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DIFFUSION LIMIT OF THE BOLTZMANN-LANDAU-LIFSHITZ-GILBERT SYSTEM IN FERROMAGNETIC MATERIALS*

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Abstract. In this paper, we continue the study initiated in our previous work on the semiclassical limit for the Schrödinger-Poisson-Landau-Lifshitz-Gilbert system in [3]. Specifically, we consider the s-wave form spin dynamics coupled with the magnetization dynamics governed by the Landau-Lifshitz-Gilbert system, and rigorously obtain the diffusion limit of the coupled system.

Keywords. Diffusion limit; Boltzmann-Landau-Lifshitz-Gilbert system; ferromagnetic materials.

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1. Introduction

The magnetization reversal process in ferromagnetic materials has important technological applications. The idea of switching the orientation of the magnetization in a ferromagnetic multilayer using currents perpendicular to the magnetic layers was introduced by Slonczewski [16] and Berger [2]. As a consequence of the conservation of spin angular momentum, there is a transfer of spin angular momentum between the conduction electrons and the magnetization in the form of an additional torque, known as *spin-transfer torque*. Controlling the magnetization using the spin-transfer torque is an attractive alternative to more traditional ways in which magnetic fields are used, as the currents are more localized and require less power consumption. This has had significant impact in the design of magneto-electronic devices, such as magnetic random access memories (MRAMs) and high-density recording media [10].

In some models of spin-transfer torque, an additional torque is expressed explicitly in terms of the magnetization and added to the Landau-Lifshitz-Gilbert equation (LLG) [11,17,21]. Global existence of weak solutions for one such model was presented in [12]. Spatial effects have proven to be significant in the reversal process, specially in the presence of magnetic domain walls and vortices. Zhang, Levy and Fert [20] introduced a model in which the dynamics of the spin accumulation is accounted for, and coupled to the LLG. These types of models have received some attention in the mathematics community in recent years as well [1,4,6–9,14,18,19].

In earlier work, the authors studied the semiclassical limit for the Schrödinger-Poisson-Landau-Lifshitz-Gilbert system [3]. In this article, we are interested in rigorously proving the diffusion limit of the coupled spin and magnetization dynamics. For the sake of simplicity, we shall start with the introduction of the s-wave form considered in the physical studies in the diffusion regime *e.g.* [15], where the spin density

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matrix $W^\varepsilon = W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t) \in \mathbb{C}^{2\times 2}$ for $(\boldsymbol{x},\boldsymbol{v},t)\in \mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}\times\mathbb{R}^+$ satisfies the following linear Boltzmann equation

$$\begin{aligned}&\partial_t W^\varepsilon + \frac{1}{\varepsilon}\mathcal{T}(W^\varepsilon) = \mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon, W^\varepsilon] + \frac{1}{\varepsilon^2}\mathcal{Q}(W^\varepsilon) + \mathcal{Q}_{\mathrm{sf}}(W^\varepsilon),\\ &W^\varepsilon(\boldsymbol{x},\boldsymbol{v},0) = W^\varepsilon_{\mathrm{in}}(\boldsymbol{x},\boldsymbol{v}), \qquad \text{for } (\boldsymbol{x},\boldsymbol{v})\in\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}\end{aligned} \tag{1.1}$$

with $\varepsilon \ll 1$ as the Knudsen number, $[\cdot,\cdot]$ gives the commutator, *i.e.* , $[A,B]=AB-BA$, the triplet of the Pauli matrices $\hat{\boldsymbol{\sigma}} = (\sigma_1,\sigma_2,\sigma_3)^T$ is defined as follows:

$$\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2 = \begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.2}$$

and the transport and collision terms are given by

$$\mathcal{T}(W) = \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} W(\boldsymbol{x},\boldsymbol{v}) - \nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})\cdot\nabla_{\boldsymbol{v}} W^\varepsilon(\boldsymbol{x},\boldsymbol{v}) \tag{1.3}$$

$$\mathcal{Q}(W) = \int_{\mathbb{R}^3} \alpha(\boldsymbol{v},\boldsymbol{v}')\left(\mathcal{M}(v)W(\boldsymbol{v}') - \mathcal{M}(\boldsymbol{v}')W(\boldsymbol{v})\right)\mathrm{d}\boldsymbol{v}', \tag{1.4}$$

$$\mathcal{Q}_{\mathrm{sf}}(W) = \frac{I}{2}\mathrm{Tr}W - W, \tag{1.5}$$

where we have used Tr to denote the matrix trace. In (1.3), $\boldsymbol{E}(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}}\phi(\boldsymbol{x})$ is the applied electric field, and we assume that ϕ is a given potential that is bounded from below. Since a constant shift in the potential does not change the electric field, we may assume $\phi\geq 0$. We also assume that $0\leq\alpha_0\leq\alpha(\boldsymbol{v},\boldsymbol{v}')\leq\alpha_1$, and $\alpha(\boldsymbol{v},\boldsymbol{v}')=\alpha(\boldsymbol{v}',\boldsymbol{v})$. The Maxwellian $\mathcal{M}(\boldsymbol{v})$ is defined by

$$\mathcal{M}(\boldsymbol{v}) = \frac{1}{(2\pi)^{3/2}}\exp\left(-\frac{|\boldsymbol{v}|^2}{2}\right).$$

We also define

$$\rho^\varepsilon = \int_{\mathbb{R}^3}\mathrm{Tr}W^\varepsilon\,\mathrm{d}\boldsymbol{v}, \quad \text{and} \quad s^\varepsilon_k = \int_{\mathbb{R}^3}\mathrm{Tr}(\sigma_k W^\varepsilon)\,\mathrm{d}\boldsymbol{v},\, k=1,2,3, \tag{1.6}$$

where ρ^ε is the position density and $\boldsymbol{s}^\varepsilon = (s^\varepsilon_1, s^\varepsilon_2, s^\varepsilon_3)^T$ is the spin density.

