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ANALYTICAL VALIDATION OF A 2+1 DIMENSIONAL CONTINUUM MODEL FOR EPITAXIAL GROWTH WITH ELASTIC SUBSTRATE*

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Abstract. We consider the evolution equation

$$h_t = \Delta[\mathcal{F}^{-1}(-aE\mathcal{F}(h)) - r/h^2 - \Delta h], \tag{0.1}$$

introduced in [Wondimu Taye Tekalign and B.J. Spencer, J. Appl. Phys., 96(10):5505–5512, 2004] by Tekalign and Spencer to describe the heteroepitaxial growth of a two-dimensional thin film on an elastic substrate. In the expression above, h denotes the surface height of the film, $\mathcal{F}$ is the Fourier transform, and a, E, r are positive material constants. For simplicity, we set $aE = r = 1$. As this equation does not have any particular structure, its analysis is quite challenging. Therefore, we introduce the auxiliary equation (with c being a given constant)

$$u_t = \nabla[-\nabla\cdot u - (\nabla\cdot u + c)^{-2} - \Delta\nabla\cdot u], \tag{0.2}$$

which has a variational structure. Equivalency between (0.1) and (0.2) will hold under sufficient regularity on the solution. The main aim of this paper is to provide an analytical validation to (0.2), by proving existence and regularity properties for weak solutions, under suitable assumptions on the initial datum.

Keywords. epitaxial growth; wetting; maximal monotone operators.

AMS subject classifications. 35K55; 35K67; 44A15; 74K35.

1. Introduction

Epitaxial growth, a process in which a relatively thin film is deposited on a thick substrate, has gained interest in the recent years due to its applications to semiconductor electronics and quantum dots. Roughly speaking, the morphology of the film is known to be the result of a competition between the elastic energy associated with the mismatch between film and substrate, and the surface mass transport due to the film deposition. An extensive mathematical analysis of the mechanism associated with epitaxial growth has been carried out in the context of plane linear elasticity, and regularity results have been established for volume-constrained minimizers. For a nonexhaustive list, we cite the papers by Bella, Goldman and Zwicknagl [3], Bonnetier and Chambolle [4], Fonseca, Pratelli and Zwicknagl [10], and Fonseca *et al.* [5,7]. Also related are the works by Gao, Liu and Lu [11,12].

Short-time existence for a surface diffusion type geometric evolution equation, keeping into account elasticity, has first been analyzed by Fonseca *et al.* [6] for a two-dimensional setting (see also Piovano [18]). Subsequently, it has been recently extended in [8] to the three-dimensional case.

The central aim of this work is to study existence and regularity of solutions to the $2+1$ dimensional evolution equation

$$h_t = \Delta[\mathcal{F}^{-1}(-aE\mathcal{F}(h)) - r/h^2 - \Delta h] \tag{1.1}$$

derived in Tekalign and Spencer [20], with spatial domain $\mathbb{R}^2$, and time domain $[0,T]$, for given $T > 0$. Here $h : [0,T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ denotes the surface height of the film, and $\mathcal{F}$ is the Fourier transform. The notation h_t denotes the derivative with respect to the

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time variable t, while Δ denotes the Laplacian with respect to the space coordinates. The quantities a, E, r are positive material constants. More precisely, a (respectively r) is the wavenumber (respectively wetting coefficient) associated with the equation, and $E := \dfrac{2\mu^F(1+\nu^F)(1-\nu^S)}{(1-\nu^F)\mu^S}$, where μ^F and ν^F (respectively μ^S and ν^S) are the elastic shear modulus and the Poisson's ratio of the film (respectively substrate). Equation (1.1) arises in the context of growth of an epitaxially strained, dislocation-free, thin solid film on a deformable substrate in the absence of vapor deposition, and under the assumption of thin-film approximation. In Equation (1.1), we wrote the Fourier transform (and its inverse) purely for fidelity with the original equation derived in [20]. Moreover, the constants a, E and r, while important from a modeling perspective, will play no role in our analysis. Hence, for simplicity we set $aE = r = 1$, and (1.1) reads

$$h_t = \Delta[-h - h^{-2} - \Delta h]. \tag{1.2}$$

Equation (1.2) is a fourth order nonlinear equation, with no particular structure, thus making its analysis quite challenging. To avoid the excessive technical complexities of working on an unbounded domain, where crucial inequalities (such as the Poincaré inequality) do not hold, we restrict our spatial domain to be the unit disk

$$D := \{x \in \mathbb{R}^2 : |x| \le 1\},$$

and consider (1.2) with spatial domain D. Then, we introduce the auxiliary parabolic equation

$$\begin{cases} u_t = \nabla[-\nabla\cdot u - (\nabla\cdot u + c)^{-2} - \Delta\nabla\cdot u] & \text{in } (0,T)\times D, \\ \langle u,\nu\rangle = \nabla\cdot u = 0 & \text{on } (0,T)\times\partial D, \\ u(0,\cdot) = u^0 & \text{on } \{t=0\}\times D, \end{cases} \tag{1.3}$$

where $c > 0$ is a given constant, u^0 is a given initial datum, and ν denotes the exterior unit normal to D. Here $u : [0,T]\times D \longrightarrow \mathbb{R}^2$ is a suitable function such that $h = \nabla\cdot u + c$, while u_t denotes the time derivative, and ∇, $\nabla\cdot$, Δ are respectively the gradient, divergence and Laplacian operators with respect to the spatial coordinates. The main functional space is

$$\begin{aligned} V := \{&u \in L^2(D) : \nabla\cdot u \in L^2(D),\ \nabla\nabla\cdot u \in L^2(D), \\ &u = \nabla\psi \text{ for some function } \psi \in W^{1,2}(D),\ \langle u,\nu\rangle = \nabla\cdot u = 0 \text{ on } \partial D\}. \end{aligned} \tag{1.4}$$

We endow V with the norm

$$\|u\|_V := \|u\|_{L^2(D)} + \|\nabla\cdot u\|_{L^2(D)} + \|\nabla\nabla\cdot u\|_{L^2(D)}. \tag{1.5}$$

Here, and for future reference, for brevity, we will use $L^2(D)$ to denote both $L^2(D;\mathbb{R})$ and $L^2(D;\mathbb{R}^2)$. There is no risk of confusion since clearly no function can belong to both the spaces.

The relation between (1.2) and (1.3) is the following: if there exists a sufficiently regular solution u (e.g., if $u \in L^1(0,T;C_c^\infty(D))$ with $\nabla\cdot u + c$ *uniformly* bounded away from zero) such that (1.3) holds for a.e. x, t, then it can be straightforwardly checked that $h := \nabla\cdot u + c$ would be solution of (1.2). The presence of the constant c in $h = \nabla\cdot u + c$ is necessary due to the fact that we imposed $\int_D \nabla\cdot u\,dx = 0$ in the definition of V to use Poincaré inequality (which allows us to use the norm (2.1) below, instead of (1.5)), while our physical model imposes $h > 0$ a.e.. Consequently, $c > 0$.

However, our main result (Theorem 1.1 below) does *not* prove that $\nabla\cdot u+c$ is uniformly bounded away from zero. Thus we cannot infer the full equivalence between (1.2) and (1.3). Nonetheless, (1.3) could be considered a "weaker" version of the original (1.2).

