SOME MATHEMATICAL PROPERTIES OF
THE WEERTMAN EQUATION*

MARC JOSIEN†

Abstract. We derive here some mathematical properties of the Weertman equation and show that it is the limit of an evolution equation. The Weertman equation is a semilinear integrodifferential equation involving a fractional Laplacian. In addition to this purely theoretical interest, the results proven here give a solid ground to a numerical approach that we have implemented elsewhere.

Keywords. Reaction-advection-diffusion equation; traveling waves; integrodifferential equation; the Weertman equation; fractional Laplacian.

AMS subject classifications. 26A33; 35K57; 35R11; 45E05.

1. Introduction

1.1. Motivation. We derive here some mathematical properties of the Weertman equation and show that it is the limit of an evolution equation. Our motivation comes from our interest in materials science. The problem we consider, however classical, enjoys the following specificity that it involves the dissipative integrodifferential operator $-|\partial_x|$ (also denoted as $-(-\Delta)^{1/2}$), which has $-|k|$ as Fourier symbol. In addition to this purely theoretical interest, the results proven here give a solid ground to a numerical approach that we have implemented in [13].

Our starting point is the so-called Weertman equation (see [18]):

$$-|\partial_x|\eta(x) + c\eta'(x) = F'(\eta(x)) \qquad \text{for} \quad x \in \mathbb{R}, \tag{1.1}$$

with boundary conditions

$$\lim_{x \to -\infty} \eta(x) = \eta_l \quad \text{and} \quad \lim_{x \to +\infty} \eta(x) = \eta_r, \tag{1.2}$$

where *both* the scalar $c \in \mathbb{R}$ (called velocity) and the function $\eta \in C^2(\mathbb{R})$ are the unknowns, and where $\eta_l < \eta_r$. The function $F \in C^3(\mathbb{R})$ is a bistable potential; namely, it satisfies

$$F'(\eta_l) = F'(\eta_r) = 0, \quad F''(\eta_l) > 0, \quad \text{and} \quad F''(\eta_r) > 0. \tag{1.3}$$

From a physical point of view, equation (1.1) can be seen as a nondimensionalized form of the Weertman equation for steadily-moving dislocations in materials science (see [18]). The latter equation is a generalization of the classical Peierls-Nabarro equation [16], which is (1.1) with the assumption that the speed c vanishes (in the case of a balanced potential, which satisfies $F(\eta_l) = F(\eta_r)$). Dislocations are linear defects in crystals, the motion of which is responsible for the plasticity of metals. From a physical standpoint, the function η represents a discontinuity between the local relative material displacement $u(x,y)$ in the upper and in the lower half-spaces surrounding the glide plane on which moves the dislocation line (see figure 1.1); see, *e.g.*, [11] for a classical

†CERMICS, Ecole des Ponts ParisTech, 6 & 8 Avenue Blaise Pascal, 77455 Marne-la-Valle CEDEX 2, France, (marc.josien@enpc.fr).
https://marcjosien.wordpress.com/

reference on dislocations. In (1.1), the term $|\partial_x|\eta$ accounts for the long-range elastic self-interactions that tend to spread the core. This repulsive interaction is counterbalanced by the nonlinear pull-back force $F'(\eta)$, which binds together the upper and lower half-spaces. Moreover, the moving dislocation is subjected to various drag mechanisms encoded into the term $c\eta'$.
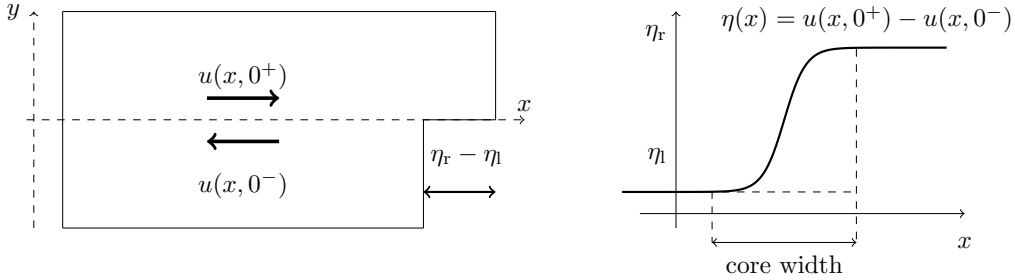

FIG. 1.1. *Typical shape of $\eta(x)$, solution to (1.1); here, $u(x,y)$ is the material displacement.*

From a broader perspective, the function η can be understood as a moving phase-transformation front between the states η_l and η_r (see Figure 1.1), which are local minimizers of the potential F. In this regard, equation (1.1) can also be found in other fields of physics. For example, it can be used to model complex media such as living cells, in which the operator $|\partial_x|$ accounts for an anomalous diffusion. See, *e.g.*, [15], where the more general operator $|\partial_x|^\alpha$ is considered (the latter operator has $|k|^\alpha$ as Fourier symbol).

1.2. Traveling wave of reaction-diffusion equation.   Equation (1.1) is a special case of the general equation

$$\begin{cases} \mathcal{A}[\eta](x) + c\eta'(x) = 0 & \text{for} \quad x \in \mathbb{R}, \\ \eta(-\infty) = \eta_\mathrm{l} \quad \text{and} \quad \eta(+\infty) = \eta_\mathrm{r}, \end{cases} \tag{1.4}$$

where $\mathcal{A}[\eta] = L\eta - F'(\eta)$ is a nonlinear operator, in which L is a diffusive operator and F is a bistable potential. As is easily seen, equation (1.4) describes the traveling waves of the following reaction-diffusion equation:

$$\begin{cases} \partial_t u(t,x) = \mathcal{A}[u(t,\cdot)](x) & \text{for} \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{1.5}$$

in the sense that, if $u(t,x) = \eta(x-ct)$ is a traveling wave satisfying (1.5), then (η,c) solves (1.4). *Ipso facto*, finding a solution to (1.4) amounts to finding traveling wave solutions to (1.5). Natural questions thus arise:

  (i) Does equation (1.4) have one and only one solution (η,c)?
  (ii) Which properties does the solution to equation (1.4) enjoy?
  (iii) Is the traveling wave $\eta(x-ct)$, for (η,c) solution to (1.4), an attractor of the dynamical system (1.5)?

These questions have been addressed by many authors for various operators L and for bistable potential F satisfying (most of the times) the extra condition that F does not admit any local minimum between η_l and η_r. Other types of nonlinearities, not

considered here, have attracted much attention. See [20] for the classification of traveling waves and an overview of reaction-diffusion equations.

In the seminal article [19], Sattinger remarked that if (η, c) is a solution to (1.4), then $(\eta(\cdot + \xi), c)$ is also a solution to (1.4), for arbitrary $\xi \in \mathbb{R}$. Therefore, solutions to (1.4) can at most be unique *up to a translation*. In this regard, he introduced the notion of asymptotic stability of traveling waves and proved that the solution (η, c) to (1.4), if it exists, is asymptotically stable under general assumptions about the spectrum of $\mathcal{A}$.

In the celebrated article [9], Fife and McLeod answered the questions (i) and (iii) for the case where L is the Laplacian. They proved indeed that if F satisfies (1.3) and has no local minimum between η_l and η_r, then there exists a solution (η, c) to (1.4), which is unique up to a translation, and that this solution is globally asymptotically stable. Namely, for all initial conditions u_0 taking values in $[\eta_l, \eta_r]$ such that $u_0(-\infty) = \eta_l$ and $u_0(+\infty) = \eta_r$, there exist $\xi \in \mathbb{R}$, $K > 0$ and $\kappa > 0$ such that the solution u of (1.5) satisfies

$$\|u(t, \cdot) - \eta(\cdot - \xi - ct)\|_{L^\infty(\mathbb{R})} \leq K e^{-\kappa t}, \tag{1.6}$$

for all $t \in \mathbb{R}_+$. Among other important concepts, all amenable to a wide class of dissipative operators, it is observed in [9] that $\mathcal{A}$ satisfies a comparison principle. Thus, any solution $u(t, x)$ of (1.5) can be squeezed between a super-solution $w_{+1}(t, x)$ and a sub-solution $w_{-1}(t, x)$, both at a controlled distance from $\eta(x - \xi - ct)$.

In a more recent article [7], Chen combined this squeezing approach with an iterative technique. Under technical assumptions about the operator $\mathcal{A}$, he proved the global asymptotic stability of the traveling waves of (1.5), provided that there exists a monotonic solution η to (1.4). In this context, a positive answer to question (i) and technical assumptions imply, using Chen's *squeezing* technique, a positive answer to question (iii). We use Chen's approach in the present article.

The article [7] also provides tools for establishing the existence and the uniqueness of a solution to (1.4). They have been used in [8] to positively answer question (i) in the case where L is the fractional Laplacian $-|\partial_x|^\alpha$, for $\alpha \in (0, 2)$. Also, in [1], the authors have adapted Chen's squeezing technique to prove that the solution (η, c) to (1.4) is globally asymptotically stable in the sense of (1.6), in a general framework including the case $L = -|\partial_x|^\alpha$, for $\alpha \in (1, 2)$. However, they underlined the fact that the case $\alpha \leq 1$ (and in particular $\alpha = 1$), is still an open question. This motivates our study.