We assume that the ferromagnetic material occupies a compact domain $\Omega\subset\mathbb{R}^3_{\boldsymbol{x}}$ with smooth boundary, and over the domain $\Omega\times\mathbb{R}^+$ the magnetization $\boldsymbol{m}^\varepsilon = \boldsymbol{m}^\varepsilon(\boldsymbol{x},t) = (m^\varepsilon_1, m^\varepsilon_2, m^\varepsilon_3)$ satisfies the Landau-Lifshitz-Gilbert (LLG) equation

$$\begin{aligned}&\partial_t\boldsymbol{m}^\varepsilon = -\gamma\boldsymbol{m}^\varepsilon\times(\boldsymbol{H}_{\mathrm{eff}}[\boldsymbol{m}^\varepsilon] + \boldsymbol{s}^\varepsilon) + \boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon,\\ &\boldsymbol{m}^\varepsilon(\boldsymbol{x},0) = \boldsymbol{m}_{\mathrm{in}}(\boldsymbol{x}), \qquad \text{for } \boldsymbol{x}\in\Omega,\\ &\partial_{\boldsymbol{\nu}}\boldsymbol{m}^\varepsilon(\boldsymbol{x},t) = 0, \qquad \text{for } (\boldsymbol{x},t)\in\partial\Omega\times\mathbb{R}^+\end{aligned} \tag{1.7}$$

with the effective field $\boldsymbol{H}_{\mathrm{eff}}$ given by

$$\boldsymbol{H}_{\mathrm{eff}}[\boldsymbol{m}^\varepsilon] = \Delta\boldsymbol{m}^\varepsilon + \boldsymbol{H}_{\mathrm{s}}[\boldsymbol{m}^\varepsilon], \text{ and } \boldsymbol{H}_{\mathrm{s}}[\boldsymbol{m}^\varepsilon] = -\nabla\sum_{j=1}^3\left(\frac{\partial}{\partial x_j}\frac{1}{4\pi|x|}*m^\varepsilon_j\right), \tag{1.8}$$

and in the domain $(\mathbb{R}^3_{\boldsymbol{x}}\backslash\Omega)\times\mathbb{R}^+$, $\boldsymbol{m}^\varepsilon\equiv\boldsymbol{0}$. The stray field, $\boldsymbol{H}_{\mathrm{s}}$, can be written as $\boldsymbol{H}_{\mathrm{s}} = -\nabla u$, where $u = \nabla N * \boldsymbol{m} = \sum_{j=1}^3\left(\frac{\partial}{\partial x_j}\frac{1}{4\pi|x|}*m^\varepsilon_j\right)$ is the magnetostatic potential, and solves the equation $\mathrm{div}\boldsymbol{H}_{\mathrm{s}} = \mathrm{div}\ \boldsymbol{m}$ in $\mathbb{R}^3$ in the sense of distributions, i.e.,

$$\int_{\mathbb{R}^3}\boldsymbol{H}_{\mathrm{s}}\cdot\nabla\varphi = \int_\Omega\boldsymbol{m}\cdot\nabla\varphi, \quad \forall\varphi\in H^1(\mathbb{R}^3). \tag{1.9}$$

The coupling of spin and magnetization is via the terms $[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon, W^\varepsilon]$ in (1.1) and $\boldsymbol{s}^\varepsilon$ in (1.7). In this paper we are interested in asymptotically deriving and rigorously proving the diffusion limit of the coupled Boltzmann-Landau-Lifshitz-Gilbert system (1.1)-(1.7).

2. Preliminary

In this section, we assume W^ε and $\boldsymbol{m}^\varepsilon$ are solutions to (1.1)-(1.7), and analyze basic properties of collision operators and the solutions. We denote the space of hermitian 2×2 matrices by $\mathbb{H}^{2\times 2}$, and give the following definition:

Definition 2.1. *Define the weighted space $\mathcal{L}^2_{\mathcal{M}}$ as*

$$\mathcal{L}^2_{\mathcal{M}} := \left\{ W:\mathbb{R}^3_{\boldsymbol{v}} \to \mathbb{H}^{2\times 2} \,\middle|\, \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr} W^2 \mathcal{M}^{-1}\,\mathrm{d}\boldsymbol{v} < \infty \right\}. \tag{2.1}$$

Proposition 2.1 ([13]). *Consider the collision operator $\mathcal{Q}:\mathcal{L}^2_{\mathcal{M}} \to \mathcal{L}^2_{\mathcal{M}}$ defined in (1.4). The following properties are satisfied:*

(1) $\mathcal{Q}$ is a linear, self-adjoint, continuous and non-positive operator;

(2) For all $W\in\mathcal{L}^2_{\mathcal{M}}$, it holds that

$$\int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathcal{Q}(W)\,\mathrm{d}\boldsymbol{v} = 0; \tag{2.2}$$

(3) The kernel of $\mathcal{Q}$

$$\ker\mathcal{Q} = \left\{ W\in\mathcal{L}^2_{\mathcal{M}} \,\middle|\, W = N\mathcal{M}(\boldsymbol{v}),\, N\in\mathbb{C}^{2\times 2} \right\}, \tag{2.3}$$

$$(\ker\mathcal{Q})^{\perp} = \left\{ W\in\mathcal{L}^2_{\mathcal{M}} \,\middle|\, \int_{\mathbb{R}^3_{\boldsymbol{v}}} W\,\mathrm{d}\boldsymbol{v} = \mathbf{0}_{2\times 2} \right\}; \tag{2.4}$$

(4) Let $\Pi W = \mathcal{M}\int_{\mathbb{R}^3_{\boldsymbol{v}}} W\,\mathrm{d}\boldsymbol{v}$, then

$$-(\mathcal{Q}(W),W)_{\mathcal{L}^2_{\mathcal{M}}} \geq \frac{\alpha_0}{2}\|\Pi W - W\|_{\mathcal{L}^2_{\mathcal{M}}}; \tag{2.5}$$

(5) The image of $\mathcal{Q}$ is closed and $\mathrm{Im}\mathcal{Q} = (\ker\mathcal{Q})^{\perp}$. The equation $Q(W)=G$ has a solution in $\mathcal{L}^2_{\mathcal{M}}$ if and only if $G\in\mathrm{Im}\mathcal{Q}$.

