RIGOROUS ACCURACY AND ROBUSTNESS ANALYSIS FOR TWO-SCALE REDUCED RANDOM KALMAN FILTERS IN HIGH DIMENSIONS[*]

ANDREW J. MAJDA[†] AND XIN T. TONG[‡]

Abstract. Contemporary data assimilation often involves millions of prediction variables. The classical Kalman filter is no longer computationally feasible in such a high dimensional context. This problem can often be resolved by exploiting the underlying multiscale structure, applying the full Kalman filtering procedures only to the large scale variables, and estimating the small scale variables with proper statistical strategies, including multiplicative inflation, representation model error in the observations, and crude localization. The resulting two-scale reduced filters can have close to optimal numerical filtering skill based on previous numerical evidence. Yet, no rigorous explanation exists for this success, because these modifications create unavoidable bias and model error. This paper contributes to this issue by establishing a new error analysis framework for two different reduced random Kalman filters, valid independent of the large dimension. The first part of our results examines the fidelity of the covariance estimators, which is essential for accurate uncertainty quantification. In a simplified setting, this is demonstrated by showing the true error covariance is dominated by its estimators. In general settings, the Mahalanobis error and its intrinsic dissipation can be used as simplified quantification of the same property. The second part develops upper bounds for the covariance estimators by comparing with proper Kalman filters. Combining both results, the classical tools for Kalman filters can be used as a-priori performance criteria for the reduced filters. In applications, these criteria guarantee the reduced filters are robust, and accurate for small noise systems. They also shed light on how to tune the reduced filters for stochastic turbulence.

Keywords. Model reduction; reduced Kalman filters; filter robustness; filter accuracy.

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1. Introduction

Data assimilation, the numerical prediction procedure for partially observed processes, has been a central problem for science and engineering for decades. In this new age of technology, the dimensions of filtering problems have grown exponentially, as a result of the increasingly abundant observations and ever growing demand for prediction accuracy. In geophysical applications such as numerical weather forecasting, the dimensions are staggeringly high, often exceeding $d = 10^6$ for the prediction variables, and $q = 10^4$ for the observations. In such a context, the well known Kalman filter is no longer computationally feasible. Its direct implementation requires high dimensional matrices product and inversion, resulting a computation complexity of $O(d^2 q)$, which far exceeds modern computing capability.

One important strategy for high dimensional filtering is dimension reduction. Many geophysical and engineering problems have intrinsic multiscale structures [1–3], where the large scale variables have more uncertainty and of more prediction importance. In comparison, the small scale variables are driven by strong dissipation and fast oscillation, their values are more predictable but of less significance. Intuitively, one would like to apply the full filtering procedures for the large scale variables, while estimating the

[†]Courant Institute of Mathematical Sciences, New York University, Warren Weaver Hall, 251 Mercer Street, New York, NY 10012, USA (jonjon@cims.nyu.edu).
http://www.math.nyu.edu/faculty/majda/
[‡]Department of mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Singapore (mattxin@nus.edu.sg).
http://www.math.nus.edu.sg/mattxin/

small scale variables with some simplified strategy. This paper investigates two such general strategies: estimate the small scale variables by their statistical equilibrium state, or use a constant statistical state as the prior distribution in each filtering step for the small scale. The resulting two-scale reduced filters will be called the dynamically decoupled reduced Kalman filter (DRKF) and general reduced Kalman filter (RKF) respectively. These ideas have been applied earlier to stochastic turbulence, and known as the reduced Fourier domain Kalman filter (RFDKF) and variance strong damping approximate filter (VSDAF), see chapters 3 and 7 of [4]. Numerous numerical tests on these reduced filters [4,5] have shown their performances are close to optimal in various regimes. And because only the large scale variable of dimension p is fully filtered, the complexity is reduced significantly to $O(dq^2 + p^2 q)$.

While the two-scale reduced filters have simple intuitions and successful applications, there is no rigorous nonasymptotic analysis framework for its performance. Precisely speaking, we are interested in the statistical and dynamical features of the filter error e_n. In the classical Kalman filtering context, we have complete knowledge of e_n, as its covariance is correctly estimated by the optimal filter, and follows a Riccati equation that quickly converges to an equilibrium state [6]. As for the reduced filters, the filter error covariance $\mathbb{E}e_n \otimes e_n$ no longer matches its reduced estimator C_n because of unavoidable model errors, which create bias through multiplicative inflation, representation error in the observations, and crude localization. Instead, it follows an online recursion where model reduction procedures constantly introduce structural biases. As a consequence, there is an intrinsic barrier between the reduced filters and the optimal one [5,7], and the classical approach which shows approximate filters are close to the optimal one is not applicable in this scenario [8–11].

This paper proposes and applies a new performance analysis strategy for the reduced filters in the subtle context of Kalman filters with random coefficients ([6], and chapter 8 of [4] for an application in large dimensions). It consists of two parts. The first part examines the fidelity of the reduced covariance estimator C_n, and aims to show the true error covariance is not underestimated, which is essential for rigorous uncertainty quantification. The direct approach, showing $\mathbb{E}e_n \otimes e_n \preceq C_n$, is applicable to RKF if the dimension reduction procedure preserves this inequality, while the system noises are uncorrelated with the system coefficients. Another more general but weaker approach is to consider the Mahalanobis error $\|e_n\|_{C_n}^2 = e_n^T C_n^{-1} e_n$. By showing that $\frac{1}{d}\mathbb{E}\|e_n\|_{C_n}^2$ is bounded by a dimension free constant, we show the error covariance estimator is not far off from the true value. This can be established by verifying the Mahalanobis error is dissipative, which is an intrinsic dynamical property of Kalman type updates. It holds for both RKF and DRFK even with correlated system noises.

The second objective is to find a bound for the covariance estimator C_n. Two signal observation systems with augmented coefficients are considered, and we show their Kalman filter covariances are respectively the covariance estimator in DRKF and an upper bound for the covariance estimator in RKF. By building this connection, we transform the original error control problem of reduced filters to one of standard classical Kalman filters. The latter has a rich literature that we can rely on, so there are multiple ways to bound C_n.

In combination, the previous results provide a theoretical framework to discuss the accuracy of reduced filters. In applications, many models, such as stochastic turbulences, could have various ways to do the two-scale separation [12,13]. Moreover, most practical reduced filters employ various covariance inflation techniques to ensure no covariance underestimation [14–16]. Which dimension reduction method is better, and how to tune

the filter parameters, are important practical questions. But previously, they can only be studied through extensive numerical experiments. In this perspective, our framework can be used as a priori estimates, or rigorous support for previous numerical findings.

The remainder of this section intends to give a quick overview of our results, while the detailed formal statements along with the proofs are left in the later sections.

1.1. Kalman filtering in high dimension. Consider a signal-observation system with random coefficients [6]:

$$
\begin{aligned}
X_{n+1} &= A_n X_n + B_n + \xi_{n+1}, \\
Y_{n+1} &= H_n X_{n+1} + \zeta_{n+1}.
\end{aligned}
\tag{1.1}
$$

In (1.1), ξ_{n+1} and ζ_{n+1} are two sequences of independent Gaussian noises, $\xi_{n+1} \sim \mathcal{N}(0, \Sigma_n)$ and $\zeta_{n+1} \sim \mathcal{N}(0, \sigma_n)$. We assume the signal variable X_n is of dimension d, the observation variable Y_n is of dimension $q \leq d$, and the observation noise matrix σ_n is nonsingular to avoid ill-posed inverse problems. The realizations of the dynamical coefficients (A_n, B_n, Σ_n), the observation coefficients (H_n, σ_n), as long as Y_n are assumed to be available, and the objective is to estimate X_n. By considering general random coefficients, many interesting models involving intermittent dynamical regimes or observations can be formulated as (1.1). Details will be discussed in Section 6.

The optimal filter for system (1.1) is the Kalman filter [4,6,17], assuming (X_0, Y_0) is Gaussian distributed. It estimates X_n with a Gaussian distribution $\mathcal{N}(m_n, R_n)$, where the mean and covariance follow a well known recursion:

$$
\begin{aligned}
m_{n+1} &= A_n m_n + B_n + K_{n+1}(Y_{n+1} - H_n m_n), \quad R_{n+1} = \mathcal{K}(\widehat{R}_{n+1}), \\
\widehat{R}_{n+1} &= A_n R_n A_n^T + \Sigma_n, \quad K_{n+1} = \widehat{R}_{n+1} H_n (\sigma_n + H_n \widehat{R}_{n+1} H_n^T)^{-1}, \\
\mathcal{K}(C) &= C - C H_n (\sigma_n + H_n C H_n^T)^{-1} H_n^T C.
\end{aligned}
\tag{1.2}
$$

The Kalman filter has found a wide range of applications in various fields. This is due to its theoretical optimality, robustness and stability in the classical low dimensional setting. However, in many modern day applications where the system dimension reaches 10^6, direct application of (1.2) is no longer feasible, because the computation complexity of (1.2) is roughly $O(d^2 q) = 10^{16}$, which is far beyond the speed of standard high performance computing, 10^{12}. This is briefly discussed by a complexity analysis in Section A.

Besides the dimension reduction strategies discussed below, there are various ways to approximate the Kalman filter by random sampling. These methods are known as the ensemble Kalman filters (EnKF) [18,19] which are commonly used in numerical weather forecast. They require various ad-hoc tuning techniques [14–16, 20]. Theoretical properties of EnKF, such as well-posedness, nonlinear stability, and geometric ergodicity, can be studied in nonlinear settings [21–24], while their performance can be studied in linear settings through similar frameworks as the one introduced here [25, 26]. Quantitative analysis of the filter error can also be done in the limit of a large sample size [10, 11, 27].

1.2. Two-scale separation and reduced Kalman filters. Dynamical features of system (1.1) can often be exploited for dimension reduction and fast computation of Kalman filters. In this paper, we focus on scenarios where system (1.1) has a two-scale separation

$$
X_n = \begin{bmatrix} X_n^L \\ X_n^S \end{bmatrix}.
$$

Here, X_n^L consists of $p(\ll d)$ large scale variables, and X_n^S consists of $d-p$ small scale variables. Throughout, $\mathcal{P}_L$ and $\mathcal{P}_S$ will denote the associated subspaces, $\mathbf{P}_L$ and $\mathbf{P}_S$ will denote the associated projections.

In Section 6.3.2, we will consider a simple stochastic turbulence model (see [4]), where A_n is a constant matrix that consists of 2×2 diagonal sub-blocks with spectral norm $\exp(-\nu h|k|^2)$. To the small scale Fourier modes with large wavenumbers $|k|$, A_n applies a strong damping. As a consequence, the small scale variables have very little uncertainty. Their exact values are of little importance, but they are difficult to compute numerically because of the stiffness involved. This simple example captures features shared by many turbulence models. In such a scenario, it is intuitive to apply dimension reduction and try to filter only the large scale part.

One naive way of dimension reduction would be directly ignoring the small scale part. But this is usually problematic. Despite that X_n^S has little uncertainty in each coordinate, the observation operator $H_n = [H_n^L, H_n^S]$ involves all coordinates, and so does the observation:

$$Y_{n+1} = H_n^L X_n^L + H_n^S X_n^S + \zeta_{n+1}. \tag{1.3}$$

X_n^S could have significant contribution to the observation Y_n through H_n^S. Directly ignoring the small scale part may create a big bias, as the filter will try to interpret the contribution of X_n^S in terms of X_n^L, which is called representation error [12, 13]. The correct filter reduction requires some simple but educated estimations of the filter impact from the small scale variables.

1.2.1. Dynamical decoupled reduced Kalman filter (DRKF). One simple estimate of the small scale variables would be their statistical equilibrium states. This idea was applied for stochastic turbulences in chapter 7 of [4] and named the RFDKF. To generalize it, we consider a simplified setting where the dynamics of the signal variable X_n is decoupled between the two scales. In other words, the system coefficients of (1.1) have the following block structure:

$$A_n = \begin{bmatrix} A_n^L & 0 \\ 0 & A_n^S \end{bmatrix}, \quad B_n = \begin{bmatrix} B_n^L \\ B_n^S \end{bmatrix}, \quad \xi_n = \begin{bmatrix} \xi_n^L \\ \xi_n^S \end{bmatrix}, \quad \Sigma_n = \begin{bmatrix} \Sigma_n^L & 0 \\ 0 & \Sigma_n^S \end{bmatrix}. \tag{1.4}$$

The diagonal A_n used for stochastic turbulence in Section 6.3.2 follows this description. Notice that with observation mixing the two scales (1.3), the optimal Kalman filter does not necessarily have a block diagonal structure, so we cannot directly apply a large scale projection to (1.2).

The DRKF filtering strategy comes as a combination of two ideas. First, if the small scale part has very little fluctuations, then its unfiltered mean, μ_n^S, is a good estimator. Second, the small scale observation $H_n^S X_{n+1}^S$ is interpreted as a noisy perturbation to the large scale observation. We can remove the mean of this perturbation by letting

$$Y_n^L = Y_n - H_n^S \mu_{n+1}^S.$$

We also need to consider the fluctuation at the small scale $\Delta X_{n+1}^S = X_{n+1}^S - \mu_{n+1}^S$. By interpreting it as a mean zero Gaussian noise, we need to include the representative error covariance:

$$\sigma_n^L = \sigma_n + H_n^S V_{n+1}^S (H_n^S)^T.$$

Here V_n^S is the unfiltered covariance of X_n^S conditioned on the system coefficients. In this way, we treat $\{\Delta X_{n+1}^S\}$ as an independent sequence. Unfortunately this is not the

case in reality, and model errors are hence created. We remedy this by inflating the covariance in the end with a factor $r > 1$.

In summary, DRKF estimates X_n^L and X_n^S by Gaussian distributions (μ_n^L, C_n^L) and (μ_n^S, V_n^S) respectively. The mean and covariance sequences are updated as below:

$$
\begin{aligned}
&\mu_{n+1}^L = A_n^L \mu_n^L + B_n^L + K_{n+1}^L (Y_n^L - H_n^L (A_n^L \mu_n^L + B_n^L)), \\
&C_{n+1}^L = r\mathcal{K}_L(\widehat{C}_{n+1}^L), \quad \widehat{C}_{n+1}^L = A_n^L C_n^L (A_n^L)^T + \Sigma_n^L, \\
&K_{n+1}^L = \widehat{C}_{n+1}^L (H_n^L)^T (\sigma_n^L + H_n^L \widehat{C}_{n+1}^L (H_n^L)^T)^{-1}, \\
&\mathcal{K}_L(\widehat{C}_{n+1}^L) = \widehat{C}_{n+1}^L - \widehat{C}_{n+1}^L (H_n^L)^T (\sigma_n^L + H_n^L \widehat{C}_{n+1}^L (H_n^L)^T)^{-1} H_n^L \widehat{C}_{n+1}^L, \\
&Y_n^L = Y_n - H_n^S \mu_{n+1}^S, \quad \sigma_n^L = \sigma_n + H_n^S V_{n+1}^S (H_n^S)^T, \\
&\mu_{n+1}^S = A_n^S \mu_n^S + B_n^S, \quad V_{n+1}^S = A_n^S V_n^S (A_n^S)^T + \Sigma_n^S.
\end{aligned} \tag{1.5}
$$

Since the filter essentially works only in the large scale subspace, the computational complexity is reduced to $O(q^3 + p^2 q)$ or $O(q^3 + p^2 q + d^2)$, see Section A for details. Also see chapter 8 of [4] for an application of DRKF to random filtering of geophysical turbulence.

1.2.2. General RKF. When the two scales are not dynamically decoupled, the DRKF (1.5) may have a bad performance. This is because DRKF does not filter the small scale part, while the small scale error enters the large scale estimation through the cross scale dynamics (see page 43 of [4] for an example). Another more appropriate reduced filtering strategy would be filtering the small scale with a constant prior covariance D_S as an estimate of the small scale dynamics. This will be called a general reduced Kalman filter (RKF). It has been applied to stochastic turbulence in chapter 7 of [4] and called VSDAF.