The main aim of this paper is to prove existence and regularity properties of solutions to (1.3), under suitable assumptions on the initial datum. For future reference, the notation $\langle,\rangle$ (without any subscript) will denote the Euclidean scalar product of $\mathbb{R}^2$. The key advantage of working with u instead of h is that (1.3) has a variational structure: it can be written as $u_t=-\partial\phi(u)-Bu$, with $Bu=\nabla\Delta\nabla\cdot u+\nabla\nabla\cdot u$, and ϕ defined by

$$\Phi:\mathbb{R}\longrightarrow\mathbb{R}\cup\{+\infty\},\qquad \Phi(\xi):=\begin{cases}+\infty \text{ if } \xi\le 0,\\ \xi^{-1} \text{ if } \xi>0,\end{cases}$$
$$\phi:V\longrightarrow\mathbb{R}\cup\{+\infty\},\qquad \phi(u):=\int_D\Phi(\nabla\cdot u+c)\mathrm{d}x. \tag{1.6}$$

Lemma 2.3 (respectively Lemma 2.4) will prove that $\partial\phi(u)$ (respectively Bu) is maximal monotone.

The main result is:

THEOREM 1.1. *Given $T,c>0$, let $u^0\in V$ be an initial datum such that $\nabla\cdot u^0+c>0$ for a.e. $x\in D$, and assume there exists $z^0\in L^2(D)$ such that*

$$\int_D[\langle z^0,v-u^0\rangle+\langle\nabla\nabla\cdot u^0,\nabla\nabla\cdot(v-u^0)\rangle-(\nabla\cdot u^0)(\nabla\cdot(v-u^0))]\,dx+\phi(v)-\phi(u^0)\ge 0 \tag{1.7}$$

for any $v\in V$. Here ϕ is the functional defined in (1.6)*. Then there exists a unique weak solution*

$$u\in C^0([0,T];L^2(D))\cap L^\infty(0,T;V) \tag{1.8}$$

in the sense that

$$u(0)=u^0,\qquad u_t\in L^\infty(0,T;L^2(D)),$$

and

$$\int_D\left[\langle u_t(t),\varphi\rangle+\langle\nabla\nabla\cdot u(t),\nabla\nabla\cdot\varphi\rangle-(\nabla\cdot u(t))(\nabla\cdot\varphi)-\frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\right]dx=0 \tag{1.9}$$

for a.e. $t\in[0,T]$ and any $\varphi\in V\cap C_c^\infty(D)$.

Further discussion about condition (1.7) will be carried out in Lemma 3.1. We do not expect u to be a strong solution, since its regularity (1.8) is not sufficient to ensure that $\nabla\Delta\nabla\cdot u$ is well defined as a function.

There are two main difficulties in our analysis, both due to the nonlinear term $\nabla(\nabla\cdot u+c)^{-2}$:

(1) first, $(\nabla\cdot u+c)^{-2}$ is unbounded as $\nabla\cdot u+c$ approaches zero.
(2) Second, and more crucially, $\nabla(\nabla\cdot u+c)^{-2}$ is *not* the highest order term in (1.3). This makes it difficult to characterize it as *unique* sub-gradient of some proper, convex, lower-semicontinuous functional, since, despite our attempts, such arguments (see for instance [13, Proposition 15]) rely on density arguments which are difficult to apply unless the nonlinear term contains the highest order term. This prevents

us from using the theory of maximal monotone operators in Banach/Hilbert spaces, and the techniques in [13, 16]. This is because such techniques would only give a solution of

$$v_t(t)+\partial\phi(v(t))+\nabla\Delta\nabla\cdot v(t)+\nabla\nabla\cdot v(t)\ni 0,$$

for a.e. t, and it is unclear if such solution satisfies (1.3), as $\partial\phi(\cdot)$ could be potentially multivalued.

To overcome these issues, we will use the approach from [9], based on the theory of parabolic variational inequalities in reflexive Banach spaces. However, working in a two-dimensional spatial domain D makes the truncation argument from [9] much more delicate.

2. Preliminaries

The aim of this section is to present the setting of our problem, and to prove basic monotonicity estimates. We first check that V is a Banach space.

LEMMA 2.1. *The space V is a Banach space.*

Proof. Consider an arbitrary Cauchy sequence $(v_n)_n\subseteq V$. It is straightforward to check that the space

$$X:=\{u\in L^2(D):\nabla\cdot u\in L^2(D),\ \nabla\nabla\cdot u\in L^2(D),\ \langle u,\nu\rangle=\nabla\cdot u=0 \text{ on } \partial D\},$$

endowed with the norm

$$\|u\|_X:=\|u\|_{L^2(D)}+\|\nabla\cdot u\|_{L^2(D)}+\|\nabla\nabla\cdot u\|_{L^2(D)},$$

is a Banach space. Hence there exists $v\in X$ such that $\|v_n-v\|_X\to 0$. Note that a function $u\in X$ also satisfies $u\in V$ if and only if $u=\nabla\psi$ for some $\psi\in W^{1,2}(D)$. Now to prove $v\in V$, it suffices to check $v=\nabla\psi$ for some $\psi\in W^{1,2}(D)$. For any n it holds that $v_n\in V$, thus there exists ψ_n such that $v_n=\nabla\psi_n$. Without loss of generality, ψ_n can be assumed to have zero-average, hence, by Poincaré inequality, the sequence $(\psi_n)_n$ is bounded in $W^{1,2}(D)$. Thus, there exists $\eta\in W^{1,2}(D)$ such that $\nabla\psi_n\rightharpoonup\nabla\eta$ in $L^2(D)$, i.e. $v_n=\nabla\psi_n\rightharpoonup\nabla\eta$. Since $v_n\to v$ strongly in X (and hence in $L^2(D)$), it follows $v=\nabla\eta$, hence $v\in V$. ∎

LEMMA 2.2. *There exists a constant $C>0$ such that*

$$\|\nabla\cdot v\|_{L^2(D)}\geq C\|v\|_{L^2(D)}$$

for any $v\in V$.

Proof. The thesis follows from the fact that we required all functions $u\in V$ to be gradients of some $\psi\in W^{1,2}(D)$, hence we can apply [1, Theorem 5.4]. ∎

As functions $u\in V$ are required to have zero-average, by Poincaré inequality and Lemma 2.2, it follows that the norm $\|u\|_V$ is equivalent to $\|\nabla\nabla\cdot u\|_{L^2(D)}$. Thus, for future reference, we will use the new norm (equivalent to (1.5), but formally simpler)

$$\|u\|_V:=\|\nabla\nabla\cdot u\|_{L^2(D)}. \tag{2.1}$$

It is straightforward to check that V is reflexive. Moreover, the embeddings $V\hookrightarrow L^2(D)\hookrightarrow V'$ are both dense and continuous. Here, we identified $L^2(D)$ with its dual. Endow $L^2(D)$ with the standard scalar product

$$(\forall f,g\in L^2(D))\qquad \langle f,g\rangle_{L^2(D)}:=\int_D fg\,\mathrm{d}x.$$

Then, we can define, via integration by parts, a duality pairing between V' and V

$$(\forall v' \in V', v \in V) \qquad \langle v', v\rangle_{V',V} := \int_D v' v \mathrm{d}x.$$

We recall some useful definitions (see for instance [2]). Given a Banach space Y, denote by $\langle,\rangle_{Y',Y}$, the duality pairing between Y' and Y. Such a pairing can be defined by an inner product if there exists an Hilbert space Z such that $Y \subseteq Z = Z' \subseteq Y'$, and the embeddings $Y \hookrightarrow Z$ and $Z' \hookrightarrow Y'$ are dense and continuous. An operator $A : Y \longrightarrow Y'$ is:

(1) **monotone**, if for any $u, v \in Y$, it holds

$$\langle Au - Av, u - v\rangle_{Y',Y} \geq 0.$$

Similarly, a set $G \subseteq Y \times Y'$ is *"monotone"* if for any pair (u,u'), $(v,v') \in G$, it holds

$$\langle u' - v', u - v\rangle_{Y',Y} \geq 0.$$

(2) **maximal monotone**, if the graph

$$\Gamma_A := \{(u, Au) : u \in Y\} \subseteq Y \times Y'$$

is not a proper subset of any monotone set of $Y \times Y'$.