Proof. Here we only give the proof to the fourth item, and the others can be verified in a straightforward way.

$$\begin{aligned}
&\mathrm{Tr}\int_{\mathbb{R}^3} \mathcal{Q}(W)\frac{W}{\mathcal{M}}\,\mathrm{d}\boldsymbol{v} = \mathrm{Tr}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \alpha(\mathcal{M}W' - \mathcal{M}'W)\frac{W}{\mathcal{M}}\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{v} \\
&= -\frac{1}{2}\mathrm{Tr}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \alpha(\mathcal{M}W' - \mathcal{M}'W)\left(\frac{W'}{\mathcal{M}'} - \frac{W}{\mathcal{M}}\right)\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{v} \\
&= -\frac{1}{2}\mathrm{Tr}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \alpha\mathcal{M}\mathcal{M}'\left(\frac{W'}{\mathcal{M}'} - \frac{W}{M}\right)^2\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{v} \\
&\leq -\frac{\alpha_0}{2}\mathrm{Tr}\int_{\mathbb{R}^3}\mathcal{M}\left(N - \frac{W}{\mathcal{M}}\right)^2\mathrm{d}\boldsymbol{v} = -\frac{\alpha_0}{2}\mathrm{Tr}\int_{\mathbb{R}^3}\frac{(N\mathcal{M}-W)^2}{\mathcal{M}}\,\mathrm{d}\boldsymbol{v},
\end{aligned}$$

where $N = \displaystyle\int_{\mathbb{R}^3} W\,\mathrm{d}\boldsymbol{v}$, and we have used the Jensen inequality. ∎

DEFINITION 2.2. *Define the weighted space* $\mathbb{L}^2_{\mathcal{M}}$ *as*

$$\mathbb{L}^2_{\mathcal{M}} := \left\{ W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{H}^{2\times 2} \,\middle|\, \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr} W^2 \mathcal{M}^{-1} \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} < \infty \right\}. \tag{2.6}$$

PROPOSITION 2.2. *Let* $W^\varepsilon \in \mathbb{L}^2_{\mathcal{M}}$ *be a solution of* (1.1)*, then* $\|s^\varepsilon_k\|_{L^2(\mathbb{R}^3)} \le C \|W^\varepsilon\|_{\mathbb{L}^2_{\mathcal{M}}}$.

Proof. The conclusion follows by observing that

$$\begin{aligned}
\|s^\varepsilon_k\|^2_{L^2(\mathbb{R}^3)} &= \int_{\mathbb{R}^3_{\boldsymbol{x}}} \left| \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}(\sigma_k W^\varepsilon) \mathrm{d}\boldsymbol{v} \right|^2 \mathrm{d}\boldsymbol{x} \\
&\le \int_{\mathbb{R}^3_{\boldsymbol{x}}} \left(\int_{\mathbb{R}^3_{\boldsymbol{v}}} |\mathrm{Tr}(\sigma_k W^\varepsilon)|^2 \mathcal{M}^{-1} \mathrm{d}\boldsymbol{v} \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathcal{M} \mathrm{d}\boldsymbol{v} \right) \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} |\mathrm{Tr}(\sigma_k W^\varepsilon)|^2 \mathcal{M}^{-1} \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \\
&\le C \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3} \mathrm{Tr}(\sigma_k^2) \mathrm{Tr}\left[(W^\varepsilon)^2 \right] \mathcal{M}^{-1} \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \\
&\le C \|W^\varepsilon\|^2_{\mathbb{L}^2_{\mathcal{M}}}.
\end{aligned}$$

∎

Let $\mathcal{E}(\boldsymbol{x}) = \exp(-\phi(\boldsymbol{x}))$ and $\mathcal{F}(\boldsymbol{x},\boldsymbol{v}) = \mathcal{E}(\boldsymbol{x})\mathcal{M}(\boldsymbol{v})$. Then one can check that $\mathcal{F}$ satisfies

$$(\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} + \boldsymbol{E} \cdot \nabla_{\boldsymbol{v}}) \mathcal{F} = 0. \tag{2.7}$$

This property motivates us to consider the solutions in the following weighted L^2 space:

DEFINITION 2.3. *Define the weighted space* $\mathbb{L}^2_{\mathcal{F}}$ *as*

$$\mathbb{L}^2_{\mathcal{F}} := \left\{ W : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{H}^{2\times 2} \,\middle|\, \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr} W^2 \mathcal{F}^{-1} \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} < \infty \right\}. \tag{2.8}$$

And then we have the following estimates of the solutions:

PROPOSITION 2.3 (Uniform bounded solutions). *Let* $W^\varepsilon_{\mathrm{in}} \in \mathbb{L}^2_{\mathcal{F}}$ *and* $(W^\varepsilon, \boldsymbol{m}^\varepsilon)$ *be a smooth solution to* (1.1) – (1.7) *for* $t \in [0,T]$*. Then there is a constant* C *independent of* ε *and* T*, such that*

$$\|W^\varepsilon\|_{L^\infty([0,T];\mathbb{L}^2_{\mathcal{F}})} \le C, \tag{2.9}$$

$$\|W^\varepsilon - \Pi W^\varepsilon\|_{L^2([0,T];\mathbb{L}^2_{\mathcal{F}})} \le C\varepsilon, \tag{2.10}$$

$$\|\rho^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^3))} + \|\boldsymbol{s}^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \le C, \tag{2.11}$$

$$\|\partial_t \boldsymbol{m}^\varepsilon\|_{L^2([0,T];L^2(\Omega))} + \|\boldsymbol{m}^\varepsilon\|_{L^\infty([0,T];H^1(\Omega))} \le C. \tag{2.12}$$

Proof. By multiplying (1.1) by $W^\varepsilon/\mathcal{F}$, taking trace, integrating over $\boldsymbol{x}$ and $\boldsymbol{v}$, using (2.7), (2.5), $\mathrm{Tr}([\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{m}^\varepsilon, W^\varepsilon], W^\varepsilon) = 0$, and $\frac{1}{2}(\mathrm{Tr} W^\varepsilon)^2 \le \mathrm{Tr}[(W^\varepsilon)^2]$, we get the following inequality

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|W^\varepsilon\|^2_{\mathbb{L}^2_{\mathcal{F}}} \le -\frac{\alpha_0}{2\varepsilon^2} \|N^\varepsilon \mathcal{M} - W^\varepsilon\|^2_{\mathbb{L}^2_{\mathcal{F}}} \le 0,$$

with

$$N^\varepsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_{\boldsymbol{v}}} W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\,\mathrm{d}\boldsymbol{v}, \tag{2.13}$$

which implies (2.9) and (2.10). Also, by Proposition 2.2, we get (2.11).