To be specific, a fixed $\mathcal{P}_S \otimes \mathcal{P}_S$ matrix D_S will be used as the prior for the small scale variables. So given a covariance estimator C_n for X_n^L, the effective covariance of X_n will be

$$
C_n^+ := C_n + D_S.
$$

In many applications, D_S can be chosen as a multiple of the unfiltered equilibrium covariance of X_n^S. But it can also take other general matrix values. In summary, the RKF estimates X_n by a Gaussian distribution $\mathcal{N}(\mu_n, C_n + D_S)$, with the mean and covariance generated by a recursion:

$$
\begin{aligned}
&\mu_{n+1} = A_n \mu_n + B_n + \widehat{K}_{n+1}(Y_{n+1} - H_n \mu_n), \\
&\widehat{C}_{n+1} = A_n C_n^+ A_n^T + \Sigma_n, \quad \widehat{K}_{n+1} = \widehat{C}_{n+1} H_n (\sigma_n + H_n \widehat{C}_{n+1} H_n^T)^{-1}, \\
&C_{n+1} = r\mathbf{P}_L \mathcal{K}(\widehat{C}_{n+1}) \mathbf{P}_L.
\end{aligned} \tag{1.6}
$$

Furthermore, using the complexity analysis in Section A, we see RKF reduces the complexity to $O(d^2 + dq^2 + dp^2)$.

Unlike DRKF, RKF applies a large scale covariance projection in the final step. This ensures the prior covariance of small scale variables in the next step is still D_S. Its practical effect is similar to the localization techniques in EnKF that are widely applied, as both simplify the covariance structures. On the other hand, this projection may underestimate the error covariance for the new update. To offset this effect, a multiplicative inflation with $r > 0$ is applied, and in the effective covariance estimator

we also include the constant covariance D_S. Ideally, these inflations will remedy the possible covariance underestimation, so that

$$\mathcal{K}(\widehat{C}_{n+1}) \preceq C_{n+1}^+ = r\mathbf{P}_L\mathcal{K}(\widehat{C}_n)\mathbf{P}_L + D_S. \tag{1.7}$$

To be pragmatic, (1.7) may not hold for all n, and we need to introduce a sequence for the ratio between both sides. This will be formalized in Definition 2.1 in Section 2.

1.3. Filter error bounds through filter covariance.   Although the reduced Kalman filters produce good estimates in various numerical tests, there is no good rigorous explanation of their successes. We intend to close this gap by developing a quantitative analysis. In our context, the filter errors of RKF and DRKF are given respectively by

$$e_n = X_n - \mu_n, \quad e_n^L = X_n^L - \mu_n^L.$$

Notice that we do not consider the small scale estimator error for DRKF, as the small scale variables are not filtered there.

Error analyses for reduced filters are usually more difficult than error analyses for the optimal filter. For the optimal filter (1.2), the covariance of the error $X_n - m_n$ is exactly R_n, which can be easily studied by the associated Riccati equation [6]. For the reduced filters, the covariance estimators C_n^+ and C_n^L do not necessarily match the real filter error covariance, since the dimension reduction procedures may create biases.

One classical error analysis strategy for non-optimal filters is to compare them with the optimal filter and show the differences are small [8–11, 28]. Roughly speaking, this strategy assumes the non-optimal filter is very close to the optimal filter at one time, and then exploits the intrinsic ergodicity and continuity of the optimal filter, or an asymptotic expansion, to show the difference remains small afterwards. Unfortunately, this strategy is invalid for our reduced filters, because they are structurally different from the optimal filter (1.2). Evidently, R_n may not have a block diagonal structure like C_n^+ does, it may not have its $\mathcal{P}_S \otimes \mathcal{P}_S$ sub-block being exactly D_S, and this sub-block can never be zero as in the case for DRKF. This is also known as the information barrier for reduced filters, investigated by [5, 7].

A more pragmatic strategy would be looking for intrinsic statistical properties of filter errors. In particular, it is important to check whether the reduced covariance estimators dominate the real error covariance, as underestimating error covariance often causes severe filter divergence (see chapter 2 of [4]). The direct way will be verifying bounds such as $\mathbb{E}e_n \otimes e_n \preceq C_n^+$. This is applicable for RKF if the system noises are independent of the system coefficients, for example when the latter are deterministic. But for general scenarios and DRKF, the error covariance matrix $\mathbb{E}e_n \otimes e_n$ is hard to track, as nonindependent system noises are involved in the recursion. For these difficult situations, we need to look at other statistical quantities.

One natural choice would be the mean square error (MSE), $\mathbb{E}|e_n|^2$. But MSE works best when the error is isotropic, in other words the error has similar strength in each component. Our two-scale setting clearly does not fit into this description, as the small scale error is much weaker. In comparison, the Mahalanobis norm is a better error measurement. Given a nonsingular $d \times d$ positive definite (PD) matrix C, it generates a Mahalanobis norm on $\mathbb{R}^d$:

$$\|v\|_C^2 := v^T[C]^{-1}v. \tag{1.8}$$

This norm is central in many Bayesian inverse problems. For example, given the prior distribution of X as $\mathcal{N}(b,C)$, and a linear observation $Y = HX + \xi$ with Gaussian noise

$\xi \sim \mathcal{N}(0, \Sigma)$, the optimal estimate is the minimizer of $\|x - b\|_C^2 + \|Y - Hx\|_\Sigma^2$. In our context, it is natural to look at the Mahalanobis error $\frac{1}{d}\|e_n\|_{C_n^+}^2$ and $\frac{1}{p}\|e_n^L\|_{C_n^L}^2$. Based on our RKF formulation, the true state is estimated by $\mathcal{N}(\mu_n, C_n^+)$. A natural statistics that verifies this hypothesis is simply $\frac{1}{d}\mathbb{E}\|e_n\|_{C_n^+}^2$. If the hypothesis holds, this statistic should roughly be of constant values. Comparing with the MSE, the Mahalanobis error puts less weights in the large scale variables, and more weights in the small scale. By showing the Mahalanobis error is bounded, we show the error covariance estimator C_n^+ more or less captures the real error covariance.

The Mahalanobis error also has surprisingly good dynamical properties. In short, $\|e_n\|_{C_n^+}^2$ is a dissipative (also called exponentially stable) sequence. This is actually carried by an intrinsic inequality induced by the Kalman covariance update operator $\mathcal{K}$. It was exploited by previous works in the literature [6, 29] to show robustness of Kalman filters and extended Kalman filters (although the name Mahalanobis error is not explicitly used, but readers can identify it easily in the proofs).

One major result of this paper is informally stated as below:

THEOREM 1.1. *When applying DRKF (1.5) to a dynamically decoupled system (1.4), the non-dimensionalized Mahalanobis filter error* $\frac{1}{p}\mathbb{E}\|e_n^L\|_{C_n^L}^2$ *decays exponentially fast and is eventually bounded by a dimension free constant.*

When applying RKF (1.6) to a general system described by (1.1), if the large scale projection does not decrease covariance estimate so (1.7) holds, the non-dimensionalized Mahalanobis filter error $\frac{1}{d}\mathbb{E}\|e_n\|_{C_n^+}^2$ *decays exponentially fast and is eventually bounded by a dimension free constant. In addition, if the system noises are independent of all system coefficients, the second moment of error is dominated by its estimator:* $\mathbb{E}e_n \otimes e_n \preceq C_n^+$.

The formal description is given by Theorems 2.1, 2.2, and 3.1. The requirement of (1.7) will be replaced by a concrete version, Definition 2.1, which can be verified in Section 4 provided that $\mathbf{P}_S \widetilde{R} \mathbf{P}_S \preceq C D_S$ for a suitable constant $C > 0$ depending on r, where $\widetilde{R}$ is the stationary asymptotic covariance for (1.6).

1.4. Intrinsic filter performance criteria. Theorem 1.1 essentially shows that the Mahalanobis error is a natural and convenient statistic to assess reduced filter performance. On the other hand, it raises two new questions to address:

- Bounds of the Mahalanobis error are informative only if the covariance estimator C_n^L or C_n^+ is bounded. So how can we bound these covariance estimators?

- A large scale projection is applied for RKF in the assimilation step. It may lead to covariance underestimation. Ideally this can be offset by the covariance inflations so (1.7) holds. In principle, (1.7) requires online verifications during the implementation of RKF. Yet, offline a priori criteria that depend only on the system coefficients are more desirable.

Let us consider DRKF first. We only need to consider the first question. In fact, the answer is quite straightforward. Consider the following augmented signal-observation system:

$$X_{n+1}^{L'} = A_n' X_n' + B_n + \xi_{n+1}^L, \quad Y_{n+1}' = H_n^L X_{n+1}^{L'} + \zeta_{n+1}',$$
$$A_n' = \sqrt{r} A_n^L, \quad \xi_{n+1}^L \sim \mathcal{N}(0, \Sigma_n'), \quad \Sigma_n' = \Sigma_n^L, \quad \zeta_{n+1} \sim \mathcal{N}(0, \sigma_n^L). \tag{1.9}$$

The optimal filter of the above system is a Kalman filter $\mathcal{N}(m_n^L, R_n^L)$. It is easy to verify that $R_n^L = C_n^L/r$ if it holds at $n = 0$, because R_n^L follows a Riccati recursion just

like (1.5). The advantage we gain from this observation is that, as a Kalman filter covariance, R_n^L converges to a unique stationary solution $\widetilde{R}_n^L$, assuming the system (1.9) is stationary, ergodic, weakly observable and controllable (See [6] and Theorem C.1). This stationary solution reflects the intrinsic filtering skills of (1.9). It is clearly bounded and independent of the filter initialization, and in many cases, it can be computed or admits simple concrete upper bounds.

A similar idea also works for RKF. The corresponding inflated signal-observation system is slightly different from (1.1) with an inflation $r' > r$:

$$X'_{n+1} = A'_n X'_n + B_n + \xi'_{n+1}, \quad Y'_{n+1} = H_n X'_{n+1} + \zeta'_{n+1},$$
$$A'_n = \sqrt{r'} A_n, \quad \xi_{n+1} \sim \mathcal{N}(0, \Sigma'_n), \quad \Sigma'_n = r'\Sigma_n + r' A_n D_S A_n^T, \quad \zeta_{n+1} \sim \mathcal{N}(0, \sigma_n). \tag{1.10}$$

If $\widetilde{R}_n$ denotes the associated stationary Kalman covariance sequence, then it possesses all the theoretical and computational advantages mentioned for $\widetilde{R}_n^L$. Theorem 4.2 in below transfers these advantages to RKF by showing that $C_n^+ \preceq r\widetilde{R}_n + D_S$. Moreover, Theorem 4.2 shows that the online condition (1.7) can be verified by a similar version for the stationary solution $\widetilde{R}_n$, which will be Assumption 4.1; but because $\widetilde{R}_n$ can be estimated a priori, we find an a priori criterion that guarantees the performance of RKF.

In Section 6, we will discuss some scenarios when $\widetilde{R}_n^L$ and $\widetilde{R}_n$ can be bounded explicitly in spectral norms or with respect to the optimal covariance R_n. Then Theorem 1.1 implies the MSE $\mathbb{E}\|e_n\|^2$ is bounded, or the reduced filter performance is comparable with the optimal one. In many practical scenarios where the observation is frequent, the system noise Σ_n and observation noise σ_n are of scale ϵ comparing to other system coefficients. Then it is easy to verify in such a setting $\widetilde{R}_n$ and $\widetilde{R}_n^L$ scale like ϵ^2, and so will the reduced filter errors. This is a nontrivial property for the reduced filters and evidently very useful in practice. This is usually framed as filter accuracy [30, 31].

Other than accuracy, another important application for our framework is finding the transition point for two-scale separation, and how to setup the small scale covariance D_S for RKF. These questions can be answered by studying the Kalman filters for (1.9) and (1.10). Section 6 discusses these issues with concrete examples in stochastic turbulence.

1.5. Preliminaries.   The remainder of this paper is arranged as follows. Although RKF (1.6) applies to more general systems, its error analysis is structurally simpler than the one of DRKF. Section 2 starts our discussion by first showing the second moment of RKF error is bounded by C_n^+ in Theorem 2.1, and then the dissipation of the Mahalanobis error in Theorem 2.2, where Definition 2.1 formalizes (1.7). A direct Corollary 2.1 shows that the filter is exponentially stable for the mean sequence. The additional structural complexity of DRKF comes from the fact that the small scale fluctuation sequence is not an independent one. Section 3 resolves this issue by proving Theorem 3.1. Section 4 introduces some intrinsic performance criteria for the reduced filters. This is carried out by a comparison argument using the Kalman filters for the inflated systems (1.9) and (1.10). The details are in Proposition 4.1 and Theorem 4.2. Immediate corollaries for MSE and accuracy are also drawn there. Section 5 generalizes this idea to more general stochastic settings. Finally, Section 6 reviews some classical methods to control the Kalman filter covariance, and applies them to stochastic turbulences in Fourier domain in various dynamical and observational settings. The related complexity estimates, convergence to a stationary Kalman covariance, and some matrix inequalities are discussed in the appendix.

Before we start the discussion, here are a few standard notations we will use in the following. $\|C\|$ denotes the spectral norm of a matrix C, and $|x|$ is the l^2 norm of a

vector x. We use $x \otimes x$ to denote the rank 1 $n \times n$ matrix xx^T generated by a column vector x. We use $C \in PD(PSD)$ or simply C is PD (PSD) to indicate a symmetric matrix C is positive definite (semidefinite). $[C]_{j,k}$ denotes the (j,k)-th entry of a matrix C, and $[C]_{I^2}$ is the sub-matrix with both indices in a set I. And $A \preceq B$ indicates that $B - A \in PSD$. $\lceil a \rceil$ is the smallest integer above a real number a.

We assume the filter initializations are known and of deterministic values. Generally speaking, there are no specific requirements for their values. But some results implicitly rely on the invertibility of the covariance matrices.

Following [6], we say a random sequence $Z_0, Z_1, \ldots$ is *stationary*, if $(Z_0, Z_1, \ldots)$ and $(Z_k, Z_{k+1}, \ldots)$ have the same distribution. We say such a sequence is *ergodic*, if there is only one invariant measure for the shifting map $(Z_0, Z_1, \ldots) \mapsto (Z_1, Z_2, \ldots)$.

There will be three filtrations in our discussion. The first one contains all the information of system coefficients up to time n, and the initial covariance matrices for the filters:

$$\mathcal{F}_n^c = \sigma\{A_k, B_k, \Sigma_k, H_k, \sigma_k, k \leq n\} \vee \sigma\{R_0, C_0, C_0^L, V_0^S, \widetilde{R}_0, \widetilde{R}_0^L\}.$$

Noticeably, all the filter systems have their covariance estimators inside this filtration:

$$\sigma\{R_k, C_k^L, \widetilde{R}_k^L, C_k, \widetilde{R}_k, k \leq n+1\} \subset \mathcal{F}_n^c.$$

We will use $\mathcal{F}^c = \vee_{n \geq 0} \mathcal{F}_n^c$ to denote all the information regarding the system coefficients through the entire time line. When the system coefficient and initial filter covariances are deterministic, $\mathcal{F}^c$ is trivial, so $\mathbb{E}_{\mathcal{F}^c} = \mathbb{E}$.

