta>0$, there exists a positive constant C depending only on ϵ, η, $M_i (i=1,2)$, a, N_1 such that for every $v \in D^{1,2}(\mathbb{R}^2)$ satisfies

$$\|v\bar{x}^{-\eta}\|_{\frac{2+\epsilon}{\bar{\eta}}} \le C(\|\sqrt{\rho}v\|_2 + \|\nabla v\|_2), \tag{3.3}$$

with $\bar{\eta} = \min\{1,\eta\}$.

The following Beale-Kato-Majda-type inequality will be crucial to derive the L^∞-norm of $\nabla u(t,x)$.

LEMMA 3.4 ([6,10,19]). *For $2<q<+\infty$, there exists a positive constant C that may depend on q such that the following estimate holds for all $\nabla u \in W^{1,q}(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$,*

$$\|\nabla u\|_\infty \le C(\|\mathrm{div}u\|_\infty + \|\mathrm{curl}u\|_\infty)\log(e+\|\nabla^2 u\|_q) + C\|\nabla u\|_2 + C.$$

The following elliptic estimates have been frequently used in Section 4. We state the result directly and omit the proof.

LEMMA 3.5. *Let $[f(t,x,v),\rho(t,x),u(t,x)]$ be a strong solution to the coupled system (1.1). For $\nabla^\perp = (-\partial_{x_2},\partial_{x_1})$, $\dot{u} = u_t + u\cdot\nabla u$, we can rewrite the momentum equation $(1.1)_3$ as:*

$$\Delta F = \mathrm{div}(\rho\dot{u} + (u-u_f)\rho_f), \quad \mu\Delta\omega = \nabla^\perp\cdot(\rho\dot{u} + (u-u_f)\rho_f), \tag{3.4}$$

where $F = (\lambda+2\mu)\mathrm{div}u - P$, $\omega = \partial_{x_1}u_2 - \partial_{x_2}u_1$. Then we have the following L^p estimate for the above elliptic system,

$$\|\nabla F\|_p + \|\nabla\omega\|_p \le C\|\rho\dot{u}\|_p + C\|(u-u_f)\rho_f\|_p, \tag{3.5}$$

$$\|F\|_p + \|\omega\|_p \le C(\|\rho\dot{u}\|_2 + \|(u-u_f)\rho_f\|_2)^{1-\frac{2}{p}}(\|\nabla u\|_2 + \|P\|_2)^{\frac{2}{p}}, \tag{3.6}$$

$$\|\nabla u\|_p \le C(\|\rho\dot{u}\|_2 + \|(u-u_f)\rho_f\|_2)^{1-\frac{2}{p}}(\|\nabla u\|_2 + \|P\|_2)^{\frac{2}{p}} + C\|P\|_p. \tag{3.7}$$

4. *A priori* estimates and proof of Theorem 2.1

Let $[f(t,x,v),\rho(t,x),u(t,x)]$ be a strong solution to the coupled system (1.1). Suppose that (2.4) were false, that is, there exists a constant $M_0>0$ such that

$$\lim_{T\to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} \le M_0 < \infty. \tag{4.1}$$

First, we state elementary energy estimates for the coupled system without proofs. The estimate on the $L^\infty(0,T;L^p)$ norm of the density could be deduced from $(1.1)_2$.

LEMMA 4.1. *Under the conditions* (2.2) *and* (4.1)*, it holds that for any $T\in(0,T^*)$,*

$$\sup_{0\le t\le T} \|\rho\|_{L^1\cap L^\infty} \le C. \tag{4.2}$$

LEMMA 4.2. *Suppose that initial data* $[f_0,\rho_0,u_0]$ *satisfy the conditions* (2.2), *let* $[f,\rho,u]$ *be a smooth solution to system* (1.1)-(1.2) *in* $[0,T)$ *and* $T\in(0,T^*)$, *we have*

$$(i)\ \left(\int_{\mathbb{R}^4}|v|^2 f\mathrm{d}v\mathrm{d}x+\int_{\mathbb{R}^2}\rho u^2\mathrm{d}x+\int_{\mathbb{R}^2}(\rho^\gamma+\rho)\mathrm{d}x\right)(t)$$
$$+\int_{[0,t]\times\mathbb{R}^2}(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2)\mathrm{d}x\mathrm{d}\tau+\int_{[0,t]\times\mathbb{R}^4}|u-v|^2 f\mathrm{d}v\mathrm{d}x\mathrm{d}\tau\le C.$$

$$(ii)\ \|f\|_{L^\infty(0,T;L^p(\mathbb{R}^4))}+\||v-v_c|\nabla_v(f^{\frac{p}{2}})\|^{\frac{2}{p}}_{L^2(0,T;L^2(\mathbb{R}^4))}\le C,\quad 1\le p<\infty.$$

$$(iii)\ \|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^4))}\le C,\quad \inf_{0\le t\le T}\int_{B_{N_1}}\rho\mathrm{d}x\ge\frac{1}{4}\int_{\mathbb{R}^2}\rho_0\mathrm{d}x,$$

$$(iv)\ \sup_{0\le t\le T}\int_{\mathbb{R}^4}f\mathrm{d}v\mathrm{d}x\le C,\quad \sup_{0\le t\le T}\int_{\mathbb{R}^4}|x|^2 f\mathrm{d}v\mathrm{d}x\le C,\quad \sup_{0\le t\le T}|v_c|\le C.$$

for some constant $N_1>0$ *depending on* T *and and* $B_{N_1}:=\{x\in\mathbb{R}^2\big|\ |x|<N_1\}$,

Proof. For (i), we multiply $(1.1)_1$ by the $\frac{v^2}{2}$ and integrate the resulting equation with respect to x, v over $\mathbb{R}^4$ to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^4}v^2 f\mathrm{d}v\mathrm{d}x=&\int_{\mathbb{R}^8}\psi(x-y)v\cdot(v_*-v)f(t,y,v_*)f(t,x,v)\mathrm{d}v_*\mathrm{d}v\mathrm{d}y\mathrm{d}x\\&+\int_{\mathbb{R}^4}(u-v)\cdot vf\mathrm{d}v\mathrm{d}x+2\int_{\mathbb{R}^4}|v-v_c|^2 f\mathrm{d}v\mathrm{d}x\\\le&\int_{\mathbb{R}^4}(u-v)\cdot vf\mathrm{d}v\mathrm{d}x+2\int_{\mathbb{R}^4}|v-v_c|^2 f\mathrm{d}v\mathrm{d}x\\\le&C\int_{\mathbb{R}^4}v^2 f(t,x,v)\mathrm{d}v\mathrm{d}x+\int_{\mathbb{R}^4}(u-v)\cdot vf\mathrm{d}v\mathrm{d}x,\end{aligned}\tag{4.3}$$

where we use that the term $\int_{\mathbb{R}^8}\psi(x-y)v\cdot(v_*-v)f(t,y,v_*)f(t,x,v)\mathrm{d}v_*\mathrm{d}v\mathrm{d}y\mathrm{d}x$ has negative sign due to anti-symmetry.

Then, multiply $(1.1)_2$ by the $\frac{\gamma}{\gamma-1}\rho^{\gamma-1}$ and $(1.1)_3$ by the u, summing the resulting equation, we use the integration by parts to have

$$\begin{aligned}&\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^\gamma\right)_t+\mathrm{div}\left(\frac{1}{2}\rho uu^2+\frac{\gamma}{\gamma-1}\rho^\gamma u\right)\\=&\mathrm{div}\left[\frac{1}{2}\mu\nabla u^2+(\lambda+\mu)\mathrm{div}uu\right]-\mu|\nabla u|^2-(\mu+\lambda)(\mathrm{div}u)^2+\int_{\mathbb{R}^2}u\cdot(v-u)f\mathrm{d}v.\end{aligned}\tag{4.4}$$

Integrate the above equation with respect to x over $\mathbb{R}^2$ to obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2}\left(\frac{1}{2}\rho u^2+\frac{1}{\gamma-1}\rho^\gamma\right)\mathrm{d}x+\int_{\mathbb{R}^2}(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2)\mathrm{d}x=\int_{\mathbb{R}^4}u\cdot(v-u)f\mathrm{d}v\mathrm{d}x.\tag{4.5}$$

Combining (4.3) with (4.5), we get

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^4}v^2 f\mathrm{d}v\mathrm{d}x+\int_{\mathbb{R}^2}\frac{1}{2}\rho u^2\mathrm{d}x+\int_{\mathbb{R}^2}\frac{1}{\gamma-1}\rho^\gamma\mathrm{d}x\right)+\int_{\mathbb{R}^2}(\mu|\nabla u|^2+(\mu+\lambda)(\mathrm{div}u)^2)\mathrm{d}x\\&+\int_{\mathbb{R}^4}|u-v|^2 f\mathrm{d}v\mathrm{d}x\le C\int_{\mathbb{R}^4}v^2 f(t,x,v)\mathrm{d}v\mathrm{d}x.\end{aligned}\tag{4.6}$$

We use Gronwall's inequality and (4.2) to have (i).