For future reference, the set $D_Y(A)$ will denote the domain of A, i.e.

$$D_Y(A) := \{u \in Y : Au \neq \emptyset\}.$$

The next two results prove that the right-hand side operator in (1.3) is maximal monotone.

LEMMA 2.3. *Then ϕ defined in* (1.6) *is proper, convex and lower-semicontinuous.*

Proof. It follows directly from the definition that ϕ is proper. Note that Φ is convex and lower-semicontinuous. We need to check convexity and lower-semicontinuity for ϕ.

Convexity. Consider arbitrary $u_1, u_2 \in V$. We need to check

$$(1-t)\phi(u_1) + t\phi(u_2) \geq \phi((1-t)u_1 + tu_2) \qquad \text{for all } t \in [0,1]. \tag{2.2}$$

We assume $\phi(u_1), \phi(u_2) < +\infty$ (otherwise (2.2) is trivial). The function Φ is convex, hence

$$\begin{aligned} \Phi(\nabla \cdot ((1-t)u_1 + tu_2) + c) &= \Phi((1-t)(\nabla \cdot u_1 + c) + t(\nabla \cdot u_2 + c)) \\ &\leq (1-t)\Phi(\nabla \cdot u_1 + c) + t\Phi(\nabla \cdot u_2 + c) \qquad \text{for a.e. } x \in D, \end{aligned} \tag{2.3}$$

and integrating on D gives the convexity of ϕ.

Lower-semicontinuity. Consider a sequence $(u_n)_n \subseteq V$, converging to $u \in V$. We need to check

$$\phi(u) \leq \liminf_{n \to +\infty} \phi(u_n). \tag{2.4}$$

Assume $\liminf_{n\to+\infty} \phi(u_n) < +\infty$, otherwise (2.4) is trivial. Assume also $\sup_n \phi(u_n) < +\infty$, which implies $\nabla \cdot u_n + c > 0$ for a.e. x and all n. Since $u_n \to u$ strongly in V, it follows $\nabla\cdot u_n\to\nabla\cdot u$ strongly in $L^2(D)$, hence (upon subsequence, which we do not relabel), $\nabla\cdot u_n\to\nabla\cdot u$ for a.e. $x\in D$, and $\Phi(\nabla\cdot u_n+c)\to\Phi(\nabla\cdot u+c)$ for a.e. $x\in D$. As by construction, we have $\Phi\geq 0$, by Fatou's lemma, we conclude

$$\int_D \Phi(\nabla\cdot u+c)\mathrm{d}x=\int_D \liminf_{n\to+\infty}\Phi(\nabla\cdot u_n+c)\mathrm{d}x\leq\liminf_{n\to+\infty}\int_D\Phi(\nabla\cdot u_n+c)\mathrm{d}x,$$

hence (2.4). ∎

Lemma 2.4. *The operator*

$$B:V\longrightarrow V',\qquad Bu:=\nabla\Delta\nabla\cdot u+\nabla\nabla\cdot u$$

is bounded, maximal monotone, and coercive in the sense that

$$(1+4/\pi^2)\|u-v\|_V^2\geq\langle Bu-Bv,u-v\rangle_{V',V}\geq(1-4/\pi^2)\|u-v\|_V^2. \tag{2.5}$$

Proof. By construction, B is linear. Observe that, given $u,v\in V$,

$$\begin{aligned}\langle Bu-Bv,u-v\rangle_{V',V}&=\langle B(u-v),u-v\rangle_{V',V}\\&=\int_D[\langle\nabla\nabla\cdot(u-v),\nabla\nabla\cdot(u-v)\rangle-(\nabla\cdot u)(\nabla\cdot v)]\mathrm{d}x\\&=\|\nabla\nabla\cdot(u-v)\|^2_{L^2(D)}-\|\nabla\cdot(u-v)\|^2_{L^2(D)}.\end{aligned}\tag{2.6}$$

Since $\nabla\cdot(u-v)$ has zero-average (as $u,v\in V$), by Poincaré inequality, it holds

$$\|\nabla\cdot(u-v)\|_{L^2(D)}\leq C\|\nabla\nabla\cdot(u-v)\|_{L^2(D)}$$

for some constant C. In [17], it has been proven that such optimal Poincaré constant C does not exceed d/π, where d is the diameter of the (spatial) domain. Since in our case the domain is the unit disk D, we have $C\leq 2/\pi$, hence (2.6) gives

$$\begin{aligned}\langle Bu-Bv,u-v\rangle_{V',V}&=\|\nabla\nabla\cdot(u-v)\|^2_{L^2(D)}-\|\nabla\cdot(u-v)\|^2_{L^2(D)}\\&\geq(1-4/\pi^2)\|\nabla\nabla\cdot(u-v)\|^2_{L^2(D)}=(1-4/\pi^2)\|u-v\|_V^2.\end{aligned}$$

To prove the boundedness of B, note that

$$\begin{aligned}|\langle Bu,v\rangle_{V',V}|&=\left|\int_D[\langle\nabla\nabla\cdot u,\nabla\nabla\cdot v\rangle-(\nabla\cdot u)(\nabla\cdot v)]\mathrm{d}x\right|\\&\leq\|\nabla\nabla\cdot u\|_{L^2(D)}\|\nabla\nabla\cdot v\|_{L^2(D)}+\|\nabla\cdot u\|_{L^2(D)}\|\nabla\cdot v\|_{L^2(D)}\\&\leq(1+4/\pi^2)\|\nabla\nabla\cdot u\|_{L^2(D)}\|\nabla\nabla\cdot v\|_{L^2(D)}=(1+4/\pi^2)\|u\|_V\|v\|_V \qquad (2.7)\\&\Longrightarrow\|Bu\|_{V'}=\sup_{v\neq0}\frac{|\langle Bu,v\rangle_{V',V}|}{\|v\|_V}\leq(1+4/\pi^2)\|u\|_V,\end{aligned}$$

where (2.7) follows from Poincaré inequality, and the fact that our optimal Poincaré constant does not exceed $2/\pi$. Thus $B:V\longrightarrow V'$ is bounded.

As the operator B is well defined everywhere on V, monotone, continuous, linear, and by [2, Corollary 2.1] we infer that B is maximal monotone. ∎

3. Proof of Theorem 1.1

Now we are ready to prove existence and regularity of solutions to (1.3). We will use the following theorem, due to Kačur, from [14] (also available as [15, Theorem 2]).