The second filtration in addition includes information of the observations and mean initialization

$$\mathcal{F}_n^o = \sigma\{Y_k, k \leq n\} \vee \sigma\{m_0, \mu_0, \mu_0^L\} \vee \mathcal{F}_n^c.$$

This filtration also contains the filter mean sequence m_n, μ_n, μ_n^L. The last filtration contains all the information of system (1.1) up to time n, $\mathcal{F}_n = \mathcal{F}_n^o \vee \sigma\{\zeta_k, \xi_k, k \leq n\}$. We use $\mathbb{E}_n Z, \mathbb{E}_{\mathcal{F}} Z$ to denote the conditional expectation of a random variable Z with respect to $\mathcal{F}_n$ or another fixed σ-field $\mathcal{F}$ respectively.

Finally, as our discussion concerns two filters, and two scales, quite a few variables are inevitably introduced. Table 1.1 below lists some frequently used ones for future reference.

2. Performance of RKF

In the RKF formulation (1.6), the multiplicative inflation $r > 1$ in large scale, and the constant inflation in small scale D_s, intend to remedy the side effect of large scale projection $\mathbf{P}_L$ and ensure that the covariance estimate does not decrease. We measure the actual covariance change caused by this dimension reduction step, through the following sequence of ratios:

$$\beta_{n+1} = \inf\{b \geq 0, \mathcal{K}(\widehat{C}_{n+1}) \preceq bC_{n+1}^+\}. \tag{2.1}$$

Recall that $\mathcal{K}(\widehat{C}_{n+1})$ is the target posterior covariance given by the Kalman filter formulas, and $\widehat{C}_{n+1}^+$ is the posterior covariance that RKF employs. Intuitively, if β_n is eventually bounded by 1, then C_{n+1}^+ does not underestimate the error covariance. More formally, we define the following.

Symbol	Interpretation
μ_n^L, μ_n^S	DRKF large and small scale filter mean
C_n^L	DRKF large scale filter covariance
V_n^S	DRKF small scale unfiltered covariance
$\widetilde{R}_n^L$	DRKF intrinsic Kalman filter covariance of (1.9)
μ_n	RKF filter mean of both scales
C_n	RKF large scale filter covariance
D_S	RKF small scale prefixed covariance
C_n^+	RKF filter covariance of both scales
$\widetilde{R}_n$	RKF intrinsic Kalman filter covariance of (1.10)
β_n	Covariance underestimation in projection, see (2.1)
$\beta^*(<1)$	A constant that dominates β_n asymptotically
$r(>1)$	Multiplicative inflation used in both filters
$r'(>r)$	Multiplicative inflation used to define $\widetilde{R}_n^L$
$\mathcal{K}, \mathcal{K}_L$	Covariance map from forecast to posterior in RKF and DRKF
$\widehat{K}_n, K_n^L$	Kalman gain matrix in RKF and DRKF

Table 1.1: *A list of frequently occurring symbols.*

DEFINITION 2.1 (Acceptable reduction).   *We say the dimension reduction in RKF is asymptotically acceptable if there is a finite adjustment time n_0 and a $\beta^* < 1$ such that*

$$\beta_n \leq \beta^* \quad \text{for all } n \geq n_0.$$

Moreover, we say the RKF enters the acceptable reduction phase, when $n \geq n_0$.

The conditions above are online, in the sense that their verification requires an implementation of RKF. Section 4 will provide an a priori criterion Assumption 4.1, which can lead to asymptotically acceptable reductions.

2.1. Second moment of error with system independent noises.   In many scenarios, the system noises depend on the system coefficients only through Σ_n and σ_n. In specific:

$$\xi_{n+1} \sim \mathcal{N}(0, \Sigma_n), \quad \zeta_{n+1} \sim \mathcal{N}(0, \sigma_n) \quad \text{conditioned on } \mathcal{F}^c \vee \mathcal{F}_n. \tag{2.2}$$

In simple terms, (2.2) indicates that *the system noises are independent of the system coefficients.* In the classical setting of Kalman filtering, where the system coefficients are deterministic, this holds automatically. But it may fail for some conditional Gaussian systems. Assuming (2.2), the second moment of error $\mathbb{E}_{\mathcal{F}^c} e_n \otimes e_n$ can be computed explicitly, and is in fact bounded by the effective covariance estimator C_n^+.

THEOREM 2.1.   *Suppose the system noises are independent of the system coefficients, so (2.2) holds. For any fixed inflation ratio $r > 1$, consider applying the RKF (1.6) to system (1.1). Suppose the dimension reduction in RKF is asymptotically acceptable as described in Definition 2.1. Then with any fixed initial conditions, when*

$$n \geq n_0 + \max\{0, \lceil -\log \|[\mathbb{E}_{\mathcal{F}^c} e_{n_0} \otimes e_{n_0}][C_{n_0}^+]^{-1}\| / \log \beta^* \rceil\}, \tag{2.3}$$

the second moment of the error $e_n = X_n - \mu_n$ is dominated by the covariance estimator:

$$\mathbb{E}_{\mathcal{F}^c} e_n \otimes e_n \preceq C_n^+ \quad a.s.$$

If we take average of both sides, this implies that $\mathbb{E}e_n \otimes e_n \preceq \mathbb{E}C_n^+$.

Notice that the dependence of n on $\mathbb{E}_{\mathcal{F}^c}e_{n_0} \otimes e_{n_0}$ is logarithmic, so in practice the exact value of $\mathbb{E}_{\mathcal{F}^c}e_{n_0} \otimes e_{n_0}$ is not very important.

Proof. Using Lemma B.4, we define ψ_{n_0},

$$\psi_{n_0} = \max\{1, \inf\{\psi : \mathbb{E}_{\mathcal{F}^c}(e_{n_0} \otimes e_{n_0}) \preceq \psi C_{n_0}^+\}\} = \max\{1, \|\mathbb{E}_{\mathcal{F}^c}(e_{n_0} \otimes e_{n_0})[C_{n_0}^+]^{-1}\|\}.$$

We define ψ_n for $n \geq n_0$ recursively by letting

$$\psi_{n+1} = \max\{1, \psi_n \beta_{n+1}\}.$$

By our assumptions, for $n \geq n_0$, ψ_n converges to 1 geometrically with ratio β^*, and it reaches 1 when (2.3) happens. We claim that

$$\mathbb{E}_{\mathcal{F}^c}(e_n \otimes e_n) \preceq \psi_n C_n^+, \quad n \geq n_0, \tag{2.4}$$

so the claim of this theorem holds.

We show (2.4) by induction. It is clear that it holds for $n = n_0$. Suppose it holds for n, consider the forecast error $\hat{e}_{n+1} = X_{n+1} - (A_n\mu_n + B_n)$. The following recursion can be established:

$$\hat{e}_{n+1} = A_n e_n + \xi_{n+1}, \quad e_{n+1} = (I - \widehat{K}_{n+1}H_n)\hat{e}_{n+1} - \widehat{K}_{n+1}\zeta_{n+1}.$$

In combination:

$$e_{n+1} = (I - \widehat{K}_{n+1}H_n)A_n e_n + (I - \widehat{K}_{n+1}H_n)\xi_{n+1} - \widehat{K}_{n+1}\zeta_{n+1}.$$

Because $(I - \widehat{K}_{n+1}H_n)A_n \in \mathcal{F}^c$, where ξ_{n+1} and ζ_{n+1} are conditionally mean zero based on (2.2), we find

$$\begin{aligned}
&\mathbb{E}_{\mathcal{F}^c}e_{n+1} \otimes e_{n+1} \\
&= \mathbb{E}_{\mathcal{F}^c}[(I - \widehat{K}_{n+1}H_n)(A_n(e_n \otimes e_n)A_n^T + \Sigma_n)(I - \widehat{K}_{n+1}H_n)^T + \widehat{K}_{n+1}^T\sigma_n\widehat{K}_{n+1}] \\
&= [(I - \widehat{K}_{n+1}H_n)(A_n\mathbb{E}_{\mathcal{F}^c}e_n \otimes e_n A_n^T + \Sigma_n)(I - \widehat{K}_{n+1}H_n)^T + \widehat{K}_{n+1}^T\sigma_n\widehat{K}_{n+1}] \\
&\preceq (I - \widehat{K}_{n+1}H_n)(\psi_n A_n C_n^+ A_n^T + \Sigma_n)(I - \widehat{K}_{n+1}H_n)^T + \widehat{K}_{n+1}^T\sigma_n\widehat{K}_{n+1} \\
&\preceq \psi_n[(I - \widehat{K}_{n+1}H_n)(A_n C_n^+ A_n^T + \Sigma_n)(I - \widehat{K}_{n+1}H_n)^T + \widehat{K}_{n+1}^T\sigma_n\widehat{K}_{n+1}] = \psi_n\mathcal{K}(\widehat{C}_{n+1}).
\end{aligned} \tag{2.5}$$

In the penultimate step, we used that $\psi_n \geq 1$, and also the well known matrix identity for Kalman updates

$$\mathcal{K}(\widehat{C}_{n+1}) = (I - \widehat{K}_{n+1}H_n)\widehat{C}_{n+1}(I - \widehat{K}_{n+1}H_n)^T + \widehat{K}_{n+1}^T\sigma_n\widehat{K}_{n+1}.$$

By the definition of β_{n+1} (2.1), we have $\psi_n\mathcal{K}(\widehat{C}_{n+1}) \preceq \psi_n\beta_{n+1}C_{n+1}^+ \preceq \psi_{n+1}C_{n+1}^+$. So (2.4) holds for $n+1$ as well. □

REMARK 2.1. In fact, if $\mathbb{E}_{\mathcal{F}^c}e_0 = 0$, one can also show $\mathbb{E}_{\mathcal{F}^c}e_n = 0$ in this setting, so $\mathbb{E}_{\mathcal{F}^c}e_n \otimes e_n$ is actually the error covariance. But the RKF mean μ_n is a biased estimator. Unbiasedness would require $\mathbb{E}_{\mathcal{F}_n^o}e_n = 0$ a.s., which in general does not hold. In order to avoid confusion, we did not mention this fact in the theorem.

2.2. Mahalanobis error dissipation.   If the system coefficients have dependence on the system noises, the Kalman gain matrix $\widehat{K}_{n+1}$ may have correlation with the error term e_n. So the identity (2.5) no longer holds. But even in this difficult scenario, an intrinsic matrix inequality still holds. In the context of the optimal Kalman filter (1.2), it can be formulated as

$$A_n^T (I - K_{n+1} H_n)^T R_{n+1}^{-1} (I - K_{n+1} H_n) A_n \preceq R_n^{-1},$$

and for RKF it becomes (2.9) in below. From this perspective, the Mahalanobis error $\|e_n\|_{C_n^+}^2$ is a natural statistic that dissipates through time. In specific, we have

THEOREM 2.2.   *For any fixed inflation $r > 1$, consider applying the RKF (1.6) to the $d+q$ dimensional system (1.1). Suppose the dimension reduction in RKF is asymptotically acceptable as described by Definition 2.1, then*

$$\mathbb{E}\|e_n\|_{C_n^+}^2 \leq (\beta^*)^{n-n_0} \mathbb{E}\|e_{n_0}\|_{C_{n_0}^+}^2 + \frac{2d}{1 - \beta^*}.$$

In other words, the Mahalanobis error is dissipative after the adjustment time n_0.

    Proof.   We will show that given any n,

$$\mathbb{E}_n \|e_{n+1}\|_{C_{n+1}^+}^2 \leq \beta_{n+1} \|e_n\|_{C_n^+}^2 + 2d\beta_{n+1}. \tag{2.6}$$

Then the original claim of this theorem can be achieved by applying the Gronwall's inequality in discrete time. To show (2.6), recall that in the proof of Theorem 2.1, the filter error has the following recursion:

$$e_{n+1} = (I - \widehat{K}_{n+1} H_n) A_n e_n + (I - \widehat{K}_{n+1} H_n)\xi_n - \widehat{K}_{n+1}\zeta_{n+1}.$$

Since ξ_{n+1} and ζ_{n+1} are independent of $\mathcal{F}_n$ conditioned on Σ_n and σ_n, we find that

$$\mathbb{E}_n e_{n+1}^T [C_{n+1}^+]^{-1} e_{n+1} = \mathbb{E}_n e_n^T A_n^T (I - \widehat{K}_{n+1} H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1} H_n) A_n e_n \tag{2.7}$$

$$+ \mathbb{E}_n \xi_{n+1}^T (I - \widehat{K}_{n+1} H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1} H_n)\xi_{n+1}$$

$$+ \mathbb{E}_n \zeta_{n+1}^T \widehat{K}_{n+1}^T [C_{n+1}^+]^{-1} \widehat{K}_{n+1}\zeta_{n+1}. \tag{2.8}$$

For the first part (2.7), we claim that

$$A_n^T (I - \widehat{K}_{n+1} H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1} H_n) A_n \preceq \beta_{n+1} [C_n^+]^{-1}. \tag{2.9}$$

To see that, notice by (2.1), $\beta_{n+1} C_{n+1}^+ \succeq \mathcal{K}(\widehat{C}_{n+1}) \succeq (I - \widehat{K}_{n+1} H_n)\widehat{C}_{n+1}(I - \widehat{K}_{n+1} H_n)^T$. Moreover $(I - \widehat{K}_{n+1} H_n) = (I + \widehat{C}_{n+1} H_n^T \sigma_n^{-1} H_n)^{-1}$ is clearly invertible. By Lemma B.2, the inversion of the inequality above reads

$$(I - \widehat{K}_{n+1} H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1} H_n) \preceq \beta_{n+1} \widehat{C}_{n+1}^{-1}. \tag{2.10}$$

Next, notice that $\widehat{C}_{n+1} \succeq A_n C_n^+ A_n^T$, so

$$A_n^T (I - \widehat{K}_{n+1} H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1} H_n) A_n$$

$$\preceq \beta_{n+1} A_n^T \widehat{C}_{n+1}^{-1} A_n \preceq \beta_{n+1} A_n^T [A_n C_n^+ A_n^T]^{-1} A_n,$$

which by Lemma B.2 leads to (2.9). To deal with (2.8), we use the identity $a^T A a = \text{tr}(Aaa^T)$ and the independence of ξ_{n+1}, ζ_{n+1},

$$\mathbb{E}_n \xi_{n+1}^T (I - \widehat{K}_{n+1}H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1}H_n)\xi_{n+1} + \zeta_{n+1}^T \widehat{K}_{n+1}^T [C_{n+1}^+]^{-1} \widehat{K}_{n+1}\zeta_{n+1}$$
$$= \mathbb{E}_n \text{tr}[(I - \widehat{K}_{n+1}H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1}H_n)\Sigma_n + \widehat{K}_{n+1}\sigma_n \widehat{K}_{n+1}^T [C_{n+1}^+]^{-1}].$$

Note (2.10) and that by definition, $\widehat{C}_{n+1} \succeq \Sigma_n$, so Lemma B.2 implies:

$$\text{tr}[(I - \widehat{K}_{n+1}H_n)^T [C_{n+1}^+]^{-1} (I - \widehat{K}_{n+1}H_n)\Sigma_n] \le d\beta_{n+1}.$$

Also notice that

$$\mathcal{K}(\widehat{C}_{n+1}) = (I - \widehat{K}_{n+1}H_n)\widehat{C}_{n+1}(I - \widehat{K}_{n+1}H_n)^T + \widehat{K}_{n+1}\sigma_n \widehat{K}_{n+1}^T \succeq \widehat{K}_{n+1}\sigma_n \widehat{K}_{n+1}^T. \quad (2.11)$$

Then by $\beta_{n+1}C_{n+1}^+ \succeq \mathcal{K}(\widehat{C}_{n+1}) \succeq \widehat{K}_{n+1}\sigma_n \widehat{K}_{n+1}^T$, $\text{tr}(\widehat{K}_{n+1}\sigma_n \widehat{K}_{n+1}^T [C_{n+1}^+]^{-1}) \le d\beta_{n+1}$. By summing up (2.7) and (2.8), we have reached (2.6) and so ends the proof. $\square$

REMARK 2.2. In the analysis of the standard Kalman filter and extended Kalman filter, [6, 29] implicitly exploit the same mechanism but do not apply a multiplicative inflation. But they require that the covariance sequences C_n, C_n^{-1}, Σ_n and Σ_n^{-1} are bounded from above ([6] has weaker assumptions, but its results are qualitative rather than quantitative). In principle, we can also remove the multiplicative inflation by adding similar conditions. But such conditions are usually not satisfied in high dimensional settings, since Σ_n may have many small scale entries being very close to zero.