For (ii), we multiply $(1.1)_1$ by f^{p-1}, integrate the resulting equation with respect to x, v and use integration by parts to have

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\|f\|_{L^p}^p=&\int_{\mathbb{R}^8}\nabla_v(f^{p-1}(t,x,v))\cdot(v_*-v)\psi(x-y)f(t,y,v_*)f(t,x,v)\mathrm{d}v_*\mathrm{d}v\mathrm{d}y\mathrm{d}x\\&+\int_{\mathbb{R}^4}\nabla_v(f^{p-1})\cdot(u-v)f\mathrm{d}v\mathrm{d}x\\&-\int_{\mathbb{R}^4}\nabla_v(f^{p-1})\cdot(2(v-v_c)f+|v-v_c|^2\nabla f)\mathrm{d}v\mathrm{d}x\\=&\frac{2(p-1)}{p}\int_{\mathbb{R}^8}\psi(x-y)f(t,y,v_*)f^p(t,x,v)\mathrm{d}v_*\mathrm{d}v\mathrm{d}y\mathrm{d}x\\&+\frac{6(p-1)}{p}\|f\|_{L^p}^p-(p-1)\int_{\mathbb{R}^4}|v-v_c|^2f^{p-2}|\nabla f|^2\mathrm{d}v\mathrm{d}x\\\le&C\frac{p-1}{p}\|f\|_{L^p}^p-(p-1)\int_{\mathbb{R}^4}|v-v_c|^2f^{p-2}|\nabla f|^2\mathrm{d}v\mathrm{d}x.\end{aligned}\tag{4.7}$$

We use Gronwall's inequality to get (ii).

For (iii), according to (4.7), we let $p\to\infty$ and use the relation $\lim_{p\to\infty}\frac{p-1}{p}=1$ to have $\|f\|_{L^\infty(0,T;L^\infty(\mathbb{R}^4))}\le C$.

For N, we construct a smooth function ϕ_N such that $0\le\phi_N\le1$, $|\nabla\phi_N|\le\frac{2}{N}$, and for $|x|\le N$, $\phi_N=1$;for $|x|\ge2N$, $\phi_N=0$.

The mass conservation equation $(1.1)_2$ yields that

$$\int_{\mathbb{R}^2}\rho\mathrm{d}x=\int_{\mathbb{R}^2}\rho_0\mathrm{d}x.$$

There exists a N_0, such that $\int_{B_{N_0}}\rho_0\mathrm{d}x\ge\frac{1}{2}\int_{\mathbb{R}^2}\rho_0\mathrm{d}x$.

It follows from the above equality and $(1.1)_2$ that

$$\frac{d}{dt}\int_{\mathbb{R}^2}\rho\phi_N\mathrm{d}x=\int_{\mathbb{R}^2}\rho u\cdot\nabla\phi_N\mathrm{d}x\ge-\frac{2}{N}\left(\int_{\mathbb{R}^2}\rho\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\rho u^2\mathrm{d}x\right)^{\frac{1}{2}}\ge-\frac{2C}{N}\left(\int_{\mathbb{R}^2}\rho_0\mathrm{d}x\right)^{\frac{1}{2}},$$

which gives

$$\inf_{0\le t\le T}\int_{\mathbb{R}^2}\rho\phi_N\mathrm{d}x\ge\int_{\mathbb{R}^2}\rho_0\phi_N\mathrm{d}x-\frac{2CT}{N}\left(\int_{\mathbb{R}^2}\rho_0\mathrm{d}x\right)^{\frac{1}{2}}.\tag{4.8}$$

Immediately, for $N_1=N_0+\frac{16CT}{(\int_{\mathbb{R}^2}\rho_0\mathrm{d}x)^{\frac{1}{2}}}$, we use (4.8) to have

$$\inf_{0\le t\le T}\int_{B_{N_1}}\rho\mathrm{d}x\ge\inf_{0\le t\le T}\int_{\mathbb{R}^2}\rho\phi_{\frac{N_1}{2}}\mathrm{d}x\ge\int_{\mathbb{R}^2}\rho_0\phi_{\frac{N_1}{2}}\mathrm{d}x-\frac{4CT}{N_1}\left(\int_{\mathbb{R}^2}\rho_0\mathrm{d}x\right)^{\frac{1}{2}}\ge\frac{1}{4}\int_{\mathbb{R}^2}\rho_0\mathrm{d}x.$$

For (iv), integrating $(1.1)_1$ with respect to x, v over $\mathbb{R}^4$, according to the initial data (2.2), we directly obtain that $\int_{\mathbb{R}^4}f\mathrm{d}v\mathrm{d}x=\int_{\mathbb{R}^4}f_0\mathrm{d}v\mathrm{d}x\le C$.

Multiplying $(1.1)_1$ by $\frac{x^2}{2}$ and integrating the resulting equation with respect to x, v over $\mathbb{R}^4$ to get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^4}x^2f\mathrm{d}v\mathrm{d}x=\int_{\mathbb{R}^4}x\cdot vf\mathrm{d}v\mathrm{d}x\le\frac{1}{2}\left(\int_{\mathbb{R}^4}x^2f\mathrm{d}v\mathrm{d}x+\int_{\mathbb{R}^4}v^2f\mathrm{d}v\mathrm{d}x\right)\tag{4.9}$$

We combine (ii) with (4.9) and use the Gronwall inequality to get $\int_{\mathbb{R}^4}|x|^2 f \mathrm{d}v\mathrm{d}x \le C$.
Using $\int_{\mathbb{R}^4} f \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^4} f_0 \mathrm{d}v\mathrm{d}x$ and $\int_{\mathbb{R}^4} v^2 f \mathrm{d}v\mathrm{d}x \le C$, we can get

$$|v_c|^2 \le \frac{1}{2}\left(\int_{\mathbb{R}^4} v^2 f \mathrm{d}v\mathrm{d}x + \int_{\mathbb{R}^4} f \mathrm{d}v\mathrm{d}x\right) \le C. \tag{4.10}$$

□

LEMMA 4.3. *Under the conditions* (2.2), *it holds that for any* $T \in (0,T^*)$,

$$\int_{\mathbb{R}^2} \rho \bar{x}^{\frac{5a}{4}} \mathrm{d}x \le C. \tag{4.11}$$

Proof. Multiplying $(1.1)_2$ by $\bar{x}^{\frac{5a}{4}}$ and integrating the resulting equation over $\mathbb{R}^2$, we obtain that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^2} \rho \bar{x}^{\frac{5a}{4}} \mathrm{d}x &\le C\int_{\mathbb{R}^2} \rho |u| \bar{x}^{\frac{5a}{4}-1} \log^{1+\sigma}(\mathrm{e}+|x|^2) \mathrm{d}x \\
&\le C\left(\int_{\mathbb{R}^2} \rho \bar{x}^{\frac{5a}{2}-2} \log^{2(1+\sigma)}(\mathrm{e}+|x|^2)\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} \rho u^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C\left(\int_{\mathbb{R}^2} \rho \bar{x}^{\frac{5a}{4}} \mathrm{d}x\right)^{\frac{1}{2}},
\end{aligned} \tag{4.12}$$

where in the last inequality we have used $1 < a < \frac{8}{5}$ and (ii) in Lemma 4.2.
We use Gronwall's inequality to derive (4.11). □

Next, we show some momentum (velocity) estimates for the kinetic part f. For this, we set

$$m_k f(t,x) := \int_{\mathbb{R}^2} |v|^k f(t,x,v) \mathrm{d}v.$$

LEMMA 4.4. *Under the same settings as in Lemma 4.2, we have*

$$(i)\ m_{k_1} f(t,x) \le C(1+\|f\|_{L^\infty_{t,x,v}})(m_{k_2} f(t,x))^{\frac{k_1+2}{k_2+2}}, \quad \forall k_2 > k_1 \ge 0,$$

$$(ii)\ \sup_{0\le t\le T}\int_{\mathbb{R}^4}(1+|v|^k) f \mathrm{d}v\mathrm{d}x \le C\int_{\mathbb{R}^4}(1+|v|^k) f_0 \mathrm{d}v\mathrm{d}x + C,$$

where $k \ge 1$ *is a positive constant.*

Proof. (i) Note that for $R > 0$,

$$\int_{\mathbb{R}^2} |v|^{k_1} f \mathrm{d}v = \int_{|v|\le R} |v|^{k_1} f \mathrm{d}v + \int_{|v|>R} |v|^{k_1} f \mathrm{d}v \lesssim \|f\|_{L^\infty_{t,x,v}} R^{k_1+2} + \frac{1}{R^{k_2-k_1}}\int_{\mathbb{R}^2} |v|^{k_2} f \mathrm{d}v.$$