With an approach different from [7], the existence and the uniqueness of a solution to (1.4), for $L = -|\partial_x|^\alpha$ and $\alpha \in (0, 2)$, has been proved in [4, 5] in the special case where $c = 0$ (the so-called balanced case). These results have been generalized by [10]. Assuming that $F \in C^3(\mathbb{R})$ satisfies (1.3) and the following extra conditions:

$$\begin{cases} F(u) > F(\eta_l), & \forall u \in (\eta_l, \eta_r), \\ F'(u) > 0 \quad \text{or} \quad F(u) > F(\eta_r), & \forall u \in (\eta_l, \eta_r), \end{cases} \tag{1.7}$$

it is shown in [10, Th. 1.1] that there exist a unique $c \in \mathbb{R}$ and an increasing function $\eta \in C^2(\mathbb{R})$, which is unique up to a translation, that solve (1.1). Conditions (1.3) and (1.7) mean that the potential $F(u)$ has two major wells in $u = \eta_l$ and $u = \eta_r$ (the states η_l and η_r are therefore stable), and that its behavior is controlled between these wells; for example, the potential can have minor wells (see figure 1.2). The proof of [10] relies on special solutions to (1.4) built in [4, 5], which, by homotopy techniques, are used to find the solutions to the general case. The result of [10] will be our starting point for proving the global asymptotic stability of the traveling waves of (1.5) for $L = -|\partial_x|$.

Additionally, the authors of [4, 5, 10] have also studied some properties of the solutions to (1.4). In particular, when $L = -|\partial_x|$, they have shown that $\eta' > 0$ and that there exist constants $B > A > 0$ such that, for all $|x| \geq 1$,

$$A|x|^{-2} \leq \eta'(x) \leq B|x|^{-2}. \tag{1.8}$$

See [5, Th. 2.7] for the special case where $c = 0$ and [10, Prop. 3.2] for the general case. Moreover, the following identity is proved (see [10, Prop. 4.1]):

$$c = [F(\eta_r) - F(\eta_l)] \left(\int_{\mathbb{R}} |\eta'|^2 \right)^{-1}. \tag{1.9}$$

Formula (1.9) is useful because it provides the sign of c just by considering the values $F(\eta_l)$ and $F(\eta_r)$. As remarked in [4], if $c = 0$, then (1.1) can be interpreted as the
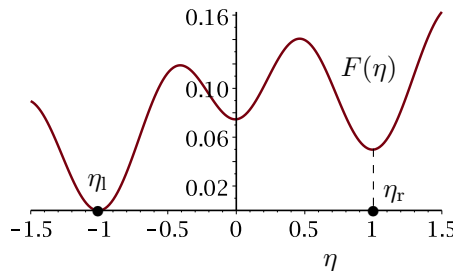

FIG. 1.2.   *A double-well "camel-hump" potential F, $\eta_l = -1$, $\eta_r = 1$.*

restriction to the boundary of an elliptic problem with Neumann boundary condition. If indeed u solves the following problem:

$$\begin{cases} \Delta u(x,y) + c\partial_x u(x,y) = 0 & \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \\ \partial_y u(x,0) = F'(u(x,0)) & \text{on} \quad \mathbb{R} \times \{0\}, \end{cases} \tag{1.10}$$

for $c = 0$, then $\eta(x) := u(x,0)$ is a solution to (1.1). However, we stress that, when $c \neq 0$, equation (1.10) describes a diffusive traveling wave in the half-space. In this case, (1.1) is *not* the restriction to the boundary of the problem (1.10), which instead reads

$$(-\Delta - c\partial_x)^{1/2} \eta(x) = -F'(\eta(x)). \tag{1.11}$$

We refer to [3] for a mathematical study of (1.11).

But, we mention for completeness that (1.1) is in fact the restriction to the boundary of the following elliptic equation

$$\begin{cases} \Delta u(x,y) = 0 & \text{in} \quad \mathbb{R} \times \mathbb{R}_+, \\ \partial_y u(x,0) + c\partial_x u(x,0) = F'(u(x,0)) & \text{on} \quad \mathbb{R} \times \{0\}. \end{cases} \tag{1.12}$$

In a physical context, the latter is envisioned as an elastic equation in the half-plane with a nonlinear boundary condition. We briefly justify it. If indeed we take the Fourier transform with respect to x, denoted as $\mathcal{F}_x$, of the first equation of (1.12), and if we restrict on bounded solutions, we obtain

$$\mathcal{F}_x \{u(\cdot,y)\}(k) = e^{-|k|y} \mathcal{F}_x \{u(\cdot,0)\}(k).$$

Injecting the above information in the second equation of (1.12) then yields (1.1) if we denote $\eta(x) = u(x,0)$ (recall that $|\partial_x|$ is an operator which has $|k|$ as Fourier symbol).

1.3. Main results. Our first result concerns the asymptotic expansion of the solution η to (1.1). The following proposition is a refinement of results of [4, 5, 10]:

PROPOSITION 1.1. *Let $\eta_l < \eta_r$ and $F \in \mathrm{C}^3(\mathbb{R})$ satisfy (1.3). Assume that (η, c) is a solution to (1.1) and (1.2) such that $\eta \in \mathrm{C}^2(\mathbb{R})$ is an increasing function satisfying $\eta' > 0$ and (1.8). Then η has the following asymptotes:*

$$\eta(x) - \eta_r \underset{x \to +\infty}{\sim} \frac{\eta_l - \eta_r}{\pi F''(\eta_r)} x^{-1}, \qquad and \qquad \eta(x) - \eta_l \underset{x \to -\infty}{\sim} \frac{\eta_l - \eta_r}{\pi F''(\eta_l)} x^{-1}. \tag{1.13}$$

In addition to their theoretical interest, these asymptotes also allow for getting more accurate numerical approximations of η, as shown in [13].

Our second result is:

PROPOSITION 1.2. *Under the hypotheses of Proposition 1.1, (c, η) satisfies the following identity:*

$$c = \frac{1}{\eta_r - \eta_l} \lim_{R \to +\infty} \int_{-R}^{R} F'(\eta). \tag{1.14}$$

The above identity is formally obtained by integrating Equation (1.1) over $\mathbb{R}$; we rigorously prove it. Notice that, by Proposition 1.1 and using a Taylor expansion, $F'(\eta) \notin \mathrm{L}^1(\mathbb{R})$.

As mentioned above in the concise form (1.5), we consider the following dynamical system:

$$\begin{cases} \partial_t u(t,x) + |\partial_x| u(t,x) = -F'(u(t,x)) & \text{for} \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \mathbb{R}, \end{cases} \tag{1.15}$$

for an initial condition $u_0 \in \mathrm{L}^\infty(\mathbb{R})$. We say that $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{L}^\infty(\mathbb{R}))$ is a weak solution to (1.15) if, for all $T > 0$, for all $\phi \in \mathrm{C}^1_{\mathrm{c}}([0,T), \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}))$, there holds

$$\int_0^T \int_{\mathbb{R}} u(t,x)(-\partial_t + |\partial_x|)\phi(t,x)\mathrm{d}x\,\mathrm{d}t$$

$$= \int_{\mathbb{R}} u_0(x)\phi(0,x)\mathrm{d}x - \int_0^T \int_{\mathbb{R}} F'(u(t,x))\phi(t,x)\mathrm{d}x\,\mathrm{d}t. \tag{1.16}$$

Our third and final result is that (1.1) is the long-time limit of (1.15), for general initial conditions u_0 with suitable behavior at infinity (see figure 1.3 for an example). We prove the following:

THEOREM 1.1. *Let $\eta_l < \eta_r$, $F \in \mathrm{C}^3(\mathbb{R})$ satisfy (1.3) and $\Delta_0 > 0$ be such that*

$$F'' > 0 \quad on \quad [\eta_l - \Delta_0, \eta_l + \Delta_0] \cup [\eta_r - \Delta_0, \eta_r + \Delta_0]. \tag{1.17}$$

Assume that $u_0 \in \mathrm{L}^\infty(\mathbb{R})$ takes values in $[\eta_l - \Delta_0, \eta_r + \Delta_0]$ and satisfies

$$\limsup_{x \to -\infty} u_0(x) < \eta_l + \Delta_0 \quad and \quad \liminf_{x \to +\infty} u_0(x) > \eta_r - \Delta_0. \tag{1.18}$$

Then:

> (i) *Equation (1.15) has a unique weak solution $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{L}^\infty(\mathbb{R}))$. Moreover, for all $T_0 > 0$,*
>
> $$u \in \mathrm{C}\left((T_0, +\infty), \mathrm{C}^2(\mathbb{R})\right) \cap \mathrm{C}^1\left((T_0, +\infty), \mathrm{C}(\mathbb{R})\right). \tag{1.19}$$

(ii) *In addition, for all $t > 0$ and $x \in \mathbb{R}$, $u(t, x) \in [\eta_{\mathrm{l}} - \Delta_0, \eta_{\mathrm{r}} + \Delta_0]$.*

(iii) *Assume that (η, c) is a solution to (1.1) and (1.2) such that $\eta \in \mathrm{C}^2(\mathbb{R})$ is an increasing function satisfying $\eta' > 0$ and*

$$\lim_{|x| \to +\infty} \eta'(x) = 0. \tag{1.20}$$

Then, there exist constants $\kappa > 0$, $K > 0$ and ξ such that

$$\|u(t, \cdot) - \eta(\cdot - ct + \xi)\|_{\mathrm{L}^\infty(\mathbb{R})} \le K e^{-\kappa t}, \tag{1.21}$$

for all $t \in \mathbb{R}_+$. In the above estimate, κ only depends on F, η and Δ_0, whereas K and ξ depend also on u_0.