In order to get the estimate for $\boldsymbol{m}^\varepsilon$, first, we can see that

$$|\boldsymbol{m}^\varepsilon(t)| = |\boldsymbol{m}_{\text{in}}|. \tag{2.14}$$

Then we multiply (1.7) by $\partial_t \boldsymbol{m}^\varepsilon$ and $\boldsymbol{H}_{\text{eff}}[\boldsymbol{m}^\varepsilon] + \boldsymbol{s}^\varepsilon$ and integrate over $\boldsymbol{x}$ to get

$$\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} = -\gamma \int_\Omega \boldsymbol{m}^\varepsilon \times (\boldsymbol{H}_{\text{eff}}[\boldsymbol{m}^\varepsilon] + \boldsymbol{s}^\varepsilon)\cdot \partial_t \boldsymbol{m}^\varepsilon\,\mathrm{d}\boldsymbol{x},$$

and

$$\int_\Omega \partial_t \boldsymbol{m}^\varepsilon \cdot (\boldsymbol{H}_{\text{eff}}[\boldsymbol{m}^\varepsilon] + \boldsymbol{s}^\varepsilon)\,\mathrm{d}\boldsymbol{x} = \int_\Omega \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \cdot (\boldsymbol{H}_{\text{eff}}[\boldsymbol{m}^\varepsilon] + \boldsymbol{s}^\varepsilon)\,\mathrm{d}\boldsymbol{x}.$$

Thus

$$\begin{aligned}\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} =& \gamma \int_\Omega \partial_t \boldsymbol{m}^\varepsilon \cdot (\boldsymbol{H}_{\text{eff}}[\boldsymbol{m}^\varepsilon] + \boldsymbol{s}^\varepsilon)\,\mathrm{d}\boldsymbol{x}\\ =& -\frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(|\nabla_{\boldsymbol{x}} \boldsymbol{m}^\varepsilon|^2 + |\boldsymbol{H}_{\text{s}}[\boldsymbol{m}^\varepsilon]|^2\right)\mathrm{d}\boldsymbol{x} + \gamma \int_\Omega \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{s}^\varepsilon\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

It follows from (1.9) that

$$\int_{\mathbb{R}^3} \partial_t \boldsymbol{H}_{\text{s}}[\boldsymbol{m}^\varepsilon]\cdot \nabla\varphi = \int_\Omega \partial_t \boldsymbol{m}^\varepsilon \cdot \nabla \varphi, \quad \forall \varphi \in H^1(\mathbb{R}^3),$$

and substituting φ for $u^\varepsilon \in H^1(\mathbb{R}^3)$, the corresponding magnetostatic potential, we get that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^3} |\boldsymbol{H}_{\text{s}}[\boldsymbol{m}^\varepsilon]|^2 = \int_\Omega \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{H}_{\text{s}}[\boldsymbol{m}^\varepsilon].$$

Therefore

$$\begin{aligned}&\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} + \frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(|\nabla_{\boldsymbol{x}} \boldsymbol{m}^\varepsilon|^2 + |\boldsymbol{H}_{\text{s}}[\boldsymbol{m}^\varepsilon]|^2\right)\mathrm{d}\boldsymbol{x}\\ =& \gamma \int_\Omega \partial_t \boldsymbol{m}^\varepsilon \cdot \boldsymbol{s}^\varepsilon\,\mathrm{d}\boldsymbol{x} \le \frac{1}{2}\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} + \frac{\gamma^2}{2}\int_\Omega |\boldsymbol{s}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.15}$$

Together with (2.11) we get

$$\frac{1}{2}\int_\Omega |\partial_t \boldsymbol{m}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} + \frac{\gamma}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(|\nabla_{\boldsymbol{x}} \boldsymbol{m}^\varepsilon|^2 + |\boldsymbol{H}_{\text{s}}[\boldsymbol{m}^\varepsilon]|^2\right)\mathrm{d}\boldsymbol{x} \le C. \tag{2.16}$$

And further we can get for any $T > 0$ fixed,

$$\|\partial_t \boldsymbol{m}^\varepsilon\|_{L^2([0,T],L^2(\Omega))} + \|\nabla \boldsymbol{m}^\varepsilon\|_{L^\infty([0,T],L^2(\Omega))} \le C. \tag{2.17}$$

$\square$

3. Existence of solutions

In this section we will establish the existence of the weak solutions to the Boltzmann-Landau-Lifshitz-Gilbert system (1.1) and (1.7), for which we use the following definition of weak solutions:

DEFINITION 3.1 (Weak solutions). *Let $W^\varepsilon_{\rm in}\in\mathbb{L}^2_{\mathcal{F}}$, $\boldsymbol{m}_{\rm in}\in H^1(\Omega)$, $|\boldsymbol{m}_{\rm in}|=1$ a.e. in Ω. We say W^ε and $\boldsymbol{m}^\varepsilon$ are weak solutions to (1.1)–(1.7) if, for all $T>0$,*

- *$W^\varepsilon\in L^\infty([0,\infty),\mathbb{L}^2_{\mathcal{F}})$, $\boldsymbol{m}^\varepsilon\in L^\infty([0,\infty),H^1(\Omega))\cap H^1([0,T]\times\Omega)$, and $|\boldsymbol{m}^\varepsilon|=1$ a.e. in Ω.*
- *For all $\eta\in C^1_c([0,T)\times\mathbb{R}^6)$ and $\chi\in H^1((0,T)\times\Omega)$, it holds that*

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}}W^\varepsilon\left(\partial_t\eta+\frac{1}{\varepsilon}\mathcal{T}(\eta)\right)-\int_{\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}}W^\varepsilon_{\rm in}\eta(t=0)\\&=-\int_0^T\int_{\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}}}\left(\mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon,W^\varepsilon]+\frac{1}{\varepsilon^2}\mathcal{Q}(W^\varepsilon)+\mathcal{Q}_{\rm sf}(W^\varepsilon)\right)\eta,\end{aligned}\tag{3.1}$$

$$\begin{aligned}&\int_0^T\int_\Omega\partial_t\boldsymbol{m}^\varepsilon\chi-\int_0^T\int_\Omega\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon\chi\\&=-\gamma\int_0^T\int_\Omega\boldsymbol{m}^\varepsilon\times(\boldsymbol{H}_{\rm s}[\boldsymbol{m}^\varepsilon]+\boldsymbol{s}^\varepsilon)\,\chi+\gamma\int_0^T\int_\Omega\boldsymbol{m}^\varepsilon\times\nabla\boldsymbol{m}^\varepsilon\cdot\nabla\chi,\end{aligned}\tag{3.2}$$

and $\boldsymbol{m}(\boldsymbol{x},0)=\boldsymbol{m}_{\rm in}$.