We now choose $R = (\int_{\mathbb{R}^2} |v|^{k_2} f \mathrm{d}v)^{\frac{1}{k_2+2}}$ in the above relation to obtain

$$\int_{\mathbb{R}^2} |v|^{k_1} f \mathrm{d}v \lesssim (\|f\|_{L^\infty_{t,x,v}}+1)\left(\int_{\mathbb{R}^2} |v|^{k_2} f \mathrm{d}v\right)^{\frac{k_1+2}{k_2+2}}.$$

(ii) We multiply $(1.1)_1$ by $(1+|v|^k)$ to have

$$\frac{d}{dt}\int_{\mathbb{R}^4}(1+|v|^k) f \mathrm{d}v\mathrm{d}x$$

$$
\begin{aligned}
&=-\int_{\mathbb{R}^4}|v|^k\nabla_v\cdot[fL[f]+(u-v)f+\nabla_v(|v-v_c|^2f)]\mathrm{d}v\mathrm{d}x\\
&=\int_{\mathbb{R}^4}v\cdot(v_*-v)\psi(x-y)k|v|^{k-2}f(y,v_*,t)f(x,v,t)\mathrm{d}v_*\mathrm{d}y\mathrm{d}v\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^4}k|v|^{k-2}v\cdot(u-v)f\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^4}k|v|^{k-2}v\cdot\nabla_v(|v-v_c|^2f)\mathrm{d}v\mathrm{d}x\\
&:=\mathcal{I}_{11}+\mathcal{I}_{12}+\mathcal{I}_{13}.
\end{aligned}\tag{4.13}
$$

Below, we estimate the terms $\mathcal{I}_{1i}(1\le i\le 3)$, separately.

• (Estimate of $\mathcal{I}_{11}$): We use the Hölder inequality and Lemma 4.2 to obtain

$$
\begin{aligned}
\mathcal{I}_{11}&\le C\int_{\mathbb{R}^4}|v|^{k-1}f\mathrm{d}v\mathrm{d}x\int_{\mathbb{R}^4}|v_*|f\mathrm{d}v_*\mathrm{d}y\\
&\le C\left(\int_{\mathbb{R}^4}|v|^kf\mathrm{d}v\mathrm{d}x\right)^{\frac{k-1}{k}}\left(\int_{\mathbb{R}^4}f\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{k}}\int_{\mathbb{R}^4}(v_*^2+1)f\mathrm{d}v_*\mathrm{d}y\\
&\le C\left(\int_{\mathbb{R}^4}(1+|v|^k)f\mathrm{d}v\mathrm{d}x\right)^{\frac{2k+2}{2k+3}}.
\end{aligned}
$$

• (Estimate of $\mathcal{I}_{12}$): Again we apply the Hölder inequality and the result (i) in Lemma 4.4 to obtain

$$
\begin{aligned}
\mathcal{I}_{12}\le&-\int_{\mathbb{R}^4}k|v|^kf\mathrm{d}v\mathrm{d}x+C\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\bar{x}^{\frac{1}{k+1}}|v|^{k-1}f\mathrm{d}v\right)^{\frac{2k+3}{2(k+1)}}\mathrm{d}x\right)^{\frac{2k+2}{2k+3}}\|u\bar{x}^{-\frac{1}{k+1}}\|_{2k+3}\\
\le&C\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}v^{\frac{k(k-1)(2k+3)}{2k^2+2k-1}}f\mathrm{d}v\right)^{\frac{2k^2+2k-1}{2k(k+1)}}\left(\int_{\mathbb{R}^2}\bar{x}^{\frac{k(2k+3)}{(k+1)^2}}f\mathrm{d}v\right)^{\frac{1}{2k}}\mathrm{d}x\right)^{\frac{2k+2}{2k+3}}\\
&\times\|u\bar{x}^{-\frac{1}{k+1}}\|_{2k+3}\\
\le&C\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}v^{\frac{k(k-1)(2k+3)}{2k^2+2k-1}}f\mathrm{d}v\right)^{\frac{2k^2+2k-1}{(2k-1)(k+1)}}\mathrm{d}x+\int_{\mathbb{R}^4}\bar{x}^{\frac{k(2k+3)}{(k+1)^2}}f\mathrm{d}v\mathrm{d}x\right)^{\frac{2k+2}{2k+3}}\\
&\times\|u\bar{x}^{-\frac{1}{k+1}}\|_{2k+3}\\
\le&C\left(\left(\int_{\mathbb{R}^4}v^kf\mathrm{d}v\mathrm{d}x\right)^{\frac{2k+2}{2k+3}}+1\right)(\|\sqrt{\rho}u\|_2+\|\nabla u\|_2),
\end{aligned}\tag{4.14}
$$

where in the last inequality we have used $\frac{k(2k+3)}{(k+1)^2}<2$, (i) in Lemma 4.2, (4.11) and (3.3).

• (Estimate of $\mathcal{I}_{13}$): We use integration by parts and the estimate $|v_c|\le C$ to get

$$
\mathcal{I}_{13}=\int_{\mathbb{R}^4}k^2|v|^{k-2}(v^2-2v\cdot v_c+v_c^2)f\mathrm{d}v\mathrm{d}x\lesssim C\int_{\mathbb{R}^4}(1+|v|^k)f\mathrm{d}v\mathrm{d}x.
$$

In (4.13), we collect estimates $\mathcal{I}_{11}$, $\mathcal{I}_{12}$, $\mathcal{I}_{13}$ to find

$$
\frac{d}{dt}\left(\int_{\mathbb{R}^4}(1+|v|^k)f\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2k+3}}\lesssim C\left(\int_{\mathbb{R}^4}(1+|v|^k)f\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{2k+3}}+C(\|\sqrt{\rho}u\|_2+\|\nabla u\|_2).
$$

Finally, we integrate the above inequality over $[0,t]$ and use (i) in Lemma 4.2 and Gronwall's inequality to derive the desired estimate. ∎

COROLLARY 4.1. *Under the conditions in Lemma 4.2, we have*

$$\begin{aligned}&(i)\ \sup_{0\le t\le T}\int_{\mathbb{R}^4}(1+|v|^k)f\mathrm{d}v\mathrm{d}x\le C,\quad 2\le k<\infty.\\&(ii)\ \sup_{0\le t\le T}\|\rho_f\|_p^p\le C,\quad \sup_{0\le t\le T}\|\rho_f u_f\|_p^p\le C,\quad \sup_{0\le t\le T}\|m_2 f\|_p^p\le C,\quad p\ge 1.\end{aligned}\tag{4.15}$$

Proof. The estimates in (4.15) are easy consequences of Lemma 4.2 and Lemma 4.4. For brevity, we omit the details. ∎

LEMMA 4.5. *Under the condition* (4.1) *and the conditions in Lemma 4.2, we have that for* $T\in(0,T^*)$,

$$\sup_{0\le t\le T}\left(\|\nabla u\|_2^2+\int_{\mathbb{R}^4}(u-v)^2 f\mathrm{d}v\mathrm{d}x\right)+\int_0^T\|\sqrt{\rho}\dot{u}\|_2^2\mathrm{d}t\le C.\tag{4.16}$$

Proof. Multiplying $(1.1)_2$ by $\dot{u}$ and integrating the resulting equation over $\mathbb{R}^2$ give that

$$\begin{aligned}&\int_{\mathbb{R}^2}\rho|\dot{u}|^2\mathrm{d}x+\frac{d}{dt}\int_{\mathbb{R}^4}\left(\frac{1}{2}u^2 f-u\cdot vf\right)\mathrm{d}v\mathrm{d}x\\&=-\int_{\mathbb{R}^2}\dot{u}\cdot\nabla P\mathrm{d}x+\mu\int_{\mathbb{R}^2}\dot{u}\cdot\Delta u\mathrm{d}x+(\lambda+\mu)\int_{\mathbb{R}^2}\dot{u}\cdot\nabla\mathrm{div}u\mathrm{d}x\\&\quad+\int_{\mathbb{R}^4}\left(\frac{1}{2}u^2 f_t-u\cdot vf_t\right)\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^2}u\cdot\nabla u(u-u_f)\rho_f\mathrm{d}x.\end{aligned}\tag{4.17}$$

Moreover, multiplying $(1.1)_1$ by $\frac{v^2}{2}$ and integrating the resulting equation over $\mathbb{R}^2\times\mathbb{R}^2$ give that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^4}v^2 f\mathrm{d}v\mathrm{d}x=\int_{\mathbb{R}^4}v\cdot(L[f]+u-v)f\mathrm{d}v\mathrm{d}x+2\int_{\mathbb{R}^4}|v-v_c|^2 f\mathrm{d}v\mathrm{d}x.\tag{4.18}$$