Theorem 1.1 suggests that simulating (1.15) is sufficient to obtain, in the long time, a numerical approximation of the solution (η, c) to (1.1). In this regard, it is significant that c, which is an unknown of (1.1), does *not* appear in (1.15). This in particular allows for constructing an approximation of the traveling wave velocity, which is unknown before the end of the simulation. We refer the reader to our study [13], where we explain the details of the numerical strategy, and to a forthcoming article [17] for the multi-dimensional case. In this regard, we shall stress that Theorem 1.1 stated above unfortunately only holds in the case where η is scalar-valued, because its proof relies on a comparison principle. However, it empirically appears that such a convergence is also achieved in many cases where η is vector-valued.
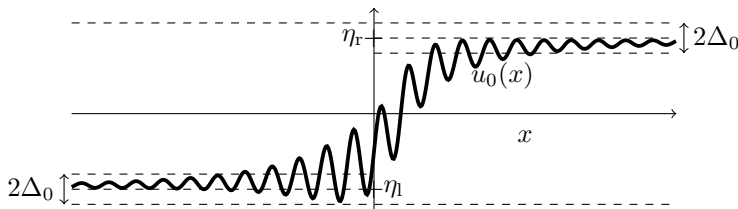

FIG. 1.3. *A possible initial condition u_0 in Theorem 1.1.*

1.4. Outline.   Our contribution is organized as follows. In Section 2, we introduce notations and give essential properties of the operator $|\partial_x|$. In Section 3, we prove Propositions 1.1 and 1.2. In Section 4, we justify the existence and the uniqueness of a weak solution to equation (1.15), which satisfies (1.19), thus establishing (i) of Theorem 1.1. In Section 5, we use Chen's approach for proving (ii) and (iii) of Theorem 1.1. The key ingredients are a comparison principle and specific sub-solutions and super-solutions. Although we could check the technical assumptions and apply Chen's theorem, we prefer to restrict Chen's proof to our special case for self-consistency and simplicity.

2. Notations and definitions

2.1. Notations.   We denote by $\mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R})$ the space of smooth functions with compact supports in $\mathbb{R}$ and by $\mathcal{D}'$ the space of distributions over $\mathbb{R}$. For $u \in \mathrm{C}_{\mathrm{c}}^\infty(\mathbb{R})$, we denote the Fourier transform by $\mathcal{F}\{u\}(k) := \int_{\mathbb{R}} e^{-ikx} u(x) \mathrm{d}x$. For two functions u and v, we denote by $*$ the convolution. Henceforth, the Fourier transform and the convolution are only taken with respect to the *space* variable x (and never with respect to the *time*

variable t). We make use of the principal value of $\frac{1}{x}$, denoted by p.v.$\left(\frac{1}{x}\right)$, which is the distribution defined by

$$\left\langle \text{p.v.}\left(\frac{1}{x}\right), u \right\rangle = \lim_{\epsilon \to 0^+} \left\{ \int_\epsilon^{+\infty} \frac{u(y)}{y} \mathrm{d}y + \int_{-\infty}^{-\epsilon} \frac{u(y)}{y} \mathrm{d}y \right\},$$

for $u \in \mathrm{C}_c^\infty(\mathbb{R})$.

2.2. Definition and properties of the operator $|\partial_x|$. For convenience, we recall some elementary properties of the operator $|\partial_x|$. The Hilbert transform $\mathcal{H}$ of $u \in \mathrm{L}^2(\mathbb{R})$ is defined by

$$\mathcal{H}\{u\} := \mathcal{F}^{-1}\left\{ -\mathrm{i}\, \text{sign}(k)\mathcal{F}\{u\}(k) \right\}. \tag{2.1}$$

It is immediate that, if $u \in \mathrm{L}^2(\mathbb{R})$, then $\mathcal{H}^2\{u\} = -u$. Next, the operator $|\partial_x|$ is defined as

$$|\partial_x|u(x) := \mathcal{H}\{u'\}(x) = \mathcal{F}^{-1}\left\{ |k|\mathcal{F}\{u\}(k) \right\}(x), \tag{2.2}$$

for $u \in \mathrm{C}_c^\infty(\mathbb{R})$. As $\mathcal{F}\{\text{p.v.}(1/x)\}(k) = -\mathrm{i}\pi\text{sign}(k)$, the operator $|\partial_x|$ can be rewritten as

$$|\partial_x|u(x) = -\frac{1}{\pi} \int_0^{+\infty} \frac{u'(x+y) - u'(x-y)}{y} \mathrm{d}y \tag{2.3}$$

$$= -\frac{1}{\pi} \int_0^{+\infty} \frac{u(x+y) - 2u(x) + u(x-y)}{y^2} \mathrm{d}y, \tag{2.4}$$

the last expression being obtained from the previous one by integrating by parts. We see from (2.2) that the operator $|\partial_x|$ is symmetric and positive, like the Laplacian. But, unlike the Laplacian, it is clear from (2.4) that $|\partial_x|u(x)$ does not only depend on u in the neighborhood of x but also on each value $u(y)$, for $y \in \mathbb{R}$; put differently, $|\partial_x|$ is *non-local*.

A straightforward computation yields that $|\partial_x|\phi \in \mathrm{L}^1(\mathbb{R})$ whenever $\phi \in \mathrm{C}_c^\infty(\mathbb{R})$. Hence, one can extend $|\partial_x|$ over $\mathrm{L}^\infty(\mathbb{R})$ by duality, defining $|\partial_x|u$ as the following distribution:

$$|\partial_x|u : \phi \in \mathrm{C}_c^\infty(\mathbb{R}) \mapsto \int_{\mathbb{R}} u(y)|\partial_x|\phi(y)\mathrm{d}y. \tag{2.5}$$

When u is sufficiently regular, explicit expressions for $|\partial_x|u$ are available. Namely, if $u \in \mathrm{L}^\infty(\mathbb{R}) \cap \mathrm{C}^2(\mathbb{R})$, then expression (2.4) is valid. In particular, $|\partial_x|u \in \mathrm{C}(\mathbb{R}) \cap \mathrm{L}^\infty(\mathbb{R})$. The proof can be done by density of $\mathrm{C}_c^\infty(\mathbb{R})$ in $\mathrm{C}_{\mathrm{loc}}^2(\mathbb{R})$, using the fact that (2.4) is true for smooth functions. If we assume furthermore that $u' \in \mathrm{L}^1(\mathbb{R})$, then expression (2.3) is also valid; this is deduced from (2.4) by integration by parts.

3. Asymptotes and an identity about velocity

The proof of Proposition 1.1 relies on the asymptotic behavior of Cauchy integrals (see [14, p. 267]) and involves the following technical lemma:

LEMMA 3.1. *Under the hypotheses of Proposition 1.1, there holds*

$$\eta'' \in \mathrm{L}^\infty(\mathbb{R}). \tag{3.1}$$

REMARK 3.1. Note that it is also possible to establish by technical arguments that there exists a constant $C > 0$ such that, for all $|x| > 1$,

$$|\eta''(x)| \leq C|x|^{-2}(1 + \ln(|x|)). \tag{3.2}$$

(We refer the reader to [12] for the proof of (3.2)). However, (3.1) is sufficient to prove Proposition 1.1. Yet, if F is sinusoidal, one can derive analytical solutions η to (1.1), as is shown in [18], which are of the form

$$\eta(x) = \eta_{\mathrm{l}} + \frac{\eta_{\mathrm{r}} - \eta_{\mathrm{l}}}{\pi} \left(\frac{\pi}{2} + \arctan(ax) \right),$$

for $a > 0$. Whence

$$\eta''(x) = -\frac{\eta_{\mathrm{r}} - \eta_{\mathrm{l}}}{\pi} \frac{2a^3 x}{(a^2 x^2 + 1)^2} \underset{x \to +\infty}{\sim} -\frac{2(\eta_{\mathrm{r}} - \eta_{\mathrm{l}})}{\pi a x^3}.$$

Thus (3.2) is probably not optimal.

We postpone the proof of Lemma 3.1 until the end of the proof of Proposition 1.1 and temporarily admit Lemma 3.1.

Proof. (**Proof of Proposition 1.1.**) We focus on the case $x \to +\infty$. Provided that

$$x |\partial_x| \eta(x) \underset{x \to +\infty}{\longrightarrow} \frac{1}{\pi} \int_{-\infty}^{+\infty} \eta'(y) \mathrm{d}y = \frac{1}{\pi} (\eta_{\mathrm{r}} - \eta_{\mathrm{l}}), \tag{3.3}$$

then, using (1.1), (1.2), (1.3) and (1.8), we get, by Taylor expansion

$$F''(\eta_{\mathrm{r}}) (\eta(x) - \eta_{\mathrm{r}}) \underset{x \to +\infty}{\sim} -\frac{1}{\pi} x^{-1} (\eta_{\mathrm{r}} - \eta_{\mathrm{l}}),$$

which is (1.13).

Let us now prove (3.3). By assumption, $\eta' \in \mathrm{C}^1(\mathbb{R})$, and by (1.8), we have $\eta' \in \mathrm{L}^1(\mathbb{R})$. As a consequence, there holds

$$x |\partial_x| \eta(x) = \frac{x}{\pi} \lim_{\epsilon \to 0^+} \left(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{+\infty} \right) \frac{\eta'(y)}{x - y} \mathrm{d}y.$$

Let $R > 0$ and $x > 2R$. We split the integral into three parts

$$\pi x |\partial_x| \eta(x) = \int_{-\infty}^{R} \frac{\eta'(y)}{1 - y/x} \mathrm{d}y + x \left(\int_{R}^{x-R} + \int_{x+R}^{+\infty} \right) \frac{\eta'(y)}{x - y} \mathrm{d}y$$

$$+ x \lim_{\epsilon \to 0^+} \left(\int_{x-R}^{x-\epsilon} + \int_{x+\epsilon}^{x+R} \right) \frac{\eta'(y)}{x - y} \mathrm{d}y. \tag{3.4}$$

The first right-hand term in (3.4) is dealt with using the dominated convergence theorem, the second one avoids the singularity of $|x - y|^{-1}$ and is bounded thanks to (1.8), and the third one is on the singularity of $|x - y|^{-1}$ and is controlled thanks to (3.1) and (1.8).