THEOREM 3.1. *Given a reflexive Banach space* $(\tilde V, \|\cdot\|_{\tilde V})$*, a Hilbert space* $(\tilde U, \langle,\rangle_{\tilde U})$ *such that the embeddings* $\tilde V \hookrightarrow \tilde U \hookrightarrow \tilde V'$ *are both dense and continuous, denote by* $\langle,\rangle_{\tilde V',\tilde V}$ *the duality pairing between* $\tilde V'$ *and* $\tilde V$.

Let $\tilde A : \tilde V \longrightarrow \tilde V'$ *be a maximal monotone operator, let* $\tilde\phi : \tilde V \longrightarrow (-\infty,+\infty]$ *be a proper, convex, lower semi-continuous functional. Let* $u^0 \in \tilde U$ *be a given initial datum, and suppose there exist:*

- v^0 *such that* $\tilde\phi(v^0) < +\infty$ *and*

$$\lim_{\|v\|_{\tilde V}\to+\infty} \frac{\langle \tilde A v, v - v^0\rangle_{\tilde V',\tilde V} + \tilde\phi(v)}{\|v\|_{\tilde V}} = +\infty, \tag{3.1}$$

- $z^0 \in \tilde U$ *such that for any* $v \in \tilde V$

$$\langle z^0, v\rangle_{\tilde U} + \langle A u^0, v\rangle_{\tilde V',\tilde V} + \tilde\phi(v) - \tilde\phi(u^0) \geq 0. \tag{3.2}$$

Then there exists a unique $u \in L^\infty(0,T;\tilde V) \cap C^0([0,T];\tilde U)$ *such that* $u_t \in L^\infty(0,T;\tilde U)$, $u(0) = u^0$, *and*

$$\langle u_t(t), v - u(t)\rangle_{\tilde U} + \langle \tilde A u(t), v - u(t)\rangle_{\tilde V',\tilde V} + \tilde\phi(v) - \tilde\phi(u(t)) \geq 0$$

for a.e. time $t \in (0,T)$*, and all* $v \in \tilde V$.

Applying Theorem 3.1 in our case gives:

THEOREM 3.2. *Given* $T > 0$ *and an initial datum* u^0 *satisfying* (1.7)*, there exists a unique function*

$$u \in L^\infty(0,T;V) \cap C^0([0,T];L^2(D))$$

such that

$$u_t \in L^\infty(0,T;L^2(D)), \qquad u(0) = u^0,$$

and

$$\langle u_t(t), v - u(t)\rangle_{L^2(D)} + \langle B u(t), v - u(t)\rangle_{V',V} + \phi(v) - \phi(u(t)) \geq 0 \tag{3.3}$$

for a.e. time $t \in (0,T)$*, and all* $v \in V$.

Proof. We will apply Theorem 3.1 with $\tilde A = B$, $\tilde\phi = \phi$, $\tilde V = V$, $\tilde U = L^2(D)$. We need to check that hypotheses (3.1) and (3.2) are satisfied.

Lemma 2.3 proved that ϕ is proper, convex, lower semi-continuous, and Lemma 2.4 proved that B is linear, bounded, and maximal monotone. Hypothesis (1.7) gives (3.2).

To check (3.1), note that, by construction, we have $\Phi \geq 0$, hence $\phi \geq 0$. Thus, taking $v^0 = 0$ (the function identically equal to 0), by (2.5) we have

$$\lim_{\|v\|_V\to+\infty} \frac{\langle Bv, v - v^0\rangle_{V',V} + \phi(v)}{\|v\|_V} \geq \lim_{\|v\|_V\to+\infty} \frac{(1-4/\pi^2)\|v\|_V^2}{\|v\|_V} = +\infty,$$

hence (3.1) is satisfied. Thus we are under the hypotheses of Theorem 3.2, hence there exists

$$u \in L^\infty(0,T;V) \cap C^0([0,T];L^2(D)) \tag{3.4}$$

such that

$$u_t \in L^\infty(0,T;L^2(D)), \qquad u(0) = u^0, \tag{3.5}$$

and

$$\langle u_t(t), v - u(t)\rangle_{L^2(D)} + \langle Bu(t), v - u(t)\rangle_{V',V} + \phi(v) - \phi(u(t)) \geq 0 \tag{3.6}$$

for a.e. time $t \in (0,T)$, and all $v \in V$. □

Now we can prove that the variational solution u given by Theorem 3.2 is a weak solution in the sense of (1.9).

Proof. (of Theorem 1.1.) We prove that the function u given by Theorem 3.2 satisfies (1.9). The idea would be to consider an arbitrary $\varphi \in V \cap C_c^\infty(D)$, and test (3.3) with test functions $u \pm \varepsilon\varphi$. However, since a priori we have only $\nabla \cdot u + c > 0$ a.e., we cannot guarantee that $\nabla \cdot (u \pm \varepsilon\varphi) + c$ is also a.e. positive. Thus, we test (3.3) with $v = u^\delta \pm \varepsilon\varphi$, for carefully chosen function u^δ and parameters $\varepsilon \ll 1$, $\delta = \delta(\varepsilon, \varphi)$, and then take the limit $\delta \to 0$. The main steps of the proof are:

(1) First, we prove that $(\nabla \cdot u(t) + c)^{-2} \in L^1(D)$.

(2) Second, we truncate $\nabla \cdot u(t)$ from below.

(3) Third, we test (3.3) with $v = u^\delta(t) \pm \varepsilon\varphi$, $\varepsilon > 0$, and take the limit $\delta \to 0^+$.

Step 1. Integrability of $(\nabla \cdot u(t) + c)^{-2}$. Fix an arbitrary t for which (3.6) holds. Note that for (3.6) to hold, it is required that

$$\phi(u(t)) = \int_D \Phi(\nabla \cdot u(t) + c)\mathrm{d}x < +\infty,$$

hence $\nabla \cdot u(t) + c > 0$ a.e., which in turn gives

$$\int_D \Phi(\nabla \cdot u(t) + c)\mathrm{d}x = \int_D (\nabla \cdot u(t) + c)^{-1}\mathrm{d}x < +\infty \qquad \text{for a.e. } t \in [0,T]. \tag{3.7}$$

Recall also that by hypothesis, we have $c > 0$. We test (3.6) with $v = (1-\varepsilon)u(t)$ (with $0 < \varepsilon \ll 1$). Since $\nabla \cdot u(t) + c > 0$ a.e., it follows

$$\nabla \cdot [(1-\varepsilon)u(t)] + c = (1-\varepsilon)(\nabla \cdot u(t) + c) + \varepsilon c > \varepsilon c \qquad \text{for a.e. } x \in D. \tag{3.8}$$

Plugging $v = (1-\varepsilon)u(t)$ in (3.6) gives

$$\langle u_t(t), -\varepsilon u(t)\rangle_{L^2(D)} + \langle Bu(t), -\varepsilon u(t)\rangle_{V',V} \geq -[\phi((1-\varepsilon)u(t)) - \phi(u(t))],$$

which implies, in view of the regularity properties (3.4) and (3.5),

$$\begin{aligned} +\infty > -\varepsilon\Big[\langle u_t(t), u(t)\rangle_{L^2(D)} + \|\nabla\nabla \cdot u(t)\|_{L^2(D)} - \|\nabla \cdot u(t)\|_{L^2(D)}\Big] \\ \geq -\phi((1-\varepsilon)u(t)) + \phi(u(t)) = -\int_D \Phi(\nabla \cdot [(1-\varepsilon)u(t)] + c)\mathrm{d}x + \int_D \Phi(\nabla \cdot u(t) + c)\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&\overset{(3.8)}{=} -\int_D [(\nabla\cdot[(1-\varepsilon)u(t)]+c)^{-1}-(\nabla\cdot u(t)+c)^{-1}]\mathrm{d}x\\
&= -\varepsilon\int_D \frac{\nabla\cdot u(t)}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\mathrm{d}x\\
&= -\varepsilon\int_D \frac{1}{\nabla\cdot[(1-\varepsilon)u(t)]+c}\mathrm{d}x+\varepsilon\int_D \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\mathrm{d}x. \qquad (3.9)
\end{aligned}
$$