We combine (4.17) and (4.18) to have

$$\begin{aligned}&\int_{\mathbb{R}^2}\rho|\dot{u}|^2\mathrm{d}x+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^4}(u-v)^2 f\mathrm{d}v\mathrm{d}x\\&=-\int_{\mathbb{R}^2}\dot{u}\cdot\nabla P\mathrm{d}x+\mu\int_{\mathbb{R}^2}\dot{u}\cdot\Delta u\mathrm{d}x+(\lambda+\mu)\int_{\mathbb{R}^2}\dot{u}\cdot\nabla\mathrm{div}u\mathrm{d}x\\&\quad+\int_{\mathbb{R}^4}\left(\frac{1}{2}u^2 f_t-u\cdot vf_t\right)\mathrm{d}v\mathrm{d}x-\int_{\mathbb{R}^2}u\cdot\nabla u(u-u_f)\rho_f\mathrm{d}x\\&\quad+\int_{\mathbb{R}^4}[v\cdot(L[f]+u-v)f+2|v-v_c|^2 f]\mathrm{d}v\mathrm{d}x\\&:=\sum_{i=1}^{6}\mathcal{I}_{2i}.\end{aligned}\tag{4.19}$$

Now, we estimate the terms $\mathcal{I}_{2i}$, $i=1,\cdots 6$, separately.

• (Estimate of $\mathcal{I}_{21}$): Integrating by parts, we derive from (4.1) that

$$\begin{aligned}\mathcal{I}_{21} &= \int_{\mathbb{R}^2} [(\mathrm{div}u)_t P - (u\cdot\nabla u)\cdot\nabla P]\mathrm{d}x \\ &= \frac{d}{dt}\int_{\mathbb{R}^2} (\mathrm{div}u)P\mathrm{d}x + \int_{\mathbb{R}^2} [(\gamma-1)(\mathrm{div}u)^2 P + \partial_j u_i \partial_i u_j P]\mathrm{d}x \\ &\leq \frac{d}{dt}\int_{\mathbb{R}^2} (\mathrm{div}u)P\mathrm{d}x + C\|\nabla u\|_2^2 + \|\nabla u\|_3^3).\end{aligned}$$

• (Estimate of $\mathcal{I}_{22}$): Integrating by parts, we can get that

$$\begin{aligned}\mathcal{I}_{22} &= \mu\int_{\mathbb{R}^2} (u_t + u\cdot\nabla u)\cdot\Delta u\mathrm{d}x \\ &= -\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \frac{\mu}{2}\int_{\mathbb{R}^2} \partial_i u_i \partial_k u_j \partial_k u_j \mathrm{d}x - \mu\int_{\mathbb{R}^2} \partial_k u_i \partial_i u_j \partial_k u_j \mathrm{d}x \\ &\leq -\frac{\mu}{2}\frac{d}{dt}\|\nabla u\|_2^2 + C\|\nabla u\|_3^3.\end{aligned}$$

• (Estimate of $\mathcal{I}_{23}$): Similar to $\mathcal{I}_{22}$, we have that

$$\mathcal{I}_{23} \quad \leq -\frac{\lambda+\mu}{2}\frac{d}{dt}\|\mathrm{div}u\|_2^2 + C\|\nabla u\|_3^3.$$

• (Estimate of $\mathcal{I}_{24}$): We have from $(1.1)_1$ that

$$\int_{\mathbb{R}^2} f_t \mathrm{d}v = -\int_{\mathbb{R}^2} v\cdot\nabla_x f\mathrm{d}v, \quad \int_{\mathbb{R}^2} v f_t \mathrm{d}v = -\int_{\mathbb{R}^2} vv\cdot\nabla_x f\mathrm{d}v + \int_{\mathbb{R}^2} (L[f]+u-v)f\mathrm{d}v. \tag{4.20}$$

Applying the integration by parts on space variables, we have that

$$\begin{aligned}|\mathcal{I}_{24}| &= \left|\int_{\mathbb{R}^4} u\cdot\nabla u\cdot vf\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^4} v\cdot\nabla u\cdot vf\mathrm{d}v\mathrm{d}x - \int_{\mathbb{R}^4} u\cdot(L[f]+u-v)f\mathrm{d}v\mathrm{d}x\right| \\ &\leq C\Big(\|\nabla u\|_2\|uu_f\rho_f\|_2 + \|\nabla u\|_2\|m_2 f\|_2 + \int_{\mathbb{R}^4} f(1+|u|^2+|v|^2)\mathrm{d}v\mathrm{d}x\Big).\end{aligned}$$

For zero-order coupled term for fluid velocity $u(t,x)$ and density distribution function $f(t,x,v)$, we only consider the term $\|uu_f\rho_f\|_2$ for simplicity. Other terms in the estimate of $\mathcal{I}_{24}$ have similar estimates.

We use Hölder's inequality, (3.3), (4.11), Corollary 4.1 and Lemma 4.2 to have that

$$\begin{aligned}\|uu_f\rho_f\|_2 &\leq C\|u\bar{x}^{-\eta}\|_{\frac{2p}{p-2}}\|\bar{x}^{\eta}u_f\rho_f\|_p \\ &\leq C\|u\bar{x}^{-\eta}\|_{\frac{2p}{p-2}}\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\bar{x}^{\eta p_1}f\mathrm{d}v\right)^{\frac{p}{p_1}}\left(\int_{\mathbb{R}^2}v^{\frac{p_1}{p_1-1}}f\mathrm{d}v\right)^{\frac{p(p_1-1)}{p_1}}\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq C\|u\bar{x}^{-\eta}\|_{\frac{2p}{p-2}}\left(\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2}\bar{x}^{\eta p_1}f\mathrm{d}v + \left(\int_{\mathbb{R}^2}v^{\frac{p_1}{p_1-1}}f\mathrm{d}v\right)^{\frac{p(p_1-1)}{p_1-p}}\right)\mathrm{d}x\right)^{\frac{1}{p}} \\ &\leq C\|u\bar{x}^{-\eta}\|_{\frac{2p}{p-2}}\left(\int_{\mathbb{R}^4}\left(\bar{x}^{\eta p_1}f + v^{\frac{p_1(3p-2)}{p_1-p}}f\right)\mathrm{d}v\mathrm{d}x\right)^{\frac{1}{p}}\end{aligned}$$

$$\le C(1+\|\nabla u\|_2), \tag{4.21}$$

when $p_1>1$, $p>2$ and $0<\eta<1$ satisfy that

$$\frac{p\eta}{p-2}>1, \quad \eta p_1<2, \quad p_1>p, \quad 3pp_1-2p-3p_1+2\ge 0.$$

Using the above inequality, we have the following further estimate for $\mathcal{I}_{24}$.

$$|\mathcal{I}_{24}|\le C\|\nabla u\|_2^2+C.$$

• (Estimate of $\mathcal{I}_{25}$ and $\mathcal{I}_{26}$) Similar to estimate of $\mathcal{I}_{24}$, we omit the details and give the estimate directly,

$$|\mathcal{I}_{25}+\mathcal{I}_{26}|\le C\|\nabla u\|_2^2+C.$$

Similar to (4.21), according to Lemma 4.1 and (3.7), it holds that

$$\|\nabla u\|_3^3\le \frac{1}{2}\|\sqrt{\rho}\dot{u}\|_2^2+C(1+\|\nabla u\|_2^4).$$

Collecting the estimates $\mathcal{I}_{2i}$, $i=1,\ldots,6$, together, we have

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^4}(u-v)^2 f\mathrm{d}v\mathrm{d}x+\frac{\mu}{2}\|\nabla u\|_2^2+\frac{\lambda+\mu}{2}\|\mathrm{div}u\|_2^2-\int_{\mathbb{R}^2}\mathrm{div}uP\mathrm{d}x\right)+\int_{\mathbb{R}^2}\rho|\dot{u}|^2\mathrm{d}x\\ &\le C\|\nabla u\|_2^2+C\|\nabla u\|_3^3+C\\ &\le \frac{1}{2}\|\sqrt{\rho}\dot{u}\|_2^2+C(1+\|\nabla u\|_2^2)\|\nabla u\|_2^2+C,\end{aligned}$$

where

$$\frac{\mu}{2}\|\nabla u\|_2^2-C\le \frac{\mu}{2}\|\nabla u\|_2^2+\frac{\lambda+\mu}{2}\|\mathrm{div}u\|_2^2-\int_{\mathbb{R}^2}\mathrm{div}uP\mathrm{d}x\le \mu\|\nabla u\|_2^2+C$$

due to Lemma 4.1.
We use (*ii*) in Lemma 4.2 and Gronwall's inequality to complete the proof. □