2.3. Exponential stability. Another useful property implied by the previous analysis is that RKF is exponentially stable. Let (μ_0, C_0) and (μ_0', C_0) be two implementations of RKF with the same covariance but different means. Then these two RKFs share the same covariance estimate, and the difference in their mean estimates is given by

$$(\mu_n - \mu_n') = U_{n,0}(\mu_0 - \mu_0'), \quad U_{n,m} = \prod_{k=m}^{n-1}(I - \widehat{K}_{k+1}H_k)A_k.$$

So if $\|U_{n,0}\|$ converges to zero exponentially fast, then so does the mean difference. In [6], this is called exponential stability.

COROLLARY 2.1. *Under the conditions of Theorem 2.2, suppose also that* $\sup_n \|C_n^+\| < \infty, \|[C_0^+]^{-1}\| < \infty$, *then the RKF filter is exponentially stable*

$$\limsup_{n \to \infty} \frac{1}{n} \log \left\| \prod_{k=0}^{n-1}(I - \widehat{K}_{k+1}H_k)A_k \right\| \le \frac{1}{2}\log \beta^*.$$

Proof. Let $U_{n,n_0} = \prod_{k=n_0}^{n-1}(I - \widehat{K}_{k+1}H_k)A_k$. By iterating (2.9) n times, we find that

$$\|C_n^+\|^{-1}U_{n,0}^T U_{n,0} \preceq U_{n,0}^T [C_n^+]^{-1}U_{n,0} \preceq \left(\prod_{k=1}^{n}\beta_n\right)[C_0^+]^{-1}.$$

Taking spectral norm on both hand side yields our claim. $\square$

Sections 4 and 6 will discuss how to bound $\|C_n^+\|$.

3. Error analysis of DRKF

When comparing DRKF with RKF in the dynamical decoupled scenario (1.4), the former has a significant advantage: since no large scale projection is applied, there is no risk of underestimating the error covariance, so online criteria like the ones in Definition 2.1 are not necessary. But there are two disadvantages. First, DRKF has a special dynamical structural requirement, second the small scale fluctuation requires more technical treatments. To see the second point, it is straight forward to have the following recursion for the filter error, just like in the proof of Theorem 2.1,

$$e_{n+1}^L = (I - K_{n+1}^L H_n^L) A_n e_n^L + (I - K_{n+1}^L H_n^L) \xi_{n+1}^L - K_{n+1}^L \zeta_{n+1} - K_{n+1}^L H_n^S \Delta X_{n+1}^S.$$

Unlike ξ_{n+1} and ζ_{n+1}, in most situations, $\Delta X_{n+1}^S = X_{n+1}^S - \mu_{n+1}^S$ has a nonzero correlation with the error e_n^L, as it is not an independent time series. Therefore the second moment matrix does not follow a recursive formula, since (2.5) no longer holds. On the other hand, the Mahalanobis error dissipation still holds since it is a more intrinsic property. In order to show that, we need additional conditions on the small scale dynamics A_n^S, and impose a (2.2) type of independence condition on the small scale system. We will see that such condition holds for many important examples in Section 6.

THEOREM 3.1.   *Consider applying DRKF (1.5) to system (1.1) with two-scale dynamical decoupling (1.4). Suppose there is a $\lambda_S < 1$ such that the small scale unfiltered covariance V_k^S satisfies*

$$A_{k,j}^S V_j^S (A_{k,j}^S)^T \preceq \lambda_S^{k-j} V_k^S, \quad A_{k,j}^S = A_{k-1}^S \cdots A_{j+1}^S A_j^S.$$

Assume also the distribution of the small scale system noise ξ_n^S is $\mathcal{N}(0, \Sigma_n^S)$, conditioned on the system coefficients σ-field $\mathcal{F}^c$. Then the following holds

$$\mathbb{E}\|e_n^L\|_{C_n^L}^2 \leq \frac{2}{r^n} \mathbb{E}\|e_0^L\|_{C_0^L}^2 + \frac{2p(1+\gamma_\sigma)}{r-1} + \frac{4\sqrt{\lambda_S} r p \gamma_\sigma}{(\sqrt{r}-1)(1-\sqrt{\lambda_S})}. \tag{3.1}$$

The last term comes from the time correlated small scale fluctuation, and the constant γ_σ is given by

$$\gamma_\sigma = \sup_{n \geq 0} \{\|[\sigma_n^L]^{-1} H_n^S V_{n+1}^S (H_n^S)^T\|\}.$$

Note that $\gamma_\sigma \leq 1$, and it has the potential to be small if $H_n^S V_{n+1}^S (H_n^S)^T$ is small.

Proof.   The filter error follows the recursion:

$$e_{n+1}^L = (I - K_{n+1}^L H_n^L) A_n^L e_n^L + K_{n+1}^L \zeta_{n+1} - K_{n+1}^L H_n^L \xi_{n+1} - K_{n+1}^L H_n^S \Delta X_{n+1}^S.$$

In order to take away the influence of ΔX_n^S, consider

$$\tilde{e}_n^L = e_n^L - \sum_{k=1}^n U_{n,k}^L Q_k^S, \quad Q_k^S := K_k^L H_{k-1}^S \Delta X_k^S,$$

and

$$U_{n,k}^L := (I - K_n^L H_{n-1}^L) A_{n-1}^L \cdots (I - K_{k+1}^L H_k^L) A_k^L.$$

$\tilde{e}_n^L$ follows the recursion

$$\tilde{e}_{n+1}^L = (I - K_{n+1}^L H_n^L) A_n^L \tilde{e}_n^L + K_{n+1}^L \zeta_{n+1} - K_{n+1}^L H_n^L \xi_{n+1}.$$

Then the same proof of Theorem 2.2 is valid for $\tilde{e}_{n+1}^L$, except that in place of (2.1), we have $C_{n+1}^L = r\mathcal{K}_L(\widehat{C}_{n+1}^L)$. So β_n will be replaced by $\frac{1}{r}$, and instead of (2.6), we have

$$\mathbb{E}_n \|\tilde{e}_{n+1}^L\|_{C_{n+1}^L}^2 \leq \frac{1}{r} \|\tilde{e}_n^L\|_{C_n^L}^2 + \frac{2d}{r}.$$

As a consequence of the Gronwall's inequality, $\mathbb{E}\|\tilde{e}_n^L\|_{C_n^L}^2 \leq \frac{1}{r^n}\|\tilde{e}_0^L\|_{C_0^L}^2 + \frac{2d}{r-1}$. Because of Young's inequality

$$\mathbb{E}\|e_n^L\|_{C_n^L}^2 \leq 2\mathbb{E}\|\tilde{e}_n^L\|_{C_n^L}^2 + 2\mathbb{E}\left\|\sum_{k=1}^n U_{n,k}^L Q_k^S\right\|_{C_n^L}^2.$$

It suffices for us to bound

$$\mathbb{E}\left\|\sum_{k=1}^n U_{n,k}^L Q_k^S\right\|_{C_n^L}^2 = \sum_{j,k \leq n} \mathbb{E}(Q_j^S)^T (U_{n,j}^L)^T [C_n^L]^{-1} U_{n,k}^L Q_k^S. \tag{3.2}$$

Following the proof of (2.9), a similar matrix inequality also holds for DRKF,

$$(A_n^L)^T (I - K_{n+1}^L H_n^L)^T [C_{n+1}^L]^{-1} (I - K_{n+1}^L H_n^L) A_n^L \preceq \frac{1}{r}[C_n^L]^{-1}.$$

Iterate this multiple times, we have $(U_{n,k}^L)^T [C_n^L]^{-1} U_{n,k}^L \preceq r^{k-n}[C_k^L]^{-1}$.

The terms in the sum (3.2) with $j = k$ can be bounded by

$$\mathbb{E}(Q_k^S)^T (U_{n,k}^L)^T [C_n^L]^{-1} U_{n,k}^L Q_k^S$$
$$\leq \frac{1}{r^{n-k}} \mathbb{E}(Q_k^S)^T [C_k^L]^{-1} Q_k^S$$
$$= \frac{1}{r^{n-k}} \mathbb{E}\mathrm{tr}((K_k^L)^T [C_k^L]^{-1} K_k^L (H_{k-1}^S \Delta X_k^S \otimes H_{k-1}^S \Delta X_k^S))$$
$$\leq \frac{1}{r^{n-k}} \mathbb{E}\mathrm{tr}((K_k^L)^T [C_k^L]^{-1} K_k^L \mathbb{E}_{\mathcal{F}_k^c}(H_{k-1}^S \Delta X_k^S \otimes H_{k-1}^S \Delta X_k^S))$$
$$= \frac{1}{r^{n-k}} \mathbb{E}\mathrm{tr}([C_k^L]^{-1} K_k^L H_{k-1}^S V_k^S (H_{k-1}^S)^T (K_k^L)^T).$$

Similar to (2.11), we have

$$C_k^L = r\mathcal{K}_L(\widehat{C}_k^L) \succeq rK_k^L \sigma_{k-1}^L (K_k^L)^T \succeq \gamma_\sigma^{-1} K_k^L H_{k-1}^S V_k^S (H_{k-1}^S)^T (K_k^L)^T. \tag{3.3}$$

As a consequence of Lemmas B.5 and B.3, the $j = k$ terms in (3.2) can be further bounded by

$$\mathbb{E}(Q_k^S)^T (U_{n,k}^L)^T [C_n^L]^{-1} U_{n,k}^L Q_k^S \leq \frac{\gamma_\sigma}{r^{n-k}} \mathbb{E}\mathrm{tr}(I_p) = \frac{p\gamma_\sigma}{r^{n-k}}. \tag{3.4}$$

The $j < k$ terms in (3.2) come from time correlations of ΔX_k^S. In order to bound them, notice that:

$$\mathbb{E}(Q_j^S)^T (U_{n,j}^L)^T [C_n^L]^{-1} U_{n,k}^L Q_k^S$$

$$=\mathbb{E}\mathrm{tr}((H_{j-1}^S)^T(K_j^L)^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,k}K_k^LH_{k-1}^S(\Delta X_k^S\otimes\Delta X_j^S))$$
$$=\mathbb{E}\mathrm{tr}(W_j^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,k}^LW_k)\mathbb{E}_{\mathcal{F}_n^c}(\Delta X_k^S\otimes\Delta X_j^S))$$

with $W_k:=K_k^LH_{k-1}^S$. The moving average representation of ΔX_k^S is:

$$\Delta X_k^S=A_{k,j}^S\Delta X_j^S+\sum_{i=j+1}^k A_{k,i}^S\xi_i^S.$$

Since for $i>j$, ξ_i^S is distributed as $\mathcal{N}(0,\Sigma_i^S)$ conditioned on $\mathcal{F}_n^c\vee\mathcal{F}_j$,

$$\mathbb{E}_{\mathcal{F}_n^c\vee\mathcal{F}_j}(\Delta X_i^S\otimes\Delta X_j^S)=\mathbb{E}_{\mathcal{F}_n^c\vee\mathcal{F}_j}(A_{k,j}^S\Delta X_j^S\otimes\Delta X_j^S)=A_{k,j}^SV_j^S.$$

Therefore, for the $j<k$ terms in (3.2)

$$\mathbb{E}(Q_j^S)^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,k}^LQ_k^S+(Q_k^S)^T(U_{n,k}^L)^T[C_n^L]^{-1}U_{n,j}^LQ_j^S$$
$$=\mathbb{E}\mathrm{tr}([W_j^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,k}^LW_kA_{k,j}^S+(A_{k,j}^S)^TW_j^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,k}^LW_k]V_j^S). \quad (3.5)$$

In order to apply Lemma B.5, we are interested in bounding the symmetric matrix

$$Z_{j,k}:=W_j^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,k}^LW_kA_{k,j}^S+(A_{k,j}^S)^TW_k^T(U_{n,k}^L)^T[C_n^L]^{-1}U_{n,j}^LW_j.$$

Notice that for any PSD matrix C, matrices A and B, and $\gamma>0$, the following holds

$$(\gamma^{-1}A-\gamma B)^TC(\gamma^{-1}A-\gamma B)\succeq 0 \quad\Rightarrow\quad \gamma^{-2}A^TCA+\gamma^2B^TCB\succeq A^TCB+B^TCA.$$

For our purpose, let $A=U_{n,k}^LW_kA_{k,j}^S, B=U_{n,j}^LW_j, C=[C_n^L]^{-1}, \gamma=(\lambda_Sr)^{\frac{k-j}{4}}$, and find

$$Z_{j,k}\preceq\gamma^{-2}(A_{k,j}^S)^TW_k^T(U_{n,k}^L)^T[C_n^L]^{-1}U_{n,k}^LW_kA_{k,j}^S+\gamma^2W_j^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,j}^LW_j.$$

To continue, recall that $(U_{n,k}^L)^T[C_n^L]^{-1}U_{n,k}^L\preceq r^{k-n}[C_k^L]^{-1}$, and the relation (3.3). Then using Lemmas B.5 and B.3,

$$\mathrm{tr}((A_{k,j}^S)^TW_k^T(U_{n,k}^L)^T[C_n^L]^{-1}U_{n,k}^LW_kA_{k,j}^SV_j^S)$$
$$\leq r^{k-n}\mathrm{tr}((A_{k,j}^S)^T(K_k^LH_{k-1}^S)^T[C_k^L]^{-1}K_k^LH_{k-1}^SA_{k,j}^SV_j^S)$$
$$=r^{k-n}\mathrm{tr}([C_k^L]^{-1}K_k^LH_{k-1}^SA_{k,j}^SV_j^S(A_{k,j}^S)^T(K_k^LH_{k-1}^S)^T)$$
$$\leq\lambda_S^{k-j}r^{k-n}\mathrm{tr}([C_k^L]^{-1}K_k^LH_{k-1}^SV_k^S(K_k^LH_{k-1}^S)^T).$$

Using (3.3) again, we find the quantity above is bounded by $\gamma_\sigma\lambda_S^{k-j}r^{k-n}p$. Likewise

$$\mathrm{tr}(W_j^T(U_{n,j}^L)^T[C_n^L]^{-1}U_{n,j}^LW_jV_j^S)\leq r^{j-n}\mathrm{tr}([C_j^L]^{-1}K_j^LH_{j-1}^SV_j^S(K_j^LH_{j-1}^S)^T)\leq r^{j-n}\gamma_\sigma p.$$

As a consequence,

$$(3.5)\leq\mathbb{E}\mathrm{tr}(Z_{j,k}V_j^S)\leq 2r^{\frac{k+j}{2}-n}\lambda_S^{\frac{k-j}{2}}\gamma_\sigma p. \quad (3.6)$$

Finally, we can bound (3.2) by (3.4) and (3.6):

$$(3.2) \leq \sum_{k=1}^{n} \frac{p}{r^{n-k}} + 2p \sum_{j<n} r^{\frac{j-n}{2}} \sum_{k \geq j+1} \lambda_S^{\frac{k-j}{2}} \leq \frac{rp\gamma_\sigma}{r-1} + \frac{2\sqrt{\lambda_S} r p \gamma_\sigma}{(\sqrt{r}-1)(1-\sqrt{\lambda_S})}.$$

$\square$

REMARK 3.1.   In the first appearance, the formulation of the result may suggest that the filter error can be reduced by picking a large inflation strength r. In fact, this is an artifact caused by the usage of Mahalanobis error. The covariance estimator C_n^L also increases with r, so $\|e_n^L\|_{C_n^L}$ being small does not imply e_n is small. This is slightly discussed in the next section. On the other hand, a larger r does lead to stronger stability.