In view of (3.8), we have $\nabla\cdot[(1-\varepsilon)u(t)]+c>(1-\varepsilon)(\nabla\cdot u(t)+c)$, hence

$$
\int_D \frac{1}{\nabla\cdot[(1-\varepsilon)u(t)]+c}\mathrm{d}x \le \frac{1}{1-\varepsilon}\int_D \frac{1}{\nabla\cdot u(t)+c}\mathrm{d}x<+\infty.
$$

Dividing by ε in (3.9) then gives

$$
\begin{aligned}
&+\infty > -[\langle u_t(t),u(t)\rangle_{L^2(D)}+\|\nabla\nabla\cdot u(t)\|_{L^2(D)}-\|\nabla\cdot u(t)\|_{L^2(D)}]+\frac{1}{1-\varepsilon}\int_D \frac{1}{\nabla\cdot u(t)+c}\mathrm{d}x\\
&\quad \ge \int_D \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\mathrm{d}x. \qquad (3.10)
\end{aligned}
$$

By the monotone convergence theorem, we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_D \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\mathrm{d}x\\
&=\lim_{\varepsilon\to 0}\int_{\{\nabla\cdot u\ge 0\}} \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\mathrm{d}x\\
&\qquad +\lim_{\varepsilon\to 0}\int_{\{\nabla\cdot u< 0\}} \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\mathrm{d}x\\
&=\int_D \frac{c}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x,
\end{aligned}
$$

and taking the limit $\varepsilon\to 0$ in (3.10) gives

$$
\begin{aligned}
&+\infty > -\Big[\langle u_t(t),u(t)\rangle_{L^2(D)}+\|\nabla\nabla\cdot u(t)\|_{L^2(D)}-\|\nabla\cdot u(t)\|_{L^2(D)}\Big]+\int_D \frac{1}{\nabla\cdot u(t)+c}\mathrm{d}x\\
&\quad \ge \int_D \frac{c}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x. \qquad (3.11)
\end{aligned}
$$

Step 2. Truncating $\nabla\cdot u(t)$. Fix an arbitrary t for which (3.6) holds. Given $\delta>0$, we truncate $\nabla\cdot u(t)$ from below as follows: let

$$
z^\delta := [\nabla\cdot u(t)-(\delta-c)]^+ +\delta-c = \begin{cases}\nabla\cdot u(t) & \text{on } D\backslash E_\delta(t),\\ \delta-c & \text{on } E_\delta(t),\end{cases} \qquad E_\delta(t):=\{\nabla\cdot u(t)\le\delta-c\}. \qquad (3.12)
$$

Here $[f]^+:=\max\{f,0\}$ denotes the positive part of f. As $\nabla\cdot u(t)+c>0$ a.e., we have $\mathcal{L}^2(E_\delta(t))\to 0$ as $\delta\to 0$, with $\mathcal{L}^2$ denoting the Lebesgue measure. Note that $\nabla\cdot u(t)=0>\delta-c$ on ∂D, hence $z^\delta=0$ on ∂D. Moreover, since $u\in V$,

$$
0=\int_D \nabla\cdot u(t)\mathrm{d}x \le J_\delta \le \delta\mathcal{L}^2(E_\delta(t)), \qquad J_\delta := \int_D z^\delta \mathrm{d}x.
$$

However, z^δ has non-zero average, hence we introduce the function (expressed in polar coordinates)

$$
w^\delta : D\longrightarrow\mathbb{R}, \qquad w^\delta(r,\theta):=(2J_\delta/\pi)(r^2-1), \qquad (3.13)
$$

which is C^1 regular, and satisfies

$$w^\delta = 0 \text{ on } \partial D, \qquad -2J_\delta/\pi \le w^\delta \le 0, \qquad \int_D w^\delta \mathrm{d}x = 4\pi \int_0^1 r J_\delta (r^2-1)/\pi \mathrm{d}r = -J_\delta. \tag{3.14}$$

The function $z^\delta + w^\delta$ then satisfies

$$z^\delta + w^\delta = 0 \text{ on } \partial D, \qquad \int_D (z^\delta + w^\delta)\mathrm{d}x = 0,$$

$$z^\delta + w^\delta + c \ge \delta - (2/\pi)J_\delta \ge \delta[1-(2/\pi)\mathcal{L}^2(E_\delta(t))] \ge \delta/2, \tag{3.15}$$

$$\nabla(z^\delta + w^\delta) = \nabla([\nabla\cdot u(t) - (\delta - c)]^+) + \nabla w^\delta = \begin{cases} \nabla\nabla\cdot u(t) + \nabla w^\delta & \text{on } D\backslash E_\delta(t), \\ \nabla w^\delta & \text{on } E_\delta(t), \end{cases} \tag{3.16}$$

in view of [19, Lemma 1.2]. Thus the Poisson equation (with Neumann boundary conditions)

$$\Delta\psi_\delta = z^\delta + w^\delta \text{ on } D, \qquad \langle \nabla\psi_\delta, \nu\rangle = 0 \text{ on } \partial D,$$

with ν denoting the exterior unit normal to D, admits a solution $\psi_\delta \in W^{1,2}(D)$, and $u^\delta := \nabla\psi_\delta \in V$ in view of (3.15) and (3.16).

Step 3. Testing (3.3) *with* $v = u^\delta \pm \varepsilon\varphi$, $\varepsilon > 0$. Consider an arbitrary test function $\varphi \in V \cap C_c^\infty(D)$. Fix an arbitrary t for which (3.6) holds. Note that (3.15) gives $\nabla\cdot u^\delta + c \ge \delta/2$, and we set

$$\varepsilon = \frac{\delta}{k}, \qquad k := 4\|\nabla\cdot\varphi\|_{L^\infty(D)}, \tag{3.17}$$

to get

$$\nabla\cdot(u^\delta + \varepsilon\varphi) + c \ge \delta/2 - \frac{\delta|\nabla\cdot\varphi|}{4\|\nabla\cdot\varphi\|_{L^\infty(D)}} \ge \delta/4. \tag{3.18}$$

Plugging $v = u^\delta + \varepsilon\varphi$ in (3.6) gives

$$\begin{aligned} &\langle u_t(t), u^\delta - u(t) + \varepsilon\varphi\rangle_{L^2(D)} + \langle Bu(t), u^\delta - u(t) + \varepsilon\varphi\rangle_{V',V} + \phi(u^\delta + \varepsilon\varphi) - \phi(u(t)) \\ =&\varepsilon[\langle u_t(t),\varphi\rangle_{L^2(D)} + \langle Bu(t),\varphi\rangle_{V',V}] \\ &\quad + \langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)} + \langle B(u(t)-u^\delta), u^\delta - u(t)\rangle_{V',V} + \langle Bu^\delta, u^\delta - u(t)\rangle_{V',V} \\ &\quad + \phi(u^\delta + \varepsilon\varphi) - \phi(u(t)) \\ \ge& 0, \end{aligned}$$

and using the monotonicity of B yields

$$\langle B(u(t)-u^\delta), u^\delta - u(t)\rangle_{V',V} \le 0,$$

hence

$$\begin{aligned} \varepsilon[\langle u_t(t),\varphi\rangle_{L^2(D)} + \langle Bu(t),\varphi\rangle_{V',V}] + \langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)} + \langle Bu^\delta, u^\delta - u(t)\rangle_{V',V} \\ + \phi(u^\delta + \varepsilon\varphi) - \phi(u(t)) \ge 0. \end{aligned} \tag{3.19}$$