Lemma 4.6. *Suppose that the condition* (4.1) *and the conditions in Lemma 4.2 hold. Then, for* $T\in(0,T^*)$, *we have*

$$\sup_{0\le t\le T}\|\sqrt{\rho}\dot{u}\|_2^2(t)+\int_0^T(\|\nabla\dot{u}\|_2^2+\|\sqrt{\rho_f}\dot{u}\|_2^2)\mathrm{d}t\le C. \tag{4.22}$$

Proof. Note that

$$\dot{u}_j[\partial_t(\rho\dot{u}_j)+\mathrm{div}(u\rho\dot{u}_j)]=\frac{1}{2}\rho\partial_t(\dot{u}_j)^2+\frac{1}{2}\rho u\cdot\nabla(\dot{u}_j)^2.$$

Then, we apply the operator $\dot{u}_j[\partial_t+\mathrm{div}(u\cdot)]$ to $(1.1)_{3,j}$ and use (4.20) to have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}\rho|\dot{u}|^2\mathrm{d}x+2\int_{\mathbb{R}^2}\rho_f|\dot{u}|^2\mathrm{d}x=&-2\int_{\mathbb{R}^2}\dot{u}_j[\partial_{jt}P(\rho)+\mathrm{div}(u\partial_jP(\rho))]\mathrm{d}x\\ &+2\mu\int_{\mathbb{R}^2}\dot{u}_j[\partial_t\Delta u_j+\mathrm{div}(u\Delta u_j)]\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+2\int_{\mathbb{R}^2}\dot u_j[\partial_{jt}((\mu+\lambda)\mathrm{div}u)+\mathrm{div}(u\partial_j((\mu+\lambda)\mathrm{div}u))]\mathrm{d}x\\
&+2\int_{\mathbb{R}^2}\dot u_j[u\cdot\nabla u_j\rho_f+u_j\mathrm{div}(u_f\rho_f)-\mathrm{div}(u(u_j\rho_f))]\mathrm{d}x\\
&+2\int_{\mathbb{R}^2}\dot u_j[\int_{\mathbb{R}^2}((L_j[f]+u_j-v_j)f-v_j(v\cdot\nabla f))\mathrm{d}v+\mathrm{div}(u(u_f\rho_f)_j)]\mathrm{d}x\\
:=&\sum_{i=1}^5\mathcal{I}_{3i}, \qquad (4.23)
\end{aligned}
$$

•(Estimate of $\sum_{i=1}^3\mathcal{I}_{3i}$): Similar to the corresponding estimates in Lemma 3.9 [16], we have

$$
\sum_{i=1}^3\mathcal{I}_{3i}\le-\frac{3\mu}{2}\|\nabla\dot u\|_2^2-\frac{3\mu}{2}\|\partial_t\mathrm{div}u+u\cdot\nabla\mathrm{div}u\|_2^2+C(1+\|\nabla u\|_4^4).
$$

•(Estimate of $\mathcal{I}_{34}$ and $\mathcal{I}_{35}$): We apply (3.3) on $\dot u$, use the integration by parts, Corollary 4.1 and (4.21)-type estimate to have

$$
\begin{aligned}
\mathcal{I}_{34}&\le C(|\dot u||u||\nabla u||\rho_f|+|\nabla\dot u||u||u_f\rho_f|+|\dot u||\nabla u||u_f\rho_f|+|\nabla\dot u||u|^2|\rho_f|)\\
&\le C(\|\sqrt{\rho}\dot u\|_2+\|\nabla\dot u\|_2)\|\nabla u\|_2(1+\|\nabla u\|_2)+C\|\nabla\dot u\|_2(\|\nabla u\|_2^2+1)\\
&\le\varepsilon\|\nabla\dot u\|_2^2+C(\|\sqrt{\rho}\dot u\|_2^2+1),
\end{aligned}
$$

$\varepsilon>0$ is a small positive constant.
Similarly, we have

$$
\begin{aligned}
\mathcal{I}_{35}&\le C|\nabla\dot u||u||u_f\rho_f|+C|\nabla\dot u||m_2f|+C|\dot u||u||\rho_f|+C|\dot u||u_f\rho_f|\\
&\le\varepsilon\|\nabla\dot u\|_2^2+C(\|\sqrt{\rho}\dot u\|_2^2+1).
\end{aligned}
$$

We collect all estimates of $\mathcal{I}_{3i}$ in (4.23) to obtain

$$
\frac{d}{dt}\|\sqrt{\rho}\dot u\|_2^2+\|\nabla\dot u\|_2^2+\|\sqrt{\rho_f}\dot u\|_2^2\le\varepsilon\|\nabla\dot u\|_2^2+C(\|\sqrt{\rho}\dot u\|_2^2+\|\nabla u\|_4^4+1).
$$

Note that

$$
\begin{aligned}
\|\nabla u\|_4^4&\le C(\|\omega\|_4^4+\|\mathrm{div}u\|_4^4)\le C(\|F\|_4^4+\|\omega\|_4^4+\|P(\rho)\|_4^4)\\
&\le C(\|\nabla F\|_2^2+\|\nabla\omega\|_2^2+1)\\
&\le C\|\sqrt{\rho}\dot u\|_2^2+C,
\end{aligned}
$$

by Lemma 3.5, (4.2), (4.16) and (4.21)-type estimate.
We can apply Gronwall's inequality to further obtain

$$
\|\sqrt{\rho}\dot u\|_2^2(t)+\int_0^t(\|\nabla\dot u\|_2^2+\|\sqrt{\rho_f}\dot u\|_2^2)\mathrm{d}\tau\le C. \qquad (4.24)
$$

□

LEMMA 4.7. *Under the conditions listed in* (2.2) *and the index q being same as in Theorem* 2.1, *for* $T\in(0,T^*)$, *it holds that*

$$
\sup_{0\le t\le T}(\|\rho\|_{H^1\cap W^{1,q}}+\|\nabla u\|_{H^1})(t)+\int_0^T\|\nabla^2u\|_q^2\mathrm{d}t\le C. \qquad (4.25)
$$

Proof. We apply the operator ∇ to $(1.1)_2$, multiply by $q|\nabla\rho|^{q-2}\nabla\rho$, and then integrate the resulting equation over $\mathbb{R}^2$ to have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla\rho|^q\mathrm{d}x=&-(q-1)\int_{\mathbb{R}^2}|\nabla\rho|^q\mathrm{divud}x\\&-q\int_{\mathbb{R}^2}|\nabla\rho|^{q-2}\nabla\rho\cdot\nabla u\cdot\nabla\rho\mathrm{d}x-q\int_{\mathbb{R}^2}|\nabla\rho|^{q-2}\rho\nabla\rho\cdot\nabla(\mathrm{divu})\mathrm{d}x.\end{aligned}$$

It yields that

$$\frac{d}{dt}\|\nabla\rho\|_q\le C(\|\nabla u\|_\infty\|\nabla\rho\|_q+\|\nabla^2u\|_q). \tag{4.26}$$

We use Sobolev's inequality, Lemma 3.5, Corollary 4.1 and (4.21)-type estimate to have

$$\begin{aligned}\|\mathrm{div}u\|_\infty+\|\omega\|_\infty&\le C(\|F\|_\infty+\|P(\rho)\|_\infty+\|\omega\|_\infty)\\&\le C(\|\nabla F\|_q^{\frac{q}{2(q-1)}}+\|\nabla\omega\|_q^{\frac{q}{2(q-1)}}+1)\\&\le C(\|\rho\dot{u}\|_q^{\frac{q}{2(q-1)}}+\|(u-u_f)\rho_f\|_q^{\frac{q}{2(q-1)}}+1)\\&\le C(\|\rho\dot{u}\|_q^{\frac{q}{2(q-1)}}+1).\end{aligned} \tag{4.27}$$

We deduce from the standard L^p-estimates for elliptic system, Lemma 4.1 and Lemma 3.5 that

$$\begin{aligned}\|\nabla^2u\|_q&\le C(\|\nabla\mathrm{div}u\|_q+\|\nabla\omega\|_q)\\&\le C(\|\nabla F\|_q+\|\nabla\omega\|_q+\|\nabla P(\rho)\|_q)\\&\le C(\|\rho\dot{u}\|_q+\|\nabla\rho\|_q)+C.\end{aligned} \tag{4.28}$$

By the Beale-Kato-Majda-type inequality, (4.27) and (4.28), it follows that

$$\begin{aligned}\|\nabla u\|_\infty&\le C(\|\mathrm{div}u\|_\infty+\|\omega\|_\infty)\log(e+\|\nabla^2u\|_q)+C\|\nabla u\|_2+C\\&\le C(\|\rho\dot{u}\|_q^{\frac{q}{2(q-1)}}+1)\log(e+\|\rho\dot{u}\|_q+\|\nabla\rho\|_q)+C.\end{aligned} \tag{4.29}$$