Then the remainder of this section is devoted to prove the following theorem:

THEOREM 3.1 (Existence of weak solutions). *Let $W^\varepsilon_{\rm in}\in\mathbb{L}^2_{\mathcal{F}}$, $\boldsymbol{m}_{\rm in}\in H^1(\Omega)$, and $|\boldsymbol{m}_{\rm in}|\equiv 1$ on Ω and $|\boldsymbol{m}_{\rm in}|\equiv 0$ on $\mathbb{R}^3\backslash\Omega$. Then for any $T>0$ there exist $W^\varepsilon\in L^\infty([0,T];\mathbb{L}^2_{\mathcal{F}})$ and $\boldsymbol{m}^\varepsilon\in L^\infty([0,T];H^1(\Omega))\cap H^1([0,T]\times\Omega)$, that are weak solutions to (1.1) and (1.7).*

Proof. Fix $T>0$. Let $W^\varepsilon_0=W^\varepsilon_{\rm in}$ and $\boldsymbol{m}^\varepsilon_0=\boldsymbol{m}_{\rm in}$. If we have defined $W^\varepsilon_n\in L^\infty([0,T];\mathbb{L}^2_{\mathcal{M}})$ and $\boldsymbol{m}^\varepsilon_n\in L^\infty([0,T];H^1(\Omega))$ with $|\boldsymbol{m}^\varepsilon_n|\equiv 1$ on Ω and $|\boldsymbol{m}^\varepsilon_n|\equiv 0$ on $\mathbb{R}^3_{\boldsymbol{x}}\backslash\Omega$, then we can define W^ε_{n+1} and $\boldsymbol{m}^\varepsilon_{n+1}$ as the solutions of

$$\begin{aligned}&\partial_t W^\varepsilon_{n+1}+\frac{1}{\varepsilon}\mathcal{T}(W^\varepsilon_{n+1})=\mathrm{i}\left[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon_n,W^\varepsilon_{n+1}\right]+\frac{1}{\varepsilon^2}\mathcal{Q}(W^\varepsilon_{n+1})+\mathcal{Q}_{\rm sf}(W^\varepsilon_{n+1}),\\&W^\varepsilon_{n+1}(t=0)=W^\varepsilon_{\rm in},\end{aligned}\tag{3.3}$$

$$\begin{aligned}&\partial_t\boldsymbol{m}^\varepsilon_{n+1}=-\gamma\boldsymbol{m}^\varepsilon_{n+1}\times\left(\boldsymbol{H}_{\rm eff}[\boldsymbol{m}^\varepsilon_{n+1}]+\boldsymbol{s}^\varepsilon_{n+1}\right)+\boldsymbol{m}^\varepsilon_{n+1}\times\partial_t\boldsymbol{m}^\varepsilon_{n+1},\\&\boldsymbol{m}^\varepsilon_{n+1}(t=0)=\boldsymbol{m}_{\rm in},\end{aligned}\tag{3.4}$$

where $\boldsymbol{s}^\varepsilon_{n+1}=\int\mathrm{Tr}(\hat{\boldsymbol{\sigma}}W^\varepsilon_{n+1})\mathrm{d}\boldsymbol{v}$. Actually since $\boldsymbol{m}^\varepsilon_n\in L^\infty([0,T]\times\mathbb{R}^3)$, we can see that there exist a (weak) solution $W^\varepsilon_{n+1}\in L^\infty([0,T];\mathbb{L}^2_{\mathcal{F}})$ such that

$$\left\|W^\varepsilon_{n+1}(t)\right\|_{\mathbb{L}^2_{\mathcal{F}}}\le C,\ \forall t\in[0,T],\tag{3.5}$$

where C is independent of n. Then by Proposition 2.2 we know $\left\|\boldsymbol{s}^\varepsilon_{n+1}\right\|_{L^2(\Omega)}\le C$ and then similarly to [3, 7, 9] it follows that there exists a weak solution $\boldsymbol{m}^\varepsilon_{n+1}\in L^\infty([0,T];H^1(\Omega))$ such that

$$\left\|\partial_t\boldsymbol{m}^\varepsilon_{n+1}\right\|_{L^2([0,T];L^2(\Omega))}+\left\|\boldsymbol{m}^\varepsilon_{n+1}\right\|_{L^\infty([0,T];H^1(\Omega))}\le C.\tag{3.6}$$

Then we have

$$W^\varepsilon_n\xrightarrow{n\to\infty}W^\varepsilon\ \text{in}\ L^\infty([0,T],\mathbb{L}^2_{\mathcal{F}})\ \text{weak*},$$

$$\begin{aligned} \boldsymbol{s}_n^\varepsilon &\xrightarrow{n\to\infty} \boldsymbol{s}^\varepsilon \text{ in } L^\infty([0,T],L^2(\mathbb{R}^3)) \text{ weak}*, \\ \boldsymbol{m}_n^\varepsilon &\xrightarrow{n\to\infty} \boldsymbol{m}^\varepsilon \text{ in } L^\infty([0,T],H^1(\Omega)) \text{ weak}*, \\ \boldsymbol{m}_n^\varepsilon &\xrightarrow{n\to\infty} \boldsymbol{m}^\varepsilon \text{ in } L^2([0,T],L^2(\Omega)) \text{ strongly}. \end{aligned}$$

And then by passing to the limit as $n\to\infty$ in the equations, we have W^ε and $\boldsymbol{m}^\varepsilon$, $|\boldsymbol{m}^\varepsilon|\equiv 1$ satisfy (1.1) and (1.7) weakly. ∎

4. Diffusion limit

In this section we study the diffusion limit as $\varepsilon\to 0$ of the system (1.1) and (1.7).

THEOREM 4.1 (Diffusion limit). *Let $W_{\text{in}}^\varepsilon\in\mathbb{L}^2_{\mathcal{F}}$, $\|W_{\text{in}}^\varepsilon\|_{\mathbb{L}^2_{\mathcal{F}}}\le C$, $\boldsymbol{m}_{\text{in}}\in H^1(\Omega)$, $|\boldsymbol{m}_{\text{in}}|=1$ a.e. in Ω, and W^ε and $\boldsymbol{m}^\varepsilon$ are weak solutions to (1.1) – (1.7). Then there exist $W\in L^\infty([0,T],\mathbb{L}^2_{\mathcal{F}})$, $W_{\text{in}}\in\mathbb{L}^2_{\mathcal{F}}$, $\rho,\boldsymbol{s}\in L^\infty([0,T],L^2(\mathbb{R}^3))$, $\boldsymbol{m}\in L^\infty([0,T],H^1(\Omega))$ such that, up to subsequences,*