We claim

$$\lim_{\delta\to 0^+}\delta^{-1}\langle Bu^\delta, u^\delta-u(t)\rangle_{V',V}=0, \tag{3.20}$$

$$\lim_{\delta\to 0^+}\delta^{-1}\langle u_t(t), u^\delta-u(t)\rangle_{L^2(D)}=0, \tag{3.21}$$

$$\lim_{\delta\to 0^+}\delta^{-1}(\phi(u^\delta+\varepsilon\varphi)-\phi(u(t)))=-k^{-1}\int_D\frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x. \tag{3.22}$$

Sub-step 3.1. Proof of (3.20). Note that (3.16) gives

$$\nabla\nabla\cdot u^\delta=\nabla w^\delta \text{ on } E_\delta(t), \qquad \nabla\nabla\cdot(u^\delta-u(t))=\nabla w^\delta \text{ on } D\backslash E_\delta(t),$$

hence

$$\begin{aligned}&\langle Bu^\delta, u^\delta-u(t)\rangle_{V',V}\\ =&\int_D\langle\nabla\nabla\cdot u^\delta,\nabla\nabla\cdot(u^\delta-u(t))\rangle\mathrm{d}x\\ =&\int_{E_\delta(t)}\langle\nabla w^\delta,\nabla(w^\delta-\nabla\cdot u(t))\rangle\mathrm{d}x+\int_{D\backslash E_\delta(t)}\langle\nabla(\nabla\cdot u(t)+w^\delta),\nabla w^\delta\rangle\mathrm{d}x.\end{aligned}$$

As, by construction (in (3.13)), we have

$$\|\nabla w^\delta\|_{L^\infty(D)}\le(4/\pi)J_\delta\le(4/\pi)\delta\mathcal{L}^2(E_\delta(t)), \qquad \nabla\nabla\cdot u(t)\in L^2(D),$$

we infer

$$\begin{aligned}&\int_{E_\delta(t)}|\langle\nabla w^\delta,\nabla(w^\delta-\nabla\cdot u(t))\rangle|\mathrm{d}x\\ \le&\|\nabla w^\delta\|_{L^2(E_\delta(t))}\|\nabla\nabla\cdot u(t)\|_{L^2(E_\delta(t))}+\|\nabla w^\delta\|^2_{L^2(E_\delta(t))},\\ \le&\|\nabla w^\delta\|_{L^\infty(E_\delta(t))}\sqrt{\mathcal{L}^2(E_\delta(t))}\|\nabla\nabla\cdot u(t)\|_{L^2(E_\delta(t))}+\|\nabla w^\delta\|^2_{L^\infty(E_\delta(t))}\mathcal{L}^2(E_\delta(t))\\ \le&(4/\pi)\delta\mathcal{L}^2(E_\delta(t))^{3/2}(\|\nabla\nabla\cdot u(t)\|_{L^2(E_\delta(t))}+(4/\pi)\delta\mathcal{L}^2(E_\delta(t))^{3/2}),\end{aligned}$$

and similarly

$$\begin{aligned}\int_{D\backslash E_\delta(t)}|\langle\nabla(\nabla\cdot u(t)+w^\delta),\nabla w^\delta\rangle\mathrm{d}x|&\le\|\nabla w^\delta\|_{L^2(D)}\|\nabla\nabla\cdot u(t)\|_{L^2(D)}+\|\nabla w^\delta\|^2_{L^2(D)}\\ &\le(4/\pi)\delta\mathcal{L}^2(E_\delta(t))(\|\nabla\nabla\cdot u(t)\|_{L^2(D)}+(4/\pi)\delta\mathcal{L}^2(E_\delta(t))).\end{aligned}$$

Thus

$$\begin{aligned}&\lim_{\delta\to 0^+}\delta^{-1}\langle Bu^\delta,u^\delta-u(t)\rangle_{V',V}\\ =&\lim_{\delta\to 0^+}\frac{1}{\delta}\Big(\int_{E_\delta(t)}\langle\nabla w^\delta,\nabla(w^\delta-\nabla\cdot u(t))\rangle\mathrm{d}x+\int_{D\backslash E_\delta(t)}\langle\nabla(\nabla\cdot u(t)+w^\delta),\nabla w^\delta\rangle\mathrm{d}x\Big)=0,\end{aligned}$$

and (3.20) is proven.

Sub-step 3.2. Proof of (3.21). Theorem 3.2 gave that $u_t\in L^\infty(0,T;L^2(D))$, and since both u^δ, $u\in V$ (hence both are gradients of some $W^{1,2}(D)$ functions), we have by [1, Theorem 5.4]

$$|\langle u_t(t),u^\delta-u(t)\rangle_{L^2(D)}|\le\|u_t\|_{L^\infty(0,T;L^2(D))}\|u^\delta-u(t)\|_{L^2(D)}$$

$$\leq \|u_t\|_{L^\infty(0,T;L^2(D))}\|\nabla\cdot(u^\delta - u(t))\|_{L^2(D)}.$$

By construction, $\nabla\cdot u^\delta = z^\delta + w^\delta$, and (see for instance (3.12))

$$|\nabla\cdot(u^\delta - u(t))| = |w^\delta| \leq (2/\pi)\delta\mathcal{L}^2(E_\delta(t)) \text{ on } D\backslash E_\delta(t), \qquad |\nabla\cdot(u^\delta - u(t))| \leq \delta \text{ on } E_\delta(t).$$

Thus

$$\begin{aligned}\|\nabla\cdot(u^\delta - u(t))\|^2_{L^2(D)} &= \int_{D\backslash E_\delta(t)} |\nabla\cdot(u^\delta - u(t))|^2 \mathrm{d}x + \int_{E_\delta(t)} |\nabla\cdot(u^\delta - u(t))|^2 \mathrm{d}x \\ &\leq \delta^2(2/\pi)^2\mathcal{L}^2(E_\delta(t))^2 + (\delta + |w^\delta|)^2\mathcal{L}^2(E_\delta(t)) \\ &\leq \delta^2[(2/\pi)^2\mathcal{L}^2(E_\delta(t))^2 + (1 + (2/\pi)\mathcal{L}^2(E_\delta(t)))^2\mathcal{L}^2(E_\delta(t))],\end{aligned}$$

hence

$$\begin{aligned}&\lim_{\delta\to 0^+} \delta^{-1}|\langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)}| \\ &\leq \lim_{\delta\to 0^+} \|u_t\|_{L^\infty(0,T;L^2(D))}\|\nabla\cdot(u^\delta - u(t))\|_{L^2(D)} \\ &\leq \lim_{\delta\to 0^+} \|u_t\|_{L^\infty(0,T;L^2(D))}[(2/\pi)^2\mathcal{L}^2(E_\delta(t))^2 + \mathcal{L}^2(E_\delta(t))]^{1/2} = 0,\end{aligned}$$

and (3.21) is proven.