Moreover, from (3.3) and (4.22), it holds that

$$\begin{aligned}\|\rho\dot{u}\|_q&\le C\|\rho\dot{u}\|_2^{\frac{2(q-1)}{q^2-2}}\|\rho\dot{u}\|_{q^2}^{\frac{q(q-2)}{q^2-2}}\\&\le C\|\rho\dot{u}\|_2^{\frac{2(q-1)}{q^2-2}}(\|\sqrt{\rho}\dot{u}\|_2+\|\nabla\dot{u}\|_2)^{\frac{q(q-2)}{q^2-2}}\\&\le C(\|\nabla\dot{u}\|_2^{\frac{q(q-2)}{q^2-2}}+1).\end{aligned} \tag{4.30}$$

We combine (4.29) and (4.30) to have

$$\|\nabla u\|_\infty\le C(1+\|\nabla\dot{u}\|_2)\log(e+\|\nabla\dot{u}\|_2+\|\nabla\rho\|_q)+C.. \tag{4.31}$$

Then, substituting (4.31), (4.28) into (4.26), we use logarithmic Gronwall's inequality and (4.22) to deduce that

$$\sup_{0\le t\le T}\|\nabla\rho\|_q\le C, \tag{4.32}$$

which along with (4.29), shows

$$\int_0^T \|\nabla u\|_\infty^2 \mathrm{d}t \le C. \tag{4.33}$$

Finally, it follows from $(1.1)_2$ and Sobolev's inequality that

$$\frac{d}{dt}\|\nabla\rho\|_2 \le C(1+\|\nabla u\|_\infty)\|\nabla\rho\|_2 + C\|\nabla^2 u\|_2. \tag{4.34}$$

For $\|\nabla^2 u\|_2$, we use (3.5) and (4.32) to have that

$$\begin{aligned}\|\nabla^2 u\|_2 &\le C(\|\nabla\omega\|_2 + \|\nabla \mathrm{div} u\|_2)\\ &\le C(\|\nabla\omega\|_2 + \|\nabla F\|_2 + \|\nabla P(\rho)\|_2)\\ &\le \frac{1}{2}\|\nabla^2 u\|_2 + C(\|\rho\dot{u}\|_2 + \|\nabla\rho\|_2 + 1).\end{aligned} \tag{4.35}$$

We use (4.33), (4.34) and (4.35) to get that

$$\sup_{0\le t\le T}(\|\nabla\rho\|_2 + \|\nabla^2 u\|_2^2) \le C. \tag{4.36}$$

Using (4.28), (4.30) and (4.32) to show that

$$\int_0^T \|\nabla^2 u\|_q^2 \mathrm{d}t \le C. \tag{4.37}$$

□

With the bound of $\|\nabla^2 u\|_{L^2(0,T;L^q)}$, $q>2$, one can get the higher-order estimates of $f(t,x,v)$ as follows.

LEMMA 4.8. *Under the conditions listed in* (2.2) *and for* $4<p<\infty$ *and* $T\in(0,T^*)$ *we have*

$$\begin{aligned}&\sup_{0\le t\le T}(\|\langle v\rangle^k\nabla_x f\|_p^p + \|\langle v\rangle^k \nabla_v f\|_p^p)(t)\\ &+\int_0^T (\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_x f|^{\frac{p}{2}})\|_2^2 + \||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_v f|^{\frac{p}{2}})\|_2^2)dt \le C.\end{aligned} \tag{4.38}$$

Proof. We apply the operator ∂_{x_i} to $(1.1)_1$ to obtain

$$\begin{aligned}&\partial_t\partial_{x_i} f + v\cdot\nabla\partial_{x_i} f + \nabla_v\cdot(\int_{\mathbb{R}^4}\partial_{x_i}(\psi(x-y))(v_*-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}y f)\\ &\quad+\nabla_v\cdot(\int_{\mathbb{R}^4}(\psi(x-y))(v_*-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}y\partial_{x_i} f) + \nabla_v\cdot((u-v)\partial_{x_i} f) + \nabla_v\cdot(\partial_{x_i} u f)\\ &=\partial_{x_i}\Delta_v(|v-v_c|^2 f).\end{aligned}$$

We multiply the above equation by $\langle v\rangle^{kp}p|\partial_{x_i} f|^{p-2}\partial_{x_i} f$, and integrate the resulting equations with respect to x, v over $\mathbb{R}^4$ to give

$$\begin{aligned}&\frac{d}{dt}\|\langle v\rangle^k\partial_{x_i} f\|_p^p\\ &=-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i} f|^{p-2}\partial_{x_i} f\nabla_v\cdot(\int_{\mathbb{R}^4}\partial_{x_i}(\psi(x-y))(v_*-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}y f)\mathrm{d}v\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i}f|^{p-2}\partial_{x_i}f\nabla_v\cdot\Big(\int_{\mathbb{R}^4}(\psi(x-y))(v_*-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}y\partial_{x_i}f\Big)\mathrm{d}v\mathrm{d}x\\
&-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i}f|^{p-2}\partial_{x_i}f\nabla_v\cdot((u-v)\partial_{x_i}f)\mathrm{d}v\mathrm{d}x\\
&-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i}f|^{p-2}\partial_{x_i}f\nabla_v\cdot(\partial_{x_i}uf)\mathrm{d}v\mathrm{d}x\\
&+\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i}f|^{p-2}\partial_{x_i}f\partial_{x_i}\Delta_v(|v-v_c|^2f)\mathrm{d}v\mathrm{d}x\\
=:&\sum_{i=1}^{5}\mathcal{I}_{4i}.
\end{aligned}
\tag{4.39}
$$

Now we deal with $\mathcal{I}_{4i}$, $i=1,\cdots,5$, as below.

• (Estimate of $\mathcal{I}_{41}$): By direct calculation, we have

$$
\begin{aligned}
\mathcal{I}_{41}=&-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i}f|^{p-2}\partial_{x_i}f\nabla_v\cdot\Big(\int_{\mathbb{R}^4}\partial_{x_i}(\psi(x-y))(v_*-v_c)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}yf\Big)\mathrm{d}v\mathrm{d}x\\
&-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i}f|^{p-2}\partial_{x_i}f\nabla_v\cdot\Big(\int_{\mathbb{R}^4}\partial_{x_i}(\psi(x-y))(v_c-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}yf\Big)\mathrm{d}v\mathrm{d}x\\
:=&\mathcal{I}_{411}+\mathcal{I}_{412}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
|\mathcal{I}_{411}|\le&\sup_{x\in\mathbb{R}^2}|\nabla_x\psi|\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial_{x_i}f|^{p-1}|\partial_{v_j}f|\Big(\int_{\mathbb{R}^4}|v_{j*}-v_{c,j}|f(v_*,y,t)\mathrm{d}v_*\mathrm{d}y\Big)\mathrm{d}v\mathrm{d}x\\
\le&C(\|\langle v\rangle^k\partial_{x_i}f\|_p^p+\|\langle v\rangle^k\nabla_vf\|_p^p),\\
|\mathcal{I}_{412}|\le&p(p-1)\int_{\mathbb{R}^4}\langle v\rangle^{kp}f|\partial_{x_i}f|^{p-2}|\partial^2_{x_iv_j}f||v_j-v_{c,j}|\mathrm{d}v\mathrm{d}x\\
&+kp^2\int_{\mathbb{R}^4}\langle v\rangle^{kp-1}f|\partial_{x_i}f|^{p-1}\mathrm{d}v\mathrm{d}x\\
\le&\varepsilon\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v|\nabla_xf|^{\frac{p}{2}}\|_2^2+C(\|\langle v\rangle^k\partial_{x_i}f\|_p^p+\|\langle v\rangle^kf\|_p^p),
\end{aligned}
$$

where $\varepsilon>0$ is a small constant. We further estimate $\mathcal{I}_{41}$ as

$$
|\mathcal{I}_{41}|\le\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_xf|^{\frac{p}{2}})\|_2^2+C(\|\langle v\rangle^k\partial_{x_i}f\|_p^p+\|\langle v\rangle^kf\|_p^p).
$$

• (Estimate of $\mathcal{I}_{4i}(i=2,3,4)$): We use integration by parts to obtain

$$
\begin{aligned}
|\mathcal{I}_{42}|+|\mathcal{I}_{43}|\le&C\|\langle v\rangle^k\partial_{x_i}f\|_p^p,\\
|\mathcal{I}_{44}|\le&\|\nabla u\|_\infty\int_{\mathbb{R}^4}\langle v\rangle^{kp}|\partial_{x_i}f|^{p-1}|\nabla_vf|\mathrm{d}v\mathrm{d}x\\
\le&C\|\nabla u\|_\infty(\|\langle v\rangle^k\partial_{x_i}f\|_p^p+\|\langle v\rangle^k\nabla_vf\|_p^p).
\end{aligned}
$$