$$\begin{aligned} W^\varepsilon &\xrightarrow{\varepsilon\to 0} W \textit{ in } L^\infty([0,T],\mathbb{L}^2_{\mathcal{F}}) \textit{ weak}*, \\ W_{\text{in}}^\varepsilon &\xrightarrow{\varepsilon\to 0} W_{\text{in}} \textit{ in } \mathbb{L}^2_{\mathcal{F}} \textit{ weakly}, \\ W^\varepsilon-\Pi W^\varepsilon &\xrightarrow{\varepsilon\to 0} \mathbf{0}_{2\times 2} \textit{ in } L^2([0,T],\mathbb{L}^2_{\mathcal{F}}) \textit{ strongly}, \\ \boldsymbol{s}^\varepsilon &\xrightarrow{\varepsilon\to 0} \boldsymbol{s} \textit{ in } L^\infty([0,T],L^2(\mathbb{R}^3)) \textit{ weak}*, \\ \boldsymbol{m}^\varepsilon &\xrightarrow{\varepsilon\to 0} \boldsymbol{m} \textit{ in } L^\infty([0,T],H^1(\Omega)) \textit{ weak}*, \\ \boldsymbol{m}^\varepsilon &\xrightarrow{\varepsilon\to 0} \boldsymbol{m} \textit{ in } L^2([0,T],L^2(\Omega)) \textit{ strongly}. \end{aligned} \tag{4.1}$$

And moreover, $W\in\ker(\mathcal{Q})$, i.e. , $W(\boldsymbol{x},\boldsymbol{v},t)=N(\boldsymbol{x},t)\mathcal{M}(k)$, and N and $\boldsymbol{m}$ satisfy the following coupled system

$$\partial_t N+\nabla_{\boldsymbol{x}}\cdot J=\mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m},N]+\mathcal{Q}_{\text{sf}}(N), \tag{4.2}$$

$$\partial_t\boldsymbol{m}=-\gamma\boldsymbol{m}\times(\boldsymbol{H}_{\text{eff}}[\boldsymbol{m}]+\boldsymbol{s})+\boldsymbol{m}\times\partial_t\boldsymbol{m}, \tag{4.3}$$

where

$$J=-D\cdot(\nabla_{\boldsymbol{x}}N+\nabla_{\boldsymbol{x}}\phi N),\quad D=\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes\boldsymbol{v}\,\mathrm{d}\boldsymbol{v}, \tag{4.4}$$

and $\boldsymbol{\theta}=\boldsymbol{\theta}(\boldsymbol{v})$ is the unique solution of $-\mathcal{Q}(\boldsymbol{\theta})=\boldsymbol{v}\mathcal{M}$ in $(\ker\mathcal{Q})^\perp$. In addition, if we define the position density $\rho=\mathrm{Tr}(N)$ and the spin density $\boldsymbol{s}=\mathrm{Tr}(\hat{\boldsymbol{\sigma}}N)$, then we get (4.2) in the physical variables:

$$\partial_t\rho-\nabla_{\boldsymbol{x}}\cdot(D\cdot(\nabla_{\boldsymbol{x}}\rho+\nabla_{\boldsymbol{x}}\phi\rho))=0, \tag{4.5}$$

$$\partial_t\boldsymbol{s}-\nabla_{\boldsymbol{x}}\cdot(D\cdot(\nabla_{\boldsymbol{x}}\boldsymbol{s}+\nabla_{\boldsymbol{x}}\phi\boldsymbol{s}))=-2\boldsymbol{m}\times\boldsymbol{s}-\boldsymbol{s}. \tag{4.6}$$

REMARK 4.1. This system is similar to the diffusion model introduced by Zhang, Levy and Fert in [20]. The first term on the right hand side of (4.6) represents the precessional motion due to the sd interaction when the magnetization directions of the spin and the local moments are not parallel; the second term on the right hand side of (4.6) represents the spin-flip relaxation.

The limits in (4.1) can be obtained from the compactness in the corresponding space, and we then look at the limit system.

4.1. Formal asymptotic expansion. In this section, we formally derive the limiting equation (4.2) using a Hilbert expansion in order to get some heuristic idea of the proof. We first expand W^ε as

$$W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t) = W^{(0)}(\boldsymbol{x},\boldsymbol{v},t) + \varepsilon W^{(1)}(\boldsymbol{x},\boldsymbol{v},t) + \varepsilon^2 W^{(2)}(\boldsymbol{x},\boldsymbol{v},t) + \dots \tag{4.7}$$

Now, plug (4.7) into (1.1) and separate in the order of ε. We then have

$$O(1): \quad \mathcal{Q}(W^{(0)}) = 0, \tag{4.8}$$

$$O(\varepsilon): \quad \mathcal{Q}(W^{(1)}) = \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} W^{(0)} - \nabla_{\boldsymbol{x}}\phi\cdot\nabla_{\boldsymbol{v}} W^{(0)}, \tag{4.9}$$

$$O(\varepsilon^2): \quad \mathcal{Q}(W^{(2)}) = \partial_t W^{(0)} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} W^{(1)} - \nabla_{\boldsymbol{x}}\phi\cdot\nabla_{\boldsymbol{v}} W^{(1)} - \mathrm{i}\left[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}, W^{(0)}\right] - \mathcal{Q}_{\mathrm{sf}}(W^{(0)}). \tag{4.10}$$

From (4.8), one obtains $W^{(0)} \in \ker(\mathcal{Q})$, thus

$$W^{(0)} = N(\boldsymbol{x},t)\mathcal{M}(\boldsymbol{v}). \tag{4.11}$$

Then $\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} W^{(0)} - \nabla_{\boldsymbol{x}}\phi\cdot\nabla_{\boldsymbol{v}} W^{(0)} \in (\ker\mathcal{Q})^\perp$, and thus (4.9) has general solution

$$W^{(1)} = -\tau\boldsymbol{\theta}(\boldsymbol{v})\cdot(\nabla_{\boldsymbol{x}} N + \nabla_{\boldsymbol{x}}\phi N) + K, \tag{4.12}$$

where $\boldsymbol{\theta}$ is the unique solution of $-\mathcal{Q}(\boldsymbol{\theta}) = \boldsymbol{v}\mathcal{M}$ in $(\ker\mathcal{Q})^\perp$, and $K \in \ker\mathcal{Q}$. Then integrating (4.10) and noticing that the K terms of $W^{(1)}$ vanish in the integrals, we obtain

$$\partial_t N + \nabla_{\boldsymbol{x}}\cdot J = \mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}, N] + \mathcal{Q}_{\mathrm{sf}}(N), \tag{4.13}$$

where $J = -D(\nabla_{\boldsymbol{x}} N + \nabla_{\boldsymbol{x}}\phi N)$ and $D = \int_{\mathbb{R}^3_{\boldsymbol{v}}} \boldsymbol{v}\otimes\boldsymbol{\theta}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}$.