Sub-step 3.3. Proof of (3.22). Note that, by construction, we have $\nabla\cdot(u^\delta + \varepsilon\varphi) + c > 0$, $\nabla\cdot u(t) + c > 0$ a.e., and

$$\phi(u^\delta + \varepsilon\varphi) - \phi(u(t)) = \int_D \frac{\nabla\cdot(u(t) - u^\delta - \varepsilon\varphi)}{(\nabla\cdot(u^\delta + \varepsilon\varphi) + c)(\nabla\cdot u(t) + c)} \mathrm{d}x. \tag{3.23}$$

To estimate this term, we first estimate the denominator: by (3.18), (3.12) and (3.14), on $E_\delta(t)$ it holds

$$\nabla\cdot(u^\delta + \varepsilon\varphi) + c \geq \delta/4 \geq (\nabla\cdot u(t) + c)/4,$$

while on $D\backslash E_\delta(t)$, it holds $\nabla\cdot u^\delta = \nabla\cdot u(t) + w^\delta$, and (due to the choice of ε in (3.17))

$$|\varepsilon\nabla\cdot\varphi| + |w^\delta| \leq \delta/3 \leq (\nabla\cdot u(t) + c)/3,$$

hence

$$\nabla\cdot(u^\delta + \varepsilon\varphi) + c = \nabla\cdot u(t) + c + \varepsilon\nabla\cdot\varphi + w^\delta \geq 2(\nabla\cdot u(t) + c)/3.$$

Thus, in both cases (on $E_\delta(t)$ and $D\backslash E_\delta(t)$) it holds

$$\nabla\cdot(u^\delta + \varepsilon\varphi) + c \geq (\nabla\cdot u(t) + c)/4.$$

Recalling that, by construction we have $|\nabla\cdot(u^\delta - u(t))| = |w^\delta|$ on $D\backslash E_\delta(t)$, and $|\nabla\cdot(u^\delta - u(t))| \leq |w^\delta| + \delta$ on $E_\delta(t)$, in view of estimates (3.18), (3.12) and (3.14), we have

$$\begin{aligned}&\frac{1}{\delta}\int_D \left|\frac{\nabla\cdot(u(t) - u^\delta)}{(\nabla\cdot(u^\delta + \varepsilon\varphi) + c)(\nabla\cdot u(t) + c)}\right| \mathrm{d}x \\ &\quad\leq \frac{1}{\delta}\int_{D\backslash E_\delta(t)} \frac{|w^\delta|}{(\nabla\cdot(u^\delta + \varepsilon\varphi) + c)(\nabla\cdot u(t) + c)} \mathrm{d}x\end{aligned}$$

$$+\frac{1}{\delta}\int_{E_\delta(t)}\frac{1+|w^\delta|}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\mathrm{d}x$$
$$\le\int_{D\setminus E_\delta(t)}\frac{4\mathcal{L}^2(E_\delta(t))}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x+\int_{E_\delta(t)}\frac{4+4\mathcal{L}^2(E_\delta(t))}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x\overset{\delta\to 0^+}{\to}0, \tag{3.24}$$

since $\mathcal{L}^2(E_\delta(t))\to 0$ and $(\nabla\cdot u(t)+c)^{-2}\in L^1(D)$ as proven in Step 1. For the other term

$$\frac{1}{\delta}\int_D\frac{\nabla\cdot(\varepsilon\varphi)}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\mathrm{d}x=\frac{1}{k}\int_D\frac{\nabla\cdot\varphi}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\mathrm{d}x,$$

note that, clearly

$$\frac{\nabla\cdot\varphi}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\to\frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\qquad\text{for a.e. } x\in D.$$

To use Lebesgue dominated convergence theorem, we need to find a dominating function.

Using $\nabla\cdot(u^\delta+\varepsilon\varphi)+c\ge(\nabla\cdot u(t)+c)/4$, we obtain

$$\left|\frac{\nabla\cdot\varphi}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\right|\le\frac{4\|\varphi\|_{L^\infty(D)}}{(\nabla\cdot u(t)+c)^2}\qquad\text{for a.e. } x\in D,$$

and $\frac{4\|\varphi\|_{L^\infty(D)}}{(\nabla\cdot u(t)+c)^2}\in L^1(D)$ by Step 1. Thus, by Lebesgue dominated convergence theorem, we infer

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\int_D\frac{\nabla\cdot(\varepsilon\varphi)}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\mathrm{d}x=\frac{1}{k}\int_D\frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x,$$

and combining with (3.24) gives (3.22).

In view of (3.20), (3.21) and (3.22), dividing by δ and taking the limit $\delta\to 0^+$ in (3.19) gives

$$\begin{aligned}\lim_{\delta\to 0^+}\delta^{-1}\Big(&\varepsilon[\langle u_t(t),\varphi\rangle_{L^2(D)}+\langle Bu(t),\varphi\rangle_{V',V}]\\&+\langle u_t(t),u^\delta-u(t)\rangle_{L^2(D)}+\langle Bu^\delta,u^\delta-u(t)\rangle_{V',V}+\phi(u^\delta+\varepsilon\varphi)-\phi(u(t))\Big)\\=k^{-1}\Big(&\langle u_t(t),\varphi\rangle_{L^2(D)}+\langle Bu(t),\varphi\rangle_{V',V}-\int_D\frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x\Big)\ge 0,\end{aligned}$$

hence (since $k>0$)

$$\langle u_t(t),\varphi\rangle_{L^2(D)}+\langle Bu(t),\varphi\rangle_{V',V}-\int_D\frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x\ge 0. \tag{3.25}$$

As the above argument can be repeated with $v=u^\delta-\varepsilon\varphi$, it follows also

$$\langle u_t(t),\varphi\rangle_{L^2(D)}+\langle Bu(t),\varphi\rangle_{V',V}-\int_D\frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\mathrm{d}x\le 0. \tag{3.26}$$

Combining (3.25) and (3.26) gives (1.9), which proves the existence part of Theorem 1.1.

To prove the uniqueness part, assume $u_1,u_2:[0,T]\longrightarrow V$ are both solutions in the sense of Theorem 1.1, with $u_1(0)=u_2(0)=u^0$. Thus (for a.e. t)

$$u_{it}(t)=-\partial\phi(u_i(t))-Bu_i(t),\qquad i=1,2,$$

hence, taking the inner product with $u_1(t)-u_2(t)$, and using the monotonicity of the operator $v\mapsto \partial\phi(v)+Bv$, we get (for a.e. t)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_1(t)-u_2(t)\|^2_{L^2(D)}=\langle u_{1t}(t)-u_{2t}(t),u_1(t)-u_2(t)\rangle$$
$$=-\langle \partial\phi(u_1(t))-\partial\phi(u_2(t))+Bu_1(t)-Bu_2(t),u_1(t)-u_2(t)\rangle\le 0,$$

hence $u_1=u_2$. ∎

We finally check that condition (1.7) on the initial datum u^0 is satisfied by a wide range of functions.