• (Estimate of $\mathcal{I}_{45}$): Again, we use integration by parts to have

$$\begin{aligned}\mathcal{I}_{45}=&\int_{\mathbb{R}^4} p\langle v\rangle^{kp}|\partial_{x_i}f|^{p-2}\partial_{x_i}f\partial_{x_i}(4f+4(v-v_c)\cdot\nabla_v f+|v-v_c|^2\Delta_v f)\mathrm{d}v\mathrm{d}x\\ =&-\int_{\mathbb{R}^4}\big[4p(\langle v\rangle^{kp}|\partial_{x_i}f|^p+kpv(v-v_c)\langle v\rangle^{kp-2}|\partial_{x_i}f|^p)+2\langle v\rangle^{kp}(v-v_c)\nabla_v(|\partial_{x_i}f|^p)\\ &+kp\langle v\rangle^{kp-2}|v-v_c|^2 v\cdot\nabla_v(|\partial_{x_i}f|^p)\big]\mathrm{d}v\mathrm{d}x-\frac{4(p-1)}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_x f|^{\frac{p}{2}})\|_2^2\\ \le&-\frac{2(p-1)}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_x f|^{\frac{p}{2}})\|_2^2+C\|\langle v\rangle^k\partial_{x_i}f\|_p^p.\end{aligned}$$

We collect all estimates of $\mathcal{I}_{4i}$ in (4.39) to find

$$\begin{aligned}&\frac{d}{dt}\|\langle v\rangle^k\nabla_x f\|_p^p+\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_x f|^{\frac{p}{2}})\|_2^2\\ &\le C(1+\|\nabla u\|_\infty)(\|\langle v\rangle^k\nabla_x f\|_p^p+\|\langle v\rangle^k\nabla_v f\|_p^p)).\end{aligned}$$

Similarly, we can obtain

$$\frac{d}{dt}\|\langle v\rangle^k\nabla_v f\|_p^p+\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_v f|^{\frac{p}{2}})\|_2^2\le C(\|\langle v\rangle^k\nabla_x f\|_p^p+\|\langle v\rangle^k\nabla_v f|_p^p).$$

Then we combine the above two estimates to have

$$\frac{d}{dt}(\|\langle v\rangle^k\nabla_x f\|_p^p+\|\langle v\rangle^k\nabla_v f\|_p^p)\le C(1+\|\nabla u\|_\infty)(\|\langle v\rangle^k\nabla_x f\|_p^p+\|\langle v\rangle^k\nabla_v f\|_p^p).$$

We further apply the Gronwall inequality and use Lemma 4.7 to derive (4.38). □

LEMMA 4.9. *Under the conditions listed in* (2.2). *Then, we have*

$$\sup_{0\le t\le T}\sum_{|\alpha_*|+|\beta_*|=2}\|\langle v\rangle^k\partial_{\beta_*}^{\alpha_*}f\|_p(t)\le C,\quad 4<p<\infty. \tag{4.40}$$

Proof. We apply $\partial^{\alpha_*}(|\alpha_*|=2)$ to $(1.1)_1$, multiply the above equation by

$$\langle v\rangle^{kp}p|\partial^{\alpha_*}f|^{p-2}\partial^{\alpha_*}f,$$

and integrate the resulting equations with respect to x, v over $\mathbb{R}^4$ to obtain

$$\begin{aligned}\frac{d}{dt}\|\langle v\rangle^k\partial^{\alpha_*}f\|_p^p=&-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial^{\alpha_*}f|^{p-2}\partial^{\alpha_*}f\\ &\times\nabla_v\cdot\Big(\int_{\mathbb{R}^4}\partial^{\alpha_*}(\psi(x-y))(v_*-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}yf\Big)\mathrm{d}v\mathrm{d}x\\ &-2\sum_{|\alpha'|=1}\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial^{\alpha_*}f|^{p-2}\partial^{\alpha_*}f\\ &\times\nabla_v\cdot\Big(\int_{\mathbb{R}^4}\partial^{\alpha'}(\psi(x-y))(v_*-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}y\partial^{\alpha-\alpha'}f\Big)\mathrm{d}v\mathrm{d}x\\ &-\int_{\mathbb{R}^4}\langle v\rangle^{kp}p|\partial^{\alpha_*}f|^{p-2}\partial^{\alpha_*}f\\ &\times\nabla_v\cdot\Big(\int_{\mathbb{R}^4}(\psi(x-y))(v_*-v)f(v_*,y,t)\mathrm{d}v_*\mathrm{d}y\partial^{\alpha_*}f\Big)\mathrm{d}v\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^4} \langle v\rangle^{kp} p|\partial^{\alpha_*} f|^{p-2}\partial^{\alpha_*} f\nabla_v\cdot((u-v)\partial^{\alpha_*} f)\mathrm{d}v\mathrm{d}x\\
&-2\sum_{|\alpha'|=1}\int_{\mathbb{R}^4} \langle v\rangle^{kp} p|\partial^{\alpha_*} f|^{p-2}\partial^{\alpha_*} f\nabla_v\cdot(\partial^{\alpha'} u\partial^{\alpha-\alpha'} f)\mathrm{d}v\mathrm{d}x\\
&-\int_{\mathbb{R}^4} \langle v\rangle^{kp} p|\partial^{\alpha_*} f|^{p-2}\partial^{\alpha_*} f\nabla_v\cdot(\partial^{\alpha_*} uf)\mathrm{d}v\mathrm{d}x\\
&+\int_{\mathbb{R}^4} \langle v\rangle^{kp} p|\partial^{\alpha_*} f|^{p-2}\partial^{\alpha_*} f\partial^{\alpha_*}\Delta_v(|v-v_c|^2 f)\mathrm{d}v\mathrm{d}x\\
=:&\sum_{i=1}^{7}\mathcal{I}_{5i}. \qquad (4.41)
\end{aligned}
$$

In the sequel, we estimate the terms $\mathcal{I}_{5i}$ separately.

• (Estimate of $\mathcal{I}_{51}$): We rewrite $\mathcal{I}_{51}$ as follows.

$$
\begin{aligned}
\mathcal{I}_{51}=&-\int_{\mathbb{R}^4} \langle v\rangle^{kp} p|\partial^{\alpha_*} f|^{p-2}\partial^{\alpha_*} f\nabla_v\cdot\Big(\int_{\mathbb{R}^4}\partial^{\alpha_*}(\psi(x-y))(v_*-v_c)f(v_*,y,)\mathrm{d}v_*\mathrm{d}yf\Big)\mathrm{d}v\mathrm{d}x\\
&-\int_{\mathbb{R}^4} \langle v\rangle^{kp} p|\partial^{\alpha_*} f|^{p-2}\partial^{\alpha_*} f\nabla_v\cdot\Big(\int_{\mathbb{R}^4}\partial^{\alpha_*}(\psi(x-y))(v_c-v)f(v_*,y,)\mathrm{d}v_*\mathrm{d}yf\Big)\mathrm{d}v\mathrm{d}x\\
:=&\mathcal{I}_{511}+\mathcal{I}_{512}.
\end{aligned}
$$

By direct computation, we have

$$
\begin{aligned}
|\mathcal{I}_{511}|\le&\sup_{x\in\mathbb{R}^2}|\nabla_x^2\psi|\int_{\mathbb{R}^4}\langle v\rangle^{kp} p|\partial^{\alpha_*} f|^{p-1}|\nabla_v f|\mathrm{d}v\mathrm{d}x\int_{\mathbb{R}^4}|v_*-v_c|f(v_*,y)\mathrm{d}v_*\mathrm{d}y)\\
\le&C(\|\langle v\rangle^k\partial^{\alpha_*} f\|_p^p+\|\langle v\rangle^k\nabla_v f\|_p^p),\\
|\mathcal{I}_{512}|\le&p(p-1)\int_{\mathbb{R}^4}\langle v\rangle^{kp} f|\partial^{\alpha_*} f|^{p-2}|\nabla_v\partial^{\alpha_*} f||v-v_c|\mathrm{d}v\mathrm{d}x\\
&+kp^2\int_{\mathbb{R}^4}\langle v\rangle^{kp-1} f|\partial^2_{x_ix_j} f|^{p-1}\mathrm{d}v\mathrm{d}x\\
\le&\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_x^2 f|^{\frac{p}{2}})\|_2^2+C(\|\langle v\rangle^k\partial^{\alpha_*} f\|_p^p+\|\langle v\rangle^k f\|_p^p).
\end{aligned}
$$

Then, we can further estimate $\mathcal{I}_{51}$ as

$$
|\mathcal{I}_{51}|\le\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v|\nabla_x^2 f|^{\frac{p}{2}}\|_2^2+C(\|\langle v\rangle^k\partial^{\alpha_*} f\|_p^p+\|\langle v\rangle^k\nabla_v f\|_p^p+\|\langle v\rangle^k f\|_p^p).
$$