As $\eta' \in \mathrm{L}^1(\mathbb{R})$ and since (recall that $x > 2R$)

$$\left| \frac{\eta'(y)}{1 - y/x} \right| \leq 2 |\eta'(y)| \quad \text{if} \quad y < R, \quad \text{and} \quad \frac{\eta'(y)}{1 - y/x} \underset{x \to +\infty}{\longrightarrow} \eta'(y),$$

then, by the dominated convergence theorem

$$\int_{-\infty}^{R} \frac{\eta'(y)}{1 - y/x} \mathrm{d}y \underset{x \to +\infty}{\longrightarrow} \int_{-\infty}^{R} \eta'(y) \mathrm{d}y. \tag{3.5}$$

Next, we split the second integral of (3.4) into three parts. Invoking (1.8), we deduce that, as $|x| > 2R$,

$$\left| \left(\int_R^{x-R} + \int_{x+R}^{+\infty} \right) \frac{\eta'(y)}{x-y} \mathrm{d}y \right|$$

$$\leq Cx^{-1} \int_R^{x/2} |\eta'(y)| \mathrm{d}y + CR^{-1} \left(\int_{x/2}^{x-R} + \int_{x+R}^{+\infty} \right) |\eta'(y)| \mathrm{d}y$$

$$\leq Cx^{-1} \int_R^{x/2} \frac{\mathrm{d}y}{y^2} + CR^{-1} \left(\int_{x/2}^{x-R} + \int_{x+R}^{+\infty} \right) \frac{\mathrm{d}y}{y^2}$$

$$\leq CR^{-1}x^{-1}. \tag{3.6}$$

Last, we split the last integral of (3.4) into two parts, namely:

$$\left| \left(\int_{x-R}^{x-\epsilon} + \int_{x+\epsilon}^{x+R} \right) \frac{\eta'(y)}{x-y} \mathrm{d}y \right| = \left| \left(\int_\epsilon^{x^{-2}} + \int_{x^{-2}}^R \right) \frac{\eta'(x-z) - \eta'(x+z)}{z} \mathrm{d}z \right|.$$

The first part of the right-hand side of the above equation is dealt with by using (3.1), and the second one by using (1.8). Whence, as $x > 2R$ and for $x^{-2} \geq \epsilon$,

$$\left| \left(\int_{x-R}^{x-\epsilon} + \int_{x+\epsilon}^{x+R} \right) \frac{\eta'(y)}{x-y} \mathrm{d}y \right| \leq Cx^{-2} \int_{x^{-2}}^R z^{-1} \mathrm{d}z + C \int_\epsilon^{x^{-2}} \mathrm{d}z$$

$$\leq Cx^{-2} \left(\ln(Rx^2) + 1 \right).$$

Therefore

$$\left| x \lim_{\epsilon \to 0^+} \left(\int_{x-R}^{x-\epsilon} + \int_{x+\epsilon}^{x+R} \right) \frac{\eta'(y)}{x-y} \mathrm{d}y \right| \leq Cx^{-1} \left(\ln(Rx^2) + 1 \right). \tag{3.7}$$

Convergence (3.5) and estimates (3.6) and (3.7) finally yield

$$\limsup_{x \to +\infty} \left| \pi x |\partial_x| \eta(x) - \int_{-\infty}^{+\infty} \eta'(y) \mathrm{d}y \right| \leq \left| \int_R^{+\infty} \eta'(y) \mathrm{d}y \right| + CR^{-1},$$

which, thanks to (1.8), implies (3.3) upon letting $R \to +\infty$. □

We then proceed with the:

Proof. **(Proof of Lemma 3.1.)** We first remark that if $g \in \mathrm{L}^2(\mathbb{R})$ and if

$$-|\partial_x| h(x) + ch'(x) = g(x), \tag{3.8}$$

then $h' \in \mathrm{L}^2(\mathbb{R})$. Indeed, the Fourier transform turns (3.8) into

$$(-|k| + c\mathrm{i}k)\mathcal{F}\{h\}(k) = \mathcal{F}\{g\}(k).$$

Therefore, $k\mathcal{F}\{h\}(k) \in \mathrm{L}^2(\mathbb{R})$, whence $h' \in \mathrm{L}^2(\mathbb{R})$. We use this result to prove that $\eta'' \in \mathrm{L}^\infty(\mathbb{R})$.

Upon differentiating (1.1), we obtain

$$-|\partial_x| \eta'(x) + c\eta''(x) = F''(\eta(x))\eta'(x). \tag{3.9}$$

As $\eta \in L^\infty(\mathbb{R})$, $F \in C^3(\mathbb{R})$ and $\eta' \in L^2(\mathbb{R})$ (thanks to (1.8)), then the right-hand side of (3.9) is in $L^2(\mathbb{R})$. Therefore $\eta'' \in L^2(\mathbb{R})$. Differentiating (3.9) yields

$$-|\partial_x|\eta''(x) + c\eta'''(x) = F''(\eta(x))\eta''(x) + F'''(\eta(x))(\eta'(x))^2. \tag{3.10}$$

As $\eta \in L^\infty(\mathbb{R})$, $F \in C^3(\mathbb{R})$, $\eta'' \in L^2(\mathbb{R})$, and, thanks to (1.8), $(\eta')^2 \in L^2(\mathbb{R})$, then the right-hand side of (3.10) is in $L^2(\mathbb{R})$. Therefore $\eta''' \in L^2(\mathbb{R})$. As a consequence, since $\eta'' \in L^2(\mathbb{R})$, we deduce by Sobolev injection that $\eta'' \in L^\infty(\mathbb{R})$, whence (3.1). $\qquad\square$

We now focus on Proposition 1.2. Both the Identities (1.9) and (1.14) are formally obtained by testing (1.1) against a certain function g, namely $g = \eta'$ for (1.9), and $g = 1$ for (1.14). We justify below this formal integration.

Proof. **(Proof of Proposition 1.2.)** Let $R > 2$. We integrate (1.1) over $[-R, R]$:

$$-\int_{-R}^{R}|\partial_x|\eta(x)\mathrm{d}x + c(\eta(R) - \eta(-R)) = \int_{-R}^{R}F'(\eta(x))\mathrm{d}x. \tag{3.11}$$

Thus, Identity (1.14) stems from (3.11), provided that

$$\lim_{R \to +\infty}\int_{-R}^{R}|\partial_x|\eta(x)\mathrm{d}x = 0. \tag{3.12}$$

We prove (3.12) using (1.8) and (3.1). As $\eta' \in L^1(\mathbb{R}) \cap C^1(\mathbb{R})$, there holds

$$\begin{aligned}|\partial_x|\eta(x) &= \frac{1}{\pi}\int_0^{+\infty}\frac{\eta'(x-y) - \eta'(x+y)}{y}\mathrm{d}y \\ &= \frac{1}{\pi}\int_{|y|<R}\frac{(\eta'(x-y) - \eta'(x))}{y}\mathrm{d}y + \frac{1}{\pi}\int_{|y|>R}\frac{\eta'(x-y)}{y}\mathrm{d}y.\end{aligned} \tag{3.13}$$

Remark that

$$\int_{|y|>R}\left|\frac{\eta'(x-y)}{y}\right|\mathrm{d}y \leq 2R^{-1}\|\eta'\|_{L^1(\mathbb{R})},$$

and that, using (3.1),

$$\int_{|y|<R}\left|\frac{(\eta'(x-y) - \eta'(x))}{y}\right|\mathrm{d}y \leq 2R\|\eta''\|_{L^\infty(\mathbb{R})} \leq CR.$$

Therefore, integrating (3.13) thanks to Fubini's theorem, yields

$$\begin{aligned}\int_{-R}^{R}|\partial_x|\eta(x)\mathrm{d}x &= \frac{1}{\pi}\int_{|y|<R}\frac{(\eta(R-y) - \eta(R)) - (\eta(-R-y) - \eta(-R))}{y}\mathrm{d}y \\ &\quad + \frac{1}{\pi}\int_{|y|>R}\frac{\eta(R-y) - \eta(-R-y)}{y}\mathrm{d}y \\ &=: T_1 + T_2.\end{aligned} \tag{3.14}$$

First, we bound T_1. If $|y| < R/2$, thanks to (1.8), we obtain

$$|\eta(R-y) - \eta(R)| \leq C|y|R^{-2} \quad \text{and} \quad |\eta(-R-y) - \eta(-R)| \leq C|y|R^{-2}.$$

As a consequence,

$$\int_{|y|<R/2} \left| \frac{(\eta(R-y)-\eta(R))-(\eta(-R-y)-\eta(-R))}{y} \right| dy \leq CR^{-1}. \qquad (3.15)$$

Note that, as underlined in [10], a consequence of (1.8) is that

$$\begin{cases} A|x|^{-1} \leq \eta_{\mathrm{r}} - \eta(x) \leq B|x|^{-1}, & \text{if} \quad x>1, \\ A|x|^{-1} \leq \eta(x) - \eta_{\mathrm{l}} \leq B|x|^{-1}, & \text{if} \quad x<-1. \end{cases} \qquad (3.16)$$

Therefore, if $|y| < R$

$$|\eta(R-y)-\eta(R)| \leq \frac{C}{R-|y|+1} \quad \text{and} \quad |\eta(-R-y)-\eta(-R)| \leq \frac{C}{R-|y|+1}.$$