Following the same proof of Corollary 2.1, the exponential stability holds for DRKF as well:

COROLLARY 3.1.   *Under the conditions of Theorem 3.1, suppose also that*

$$\sup_n \|C_n^L\| < \infty, \quad \|[C_0^L]^{-1}\| < \infty,$$

then the DRKF filter is exponentially stable as

$$\limsup_{n \to \infty} \frac{1}{n} \log \left\| \prod_{k=0}^{n-1} (I - K_{k+1}^L H_k^L) A_k^L \right\| \leq -\frac{1}{2} \log r.$$

4. Intrinsic performance criteria

Sections 2 and 3 have demonstrated that the reduced filter can estimate the real error covariance. But in order for these results to be applicable to concrete problems, there are two issues:

- The Mahalanobis error is informative about the filter error only when the covariance estimators C_n^L and C_n^+ are bounded. This is also the case for $\mathbb{E}_{\mathcal{F}^c} e_n \otimes e_n \preceq C_n^+$ in Theorem 2.1.
- Theorems 2.1 and 2.2 require conditions described in Definition 2.1. These are online or a posteriori criteria that can be verified only by running RKF. In practice, a priori criteria are more useful for verification.

To address these two issues, we consider two signal-observation systems (1.9) and (1.10) as augmentations of system (1.1), and use their Kalman filter covariance $\widetilde{R}_n^L$ and $\widetilde{R}_n$ as references. These reference filters are important for our reduced filters, because Proposition 4.1 and Theorem 4.2 below show direct connections between $\widetilde{R}_n^L, \widetilde{R}_n$ and C_n^L, C_n^+ respectively. This approach has four advantages:

1) The Kalman filter covariance R_n describes the smallest possible filter covariance for signal-observation systems like (1.1). Since the augmented systems (1.9) and (1.10) can be viewed as small perturbations of system (1.1), the associated Kalman filter covariance $\widetilde{R}_n^L, \widetilde{R}_n$ are not too different from the proper part of R_n (Proposition 6.2 explores some sufficient conditions). So if C_n^L and C_n^+ are bounded by a multiple of $\widetilde{R}_n^L, \widetilde{R}_n$, the reduced filter covariance matrices are comparable with the optimal.

2) Kalman filters are direct and intrinsic descriptions of how well systems like (1.1) can be filtered. The dependence of $\widetilde{R}_n^L$ and $\widetilde{R}_n$ on the system coefficients is very nonlinear. Imposing conditions on $\widetilde{R}_n$ instead of on the system coefficients makes our exposition much simpler.

3) Unlike reduced filters, Kalman filter has been a classical research object for decades. There is a huge literature we can exploit.

4) Kalman filter covariance matrices often converge to a unique stationary solution of the associated Riccati equation in (1.2), assuming the system coefficients are ergodic stationary sequences, and other weak conditions hold. See [6] and Section C for details. This unique solution is independent of the initial condition. By imposing conditions on the stationary solution, our results for the reduced filters are independent of the initial conditions as well. When we refer to this stationary solution in the following discussion, we implicitly assume the existence of this unique stationary solution.

4.1. Kalman filters for comparison. The connection between DRKF and the augmented system (1.9) is simple and direct:

PROPOSITION 4.1. *Consider applying DRKF (1.5) to system (1.1) with dynamical decoupling (1.4). Let $\widetilde{R}_n^L$ be the Kalman filter covariance for the large scale reference system (1.9) with the same system coefficient realization as in (1.1). If $\widetilde{R}_n^L = \frac{1}{r}C_n^L$ holds for $n = 0$, then it holds for all $n \geq 0$.*

Proof. Suppose our claim holds at time n. Then the Kalman filter covariance for system (1.9) follows:

$$\widehat{\widetilde{R}}_{n+1}^L = rA_n^L\widetilde{R}_n^L(A_n^L)^T + \Sigma_n' = A_n^L C_n^L (A_n^L)^T + \Sigma_n^L = \widehat{C}_{n+1}^L,$$

therefore our claim holds at time $n+1$ as well:

$$\widetilde{R}_{n+1}^L = \mathcal{K}_L(\widehat{\widetilde{R}}_{n+1}^L) = \mathcal{K}_L(\widehat{C}_{n+1}^L) = \frac{1}{r}C_{n+1}^L.$$

□

In the case of RKF, we need to consider system (1.10), of which the Kalman filter covariance $\widetilde{R}_n$ follows the recursion

$$\widetilde{R}_{n+1} = \mathcal{K}(\widehat{\widetilde{R}}_{n+1}), \quad \widehat{\widetilde{R}}_{n+1} = r'A_n\widetilde{R}_n A_n^T + \Sigma_n'. \tag{4.1}$$

The Kalman update operator $\mathcal{K}$ is the same as in (1.2). Unlike Proposition 4.1, where we showed $\widetilde{R}_n^L$ is directly a multiple of C_n^L, $r\widetilde{R}_n$ will be an upper bound for C_n, which leads to $r\widetilde{R}_n^+ \succeq C_n^+$. In addition, using this inequality, we can verify the online assumptions in Definition 2.1 using assumptions regarding $\widetilde{R}_n$:

ASSUMPTION 4.1. *Let $\widetilde{R}_n$ be a (the unique stationary) positive definite (PD) solution of (4.1). Assume that its small scale part is bounded as follow with $\frac{1}{r} < \beta^* < 1$*

$$\mathbf{P}_S\widetilde{R}_n\mathbf{P}_S \preceq (\beta^* r - 1)D_S.$$

As we discussed earlier in this section, the Riccati equation (4.1) has a unique stationary solution under weak conditions. Despite that Theorem 4.2 below works for any solution of (4.1), by considering the stationary solution it allows Assumption 4.1 to be independent of the initial conditions.

THEOREM 4.2. *Consider applying RKF (1.6) to system (1.1), and the Kalman filter covariance $\widetilde{R}_n$ for system (1.10). Then after a finite time n_0, the RKF error covariance estimate C_n^+ is bounded by the reference covariance $\widetilde{R}_n$*

$$C_n^+ \preceq r\widetilde{R}_n + D_S, \quad n \geq n_0 = \lceil\log(\|\widetilde{R}_0^{-1}C_0\|)/\log(r'/r)\rceil.$$

If in addition the reference projection Assumption 4.1 holds, then the conditions in Definition 2.1 also hold, and the acceptable reduction phase starts no later than n_0.

Proof. For all $n \geq 0$, denote

$$\nu_n = \inf\{\nu : C_n \preceq \nu \widetilde{R}_n\} = \|[\widetilde{R}_n]^{-1} C_n\|.$$

Following the formulation of n_0, we assume $\nu_0 < \infty$. We claim that ν_n has the following recursive relation:

$$\nu_{n+1} \leq r \max\{1, \tfrac{1}{r'}\nu_n\}. \tag{4.2}$$

This comes from a simple induction. Suppose that $C_n \preceq \nu_n \widetilde{R}_n$, we have

$$\widehat{C}_{n+1} = A_n C_n A_n^T + \Sigma_n + A_n D_S A_n^T \preceq \nu_n [A_n \widetilde{R}_n A_n^T] + \Sigma_n' \preceq \tfrac{1}{r'}\nu_n \widehat{\widetilde{R}}_{n+1}.$$

Hence by the monotonicity and concavity of the operator $\mathcal{K}$, Lemma B.1,

$$C_{n+1} \preceq r\mathcal{K}(\widehat{C}_{n+1}) \preceq r\mathcal{K}(\tfrac{1}{r'}\nu_n \widehat{\widetilde{R}}_{n+1}) \preceq r \max\{1, \tfrac{1}{r'}\nu_n\}\mathcal{K}(\widehat{\widetilde{R}}_{n+1}) = \nu_{n+1}\widetilde{R}_{n+1}, \tag{4.3}$$

which completes the induction. Then if we iterate (4.2) n_0 time, we find that

$$\nu_n \leq r, \quad C_n \preceq r\widetilde{R}_n, \quad \text{for all } n \geq n_0.$$

Next, we prove the second claim of this theorem by showing a stronger result, that is the β_n sequence defined by (2.1) is bounded by

$$\beta_n \leq \frac{\nu_n}{r^2}(\beta^* r - 1) + \frac{1}{r}. \tag{4.4}$$

Then because $\nu_n \leq r$ when $n \geq n_0$, the assumptions in Definition 2.1 are implied. To see (4.4), denote

$$\mathbf{P}_L \mathcal{K}(\widehat{C}_n)\mathbf{P}_L = K_{LL}, \quad \mathbf{P}_L \mathcal{K}(\widehat{C}_n)\mathbf{P}_S = K_{LS},$$

$$\mathbf{P}_S \mathcal{K}(\widehat{C}_n)\mathbf{P}_L = K_{SL}, \quad \mathbf{P}_S \mathcal{K}(\widehat{C}_n)\mathbf{P}_S = K_{SS}.$$

For any $\beta_n \geq \frac{1}{r}$, from

$$[\sqrt{\beta_n r - 1}\mathbf{P}_L - \tfrac{1}{\sqrt{\beta_n r - 1}}\mathbf{P}_S]\mathcal{K}(\widehat{C}_n)[\sqrt{\beta_n r - 1}\mathbf{P}_L - \tfrac{1}{\sqrt{\beta_n r - 1}}\mathbf{P}_S] \succeq 0,$$

we have

$$(\beta_n r - 1)K_{LL} + \tfrac{1}{\beta_n r - 1}K_{SS} \succeq K_{SL} + K_{LS}. \tag{4.5}$$

Finally note that, $K_{SS} \preceq \frac{1}{r}C_n \preceq \frac{1}{r}\nu_n \widetilde{R}_n \preceq \frac{1}{r}\nu_n(\beta^* r - 1)D_S$, so

$$\mathcal{K}(\widehat{C}_n) = K_{LL} + K_{LS} + K_{SL} + K_{SS}$$

$$\preceq \beta_n r K_{LL} + \frac{\beta_n r}{\beta_n r - 1}K_{SS}$$

$$\preceq \beta_n \left(r K_{LL} + \frac{\nu_n(\beta^* r - 1)}{r(\beta_n r - 1)}D_S\right).$$

Note that with the upper bound of β_n in (4.4), the coefficient before D_S is less than 1, so

$$\mathcal{K}(\widehat{C}_n) \preceq \beta_n(rK_{LL} + D_S) = \beta_n C_n^+,$$

and the proof is finished. □

REMARK 4.1.   Assumption 4.1 is not the direct replacement of the assumptions in Definition 2.1, as we need an additional constant $\beta^* r - 1$. This constant appears to be necessary for the control of the potential cross covariance between the two scales, which is achieved by a Cauchy Schwartz inequality (4.5). In certain scenarios, the cross covariance between two scales can be controlled by, say, localization structures, then the (4.5) is an overestimate, and $\beta^* r - 1$ can probably be replaced by 1. In other words, there might be scenarios where the assumptions in Definition 2.1 hold while Assumption 4.1 does not. This is why we keep two assumptions in this paper instead of combining them.

4.2. Filter error statistics.   In applications, other than the Mahalanobis error generated by the estimated covariances C_n^L or C_n^+, there are other interesting error statistics: 1) The MSE $\mathbb{E}|e_n|^2$. 2) The Mahalanobis error generated by the optimal filter covariance R_n. The latter shows a comparison between the reduced filter and the optimal filter, as the optimal filter error satisfies $\mathbb{E}\|X_n - m_n\|_{R_n}^2 = d$. In many scenarios, we may find these error statistics equivalent to the Mahalanobis error generated by C_n^L or C_n^+. To see this, we can simply combine Theorems 2.2 and 4.2,

COROLLARY 4.1.   *Suppose system* (1.1) *satisfies the reference projection Assumption 4.1, then the Mahalanobis error of RKF generated by the reference covariance is bounded uniformly in time:*

$$\limsup_{n\to\infty} \mathbb{E}\|e_n\|_{\widetilde{R}_n^+}^2 \leq \frac{2dr}{1-\beta^*}.$$

If in addition the system noises are independent of the system coefficients, (2.2)*, then*

$$\mathbb{E}e_n \otimes e_n \preceq r\mathbb{E}\widetilde{R}_n + D_S.$$

As a consequence:

- *Suppose that* $\limsup_n \|\widetilde{R}_n^+\| \leq R$, *then the MSE is bounded by* $\limsup_n \mathbb{E}|e_n|^2 \leq \frac{2Rdr}{1-\beta^*}$. *If in addition* (2.2) *holds, then* $\mathbb{E}|e_n|^2 \leq rRd$.

- *Suppose that* $\widetilde{R}_n^+ \preceq \rho^2 R_n$, *where* R_n *is the covariance sequence of the optimal filter* (1.2) *and* $\rho \geq 1$, *then the performance of RKF is comparable with the optimal filter, as* $\limsup \mathbb{E}\|e_n\|_{R_n}^2 \leq \frac{2Rdr\rho^2}{1-\beta^*}$. *If in addition* (2.2) *holds, then* $\mathbb{E}e_n \otimes e_n \preceq \rho^2 r\mathbb{E}R_n$.

The requirement that $\|\widetilde{R}_n^+\| \leq R$ or $\widetilde{R}_n^+ \preceq \rho^2 R_n$ can be verified by various ways discussed in Section 6. As for DRKF, we consider only the MSE, because $\widetilde{R}_n^L$ is not directly comparable with R_n.

COROLLARY 4.2.   *Suppose system* (1.1) *is dynamically decoupled in two scales* (1.4)*, then if the Kalman filter covariance of system* (1.9) *satisfies* $\widetilde{R}_0^L = \frac{1}{r}C_0^L$ *and* $\limsup\|\widetilde{R}_n^L\| \leq R$, *the MSE of DRKF is bounded:*

$$\limsup_{n\to\infty} \mathbb{E}|e_n^L|^2 \leq \frac{2Rp(1+\gamma_\sigma)}{r^2-r} + \frac{4\sqrt{\lambda_S}Rp\gamma_\sigma}{(r-\sqrt{r})(1-\sqrt{\lambda_S})}.$$

4.3. Reduced filter accuracy. In many applications, the observations are partial but very frequent and accurate. In such cases, one would expect the filter error to be small. This is quite easy to show for optimal Kalman filters, but not obvious for reduced filters. With our framework, we can easily obtain the filter accuracy of the reduced filters.

COROLLARY 4.3. *Suppose system* (1.1) *is dynamically decoupled in two scales with the stationary Kalman filter covariance of system* (1.9) *being bounded* $\|\widetilde{R}_n^L\| \leq R$*; or suppose the reference projection Assumption 4.1 holds with the stationary Kalman filter covariance of system* (1.10) *being bounded* $\|\widetilde{R}_n^+\| \leq R$*. In either case, assume the stationary Kalman covariance attracts other Kalman filter covariance sequence as in [6]. Then there is a DRKF, or RKF, for the following signal-observation system with small system and observation noises:*

$$X_{n+1}^\epsilon = A_n X_n^\epsilon + B_n + \epsilon \xi_{n+1}, \quad Y_{n+1}^\epsilon = H_n X_{n+1}^\epsilon + \epsilon \zeta_{n+1}. \tag{4.6}$$

The MSE of this filter scales like ϵ^2*. More precisely, there is a constant* D_R *such that*

$$\limsup_{n\to\infty} \mathbb{E}|e_n^\epsilon|^2 \leq \epsilon^2 D_R.$$

Here e_n^ϵ *stands for* $X_n^{L,\epsilon} - \mu_n^{L,\epsilon}$ *for DRKF, or* $X_n^\epsilon - \mu_n^\epsilon$ *for RKF.*

Proof. In the dynamically decoupled case, the corresponding reference system will be

$$X_{n+1}^{'L} = A_n^{'L} X_n^{'L} + B_n + \epsilon \xi_{n+1}, \quad Y_{n+1}^{'L} = H_n X_{n+1}^{'L} + \epsilon \zeta_{n+1}'.$$

The stationary Kalman filter covariance of this system will be $\widetilde{R}_n^{L,\epsilon} = \epsilon^2 \widetilde{R}_n^L$, so

$$\limsup_{n\geq 0} \|\widetilde{R}_n^{L,\epsilon}\| = \epsilon^2 \limsup_{n\geq 0} \|\widetilde{R}_n^L\| \leq \epsilon^2 R.$$

Then applying Corollary 4.2 we have our claim.