Similarly, we use integration by parts to have

$$
\begin{aligned}
|\mathcal{I}_{52}|+|\mathcal{I}_{53}|+|\mathcal{I}_{54}|\le&C(\|\langle v\rangle^k\partial^{\alpha_*} f\|_p^p+\|\langle v\rangle^k\nabla_{x,v}^2 f\|_p^p),\\
|I_{55}|+|\mathcal{I}_{56}|\le&C\|\nabla u\|_\infty\|\langle v\rangle^k\partial^{\alpha_*} f\|_p^{(p-1)}\|\langle v\rangle^k\nabla_{x,v}^2 f\|_p\\
&+C\|\nabla^2 u\|_{2p}\|\langle v\rangle^k\partial^{\alpha_*} f\|_p^{(p-1)}\|\langle v\rangle^k\nabla_v f\|_{2p}\\
\le&C(\|\nabla u\|_\infty+\|\nabla^2 u\|_{2p})(\|\langle v\rangle^k\partial^{\alpha_*} f\|_p+\|\langle v\rangle^k\nabla_{x,v}^2 f\|_p+1).
\end{aligned}
$$

Similar to the estimate of $\mathcal{I}_{45}$, we have

$$
\mathcal{I}_{57}\le-\frac{2(p-1)}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v(|\nabla_x^2 f|^{\frac{p}{2}})\|_2^2+C(p)\|\langle v\rangle^k\partial^{\alpha_*} f\|_p^p.
$$

We collect the estimates of $\mathcal{I}_{5i}$ in (4.41) to have

$$\begin{aligned}&\frac{d}{dt}\|\langle v\rangle^k\partial^{\alpha_*}f\|_p^p+\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v|\nabla_x^2f|^{\frac{p}{2}}\|_2^2\\ \le&C(\|\nabla u\|_\infty+\|\nabla^2u\|_{2p})(\|\langle v\rangle^k\partial^{\alpha_*}f\|_p+\|\langle v\rangle^k\nabla_{x,v}^2f\|_p+1).\end{aligned}\tag{4.42}$$

Similarly, we can obtain

$$\begin{aligned}&\frac{d}{dt}\|\langle v\rangle^k\nabla_x\nabla_vf\|_p^p+\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_x\nabla_v^2|\nabla_x^2f|^{\frac{p}{2}}\|_2^2\\ \le&C(\|\langle v\rangle^k\nabla_v^2f\|_p+\|\langle v\rangle^k\nabla_{x,v}^2f\|_p+1),\\ &\frac{d}{dt}\|\langle v\rangle^k\nabla_v^2f\|_p^p+\frac{p-1}{p}\||v-v_c|\langle v\rangle^{\frac{kp}{2}}\nabla_v^3|\nabla_x^2f|^{\frac{p}{2}}\|_2^2\le C(\|\langle v\rangle^k\nabla_v^2f\|_p+\|\langle v\rangle^k\nabla_{x,v}^2f\|_p+1).\end{aligned}\tag{4.43}$$

We combine (4.42) and (4.43)to have

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{|\alpha_*|+|\beta_*|=2}\|\langle v\rangle^k\partial_{\beta_*}^{\alpha_*}f\|_p\Big)\\ \le&C(\|\nabla u\|_\infty+\|\nabla^2u\|_{2p})(\|\langle v\rangle^k\nabla_x^2f\|_p+\|\langle v\rangle^k\nabla_v^2f\|_p+\|\langle v\rangle^k\nabla_{x,v}^2f\|_p+1).\end{aligned}$$

We further apply Gronwall's inequality and use Lemma 4.7 to have (4.40). ∎

LEMMA 4.10. *Suppose the conditions* (2.2) *hold and the index* q *is same as in Theorem* 2.1*, it holds that*

$$\sup_{0\le t\le T}(\|\bar{x}^a\rho\|_{D^{1,2}\cap D^{1,q}})(t)\le C.\tag{4.44}$$

Proof. First, for $0<\delta<1$, it follows from (3.3) and (4.16) that

$$\|u\bar{x}^{-\delta}\|_{\frac{4}{\delta}}\le C.$$

We use Sobolev's inequality to have that

$$\begin{aligned}\|\nabla(u\bar{x}^{-\delta})\|_q&\le C\|\nabla u\|_q+C\|u\bar{x}^{-\delta}\|_\infty\|(e+|x|^2)^{-\frac12}\|_q\\&\le\frac12\|\nabla(u\bar{x}^{-\delta})\|_q+C\|\nabla u\|_q+C\|u\bar{x}^{-\delta}\|_{\frac{4}{\delta}}.\end{aligned}$$

Combining the above estimates, we have

$$\|u\bar{x}^{-\delta}\|_\infty\le C+C\|\nabla u\|_q.\tag{4.45}$$

Then, we use $(1.1)_2$ to get that

$$(\rho\bar{x}^a)_t+u\cdot\nabla(\rho\bar{x}^a)-a\rho\bar{x}^au\cdot\nabla\log\bar{x}+\rho\bar{x}^a\mathrm{div}u=0.$$

Direct calculations give that

$$\begin{aligned}(\|\nabla(\rho\bar{x}^a)\|_q)_t\le&C(1+\|\nabla u\|_\infty+\|u\cdot\nabla\log\bar{x}\|_\infty)\|\nabla(\rho\bar{x}^a)\|_q\\&+C(\||\nabla u||\nabla\log\bar{x}|\|_q+\||u||\nabla^2\log\bar{x}|\|_q+\|\nabla^2u\|_q)\|\rho\bar{x}^a\|_\infty\\ \le&C(1+\|\nabla u\|_{W^{1,q}})\|\nabla(\rho\bar{x}^a)\|_q\end{aligned}$$

$$
\begin{aligned}
&+C\|(\rho\bar{x}^a)\|_\infty(\|\nabla u\|_q+\|u\bar{x}^{-\frac{1}{2}}\|_\infty\|\bar{x}^{-\frac{3}{2}}\|_q+\|\nabla^2 u\|_q)\\
\le& C(1+\|\nabla^2 u\|_q+\|\nabla u\|_{W^{1,q}})(1+\|\nabla(\rho\bar{x}^a)\|_q),
\end{aligned}
$$

where in the last inequality, we have used (4.45) and the following Sobolev's inequality,

$$
\begin{aligned}
\|\rho\bar{x}^a\|_\infty \le& C\|\rho\bar{x}^a\|_{\frac{5}{4}}+C\|\nabla(\rho\bar{x}^a)\|_q\\
\le& C\left(\|\rho\|_\infty^{\frac{1}{4}}\int_{\mathbb{R}^2}\rho\bar{x}^{\frac{5a}{4}}\,\mathrm{d}x\right)^{\frac{4}{5}}+C\|\nabla(\rho\bar{x}^a)\|_q\\
\le& C+C\|\nabla(\rho\bar{x}^a)\|_q. \qquad (4.46)
\end{aligned}
$$

From (4.25) and Gronwall's inequality, we get that

$$
\sup_{0\le t\le T}\|\nabla(\rho\bar{x}^a)\|_q\le C.
$$

Specially, $\sup_{0\le t\le T}\|\nabla(\rho\bar{x}^a)\|_2\le C$ also holds. □

With Lemma 4.1-4.10 at hand, we are now in a position to prove our theorem.

4.1. Proof of Theorem 2.1. We argue by contradiction. Suppose that (4.1) holds. Note that the generic constant C in Lemma 4.1-4.10 remains bounded for all $T<T^*$, so the functions $[f(T^*,x,v),\rho(T^*,x),u(T^*,x)]:=\lim_{t\to T^*}[f(t,x,v),\rho(t,x),u(t,x)]$ satisfy the conditions imposed on the initial data at the time $t=T^*$. Furthermore, standard arguments yield that $\rho\dot{u}\in C([0,T];L^2)$, which implies

$$
\rho\dot{u}(T^*,x)=\lim_{t\to T^*}\rho\dot{u}\in L^2.
$$

Hence,

$$
-\mu\Delta u-(\mu+\lambda)\nabla\mathrm{div}u+u\rho_f-u_f\rho_f+\nabla P|_{t=T^*}=\sqrt{\rho}(T^*,x)g(x)
$$

with

$$
g(x):=\begin{cases}\sqrt{\rho}(T^*,x)\dot{u}(T^*,x), & if \quad x\in\{x|\rho(T^*,x)>0\},\\ 0, & if \quad x\in\{x|\rho(T^*,x)=0\},\end{cases} \qquad (4.47)
$$

satisfying $g\in L^2$ due to Lemma 4.7. Therefore, one can take $[f(T^*,x,v),\rho(T^*,x),u(T^*,x)]$ as the initial data and extend the local strong solution beyond T^*. This contradicts the assumption on T^*. We thus finish the proof of Theorem 2.1.

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