Whence

$$\int_{R/2<|y|<R} \left| \frac{(\eta(R-y)-\eta(R))-(\eta(-R-y)-\eta(-R))}{y} \right| dy$$

$$\leq \frac{C}{R} \int_{R/2}^{R} \frac{dy}{R+1-y} \leq CR^{-1}\ln(R). \qquad (3.17)$$

We deduce from (3.15) and (3.17) that

$$|T_1(R)| \leq CR^{-1}(1+\ln(R)). \qquad (3.18)$$

Thanks to (1.8) and since $\eta \in \mathrm{L}^\infty(\mathbb{R})$, if $|y| > R$, we have

$$|\eta(R-y)-\eta(-R-y)| \leq C\min\left\{ R(|y|-R)^{-2}, 1 \right\}.$$

Whence, splitting T_2 into two parts,

$$|T_2(R)| \leq C \left(\int_R^{R+\sqrt{R}} \frac{dy}{y} + \int_{R+\sqrt{R}}^{+\infty} \frac{R}{y(y-R)^2} dy \right)$$

$$\leq C\frac{\sqrt{R}}{R} + C \int_{\sqrt{R}}^{+\infty} \frac{dz}{z^2} \leq CR^{-1/2}. \qquad (3.19)$$

Bearing (3.14) in mind, we observe that (3.18) and (3.19) imply (3.12). □

4. Existence, uniqueness and regularity of the solution to the evolution Equation (1.15)

We now justify the existence, the uniqueness and the regularity of a weak solution u to (1.15). We proceed in the classical way; as the methods as well as the type of results are well-known, we only give a few hints of the proofs. We refer the interested reader to [12] for some technical details and extra materials about the proofs, and to [6] for a reference on evolution equations involving m-dissipative operators.

Using the Fourier transform, the solution to the homogeneous linear equation

$$\partial_t u(t,x) + |\partial_x| u(t,x) = 0 \quad \text{for} \quad x \in \mathbb{R}, \qquad \text{and} \quad u(0,x) = u_0(x) \qquad (4.1)$$

is given by $u(t,x) = \{K_t * u_0\}(x)$, where the kernel K_t is defined by

$$K_t(x) = \frac{t}{\pi(t^2+x^2)} \qquad \text{if} \quad t>0 \quad \text{and} \quad K_0 = \delta_0, \qquad (4.2)$$

the Fourier transform of which is $\mathrm{e}^{-|k|t}$. Before getting to the inhomogeneous linear equation, we underline some interesting properties of the kernel K_t. First, for all $t \geq 0$, K_t is a probability measure. Moreover, for all $t > 0$, K_t is a smooth function. In particular, the space derivative of K_t satisfies

$$\left\| \frac{\mathrm{d}}{\mathrm{d}x} K_t \right\|_{\mathrm{L}^1(\mathbb{R})} \leq C t^{-1}, \tag{4.3}$$

where C is a universal constant. In all these aspects, K_t is similar to the Gaussian kernel $\mathcal{K}_t(x) = \mathrm{e}^{-\frac{x^2}{2t^2}} / (t\sqrt{2\pi})$.

The semi-group generated by K_t allows for solving the inhomogeneous equation

$$\begin{cases} \partial_t u(t,x) + |\partial_x| u(t,x) = g(t,x) & \text{for} \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x) & \text{for} \quad x \in \mathbb{R}. \end{cases} \tag{4.4}$$

Indeed, let $T > 0$, $u_0 \in \mathrm{L}^\infty(\mathbb{R})$ and $g \in \mathrm{L}^\infty([0,T] \times \mathbb{R})$. Then there exists a unique weak solution $u \in \mathrm{L}^\infty([0,T] \times \mathbb{R})$ to (4.4) in the sense that, for all $\phi \in \mathrm{C}_c^1([0,T), \mathrm{C}_c^\infty(\mathbb{R}))$, the following identity holds:

$$\int_0^T \int_{\mathbb{R}} u(t,x) (-\partial_t + |\partial_x|) \, \phi(t,x) \mathrm{d}x \mathrm{d}t - \int_{\mathbb{R}} u_0(x) \phi(0,x) \mathrm{d}x$$

$$= \int_0^T \int_{\mathbb{R}} g(t,x) \phi(t,x) \mathrm{d}x \mathrm{d}t. \tag{4.5}$$

This solution can be written, thanks to the Duhamel formula, as

$$u(t,x) = K_t * u_0(x) + \left\{ \int_0^t K_{t-s} * g(s,\cdot) \mathrm{d}s \right\}(x), \tag{4.6}$$

with the convention that $K_0 * u_0 = u_0$, even if u_0 is not regular. The existence of a solution u to (4.4) is a consequence of the fact that (4.6) is well-defined; its uniqueness is shown using the adjoint problem of (4.4).

We now turn to the semi-linear equation (1.15). Let $F \in \mathrm{W}^{2,\infty}(\mathbb{R})$ and $u_0 \in \mathrm{L}^\infty(\mathbb{R})$. Then there exists a unique weak solution $u \in \mathrm{L}_{\mathrm{loc}}^\infty(\mathbb{R}_+, \mathrm{L}^\infty(\mathbb{R}))$ to (1.15). Moreover, u can be expressed as

$$u(t,x) = \{K_t * u_0\}(x) - \left\{ \int_0^t K_{t-s} * F'(u(s,\cdot)) \mathrm{d}s \right\}(x). \tag{4.7}$$

The proof is done by a classical fixed-point argument on (4.7) (see for example [6, Sec. 4.3 p. 56]).

Finally, we justify that the evolution equation (1.15) has a regularizing effect; in other words, the weak solution to (1.15) becomes instantly a classical solution. Assume indeed that $F \in \mathrm{C}^3(\mathbb{R}) \cap \mathrm{W}^{3,\infty}(\mathbb{R})$ and $u_0 \in \mathrm{L}^\infty(\mathbb{R})$, and let $u \in \mathrm{L}_{\mathrm{loc}}^\infty(\mathbb{R}_+, \mathrm{L}^\infty(\mathbb{R}))$ be the weak solution to (1.15). Then, for all $T_0 > 0$, we have (1.19). Therefore, for all $t > 0$, there holds

$$\partial_t u(t,x) + |\partial_x| u(t,x) = -F'(u(t,x)), \tag{4.8}$$

in the strong sense. Finally

$$u \in \mathrm{C}([0,+\infty[, \text{weak-}*\text{-}\mathrm{L}^\infty(\mathbb{R})). \tag{4.9}$$

The proof of (1.19) relies on an iterative argument based on (4.7), using the fact that, for all $t > 0$, K_t is a smooth probability measure that satisfies (4.3). Last, a straightforward adaptation of the proof of [2, Ex. 4.24 p. 126] yields (4.9).

Note that, similarly to [1, Th. 2.9.], $t \mapsto u(t, \cdot)$ may not be continuous at 0 in $\mathrm{L}^\infty(\mathbb{R})$. If indeed u_0 is discontinuous, then $u(t, \cdot)$ cannot tend to u_0 in $\mathrm{L}^\infty(\mathbb{R})$ when $t \to 0$ because, invoking (1.19), $u(t, \cdot)$ is a continuous function for all $t > 0$.

5. Convergence of the evolution Equation (1.15) to the Weertman Equation (1.1)

In this section, we prove (ii) and (iii) of Theorem 1.1. The proof can be summarized in two steps: first, we show that (1.15) satisfies a comparison principle, then we use Chen's method of squeezing, establishing respectively (ii) and (iii) of Theorem 1.1. For the sake of self-consistency, simplicity and conciseness (Chen's theory being quite general), we prefer to restrict the whole proof of [7, Th. 3.1] to our specific case rather than to check that the hypotheses of Chen's theory are satisfied (precisely Hypotheses (A1), (A2), (A3), (B1), (B2) and (B3) of [7], which are indeed satisfied in our case).

We henceforth assume that $F \in \mathrm{C}^3(\mathbb{R}) \cap \mathrm{W}^{3,\infty}(\mathbb{R})$ satisfies (1.3) and (1.17). We introduce the non-linear operator $\mathcal{A}[u] := -|\partial_x| u - F'(u)$, of which, we now discuss some immediate properties. By the results of Section 4, $\mathcal{A}$ generates a semi-group on the Banach space $\mathrm{L}^\infty(\mathbb{R})$. $\mathcal{A}$ is translation invariant; namely, for all $h \in \mathbb{R}$, and for any function $u(x)$, there holds

$$\mathcal{A}[u(h + \cdot)](x) = \mathcal{A}[u](x + h), \qquad \forall x \in \mathbb{R}. \tag{5.1}$$

Moreover, $\mathcal{A}$ maps constant functions to constant functions; namely

$$\mathcal{A}[\alpha \cdot 1] = -F'(\alpha) \cdot 1,$$

for all $\alpha \in \mathbb{R}$, where 1 above denotes the function identically equal to 1.