As for the second case, we apply RKF with $D_S^\epsilon = \epsilon^2 D_S$. The corresponding reference will be

$$X_{n+1}' = A_n' X_n' + B_n + \epsilon \xi_{n+1}, \quad Y_{n+1}' = H_n X_{n+1}' + \epsilon \zeta_{n+1}.$$

The stationary solution of this system will be $\widetilde{R}_n^\epsilon = \epsilon^2 \widetilde{R}_n$, so

$$\limsup_{n\geq 0} \|\widetilde{R}_n^{\epsilon+}\| = \epsilon^2 \limsup_{n\geq 0} \|\widetilde{R}_n^+\| \leq \epsilon^2 R.$$

Then applying Corollary 4.1 we have our claim. □

5. General stochastic sequence setting

In some challenging scenarios, the reference stationary covariance $\widetilde{R}_n$ may not be a bounded sequence, then Assumption 4.1 cannot be verified. But weaker results may be obtainable, and interestingly the proofs do not need much of a change. The content of this section is not necessary for most parts of Section 6, and can be skipped in the first reading.

An assumption that is more general than the ones in Defininition 2.1 would be requiring the truncation error converges to a sequence that is stable on average:

ASSUMPTION 5.1.   *Suppose there is a stochastic sequence β_n^* with a finite adjustment time n_0 such that the sequence (2.1) satisfies $\beta_n \leq \beta_n^*$ for all $n \geq n_0$.*

The generalization of Theorem 2.2 is

THEOREM 5.2.   *For any fixed inflation $r > 1$, consider applying the RKF (1.6) to system (1.1). Suppose the large scale truncation of the RKF satisfies Assumption 5.1, then for any fixed times $n_0 \leq n$,*

$$\mathbb{E}(\beta_{n_0+1}^* \cdots \beta_n^*)^{-1} \|e_n\|_{C_n^+}^2 \leq \mathbb{E}\|e_{n_0}\|_{C_{n_0}^+}^2 + 2d\mathbb{E} \sum_{k=n_0+1}^{n} (\beta_{n_0+1}^* \cdots \beta_k^*)^{-1}. \qquad (5.1)$$

Proof. First of all, notice that the inequality (2.6) still holds. Our claim is simply an induction, because

$$\mathbb{E}(\beta_{n_0+1}^* \cdots \beta_{n+1}^*)^{-1} \|e_{n+1}\|_{C_{n+1}^+}^2 = \mathbb{E}(\beta_{n_0+1}^* \cdots \beta_{n+1}^*)^{-1} \mathbb{E}_n \|e_{n+1}\|_{C_{n+1}^+}^2$$
$$\leq \mathbb{E}(\beta_{n_0+1}^* \cdots \beta_n^*)^{-1} (\|e_n\|_{C_n^+}^2 + 2d).$$

If (5.1) holds for time n and we replace $\|e_n\|_{C_n^+}^2$ by its upperbound, then (5.1) holds also for time $n+1$. □

In order to verify the general Assumption 5.1, an a priori condition can also be derived from the reference Kalman covariance.

ASSUMPTION 5.3.   *Let $\widetilde{R}_n$ be a (stationary) PD solution of (4.1). Assume its small scale part is bounded as below with a stochastic sequence β_n^**

$$\mathbf{P}_S \widetilde{R}_n \mathbf{P}_S \preceq (\beta_n^* r - 1) D_S.$$

Since in the proof of Theorem 4.2, we used nothing about the fact that β^* is a constant, so if we replace β^* with β_n^* in that proof, it is still valid. Therefore the following claim holds:

THEOREM 5.4.   *Suppose the general referenced projection Assumption 5.3 holds, then Assumption 5.1 also holds, and the acceptable reduction phase starts no later than*

$$n_0 = \lceil \log(\|\widetilde{R}_0^{-1} C_0\|)/\log(r'/r) \rceil.$$

Moreover the covariance estimator is bounded by $C_n^+ \preceq r\widetilde{R}_n + D_S$ for $n \geq n_0$.

REMARK 5.1.   The previous discussion provides an easy generalization of our framework, but admittedly it buries some difficulties inside the result (5.1). If we want Theorem 5.2 to provide concrete Mahalanobis error dissipation and convergence like in Theorem 2.2, we roughly need to show

- $\mathbb{E}(\beta_{n_0+1}^* \cdots \beta_{n+1}^*)^{-1} \geq \exp(b^*(n-n_0))$ for a constant $b^* > 0$.
- $\mathbb{E}\sum_{k=n_0+1}^{n}(\beta_{n_0+1}^* \cdots \beta_k^*)^{-1} \leq D\exp(b^*(n-n_0))$ for the same constant $b^* > 0$, and some D.

Usually it is not difficult to establish either of these ingredients, the major difficulty is that the growth ratio b^* needs to be the same in both. Some special structures, like β_k^* being independent of each other, will make the verification straightforward, but in general it is difficult.

6. Applications and examples

Given a concrete system (1.1), there might be various ways that the two-scale separation can be done. It is of practical importance to find the minimal large scale subspace, the proper inflation ratio r, while keeping the filter error small. Based on our previous results, these problems can be solved by numerically computing the Kalman filter covariance for the augmented system with a fixed $r' > 1$, (1.9) or (1.10), then verify Assumption 4.1 for RKF. The optimal two-scale separation and inflation can be obtained by minimizing the MSE upper bound in Corollary 4.1.

In this section, we will discuss a few general principles that may facilitate the filter error quantification and the verification of Assumption 4.1. A simple stochastic turbulence model will be considered as an example, and we will apply these principles to this model in different settings [4].

6.1. Some general guidelines for covariance bounds. Section 4 uses Kalman filters to provide a priori performance criteria. One of the advantages is that Kalman filters have a huge literature, so there are many known results on how to control the Kalman filter covariance. We present in below a few simple ones. For the simplicity of illustration, we convey them only for system (1.1) and its Kalman filter covariance R_n, while the same ideas are also applicable to the augmented systems (1.9), (1.10) and filter covariances $\widetilde{R}_n^L, \widetilde{R}_n$.

6.1.1. Unfiltered covariance. In most applications, system (1.1) has a stable dynamics itself, so the covariance of X_n conditioned on the system coefficients $\mathcal{F}_n^c$ can be bounded. The computation of this covariance

$$V_n = \mathbb{E}_{\mathcal{F}_n^c}(X_n \otimes X_n) - \mathbb{E}_{\mathcal{F}_n^c}(X_n) \otimes \mathbb{E}_{\mathcal{F}_n^c}(X_n),$$

follows a straightforward iteration: $V_{n+1} = A_n V_n A_n^T + \Sigma_n$. In fact, we already used the small scale part V_n^S for the formulation of DRKF. Clearly if $V_0 \succeq R_0$, then $V_n \succeq R_n$. Although this seems trivial, it is useful as it is independent of the choice of observations, and involves very little computation.

6.1.2. Equivalent transformation on observation. Sometimes, changing the way we view the observations may simplify the computation by a lot. Mathematically speaking, we can consider a sequence of invertible $q \times q$ matrix Ψ_n, and the signal-observation system as below

$$X_{n+1} = A_n X_n + B_n + \xi_{n+1}, \quad \widetilde{Y}_{n+1} = \Psi_n H_n X_{n+1} + \Psi_n \zeta_{n+1}.$$

Intuitively, the Kalman filter performance of this system would be the same as (1.1). This is true, as one can check the Kalman covariance update operator $\mathcal{K}$ is invariant under this transformation. This equivalent transformation can be used to simplify our notation. For example, we can let $\Psi_n = \sigma_n^{-1/2}$, then the observation noise for $\widetilde{Y}_n$ is a sequence of i.i.d. Gaussian random variables.

6.1.3. Benchmark principle. Since the Kalman filter (1.2) is the optimal filter of system (1.1), for any other estimator $\widehat{X}_n$ of X_n, its error covariance is an upper bound for R_n:

$$\mathbb{E}_{\mathcal{F}_n^c}(X_n - \widehat{X}_n) \otimes (X_n - \widehat{X}_n) = \mathbb{E}_{\mathcal{F}_n^c}\mathbb{E}_{\mathcal{F}_n^o}(X_n - \widehat{X}_n) \otimes (X_n - \widehat{X}_n) \succeq R_n.$$

So if there is an estimator $\widehat{X}_n$ with a computable error covariance Q_n, then $Q_n \succeq R_n$. Although this idea is simple, it has been used many places to guarantee that R_n is

bounded, and as to the authors' knowledge, it is the only general strategy. The unfiltered covariance is actually a special application of this principle, where the estimator is simply the mean, $\widehat{X}_n = \mathbb{E}_{\mathcal{F}_n^c} X_n$.

When the observation H_n is full rank, another simple estimator could be trusting the observation: $\widehat{X}_{n+1} = H_n^{-1} Y_{n+1}$. The error covariance is $[H_n^T]^{-1} \sigma_n H_n^T$. This idea can be generalized to the scenario where system (1.10) is *detectable* through a time interval $[m,n]$. Here we provide an explicit estimate, while similar results can also be found in [32–34].

PROPOSITION 6.1.   *Denote the observability Gramian matrix as*

$$\mathcal{O}_{n,m} = \sum_{k=m}^{n} A_{k,m}^T H_{k-1}^T \sigma_{k-1}^{-1} H_{k-1} A_{k,m}, \quad A_{k,j} = A_{k-1} \cdots A_{j+1} A_j.$$

Suppose that $\mathcal{K}_{n,m} = \mathcal{O}_{n,m} + \widehat{R}_m^{-1}$ is invertible, where $\widehat{R}_m$ is the prior covariance of X_m without observing Y_m. Then

$$R_n \preceq \sum_{j=m+1}^{n} Q_{n,m}^j \Sigma_{j-1} (Q_{n,m}^j)^T + A_{n,m} \mathcal{K}_{n,m}^{-1} A_{n,m}^T,$$

where $Q_{n,m}^j = A_{n,m} \mathcal{K}_{n,m}^{-1} \mathcal{K}_{j-1,m} A_{j,m}^{-1}$. In case there is no prior knowledge of X_m, $\widehat{R}_m^{-1}$ can be set as a zero matrix, which is the inverse of the infinite covariance.

Proof.   For the simplicity of notations, in our proof, we do transform the observations, and replace H_k by $\sigma_k^{-1/2} H_k$ and σ_k by I_q. We will first build up a smoother for X_m and then propagate it through time $[m,n]$. Also, without lost of generality, we assume $X_m \sim \mathcal{N}(0, \widehat{R}_m)$ and $B_k \equiv 0$. Consider the estimator

$$\widehat{X}_m = \mathcal{K}_{n,m}^{-1} \sum_{k=m}^{n} A_{k,m}^T H_{k-1}^T Y_k, \quad \widehat{X}_n = A_{n,m} \widehat{X}_m.$$

Notice that X_k and Y_k have the following moving average formulation:

$$X_k = A_{k,m} X_m + \sum_{j=m+1}^{k} A_{k,j} \xi_j, \quad Y_k = H_{k-1} \left(A_{k,m} X_m + \sum_{j=m+1}^{k} A_{k,j} \xi_j \right) + \zeta_k.$$

The error made by this estimator, $X_n - \widehat{X}_n$, can be written as

$$X_n - \widehat{X}_n = \left[A_{n,m} - A_{n,m} \mathcal{K}_{n,m}^{-1} \sum_{k=m}^{n} A_{k,m}^T H_{k-1}^T H_{k-1} A_{k,m} \right] X_m$$

$$+ \sum_{j=m+1}^{n} \left[A_{n,j} - A_{n,m} \mathcal{K}_{n,m}^{-1} \sum_{k=j}^{n} A_{k,m}^T H_{k-1}^T H_{k-1} A_{k,j} \right] \xi_j$$

$$- A_{n,m} \mathcal{K}_{n,m}^{-1} \sum_{k=m}^{n} A_{k,m}^T H_{k-1}^T \zeta_k.$$

When $\widehat{R}_m^{-1} = 0$, one can check that the quantity above is independent of X_m.

Note that $\mathcal{K}_{n,m} = A_{j,m}^T \mathcal{O}_{n,j} A_{j,m} + \mathcal{K}_{j-1,m}$, so

$$A_{n,j} - A_{n,m}\mathcal{K}_{n,m}^{-1}\sum_{k=j}^n A_{k,m}^T H_{k-1}^T H_{k-1} A_{k,j} = A_{n,j} - A_{n,m}\mathcal{K}_{n,m}^{-1} A_{j,m}^T \mathcal{O}_{n,j}$$

$$= A_{n,m}[I - \mathcal{K}_{n,m}^{-1} A_{j,m}^T \mathcal{O}_{n,j} A_{j,m}]A_{j,m}^{-1} = A_{n,m}\mathcal{K}_{n,m}^{-1}\mathcal{K}_{j-1,m}A_{j,m}^{-1} = Q_{n,m}^j.$$

In particular

$$Q_{n,m}^m = A_{n,m}\mathcal{K}_{n,m}^{-1}\mathcal{K}_{m-1,m} = A_{n,m}\mathcal{K}_{n,m}^{-1}\widehat{R}_m^{-1}.$$

The expected error covariance $\mathbb{E}_{\mathcal{F}_n^c}(X_n - \widehat{X}_n)\otimes(X_n - \widehat{X}_n)$ is bounded by

$$A_{n,m}\mathcal{K}_{n,m}^{-1}\widehat{R}_m^{-1}\mathcal{K}_{n,m}^{-1}A_{n,m}^T + \sum_{j=m+1}^n Q_{n,m}^j \Sigma_{j-1}(Q_{n,m}^j)^T$$

$$+ A_{n,m}\mathcal{K}_{n,m}^{-1}\left(\sum_{k=m}^n A_{n,k}^T H_{k-1}^T H_{k-1} A_{n,k}\right)\mathcal{K}_{n,m}^{-1}A_{n,m}^T$$

$$= \sum_{j=m+1}^n Q_{n,m}^j \Sigma_{j-1}(Q_{n,m}^j)^T + A_{n,m}\mathcal{K}_{n,m}^{-1}\left(\widehat{R}_m^{-1} + \sum_{k=m}^n A_{n,k}^T H_{k-1}^T H_{k-1} A_{n,k}\right)\mathcal{K}_{n,m}^{-1}A_{n,m}^T$$

$$= \sum_{j=m+1}^n Q_{n,m}^j \Sigma_j(Q_{n,m}^j)^T + A_{n,m}\mathcal{K}_{n,m}^{-1}A_{n,m}^T.$$

$\square$

6.1.4. The comparison principle of Riccati equations. In order to control R_n, sometimes it suffices to find another set of system coefficients, such that its Kalman filter covariance $R_n' \succeq R_n$. One way to generate such R_n' is applying the comparison principle of Riccati equations for the forecast covariance [35].