4.2. Rigorous proof.

Proof. The estimates (2.9) and (2.10) suggest us to write

$$W^\varepsilon = N^\varepsilon\mathcal{M} + \varepsilon R^\varepsilon, \tag{4.14}$$

where $\int R^\varepsilon\,\mathrm{d}\boldsymbol{v} = \mathbf{0}_{2\times 2}$, and there exists a constant C such that

$$\|R^\varepsilon\|_{L^2([0,T];\mathbb{L}^2_{\mathcal{F}})} \le C \quad \text{and} \quad \|N^\varepsilon\|_{L^\infty([0,T];L^2(\mathbb{R}^3))} \le C. \tag{4.15}$$

Since W^ε is bounded uniformly in $L^\infty([0,T];\mathbb{L}^2_{\mathcal{F}})$, we can get that (3.1) is satisfied for all the test function of the form $\eta = \varphi(x,t) \in C^1_c([0,T)\times\mathbb{R}^3)$. That is, by substituting (4.14), we obtain

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^3_{\boldsymbol{x}}} N^\varepsilon\partial_t\varphi\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} + \int_0^T\int_{\mathbb{R}^3_{\boldsymbol{x}}} R^\varepsilon\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\varphi\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x} - \int_{\mathbb{R}^3_{\boldsymbol{x}}}\left(\int_{\mathbb{R}^3_{\boldsymbol{v}}} W^\varepsilon_{\mathrm{in}}\,\mathrm{d}\boldsymbol{v}\right)\varphi(t=0)\,\mathrm{d}\boldsymbol{x}\\ &= -\int_0^T\int_{\mathbb{R}^3_{\boldsymbol{x}}} (\mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon, N^\varepsilon] + \mathcal{Q}_{\mathrm{sf}}(N^\varepsilon))\varphi\,\mathrm{d}t\,\mathrm{d}\boldsymbol{x},\end{aligned} \tag{4.16}$$

which is nothing but the weak formulation of

$$\partial_t N^\varepsilon + \nabla_{\boldsymbol{x}}\cdot\int_{\mathbb{R}^3_{\boldsymbol{v}}} \boldsymbol{v}\otimes R^\varepsilon\,\mathrm{d}\boldsymbol{v} = \mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon, N^\varepsilon] + \mathcal{Q}_{\mathrm{sf}}(N^\varepsilon). \tag{4.17}$$

with initial condition

$$N^\varepsilon(\boldsymbol{x},0)=N^\varepsilon_{\text{in}}(x):=\int_{\mathbb{R}^3_{\boldsymbol{v}}}W^\varepsilon_{\text{in}}(\boldsymbol{x},\boldsymbol{v})\,\mathrm{d}\boldsymbol{v}. \tag{4.18}$$

Since $\|W^\varepsilon_{\text{in}}\|_{\mathbb{L}^2_{\mathcal{F}}}\le C$ uniformly, we know that $N^\varepsilon_{\text{in}}$ is uniformly bounded in $L^2(\mathbb{R}^3)$. Then there exists $N_{\text{in}}\in L^2(\mathbb{R}^3)$ such that $N^\varepsilon_{\text{in}}\xrightarrow{\varepsilon\to 0}N_{\text{in}}$ in the weak sense.

We claim that

Lemma 4.1. $N_{\text{in}}=\int_{\mathbb{R}^3_{\boldsymbol{v}}}W_{\text{in}}\,\mathrm{d}\boldsymbol{v}$.

Proof. Let $N^0=\int_{\mathbb{R}^3_{\boldsymbol{v}}}W\,\mathrm{d}\boldsymbol{v}$. Then for any $\varphi\in L^2(\mathbb{R}^3)$

$$\begin{aligned}\int_{\mathbb{R}^3_{\boldsymbol{x}}}(N^\varepsilon_{\text{in}}-N^0)\varphi\,\mathrm{d}\boldsymbol{x}&=\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}(W^\varepsilon_{\text{in}}-W_{\text{in}})\,\mathrm{d}\boldsymbol{v}\,\varphi\,\mathrm{d}\boldsymbol{x}\\&=\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{|\boldsymbol{v}|>K}(W^\varepsilon_{\text{in}}-W_{\text{in}})\,\mathrm{d}\boldsymbol{v}\,\varphi\,\mathrm{d}\boldsymbol{x}+\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}(W^\varepsilon_{\text{in}}-W_{\text{in}})\varphi I_{|v|\le K}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Since W_{in} is the $\mathbb{L}^2_{\mathcal{F}}$ weak limit of $W^\varepsilon_{\text{in}}$, the second integral converges to zeros as $\varepsilon\to 0$ for every fixed $K>0$. The first integral decays uniformly and exponentially in K since

$$\begin{aligned}&\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{|\boldsymbol{v}|>K}(W^\varepsilon_{\text{in}}-W_{\text{in}})\,\mathrm{d}\boldsymbol{v}\,\varphi\,\mathrm{d}\boldsymbol{x}\\&\le\left(\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}(W^\varepsilon_{\text{in}}-W_{\text{in}})^2\mathcal{M}^{-1}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x})\right)^{\frac12}\left(\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{|\boldsymbol{v}|>K}\mathcal{M}\,\mathrm{d}\boldsymbol{v}\,|\varphi|^2\,\mathrm{d}\boldsymbol{x})\right)^{\frac12}\\&\le C(\|W^\varepsilon_{\text{in}}\|_{\mathbb{L}^2_{\mathcal{F}}}+\|W_{\text{in}}\|_{\mathbb{L}^2_{\mathcal{F}}})\|\varphi\|_2\mathrm{e}^{-K}\le 2C\|\varphi\|_2\mathrm{e}^{-K}.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^3_{\boldsymbol{x}}}(N^\varepsilon_{\text{in}}-N^0)\varphi\,\mathrm{d}\boldsymbol{x}\to 0,$$

and then $N_{\text{in}}=N^0$. ◻

In order to pass to the limit of (4.17), we need to find the limit of $\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{v}\otimes R^\varepsilon\,\mathrm{d}\boldsymbol{v}$. Let $\boldsymbol{\theta}=\boldsymbol{\theta}(\boldsymbol{v})$ be the unique solution of $\mathcal{Q}(\boldsymbol{\theta}(\boldsymbol{v}))=-\boldsymbol{v}\mathcal{M}(v)$, then we know that [5]:

$$\left|\frac{\boldsymbol{\theta}}{\mathcal{M}}\right|\le C(1+|\boldsymbol{v}|),\qquad\left|\frac{\nabla_{\boldsymbol{v}}\boldsymbol{\theta}}{\mathcal{M}}\right|\le C(1+|\boldsymbol{v}|^2), \tag{4.19}$$

and we have the following lemma for $\boldsymbol{\theta}$:

Lemma 4.2. *It holds that for any $\eta\in C_c([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})$,*

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}}\eta\boldsymbol{v}\otimes R^\varepsilon\,\mathrm{d}\boldsymbol{v}=-\lim_{\varepsilon\to 0}\int_{\mathbb{R}^3_{\boldsymbol{v}}}\eta\boldsymbol{\theta}\otimes(\boldsymbol{v}\cdot(\nabla_{\boldsymbol{x}}N^\varepsilon+\nabla_{\boldsymbol{x}}\phi N^\varepsilon))\,\mathrm{d}\boldsymbol{v}. \tag{4.20}$$

Proof. Notice that if $W^\varepsilon\in L^\infty([0,T];\mathbb{L}^2_{\mathcal{F}})$ is a weak solution of (1.1), then (3.1) holds for any test function in the form of $\eta\boldsymbol{\theta}/\mathcal{M}$, where $\eta\in C_c([0,T)\times\mathbb{R}^3_{\boldsymbol{x}})$. Then substituting (4.14) in to (1.1) gives

$$\mathcal{Q}(R^\varepsilon)=\boldsymbol{v}\cdot(\nabla_{\boldsymbol{x}}N^\varepsilon+\nabla_{\boldsymbol{x}}\phi N^\varepsilon)\mathcal{M}$$

$$+\varepsilon(\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}+\nabla_{\boldsymbol{x}}\phi\cdot\nabla_{\boldsymbol{v}})R^{\varepsilon}+\varepsilon(\partial_t W^{\varepsilon}-\mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^{\varepsilon},W^{\varepsilon}]-\mathcal{Q}_{\mathrm{sf}}(W^{\varepsilon})). \tag{4.21}$$

On the other hand, since $\mathcal{Q}$ is a self-adjoint operator and $\mathcal{Q}(\boldsymbol{\theta}(\boldsymbol{v}))=-\boldsymbol{v}\mathcal{M}(v)$, we have

$$\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes\mathcal{Q}(R^{\varepsilon})/\mathcal{M}\,\mathrm{d}\boldsymbol{v}=\int_{\mathbb{R}^3_{\boldsymbol{v}}}\mathcal{Q}(\boldsymbol{\theta})\otimes R^{\varepsilon}/\mathcal{M}\,\mathrm{d}\boldsymbol{v}=-\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{v}\otimes R^{\varepsilon}/\mathcal{M}\,\mathrm{d}\boldsymbol{v}. \tag{4.22}$$

Multiplying (4.21) by $\boldsymbol{\theta}/\mathcal{M}$ and integrating over $\boldsymbol{v}$ gives

$$\begin{aligned}-\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{v}\otimes R^{\varepsilon}\,\mathrm{d}\boldsymbol{v}&=\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes\mathcal{Q}(R^{\varepsilon})/\mathcal{M}\,\mathrm{d}\boldsymbol{v}\\&=\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes(\boldsymbol{v}\cdot(\nabla_{\boldsymbol{x}}N^{\varepsilon}+\nabla_{\boldsymbol{x}}\phi N^{\varepsilon}))\,\mathrm{d}\boldsymbol{v}\\&\quad+\varepsilon\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes(\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}+\nabla_{\boldsymbol{x}}\phi\cdot\nabla_{\boldsymbol{v}})R^{\varepsilon}/\mathcal{M}\,\mathrm{d}\boldsymbol{v}\\&\quad+\varepsilon\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes(\partial_t W^{\varepsilon}-\mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^{\varepsilon},W^{\varepsilon}]-\mathcal{Q}_{\mathrm{sf}}(W^{\varepsilon}))/\mathcal{M}\,\mathrm{d}\boldsymbol{v}.\end{aligned} \tag{4.23}$$

Then by multiplying the above by a test function in $C_c((0,T)\times\mathbb{R}^3_{\boldsymbol{x}})$ and using the boundedness of $W^{\varepsilon},N^{\varepsilon},R^{\varepsilon}$ given in (2.9), (4.15), and the estimate (4.19) for θ, we can obtain $\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{v}\otimes R^{\varepsilon}\,\mathrm{d}\boldsymbol{v}$ converges to $-\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes\boldsymbol{v}\cdot(\nabla_{\boldsymbol{x}}N+\nabla_{\boldsymbol{x}}\phi N)\,\mathrm{d}\boldsymbol{v}$ weakly. ∎

From (4.14) and (4.15), we know thaht there exists $N\in L^{\infty}([0,T];L^2(\mathbb{R}^3_{\boldsymbol{x}}))$ such that

$$N^{\varepsilon}\xrightarrow{\varepsilon\to 0}N \text{ in } L^{\infty}([0,T],L^2(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak*}, \tag{4.24}$$

$$W^{\varepsilon}\xrightarrow{\varepsilon\to 0}N\mathcal{M} \text{ in } L^{\infty}([0,T],\mathbb{L}^2_{\mathcal{F}}) \text{ weak*}. \tag{4.25}$$

The other terms in (4.17) and (1.7) can pass to the limit directly and we obtain the limit equation

$$\partial_t N-\nabla_{\boldsymbol{x}}\cdot\int_{\mathbb{R}^3_{\boldsymbol{v}}}\boldsymbol{\theta}\otimes\boldsymbol{v}\cdot(\nabla_{\boldsymbol{x}}N+\nabla_{\boldsymbol{x}}\phi N)\,\mathrm{d}\boldsymbol{v}=\mathrm{i}[\hat{\boldsymbol{\sigma}}\cdot\boldsymbol{m},N]+\mathcal{Q}_{\mathrm{sf}}(N), \tag{4.26}$$

$$\partial_t\boldsymbol{m}=-\gamma\boldsymbol{m}\times(\boldsymbol{H}_{\mathrm{eff}}[\boldsymbol{m}]+\boldsymbol{s})+\boldsymbol{m}\times\partial_t\boldsymbol{m}. \tag{4.27}$$

This complete the proof of Theorem 4.1.
∎

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