The operator $\mathcal{A}$ satisfies the following comparison principle:

PROPOSITION 5.1.   *Let $F \in \mathrm{W}^{2,\infty}(\mathbb{R})$. Let $\overline{u}$ and $\underline{u} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{L}^\infty(\mathbb{R}))$ be such that*

$$\begin{cases} \partial_t \underline{u}(t,x) - \mathcal{A}[\underline{u}(t,\cdot)](x) = \underline{g}(t,x) \le 0, \\ \partial_t \overline{u}(t,x) - \mathcal{A}[\overline{u}(t,\cdot)](x) = \overline{g}(t,x) \ge 0, \end{cases} \tag{5.2}$$

where $\underline{g}$ and $\overline{g} \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{L}^\infty(\mathbb{R}))$, and $\underline{u}(0, \cdot) \le \overline{u}(0, \cdot)$ with $\underline{u}(0, \cdot) \ne \overline{u}(0, \cdot)$ on a non-negligible set. Then, for almost every $t > 0$, $x \in \mathbb{R}$, there holds

$$\underline{u}(t,x) < \overline{u}(t,x). \tag{5.3}$$

REMARK 5.1.   Proposition 5.1 has an immediate corollary: Assume that $u_0 \in \mathrm{L}^\infty(\mathbb{R})$ takes values in $[\eta_l - \Delta_0, \eta_r + \Delta_0]$ and let $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathrm{L}^\infty(\mathbb{R}))$ be the unique solution to (1.15) with the initial condition u_0. By (1.3), $\overline{u} := x \mapsto \eta_r + \Delta_0$ and $\underline{u} : x \mapsto \eta_l - \Delta_0$ are respectively super-solutions and sub-solutions to (1.15). Therefore, Proposition 5.1 implies that $u(t,x) \in [\eta_l - \Delta_0, \eta_r + \Delta_0]$, for almost every $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$, thus establishing (ii) of Theorem 1.1.

Proposition 5.1 is shown thanks to the Duhamel's Formula (4.6) and Grönwall's Lemma.

Proof.   Let $M > \|F''\|_{\mathrm{L}^\infty(\mathbb{R})}$. We set

$$v(t,x) := \mathrm{e}^{Mt}(\overline{u}(t,x) - \underline{u}(t,x)), \tag{5.4}$$

and prove that $v \geq 0$. In view of (5.2), we have

$$\partial_t v(t,x) + |\partial_x| v(t,x) = Mv(t,x) - e^{Mt}(F'(\overline{u}(t,x)) - F'(\underline{u}(t,x)) + g(t,x),$$

where $g(t,x) := e^{Mt}(\overline{g}(t,x) - \underline{g}(t,x)) \geq 0$. Since the right-hand side of the latter equation is in $L^\infty_{\text{loc}}(\mathbb{R}_+, L^\infty(\mathbb{R}))$, then, using (4.6), one can express $v(t,x)$ as

$$v(t,x) = \{K_t * v(0,\cdot)\}(x) + \left\{ \int_0^t K_{t-s} * \left[Mv(s,\cdot) \right. \right.$$
$$\left. \left. - e^{Ms}(F'(\overline{u}(s,\cdot)) - F'(\underline{u}(s,\cdot)) + g(s,\cdot)) \right] ds \right\}(x). \tag{5.5}$$

We introduce $v_-(t) := -\text{ess inf}\{\min(0, v(t,x)), x \in \mathbb{R}\}$. By Taylor expansion, for almost every $(s,y) \in [0,t] \times \mathbb{R}$, there holds

$$Mv(s,y) - e^{Ms}(F'(\overline{u}) - F'(\underline{u}))(s,y) \geq Mv(s,y) - \|F''\|_{L^\infty(\mathbb{R})} e^{Ms}|\overline{u}(s,y) - \underline{u}(s,y)|$$
$$\geq Mv(s,y) - M|v(s,y)|$$
$$\geq -2Mv_-(s). \tag{5.6}$$

Therefore, using (5.5), since K_t, g and $v(0,\cdot)$ are nonnegative, and since K_t is a probability measure for all $t \geq 0$, we obtain, for almost every $t \in \mathbb{R}_+$ and $x \in \mathbb{R}$,

$$-v(t,x)) \leq 2M \int_0^t v_-(s)ds,$$

whence

$$v_-(t) \leq 2M \int_0^t v_-(s)ds. \tag{5.7}$$

Since $\underline{u}, \overline{u} \in L^\infty_{\text{loc}}(\mathbb{R}_+, L^\infty(\mathbb{R}))$, then, $v_- \in L^\infty_{\text{loc}}(\mathbb{R}_+)$. Hence, by Grönwall's Lemma, we deduce from (5.7) that $v_-(t) = 0$, for almost every $t > 0$. Injecting this information in (5.6), and next in (5.5), yields $v(t,x) \geq \{K_t * v(0,\cdot)\}(x)$. As a consequence, as K_t is positive if $t > 0$ and as $v(0,\cdot) = \overline{u}(0,\cdot) - \underline{u}(0,\cdot)$ is nonnegative and positive on a non-negligible set, we deduce that $v(t,x) > 0$, for almost every $t > 0$ and $x \in \mathbb{R}$. This implies (5.3).   □

Then, we establish a stronger version of the comparison principle, the proof of which mimicks that of Proposition 5.1:

COROLLARY 5.1.   *Under the assumptions of Proposition 5.1, there exists a positive decreasing function ρ such that, for all $R > 1$,*

$$\text{ess inf}_{x \in [-R,R]}(\overline{u}(1,x) - \underline{u}(1,x)) \geq \rho(R) \int_0^1 (\overline{u}(0,y) - \underline{u}(0,y))dy. \tag{5.8}$$

Proof.   Introducing v defined by (5.4), and using (5.5) and (5.6), we obtain

$$v(t,x) \geq \{K_t * v(0,\cdot)\}(x),$$

since v is nonnegative. By definition of K_t and of v, it implies (5.8).   □

The proof of (iii) of Theorem 1.1 follows [7, Th. 3.1], the proof of which we restrict here to our particular case. By the previous steps, we already know that there exists a unique weak solution $u(t,x)$ to (1.15) (see Remark 5.1), and we aim at establishing (1.21).

Namely, we build special sub-solutions and super-solutions to (1.15) that are based on the existing solution to (1.1) (see Lemma 5.1 below). Then, we prove that both the vertical and the horizontal distances between these solutions are controlled (respectively $2\delta_j$ and $2l_j$ on figure 5, see also Lemma 5.2 below). Using the fact that (1.15) is an autonomous system, we use the established control to iteratively build successive sub-solutions w_{-1}^j and super-solutions w_{+1}^j surrounding the actual solution $u(t_j,x)$ to (1.15). At each step j, the distance between these sub-solutions and super-solutions is lowered. Thus, the solution u is *squeezed* between these sub-solutions and super-solutions. As a consequence, when t goes to infinity, the solution $u(t,\cdot)$ tends, up to a translation, to the solution to (1.1). Because of the iterative nature of the squeezing, this convergence is achieved with exponential speed.
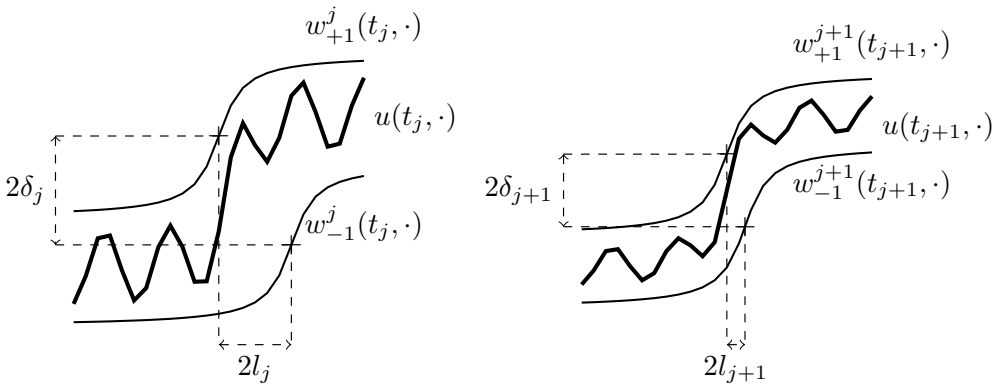

FIG. 5.1. *Squeezing of $u(t,x)$ solution to* (1.5).

LEMMA 5.1 (Lemma 2.2 of [7]). *Under the hypotheses of Theorem 1.1, let $\Delta_1 < \Delta_0$. Then, there exist positive constants σ and β such that, for all $\delta \in (0,\Delta_1)$ and $l \in \mathbb{R}$, the functions $w_{-1}(t,x)$ and $w_{+1}(t,x)$ defined by*

$$w_i(t,x) = \eta(\zeta_i(t,x)) + i\delta e^{-\beta t} \qquad for \quad i \in \{-1,+1\}, \qquad (5.9)$$

where

$$\zeta_i(t,x) = x - ct + il + i\sigma\delta\left[1 - e^{-\beta t}\right] \qquad for \quad i \in \{-1,+1\}, \qquad (5.10)$$

are respectively a sub-solution and a super-solution to (1.15).

Proof. As $\mathcal{A}$ is invariant by translation, the variable l in the definition of ζ plays no role. Hence, we take $l = 0$ in the proof below. We also impose for the moment $\sigma \leq 1$.