THEOREM 6.1 (Freiling and Jank 96). *Consider a signal-observation system*

$$X_{n+1}' = A_n'X_n' + B_n' + \xi_{n+1}', \quad Y_{n+1}' = H_n'X_{n+1}' + \zeta_{n+1}',$$

with $\xi_{n+1}' \sim \mathcal{N}(0,\Sigma_n')$ and $\zeta_{n+1}' \sim \mathcal{N}(0,\sigma_n')$. Suppose the following holds a.s. with system coefficients of (1.1)

$$\begin{bmatrix} \Sigma_n & A_n^T \\ A_n & -H_n^T \sigma_n^{-1} H_n \end{bmatrix} \preceq \begin{bmatrix} \Sigma_n' & A_n'^T \\ A_n' & -H_n'^T \sigma_n'^{-1} H_n^T \end{bmatrix}. \tag{6.1}$$

Then if the forecast covariance satisfies $\widehat{R}_1 \preceq \widehat{R}_1'$, we have $\widehat{R}_n \preceq \widehat{R}_n'$ for all $n \geq 1$.

In particular, we can compare the reference Kalman filter of (1.10) with the optimal filter (1.2):

PROPOSITION 6.2. *Suppose that there are constants c and C such that $c\Sigma_n \succeq A_n D_S A_n^T$ and $A_n \Sigma_n^{-1} A_n^T \preceq CH_n \sigma_n^{-1} H_n^T$, and there is a $\rho \geq 1$ such that*

$$\frac{1}{\sigma}\left(1 - \frac{1}{\rho^2}\right) \geq \frac{C(1-\sqrt{r'})^2}{\rho^2 - r'(1+c)}.$$

Then the stationary solution $\widetilde{R}_n$ of (4.1) is bounded by the stationary Kalman filter covariance R_n of (1.2) by the following

$$\widetilde{R}_n \preceq \rho^2 R_n.$$

It is worth noticing that if r' is close to 1 and c is close to 0, ρ can be close to 1 as well.

Proof.   We apply the equivalent observation transformation mentioned in Section 6.1.2, and assume $\sigma_n = I_q$. Let us consider the following inflation of (1.1) with $\rho \geq 1$

$$X^\rho_{n+1} = A_n X^\rho_n + B_n + \rho \xi_{n+1}, \quad Y^\rho_{n+1} = H_n X^\rho_{n+1} + \rho \zeta_{n+1}. \tag{6.2}$$

Let R^ρ_n be the stationary filter covariance sequence of the associated Kalman filter, and R_n be the one for (1.1). Evidently, the stationary solution of this system satisfies $R^\rho_n = \rho^2 R_n$, and so are the forecast covariances $\widehat{R}^\rho_n = \rho^2 \widehat{R}_n$. In order to apply Theorem 6.1 to the previous system and (1.10), we consider the following matrix difference

$$\begin{bmatrix} \rho^2 \Sigma_n & (A_n)^T \\ A_n & -\frac{1}{\rho^2\sigma} H_n^T H_n \end{bmatrix} - \begin{bmatrix} \Sigma'_n & (A'_n)^T \\ A'_n & -\frac{1}{\sigma} H_n^T H_n \end{bmatrix} = \begin{bmatrix} (\rho^2 - r')\Sigma_n - r' A_n D_S A_n^T & (1-\sqrt{r'})A_n^T \\ (1-\sqrt{r'})A_n & (\frac{1}{\sigma} - \frac{1}{\sigma\rho^2})H_n^T H_n \end{bmatrix}$$

$$\preceq \begin{bmatrix} (\rho^2 - (1+c)r')\Sigma_n & (1-\sqrt{r'})A_n^T \\ (1-\sqrt{r'})A_n & (\frac{1}{\sigma} - \frac{1}{\sigma\rho^2})H_n^T H_n \end{bmatrix}.$$

With the conditions in the proposition, the matrix above is PSD. Therefore $\widehat{R}^\rho_n \succeq \widehat{\widetilde{R}}_n$, then because $\rho > 1$ stands for a worse observation, it is straight forward to verify that

$$\widetilde{R}_n = \mathcal{K}(\widehat{\widetilde{R}}_n) \preceq \mathcal{K}_\rho(\widehat{\widetilde{R}}_n) \preceq \mathcal{K}_\rho(\widehat{R}^\rho_n) = \rho^2 \mathcal{K}(\rho^{-2}\widehat{R}^\rho_n) = \rho^2 \mathcal{K}(\widehat{R}_n) = \rho^2 R_n.$$

Here K_ρ denotes the forecast-posterior Kalman covariance update for the system (6.2).

□

6.2. Different signal observation systems.
The analysis framework of this paper can address system (1.1) with very general setups. When applying it to particular dynamical and observation settings, there are special ways to verify the conditions. In this section, we briefly discuss a few settings commonly used in various scientific and engineering fields. An explicit example will be given in Section 6.3.

6.2.1. Classical setting.
In the classical setting, the system coefficients are deterministic and time homogenous, in other words they are of constant values. In this case, the stationary Kalman filter covariance matrices are also constant $\widetilde{R}^L_n = \widetilde{R}^L, \widetilde{R}_n = \widetilde{R}$. Each of them solves an algebraic Riccati equation(ARE) equation

$$\widetilde{R} = \mathcal{K}(\widehat{R}) = \widehat{R} - \widehat{R}H^T(\sigma + H\widehat{R}H^T)^{-1}H\widehat{R}, \quad \widehat{R} = r'A\widetilde{R}A^T + r'AD_S A^T + r'\Sigma,$$
$$\widetilde{R}^L = \mathcal{K}_L(\widehat{R}^L), \quad \widehat{R}^L = rA^L\widetilde{R}^L(A^L)^T + \Sigma^L. \tag{6.3}$$

In general, these solutions require numerical methods to compute.

6.2.2. Intermittent dynamical regimes.
One challenge that practical filters often face is that the dynamical coefficient A_n is not always stable with spectral norm less than 1. This is usually caused by the large scale chaotic dynamical regime transitions. One simple way of modelling this phenomenon, is letting A_n be a Markov jump process

on two states $\{A_+, A_-\}$, where $\|A_-\| \leq 1$ and $\|A_+\| > 1$. Chapter 8 of [4] has shown that this model could generate intermittent turbulence signals as seen in nature. Chapter 8 of [4] has also numerically tested the DRKF for the related filtering problem, showing close to optimal performance.

Our analysis framework naturally applies to these scenarios. The only difficulty is that Assumption 5.1 may require additional works to verify. In general, one may need the general results in Section 5 or even other mechanisms mentioned in Remark 5.1.

On the other hand, in many practical situations, the random dynamical regime switchings occur only on part of the model. If the large scale subspace includes this random part as in [4], the coefficients for small scale part are deterministic. This may make the conditions in our theorems the same as the deterministic case. For example, the formulation of Theorem 3.1 for DRKF is independent of the large scale coefficients. For another example, if the large scale variables have no impact on the small scales, $\mathbf{P}_S A_n \mathbf{P}_L \equiv 0$, then when computing the unfiltered covariance for small scale V^S, the large scale coefficients also are irrelevant.

6.2.3. Conditional Gaussian systems. If the system coefficients are functions of the observation, that is $A_n = A(Y_n)$ and likewise for the other terms, system (1.1) is a conditional Gaussian system. Although the evolution of (X_n, Y_n) in this case can be very nonlinear, the optimal filter is still given by (1.2) according to [17]. Such structure rises in many practical situations, like Lagrangian data assimilation, and turbulent diffusion with a mean flow. The conditional Gaussian structure can be exploited in these situations to gain significant advantages [36,37]. In particular, dynamical structures like geostrophic balance can yield other types of reduced filters [38].

In our context of reduced filtering, one caveat of conditional Gaussian system is that the system noises are in general not independent of the future system coefficients. For example, Y_{n+1} may depend on ξ_n, and so does $A_{n+1} = A(Y_{n+1})$. As a consequence, Theorems 2.1 and 3.1 may not apply, while Theorem 2.2 still holds.

6.2.4. Intermittent observations. Due to equipment problems, observations may not come in constantly but randomly. [39] models this feature by letting $H_n = \gamma_n H$ where γ_n is a sequence of independent Bernoulli random variables with $\mathbb{E}\gamma_n = \bar{\gamma}$. When the signal dynamics is unstable, [39] has shown that there is a critical frequency γ_c, such that the average Kalman filter covariance $\mathbb{E}R_n$ has a time uniform upper bound if and only if $\bar{\gamma} < \gamma_c$. Such results can be directly applied to the reference Kalman filters of systems (1.9) and (1.10), which leads to upper bounds for the reduced filter errors. On the other hand, if the system dynamics is stable, the reduced filter error can also be bounded using methods in Section 5. This will also be discussed in Section 6.3.5.

6.3. Stochastic turbulence examples. One of the most important applications of filtering is numerical weather forecast. The computational problems in this field are challenging, because the system dimensions are extremely high, and the system parameters are changing constantly. One simple way to model a planetary turbulence flow is linearizing a stochastic dynamics in the Fourier domain. In order to apply the reduced filters to these models, we are interested in finding the minimal amount of Fourier modes for the large scale subspace, and how to set up the small scale covariance D_S for RKF.

6.3.1. Linearized stochastic turbulence in Fourier domain. Consider the following stochastic partial differential equation [4,40]

$$\partial_t u(x,t) = \Omega(\partial_x) u(x,t) - \gamma(\partial_x) u(x,t) + F(x,t) + \dot{W}(x,t). \qquad (6.4)$$

For the simplicity of discussion, the underlying space is assumed to be an one dimensional torus $\mathbb{T} = [0, 2\pi]$, while the generalization to higher dimensions is quite straight forward. The terms in (6.4) have the following physical interpretations:

1) Ω is an odd polynomial of ∂_x. This term usually comes from the Coriolis effect from earth's rotation, or the advection by another turbulence flow.

2) γ is a positive and even polynomial of ∂_x. This term models the general diffusion and damping effects in stochastic turbulences.

3) $F(x,t)$ is a deterministic forcing and $W(x,t)$ is a stochastic forcing.

In this paper, we assume both forcings have a Fourier decomposition

$$F(x,t) = \sum_{k \in \mathcal{I}} f_k(t) e^{\mathbf{i}k \cdot x}, \quad W(x,t) = \sum_k \sigma_k^u W_k(t) e^{\mathbf{i}k \cdot x}.$$

Here $W_k(t) = \frac{1}{\sqrt{2}} W_{k,r}(t) + \frac{\mathbf{i}}{\sqrt{2}} W_{k,i}(t)$ is a standard Wiener process on $\mathbb{C}$, and the conjugacy condition is imposed to ensure terms in (6.4) are of real values: $f_k(t) = f_{-k}^*(t), \sigma_k^u = (\sigma_{-k}^u)^*, W_k(t) = W_{-k}^*(t)$. Suppose $P(\partial_x) e^{\mathbf{i}k \cdot x} = i\omega_k e^{\mathbf{i}k \cdot x}, \gamma(\partial_x) e^{\mathbf{i}k \cdot x} = \gamma_k e^{\mathbf{i}k \cdot x}$ with $\gamma_k > 0$. Then the solution of (6.4) can be written in terms of its Fourier coefficients, $u(x,t) = \sum_k u_k e^{\mathbf{i}k \cdot x}$, where the real and imaginary parts follow

$$d \begin{bmatrix} u_k^r(t) \\ u_k^i(t) \end{bmatrix} = \begin{bmatrix} -\gamma_k & -\omega_k \\ \omega_k & -\gamma_k \end{bmatrix} \begin{bmatrix} u_k^r(t) \\ u_k^i(t) \end{bmatrix} dt + \begin{bmatrix} f_k^r(t) \\ f_k^i(t) \end{bmatrix} dt + \frac{\sigma_k^u}{\sqrt{2}} \begin{bmatrix} dW_k^r(t) \\ dW_k^i(t) \end{bmatrix}. \tag{6.5}$$

To transform (6.4) to a discrete time formulation like (1.1) in the real domain, we assume the intervals between observations are of constant length $h > 0$, and pick a Galerkin truncation range $K \in \mathbb{N}$. Let X_n in (1.1) be a $(2K+1)$-dim column vector, with coordinates being:

$$[X_n]_0 = u_0(nh), \quad [X_n]_k = u_k^r(nh), \quad [X_n]_{-k} = u_k^i(nh), \quad k = 1, \ldots, K. \tag{6.6}$$

The system coefficients for the dynamic part of (1.1) then can be formulated as follows, where $A_n = A$ is diagonal with 2×2 sub-blocks, and $\Sigma_n = \Sigma$ is diagonal. Their entries are given below:

$$[A]_{\{k,-k\}^2} = \exp(-\gamma_k h) \begin{bmatrix} \cos(\omega_k h) & \sin(\omega_k h) \\ -\sin(\omega_k h) & \cos(\omega_k h) \end{bmatrix}, \quad [B]_k = f_k^r(nh)h, \quad [B]_{-k} = f_k^i(nh)h,$$

$$[\Sigma]_{k,k} = \frac{(\sigma_k^u)^2}{2} \int_{nh}^{(n+1)h} \exp(-2\gamma_k s) ds = \frac{1}{2} E_k^u (1 - \exp(-2\gamma_k h)). \tag{6.7}$$

$E_k^u = \frac{1}{2\gamma_k} (\sigma_k^u)^2$ is the stochastic energy of the k-th Fourier mode, it is also the sum of stochastic energy of $[X_n]_k$ and $[X_n]_{-k}$.

In applications, the damping often grows and the energy decays like polynomials of the wavenumber $|k|$

$$\gamma_k = \gamma_0 + \nu |k|^\alpha, \quad E_k^u = E_0 |k|^{-\beta}, \quad \alpha > 0, \ \beta \geq 0. \tag{6.8}$$

As we will see in our discussion below, such formulation guarantees the existence of a large scale separation with good reduced filter performance. To show that our framework is explicitly computable, we will consider the following specific set of physical parameters for (6.8) with a Kolmogorov energy spectrum used in [41]:

$$\alpha = 2, \quad \beta = \frac{5}{3}, \quad r = 1.2, \quad r' = 1.21, \quad h = 0.1, \quad \nu = 0.01, \quad \beta^* = 0.9, \quad E_0 = 1. \tag{6.9}$$

6.3.2. Setups for reduced filters. Since the system coefficients of (6.7) are all block diagonal, both DRKF and RKF can be applied for reduced filters. Naturally, the large scale set consists of modes with low wavenumbers $\{|k| < N\}$ for a certain N. And for RKF, D_S should be a diagonal matrix with entries $\{\delta_k\}_{|k| \geq N}$. The question is how to pick these reduced filter parameters, and how do they depend on the system coefficients.

Based on Theorem 3.1, DRKF does not require additional constraints. But in order to have good practical performances, intuitively the error caused by small scale time correlation should be small comparing with the other terms. In other words, for some $\epsilon > 0$,

$$\frac{2\sqrt{\lambda_S} r (\sqrt{r} + 1) \gamma_\sigma}{(1 - \sqrt{\lambda_S})(1 + \gamma_\sigma)} \leq \epsilon, \tag{6.10}$$

The λ_S in this model is $\max_{|k| \geq N} \exp(-\gamma_k h) = \exp(-\gamma_N h)$. If we approximate $(1 - \sqrt{\lambda_S})$ with 1, and bound γ_σ with 1, we find that $\gamma_N \geq -\frac{2}{h} \log(\epsilon/\sqrt{r(r+1)})$. This relation is independent of the energy spectrum, and if the dissipation has a polynomial growth as in (6.8) with $\gamma_0 = 0$, we find that approximately

$$N \geq \left[-\frac{2}{h\nu} \log(\epsilon/\sqrt{r(r+1)}) \right]^{\frac{1}{a}}.$$

In the physical setup of (6.9) with $\epsilon = 0.2$, we find that $N \approx 65$.