LEMMA 3.1. *Condition* (1.7) *is satisfied for all* $u^0\in V\cap W^{4,2}(D)$ *such that* $\nabla\cdot u^0+c$ *is uniformly bounded away from zero.*

Proof. Lemma 2.3 proved that ϕ is proper, convex and lower-semicontinuous. We claim that for any u^0 such that $\nabla\cdot u^0+c\ge\alpha$ a.e. (for some $\alpha>0$), it holds

$$\partial\phi(u^0)=\{\nabla(\nabla\cdot u^0+c)^{-2}\},$$

where $\partial\phi(u^0)\subseteq V'$ denotes the sub-differential of ϕ at u^0, i.e.,

$$\xi\in\partial\phi(u^0)\Longleftrightarrow \phi(u^0+v)-\phi(u^0)\ge\langle\xi,v\rangle_{V',V}\qquad\text{for any } v\in V.$$

We first prove $\partial\phi(u^0)\supseteq\{\nabla(\nabla\cdot u^0+c)^{-2}\}$, provided that $\nabla(\nabla\cdot u^0+c)^{-2}\in V'$. The goal is to show

$$\phi(u^0+v)-\phi(u^0)\ge\langle\nabla(\nabla\cdot u^0+c)^{-2},v\rangle_{V',V}\qquad\text{for any } v\in V. \tag{3.27}$$

Consider an arbitrary $v\in V$. If $\nabla(u^0+v)+c$ is not a.e. positive, then $\phi(u^0+v)=+\infty$ and (3.27) is trivial. If $\nabla(u^0+v)+c>0$ a.e., then using the fact that $\Phi(\xi)=\xi^{-1}$ on $(0,+\infty)$ (which is convex) gives

$$\Phi'(\nabla\cdot u^0+c)\nabla\cdot v\le\Phi(\nabla\cdot(u^0+v)+c)-\Phi(\nabla\cdot u^0+c),$$

hence integrating on D gives

$$\int_D\Phi'(\nabla\cdot u^0+c)\nabla\cdot v\mathrm{d}x\le\int_D[\Phi(\nabla\cdot(u^0+v)+c)-\Phi(\nabla\cdot u^0+c)]\mathrm{d}x=\phi(u^0+v)-\phi(u^0),$$

i.e.,

$$-\nabla\Phi'(\nabla\cdot u^0+c)=\nabla(\nabla\cdot u^0+c)^{-2}\in\partial\phi(u^0).$$

To prove that $\partial\phi(u^0)$ contains *only* $\nabla(\nabla\cdot u^0+c)^{-2}$, we choose v such that $\nabla\cdot v\in L^\infty(D)$, and compute the Gâteaux derivative

$$\lim_{\varepsilon\to 0}\frac{\phi(u^0+\varepsilon v)-\phi(u^0)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_D\frac{-\varepsilon\nabla\cdot v}{(\nabla\cdot(u^0+\varepsilon v)+c)(\nabla\cdot u^0+c)}\mathrm{d}x.$$

It is clear that

$$\frac{\nabla\cdot v}{(\nabla\cdot(u^0+\varepsilon v)+c)(\nabla\cdot u^0+c)}\to\frac{\nabla\cdot v}{(\nabla\cdot u^0+c)^2}\qquad\text{for a.e. } x\in D.$$

To find a dominating function, note that since $\nabla\cdot u^0+c\geq\alpha>0$ a.e., and $\nabla\cdot v\in L^\infty(D)$, for all ε such that $\varepsilon\|\nabla\cdot v\|_{L^\infty(D)}<\alpha/2$ it holds $\nabla\cdot(u^0+\varepsilon v)+c\geq\alpha/2$, hence

$$\left|\frac{\nabla\cdot v}{(\nabla\cdot(u^0+\varepsilon v)+c)(\nabla\cdot u^0+c)}\right|\leq\frac{2\|\nabla\cdot v\|_{L^\infty(D)}}{\alpha(\nabla\cdot u^0+c)}\in L^1(D)$$

in view of (3.11). Thus by Lebesgue dominated convergence theorem, we infer

$$\lim_{\varepsilon\to0}\frac{\phi(u^0+\varepsilon v)-\phi(u^0)}{\varepsilon}=\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\int_D\frac{-\varepsilon\nabla\cdot v}{(\nabla\cdot(u^0+\varepsilon v)+c)(\nabla\cdot u^0+c)}\mathrm{d}x=-\int_D\frac{\nabla\cdot v}{(\nabla\cdot u^0+c)^2}\mathrm{d}x.$$

Thus the Gâteaux derivative is well defined for all $v\in V$ with $\nabla\cdot v\in L^\infty(D)$. If there were another element $\eta\in\partial\phi(u^0)$, then for all $v\in V$ with $\nabla\cdot v\in L^\infty(D)$, we have

$$\begin{gathered}\lim_{\varepsilon\to0}\frac{\phi(u^0+\varepsilon v)-\phi(u^0)}{\varepsilon}\geq\langle-\nabla\Phi'(\nabla\cdot u^0+c),v\rangle_{V',V},\\ \lim_{\varepsilon\to0}\frac{\phi(u^0-\varepsilon v)-\phi(u^0)}{\varepsilon}\geq\langle\nabla\Phi'(\nabla\cdot u^0+c),v\rangle_{V',V},\\ \lim_{\varepsilon\to0}\frac{\phi(u^0+\varepsilon v)-\phi(u^0)}{\varepsilon}\geq\langle\eta,v\rangle_{V',V},\qquad\lim_{\varepsilon\to0}\frac{\phi(u^0-\varepsilon v)-\phi(u^0)}{\varepsilon}\geq\langle-\eta,v\rangle_{V',V},\end{gathered}$$

hence $\eta=-\nabla\Phi'(\nabla\cdot u^0+c)$ for all $v\in V$ with $\nabla\cdot v\in L^\infty(D)$. As

$$\{v\in V:\nabla\cdot v\in L^\infty(D)\}\supseteq V\cap C_c^\infty(D),$$

it is dense in V, thus we infer $\eta=-\nabla\Phi'(\nabla\cdot u^0+c)$ as elements of V'.

Hence $\partial\phi(u^0)=\{\nabla(\nabla\cdot u^0+c)^{-2}\}$. Thus, if u^0 also has sufficient regularity (e.g. if $u^0\in W^{4,2}(D)\cap V$) such that

$$\nabla\Delta\nabla\cdot u^0+\nabla\nabla\cdot u^0+\nabla(\nabla\cdot u^0+c)^{-2}\in L^2(D),$$

choosing

$$z^0:=-[\nabla\Delta\nabla\cdot u^0+\nabla\nabla\cdot u^0+\nabla(\nabla\cdot u^0+c)^{-2}]$$

gives

$$\begin{aligned}0&=\int_D\langle z^0+\nabla\Delta\nabla\cdot u^0+\nabla\nabla\cdot u^0+\nabla(\nabla\cdot u^0+c)^{-2},v-u^0\rangle\mathrm{d}x\\&=\int_D[\langle z^0,v-u^0\rangle+\langle\nabla\nabla\cdot u^0,\nabla\nabla\cdot(v-u^0)\rangle\\&\qquad-(\nabla\cdot u^0)(\nabla\cdot(v-u^0))-(\nabla\cdot(v-u^0))(\nabla\cdot u^0+c)^{-2}]\mathrm{d}x\\&\leq\int_D[\langle z^0,v-u^0\rangle+\langle\nabla\nabla\cdot u^0,\nabla\nabla\cdot(v-u^0)\rangle-(\nabla\cdot u^0)(\nabla\cdot(v-u^0))]\mathrm{d}x+\phi(v)-\phi(u^0),\end{aligned}$$

for any $v\in V$, i.e. (1.7). □

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