A straightforward computation yields

$$(\partial_t + |\partial_x|)w_i(t,x) = \left(i\sigma\beta\delta e^{-\beta t/2} - c\right)\eta'(\zeta_i(t,x)) - i\delta\beta e^{-\beta t} + |\partial_x|\eta(\zeta_i(t,x)),$$

and, as η satisfies (1.1),

$$(\partial_t + |\partial_x|)\, w_i(t,x) = i\sigma\beta\delta e^{-\beta t}\eta'\left(\zeta_i(t,x)\right) - i\beta\delta e^{-\beta t} - F'\left(\eta\left(\zeta_i(t,x)\right)\right).$$

Thus

$$(\partial_t - \mathcal{A})\, w_i(t,x) = i\beta\delta e^{-\beta t}\left[\sigma\eta'\left(\zeta_i(t,x)\right) - 1\right] + F'\left(w_i(t,x)\right) - F'\left(\eta\left(\zeta_i(t,x)\right)\right).$$

By Taylor-Lagrange expansion, there exists a convex combination $\theta^i(t,x)$ of $\eta(\zeta_i(t,x))$ and $w_i(t,x)$ such that

$$(\partial_t - \mathcal{A})\, w_i(t,x) = i\delta e^{-\beta t}\left[\beta\sigma\eta'\left(\zeta_i(t,x)\right) - \beta + F''\left(\theta^i(t,x)\right)\right]. \tag{5.11}$$

Recall that $\Delta_0 > \Delta_1$. Then, by (1.2), there exists $R_0 > 0$ such that

$$\begin{cases} |\eta(y) - \eta_{\mathrm{l}}| < (\Delta_0 - \Delta_1)/2 & \text{if} \quad y < -R_0, \\ |\eta(y) - \eta_{\mathrm{r}}| < (\Delta_0 - \Delta_1)/2 & \text{if} \quad y > R_0. \end{cases}$$

Therefore, as $\sigma\delta < \Delta_1$, we also have

$$\begin{cases} |w_i(t,x) - \eta_{\mathrm{l}}| \le (\Delta_0 + \Delta_1)/2 < \Delta_0 & \text{if} \quad \zeta_i(t,x) < -R_0, \\ |w_i(t,x) - \eta_{\mathrm{r}}| \le (\Delta_0 + \Delta_1)/2 < \Delta_0 & \text{if} \quad \zeta_i(t,x) > R_0. \end{cases}$$

Let

$$\beta := \inf\left\{F''(v), |v - \eta_{\mathrm{r}}| \le \Delta_0 \quad \text{or} \quad |v - \eta_{\mathrm{l}}| \le \Delta_0\right\} \tag{5.12}$$

Thus, if $|\zeta_i(t,x)| > R_0$, by definition of β and of $\theta^i(t,x)$, $F''\left(\theta^i(t,x)\right) - \beta \ge 0$. Moreover, $\eta' > 0$. Therefore

$$\beta\sigma\eta'\left(\zeta_i(t,x)\right) + F''\left(\theta^i(t,x)\right) - \beta \ge 0. \tag{5.13}$$

Now, we set

$$\sigma := \min\left\{\beta^{-1}\left(\inf_{|y| < R_0}\eta'(y)\right)^{-1}\left(\|F''\|_{\mathrm{L}^\infty(\mathbb{R})} + \beta\right), 1\right\}. \tag{5.14}$$

Therefore, if $|\zeta_i(t,x)| \le R_0$, we also have (5.13). As a conclusion, in any case, (5.11) and (5.13) yield

$$i\left(\partial_t - \mathcal{A}\right) w_i(t,x) \ge 0,$$

which implies that w_{+1} and w_{-1} are respectively a super-solution and a sub-solution to (1.15).   □

Using Lemma 5.1, it is eventually possible to squeeze a solution $u(t,x)$ of (1.15) between a sub-solution and a super-solution. The following Lemma explains how this squeezing is tightened:

LEMMA 5.2 (Lemma 3.3 of [7]).   *Under the hypotheses of Theorem 1.1, let $\Delta_1 < \Delta_0$. Assume that there exist $\xi \in \mathbb{R}$, $\delta \in (0, \Delta_1)$ and $l \in [0, L]$ for fixed L such that, for all $x \in \mathbb{R}$,*

$$\eta(x - l) - \delta \le u(0,x) \le \eta(x + l) + \delta. \tag{5.15}$$

Then, taking β and σ as in Lemma 5.1, there exist a positive constant ϵ_* depending only on η, F', and L, and parameters $\tilde{\xi}$, $\tilde{\delta}$, $\tilde{l}$ satisfying

$$\left|\tilde{\xi}\right| \leq l, \quad \tilde{\delta} = \delta + \epsilon_* \min(1,l) \mathrm{e}^\beta, \quad and \quad 0 \leq \tilde{l} \leq l + \sigma\delta - \frac{\sigma\epsilon_* \min(1,l)}{2},$$

such that, for all $t \geq 1$ and $x \in \mathbb{R}$,

$$\eta(x - \tilde{\xi} - \tilde{l} - ct) - \tilde{\delta}\mathrm{e}^{-\beta t} \leq u(t,x) \leq \eta(x - \tilde{\xi} + \tilde{l} - ct) + \tilde{\delta}\mathrm{e}^{-\beta t}. \tag{5.16}$$

Proof. Thanks to Lemma 5.1, the functions w_{+1} and w_{-1} defined by (5.9), for ζ_i defined by (5.10), are respectively a super-solution and a sub-solution to (1.15). Using (5.15), it follows from Proposition 5.1 that, for all $t \geq 0$ and $x \in \mathbb{R}$,

$$w_{-1}(t,x) \leq u(t,x) \leq w_{+1}(t,x). \tag{5.17}$$

Let $\hat{l} := \min(1,l)$ and $\epsilon_1 := \inf_{x \in [-1,2]} \eta'(x)$. Since η is increasing, a Taylor expansion yields

$$\int_0^1 (\eta(x+l) - \eta(x-l)) \, \mathrm{d}x \geq \int_0^1 \left(\eta\left(x + \hat{l}\right) - \eta\left(x - \hat{l}\right) \right) \mathrm{d}x \geq 2\epsilon_1 \hat{l}.$$

Therefore, at least one of the following estimates is true

$$\int_0^1 (u(0,x) - \eta(x-l)) \, \mathrm{d}x \geq \epsilon_1 \hat{l} \quad \text{or} \quad \int_0^1 (\eta(x+l) - u(0,x)) \, \mathrm{d}x \geq \epsilon_1 \hat{l}.$$

Hereafter, we only consider the first case, as the second one is similar. First, using (1.20), there exists R_1 such that

$$2\sigma\eta'(x) \leq 1 \quad \text{if} \quad |x| > R_1 \tag{5.18}$$

Let $R_2 := R_1 + L + |c| + 1 + \sigma\Delta_0$. On the one hand, invoking Proposition 5.1, we compare u and w_{-1} on $[-R_2, R_2]$

$$\inf_{x \in [-R_2, R_2]} \left\{ u(1,x) - \eta(\zeta_{-1}(1,x)) + \delta\mathrm{e}^{-\beta} \right\} \geq \rho(R_2) \int_0^1 [u(0,y) - \eta(y-l) + \delta] \, \mathrm{d}y$$

$$\geq \rho(R_2)\epsilon_1 \hat{l}. \tag{5.19}$$

We define

$$\epsilon_* := \min\left(\Delta_1 \left(1 - \mathrm{e}^{-\beta}\right), \frac{1}{2\sigma}, \frac{\rho(R_2)\epsilon_1}{2\sigma} \left\|\eta'\right\|_{\mathrm{L}^\infty(\mathbb{R})}^{-1} \right). \tag{5.20}$$

As a consequence, if $|x| < R_2$, (5.19) yields

$$u(1,x) - \eta\left(\zeta_{-1}(1,x) + 2\epsilon_*\sigma\hat{l}\right) + \delta\mathrm{e}^{-\beta} \geq \rho(R_2)\epsilon_1 \hat{l} - 2\epsilon_*\sigma\hat{l}\left\|\eta'\right\|_{\mathrm{L}^\infty(\mathbb{R})} \geq 0. \tag{5.21}$$

On the other hand, if $|x| > R_2$, then $|\zeta_{-1}(1,x)| \geq R_1 + 1$ whence, by definition of ϵ^*, $\left|\zeta_{-1}(1,x) + 2\epsilon_*\sigma\hat{l}\right| \geq R_1$. Inequality (5.17) and Definition (5.18) then imply that

$$u(1,x) - \eta\left(\zeta_{-1}(1,x) + 2\epsilon_*\sigma\hat{l}\right) + \delta\mathrm{e}^{-\beta} \geq u(1,x) - w_{-1}(1,x) - \epsilon_*\hat{l} \quad \geq -\epsilon_*\hat{l}. \tag{5.22}$$

Therefore, from (5.21) and from (5.22), it appears that, for all $x \in \mathbb{R}$, there holds

$$u(1,x) \geq \eta \left(\zeta_{-1}(1,x) + 2\epsilon_*\sigma\hat{l} \right) - \left(\epsilon_*\hat{l} + \delta\mathrm{e}^{-\beta} \right).$$

We set

$$\tilde{\delta} := \left(\delta\mathrm{e}^{-\beta} + \epsilon_*\hat{l} \right) \mathrm{e}^{\beta},$$

which, thanks to (5.20), satisfies $\tilde{\delta}\mathrm{e}^{-\beta} \leq \Delta_1$. Applying once more Lemma 5.1 yields, for all $t \geq 1$,

$$u(t,x) \geq \eta \left(\zeta_{-1}(1,x) - c(t-1) + 2\epsilon_*\sigma\hat{l} - \sigma\tilde{\delta}\mathrm{e}^{-\beta} \left[1 - \mathrm{e}^{-\beta(t-1)} \right] \right) - \tilde{\delta}\mathrm{e}^{-\beta t}. \qquad (5.23)$$

By definition of $\tilde{\delta}$ and ζ_{-1}, the argument of η in the above estimate is

$$x - ct - l - \sigma\delta \left[1 - \mathrm{e}^{-\beta} \right] + 2\epsilon_*\sigma\hat{l} - \sigma\tilde{\delta}\mathrm{e}^{-\beta} \left[1 - \mathrm{e}^{-\beta(t-1)} \right]$$
$$\geq x - ct - \left[l + \sigma\delta - \sigma\epsilon_*\hat{l} \right]. \qquad (5.24)$$

Defining now

$$\tilde{\xi} := -\frac{\sigma\epsilon_*\hat{l}}{2}, \quad \text{and} \quad \tilde{l} := l + \sigma\delta - \frac{\sigma\epsilon_*\hat{l}}{2},$$

and bearing in mind that η is increasing, we deduce from (5.23) and (5.24) that

$$u(t,x) \geq \eta \left(x - ct - \tilde{\xi} - \tilde{l} \right) - \tilde{\delta}\mathrm{e}^{-\beta t}. \qquad (5.25)$$

Moreover, recalling (5.17), we have

$$u(t,x) \leq \eta \left(x - ct + l + \sigma\delta \right) + \delta\mathrm{e}^{-\beta t} \leq \eta \left(x - ct + \tilde{l} - \tilde{\xi} \right) + \tilde{\delta}\mathrm{e}^{-\beta t}. \qquad (5.26)$$

As a consequence, we obtain the desired result (5.16) from (5.25) and (5.26). $\qquad \square$

We are now in a position to finish the proof of Theorem 1.1. The proof is done while iterating Lemma 5.2, which gradually tightens the squeezing around $u(t,x)$.

Proof. **(Proof of (iii) of Theorem 1.1 – restriction of the proof of Theorem 3.1 of [7].)** We proceed in four steps, lowering iteratively in time the values δ and l such that, for all $x \in \mathbb{R}$, there holds

$$\eta \left(x - ct - \xi - l \right) - \delta \leq u(t,x) \leq \eta \left(x - ct - \xi + l \right) + \delta. \qquad (5.27)$$

Step 1.   By assumption (1.18) and since η is increasing from η_l to η_r, there exist $\Delta_1 < \Delta_0$, and $L > 1$ sufficiently large such that (5.27) holds with

$$t = t_1 := 0, \quad \delta = \delta_1 := \Delta_1, \quad \xi = \xi_1 := 0, \quad \text{and} \quad l = l_1 := L - \sigma\Delta_0.$$

Step 2. We define

$$\delta_* := \min(\Delta_1, \epsilon_*/4) \quad \text{and} \quad \kappa_* := \sigma \epsilon_*/2 - \sigma \delta_* \geq \sigma \epsilon_*/4 > 0. \tag{5.28}$$

Also, we set $t_* \geq 2$ such that

$$\mathrm{e}^{-\beta t_*}(1 + \epsilon_*/\delta_*)\mathrm{e}^{\beta} \leq 1 - \kappa_*. \tag{5.29}$$

Using Lemma 5.1, we deduce from the previous step that there exists t_2 sufficiently high such that (5.27) holds with $t = t_2$, $\delta = \delta_2 = \delta_*$ and for a certain $\xi \in \mathbb{R}$ and $l = l_2 \leq L$ (as ϵ_* implicitly depends on L, we further ensure that $l_j \leq L$, for all j).

If $l_2 \leq 1$, one directly goes to Step 3. Otherwise, as long as $l_j > 1$, one applies Lemma 5.2 at time $t_j = t_2 + (j - 2)t_*$ (recall that (1.15) is an autonomous evolution equation), $\delta_j = \delta_*$ and get, by (5.28) and (5.29), that (5.27) holds for $t = t_{j+1}$ with $l \leq l_j - \kappa_*$ and $\delta \leq (1 - \kappa_*)\delta_*$. Therefore, one can take $\delta_{j+1} := \delta_*$, $l_{j+1} := l_j - \kappa_*$ and iterate until $l_j < 1$.

Step 3. By Step 2, we have an index j_0 such that (5.27) holds for $t = t_{j_0}$, $\delta = \delta_{j_0} = \delta_*$, $\xi_{j_0} \in \mathbb{R}$ and $l = l_{j_0} = 1$. Using Lemma 5.2 and Definitions (5.28) and (5.29), a straightforward computation inductively shows that, for all $j \geq 0$, Inequalities (5.27) hold for $t = t_{j+j_0}$, $\delta = \delta_{j+j_0}$, and $l = l_{j+j_0}$ being defined by

$$t_{j+j_0} := t_{j_0} + jt_*, \quad \delta_{j_0+j} := (1 - \kappa_*)^j \delta_* \quad \text{and} \quad l_{j_0+j} := (1 - \kappa_*)^j, \tag{5.30}$$

and for $\xi = \xi_j$ such that $|\xi_{j_0+j+1} - \xi_{j_0+j}| \leq (1 - \kappa_*)^j$.

Step 4. We have shown that (5.27) holds for $(t, \delta, l) = (t_j, \delta_j, l_j)$, for all $j \geq 0$. For $t > 0$, we associate j implicitly defined by $t \in [t_j, t_{j+1})$. Thus, we deduce from Lemma 5.1 that (5.27) also holds for t, $\delta = \delta_j$, $\xi = \xi_j$, and $l = l_j + \sigma \delta_j$. Taking Step 3 into account yields, for all $j > j_0$,

$$\delta \leq \delta_*(1 - \kappa_*)^{j - j_0} \quad \text{and} \quad l \leq (1 + \sigma \delta_*)(1 - \kappa_*)^{j - j_0}.$$

Moreover, ξ_j converges to ξ_∞ and, for all $j > j_0$,

$$|\xi_j - \xi_\infty| \leq \kappa_*^{-1}(1 - \kappa_*)^{j - j_0}.$$

Yet, a simple calculation shows

$$(1 - \kappa_*)^{j - j_0} = (1 - \kappa_*)^{-j_0 - t_{j_0}/t_*} \exp(t \ln(1 - \kappa_*)/t_*).$$

Setting $\kappa := -\ln(1 - \kappa_*)/t_* > 0$ and

$$K := \left[\delta_* + \left(1 + \sigma \delta_* + \kappa_*^{-1}\right) \|\eta'\|_{\mathrm{L}^\infty(\mathbb{R})} \right] (1 - \kappa_*)^{-j_0 - t_{j_0}/t_*},$$

we obtain, for all $t \geq t_{j_0}$, $t \in [t_j, t_{j+1})$,

$$\sup_{x \in \mathbb{R}} |\eta(x - ct - \xi_\infty) - u(t, x)| \leq \delta_j + (|l_j| + |\xi_j - \xi_\infty|) \|\eta'\|_{\mathrm{L}^\infty(\mathbb{R})} \leq K\mathrm{e}^{-\kappa t}.$$

This concludes the proof of Theorem 1.1. □

REFERENCES

[1] F. Achleitner and C. Kuehn, *Traveling waves for a bistable equation with nonlocal diffusion*, Adv. Diff. Eqs., 20(9-10):887–936, 2015. 1.2, 4

[2] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer Science & Business Media, 2010.

[3] X. Cabré, N. Cónsul, and J. Mandé, *Traveling wave solutions in a half-space for boundary reactions*, Analysis & PDE, 8(2):333–364, 2015. 4

[4] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians I*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31(1):23–53, 2014. 1.2

[5] X. Cabré and Y. Sire, *Nonlinear equations for fractional Laplacians II*, Trans. Amer. Math. Soc., 367(2):911–941, 2015. 1.2, 1.2, 1.2, 1.3

[6] T. Cazenave and A. Haraux, *An Introduction to Semilinear Evolution Equations*, Oxford Lecture Series in Mathematics and its Applications, The Clarendon Press, Oxford University Press, New York, 13, 1998. 1.2, 1.2, 1.2, 1.3

[7] X. Chen, *Existence, uniqueness, and asymptotic stability of traveling waves in nonlocal evolution equations*, Adv. Diff. Eqs., 2(1):125–160, 1997. 4, 4

[8] A. Chmaj, *Existence of traveling waves in the fractional bistable equation*, Arch. Math. (Basel), 100(5):473–480, 2013. 1.2, 5, 5, 5.1, 5.2, 5

[9] P.C. Fife and J.B. McLeod, *The approach of solutions of nonlinear diffusion equations to travelling front solutions*, Arch. Rat. Mech. Anal., 65(4):335–361, 1977. 1.2

[10] C. Gui and M. Zhao, *Traveling wave solutions of Allen-Cahn equation with a fractional Laplacian*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 32(4):785–812, 2015. 1.2, 1.2

[11] J.P. Hirth and J. Lothe, *Theory of Dislocations*, John Wiley & Sons, 1982. 1.2, 1.2, 1.2, 1.3, 3

[12] M. Josien, *Thèse de l'Université Paris Est*, in preparation. 1.1

[13] M. Josien, Y.-P. Pellegrini, F. Legoll, and C. Le Bris, *Fourier-based numerical approximation of the Weertman equation for moving dislocations*, International Journal for Numerical Methods in Engineering, 113:1827–1850, 2018. 3.1, 4

[14] N.I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Noordhoff International Publishing, Leiden, 1977. 1.1, 1.3, 1.3

[15] Y. Nec, A.A. Nepomnyashchy, and A.A. Golovin, *Front-type solutions of fractional Allen-Cahn equation*, Phys. D, 237(24):3237–3251, 2008. 3

[16] R. Peierls, *The size of a dislocation*, Proceedings of the Physical Society, 52(1):34, 1940. 1.1

[17] Y.-P. Pellegrini and M. Josien, In preparation. 1.1

[18] P. Rosakis, *Supersonic dislocation kinetics from an augmented Peierls model*, Phys. Rev. Lett., 86(1):95, 2001. 1.3

[19] D.H. Sattinger, *On the stability of waves of nonlinear parabolic systems*, Advances in Math., 22(3):312–355, 1976. 1.1, 1.1, 3.1

[20] A.I. Volpert, V.A. Volpert, and V.A. Volpert, *Traveling wave solutions of parabolic systems*, Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 140, 1994. 1.2
    1.2