RKF requires the verification of Assumption 4.1. Here we use the unfiltered covariance $\widetilde{V}$ of the inflated system (1.10) as an upper bound for $\widetilde{R}_n$ as in Section 6.1.1. It is easy to find that $\widetilde{V}$ is diagonal with entries

$$[\widetilde{V}]_{k,k} = \tilde{v}_k = \frac{r' E_k^u (1 - r' \exp(-2\gamma_k h)) + \delta_k r' \exp(-2\gamma_k h)}{2 - 2r' \exp(-2\gamma_k h)}. \tag{6.11}$$

In order for Assumption 4.1 to hold, we need that for some $\beta^* \leq 1$

$$\tilde{v}_k \leq (\beta^* r - 1)\delta_k \quad k \geq N. \tag{6.12}$$

In order to achieve this, we need $\beta^* r \geq \beta^* r r' \exp(-2\gamma_k h) + 1$ and $\delta_k \geq \frac{r' E_k^u}{\beta^* r - \beta^* r r' \exp(-2\gamma_k h) - 1}$. In the setting of (6.8), $\exp(-2\gamma_k h) \to 0$ for large $|k|$, so we roughly require

$$\delta_k \geq \frac{r' E_k^u}{\beta^* r - 1} \quad \Rightarrow \quad D_S \approx \frac{r' \mathbf{P}_S E_k^u \mathbf{P}_S}{\beta^* r - 1}. \tag{6.13}$$

The small scale truncation requires $\gamma_N \geq \frac{1}{2h} \log\left(\frac{1}{r'} - \frac{1}{\beta^* r r'}\right)$. In the polynomial dissipation setting (6.8), this implies $N \geq [\frac{1}{2h\nu} \log\left(\frac{1}{r'} - \frac{1}{\beta^* r r'}\right)]^{\frac{1}{\alpha}}$. In particular with the physical parameters (6.9), $N \approx 25$.

6.3.3. Intermittent physical environment. A simple way to model intermittent physical environment for stochastic turbulence (6.7) is letting A_n be a Markov chain, while maintaining the sub-block structure: $[A_n]_{\{k,-k\}^2} = [\lambda_n]_k [A]_{\{k,-k\}^2}$. Here λ_n is a Markov chain taking values in $\mathbb{R}^{K+1}$. Then the system random instability can be modeled as the random fluctuation of $[\lambda_n]_k$, so that occasionally $\|[A_n]_{\{k,-k\}^2}\| > 1$ for some k.

In many situations, such instability only occur on the a small subset I of Fourier modes. This is because when the wave numbers are high, the dissipation force is much stronger than the random environmental force. So for $k \in I^c$, $[A_n]_{\{k,-k\}}$ remains a constant value like in (6.7). Then it suffices to let the large scale mode set include subset I, and the discussion in Section 6.3.2 remains the same.

6.3.4. Advection from a strong jet flow. One major nonlinearity source for planetary or engineering turbulence takes the form of a jet flow advection. For example, the meridional flows on earth are often advected by a eastward zonal flow [37, 41]. (6.4) can be extended to this scenario, by adding an auxiliary process $w_t \in \mathbb{R}$ to describe the jet flow, with B_t being an independent standard Wiener process in $\mathbb{R}$,

$$
\begin{aligned}
dw_t &= G_{w_t}(u_t)dt + g_t dt + \sigma_w dB_t, \\
\partial_t u(x,t) &= (\Omega(\partial_x) + w_t \partial_x - \gamma(\partial_x)) u(x,t)dt + F(x,t)dt + dW(x,t).
\end{aligned}
\tag{6.14}
$$

The feedback of $u_t = u(\cdot,t)$ on w_t, $G_{w_t}(u_t)$, is assumed to be linear on u_t, but may have nonlinear dependence on w_t. Since strong jet flows often have close to accurate observations, we assume w_t can be directly observed. The resulting system will be conditionally Gaussian. A time discretization like in Section 6.3.1 would lead to the same dynamical formulation as (6.7), except that the phase speed ω_k is replaced by $\omega_k + k\tilde{w}_n$. $\tilde{w}_n = w_{nh}$ is the time discretization of w_t, and it follows

$$
\tilde{w}_{n+1} = \widetilde{G}_{\tilde{w}_n} X_{n+1} h + \tilde{w}_n + g_{nh} h + \sigma_w \sqrt{h} \zeta^v_{n+1}.
$$

Here $\widetilde{G}_w X = G_w(u)$ if X consists of the Fourier modes of field u like in (6.6). $\tilde{w}_{n+1}$ can be seen as the $q+1$-th dimension of the observation vector Y_{n+1}, and ζ^v_{n+1} is its observation noise.

Jet flow advection in fact is a good example to show that the system independent noise condition (2.2) may fail, since the observation noise ζ^v_{n+1} is correlated with coefficients A_{n+1} and H_{n+1} through $\tilde{w}_{n+1}$. As a consequence, Theorems 2.1 and 3.1 no longer apply, but Theorem 2.2 still holds.

6.3.5. Intermittent observations. Observations of turbulence in practice often come from a network of sensors, that are located at a group of points $x_j \in \mathbb{T}$, and the observation noise can be modeled by i.i.d. $\mathcal{N}(0,\sigma)$ random variables:

$$
[H]_{j,0} = 1, \quad [H]_{j,k} = 2\cos(kx_j), \quad [H]_{j,-k} = 2\sin(kx_j), \quad \sigma = \sigma^o I_q.
\tag{6.15}
$$

One particular choice of sensor location will be equal spacing, $x_j = \frac{2\pi j}{2J+1}, j = 0,1,\ldots,2J$, studied by chapter 7 of [4]. Consider an equivalent observation transformation

$$
[\Psi]_{0,j} = \frac{1}{2J+1}, \quad [\Psi]_{i,j} = \frac{\cos(\frac{2\pi ij}{2J+1})}{2J+1}, \quad [\Psi]_{-i,j} = \frac{\sin(\frac{2\pi ij}{2J+1})}{2J+1},
$$

so the transformed observation coefficients satisfies $[\Psi H]_{j,k} = \delta_{j \equiv k \bmod 2J+1}$, and $\Psi \sigma \Psi^T = \frac{\sigma^o I_{2J+1}}{2J+1}$. When $J < K$, such observation network introduces aliasing effect among the Fourier modes, which is carefully studied in [4, 42]. Here after we focus only on the simple case where $K = J$ so $\Psi H = I$.

In real applications, turbulence observations may not be available at each time step. Following the example of [39], we model this problem by letting $H_n = \gamma_n H$, where γ_n is a sequence of Bernoulli random variables with average $\mathbb{E}\gamma_n = \bar{\gamma}$. We will investigate how does this observation setting change the reduced filter setup.

For DRKF, the small scale unfiltered covariance V is diagonal with entries $[V]_{k,k} = \frac{1}{2}E_k^u$. Then $\gamma_\sigma = \sup_n \|(\sigma_n^L)^{-1}H_n V^S H_n^T\| = \sup_{|k| \geq N} \frac{(2K+1)E_k^u}{(2K+1)E_k^u + 2\sigma^o}$. Following the discussion in Section 6.3.2, we are interested in maintaining (6.10). In the polynomial dissipation regime (6.8), if we approximate $1 - \sqrt{\lambda_S}$ by 1, and replace $1 + \gamma_\sigma$ by a lower bound 1, we find that

$$\frac{\exp(-\frac{1}{2}h\nu N^\alpha)E_0 N^{-\beta}}{E_0 N^{-\beta} + \frac{2\sigma^o}{2K+1}} \leq \frac{\epsilon}{\sqrt{r(r+1)}}$$

In the physical setup of (6.9) with $\epsilon = 0.2, \sigma^o = 0.1, K = 200$ and by numerical computing the quantities above, we find that $N \approx 59$.

As for RKF, at any fixed time n, the unfiltered covariance $\widetilde{V}$ is given by (6.11), and we know the reference Kalman filter covariance $\widetilde{R}_n \preceq \widetilde{V}$. If at time n, the observations are available, $\gamma_n = 1$, note that $\widehat{\widetilde{R}}_n \preceq A\widetilde{V}A^T + \Sigma = \widetilde{V}$, by Proposition 6.1 with $m = n$,

$$[\widetilde{R}_n]_{k,k} \leq v_k' = \frac{\tilde{v}_k \sigma^o}{\sigma^o + (2K+1)\tilde{v}_k} < \tilde{v}_k.$$

Denote $\beta_o = \max_{k \geq N} \frac{\tilde{v}_k}{r\delta_k} + \frac{1}{r}, \beta_u = \max_{k \geq N} \frac{\tilde{v}_k}{r\delta_k} + \frac{1}{r}$. Clearly $\beta_o < \beta_u$. So in Assumption 5.1, we can let

$$\beta_n^* = \gamma_n \beta_o + (1 - \gamma_n)\beta_u,$$

which is an independent sequence.

In order for Theorem 5.2 to give a meaningful upper bound, it suffices to require

$$\bar{\beta}^* = \mathbb{E}\beta_n^* = \bar{\gamma}\beta_o + (1 - \bar{\gamma})\beta_u < 1. \tag{6.16}$$

With (6.16), we will have $\mathbb{E}\|e_n\|_{C_n}^2 \leq \bar{\beta}^{*(n-n_0)}\mathbb{E}\|e_{n_0}\|_{C_{n_0}}^2 + \frac{2d}{1-\bar{\beta}^*}$. Since (6.12) is equivalent to $\beta_u < 1$, so (6.16) is a weaker requirement and ends up with a smaller N. In particular, if we pick D_S as in (6.13), the parameters as in (6.9), and let $\bar{\gamma} = 0.9, \sigma^o = 0.1, K = 200$, we find $N \approx 14$.

7. Conclusion and discussion

High dimensionality is an important challenge for modern day numerical filtering, as the classical Kalman filter is no longer computationally feasible. This problem can sometimes be resolved by proper dimension reduction techniques, such as exploiting intrinsic multiscale structures. This paper considers two reduced filters. The DRKF works for dynamically decoupled systems, it estimates the small scale variables with their equilibrium statistical state. The RKF uses a constant statistical state as the small scale filtering prior, and requires the large scale projection not to decrease the error covariance. Both methods have been studied by [4] for stochastic turbulence filtering, and they have close to optimal performances in various regimes. On the other hand, rigorous error analysis of these reduced filter has been an open problem, since the dimension reduction techniques bring in unavoidable biases, just like in many other practical uncertainty quantification procedures. This paper fills in this gap by developing a two-step framework. The first step examines the fidelity of the reduced covariance estimators, showing that the real filter error covariance is not underestimated. For RKF with system independent noises, this can be verified by tracking the covariance matrix. For DRKF and more general scenarios, the covariance fidelity can be demonstrated by the intrinsic dissipation mechanism of the Mahalanobis error. The second step bounds

the reduced filter covariance estimators, by building a connection between them and proper Kalman filter covariances. The combination of these two steps yields an error analysis framework for the reduced filters, with exponential stability and accuracy for small system noises as simple corollaries. When applied to a linearized stochastic turbulence, this framework provides a priori guidelines for large scale projection range and reduced filter parameterizations.

Besides the major themes mentioned above, there are two related issues we have not focused on:

- The multiplicative inflation is applied in our reduced filters to avoid covariance underestimation. This technique has been applied widely for various practical filters, but its theoretical importance has never been studied except in one dimension [43]. The error analysis of this paper implicitly studies this issue, as the inflation plays an important role in our proof. Based on the formulation of Theorems 2.2, 3.1, and 4.2, stronger inflation provides better filter stability. Moreover, as mentioned in Remark 2.2, this inflation is an essential high dimension replacement of the classical uniform bounded conditions in [29].

- For RKF, Theorem 2.1 has a much stronger result comparing with Theorem 2.2, while the additional condition on system independent noises often holds. But for many other practical filters such as the ensemble Kalman filter, the second moment of the filter error cannot be expressed through recursive formulas, as the Kalman gain matrix is correlated with the filter error. The Mahalanobis error dissipation on the other hand still holds as it is a more intrinsic property.

Appendix A. Complexity estimates. In this section we do some simple computational complexity estimates for the Kalman filter (1.2) and the reduced Kalman filters (1.5) and (1.6). Through these estimations, we find that the reduced filters reduce computation complexity from $O(d^2q)$ to $O(d^2 + p^2q + p^3)$ and $O(d^2 + dp^2 + dq^2)$, which is a significant reduction when the state space dimension d is much larger than the observation dimension q and large scale dimension p. For simplicity, we only consider the most direct numerical implementation of the related formulas, although there are many alternative implementation methods with improved numerical stability and accuracy [44]. We assume the complexity of matrix product of $[A]_{a \times b}$ and $[B]_{b \times c}$ is abc, and the complexity of the inversion and Cholesky decomposition of a general $[A]_{a \times a}$ matrix is a^3 [45]. There are also a few additional assumptions that hold for most applications, while without them similar qualitative claims hold as well.

1) We focus mostly on the online computational cost, which is the cost for the computation of filter iteration. This is the most significant cost in the long run.

2) When the system coefficients are deterministic, the Kalman gain matrix sequence in principle can be computed offline [44]. We do not consider this scenario as it oversimplifies the problem.

3) A_n is a sequence of sparse matrices. This holds for the stochastic turbulence model in Section 6. It rises in various differential equation context as most physical interaction involves only close neighbors. As a consequence, $\widehat{C}_{n+1} = A_n C_n A_n^T + \Sigma_n$ involves only

$O(d^2)$ complexity instead of $O(d^3)$. On the other hand, if this assumption is not true, then the leading computational cost is $O(d^3)$ and comes from the prescribed forecast step, while the reduced filters obviously reduce the cost to $O(p^2 d)$, so there is no need of further discussion.

4) H_n and σ_n are also sparse matrices with relatively time invariant structure, so matrix product like $H_n \widehat{C}_n H_n^T$ involves only $O(d^2)$ computation. This assumption holds in many applications, where the observation noises are independent.

Based on these assumptions, the complexity of Kalman filter is given by Table A.1.

Operation	Complexity order
$\widehat{R}_n = A_n R_{n+1} A_n^T + \Sigma_n$	d^2
$(\sigma_n + H_n \widehat{R}_n H_n^T)^{-1}$	$d^2 + q^3 + d^2 q$
$K_{n+1} = \widehat{R}_n H_n^T (\sigma_n + H_n \widehat{R}_{n+1} H_n^T)^{-1}$	$d^2 q$
$m_{n+1} = A_n m_n + B_n - K_{n+1}(Y_{n+1} - H_n(A_n m_n + B_n))$	$d + dq$
$R_{n+1} = \widehat{R}_{n+1} - \widehat{R}_{n+1} H_n^T (\sigma_n + H_n \widehat{R}_{n+1} H_n^T)^{-1} H_n \widehat{R}_{n+1}$	$d^2 q$
total	$d^2 q$

Table A.1: *Complexity estimate of the Kalman filter* (1.2).

A.1. DRKF. DRKF essentially is applying a Kalman filter in the large scale subspace with dimension p. The additional computation involves estimating the unfiltered small scale covariance V_n^S which requires an $O(d^2)$ complexity. When the system coefficients are constants, V_n^S is of constant value and there is no need to update it. We put such savable cost in brackets in Table A.2.

Operation	Complexity order
$\widehat{C}_{n+1}^L = A_n^L C_n^L (A_n^L)^T + \Sigma_n^L$	p^2
$V_{n+1}^S = A_n V_n^S A_n^T + \Sigma_n^S, \ \mu_{n+1}^S = A_n^S \mu_n^S + B_n^S$	(d^2)
$(\sigma_n^L + H_n \widehat{C}_{n+1}^L H_n^T)^{-1}$	$(d^2) + q^3$
$K_{n+1}^L = \widehat{C}_{n+1}^L (H_n^L)^T (\sigma_n^L + H_n^L \widehat{C}_{n+1}^L (H_n^L)^T)^{-1}$	$p^2 q$
$\mu_{n+1}^L = A_n^L \mu_n^L + B_n^L - K_{n+1}^L (Y_{n+1} - H_n^S \mu_{n+1}^S - H_n^L (A_n^L \mu_n^L + B_n^L))$	pq
$C_{n+1}^L = r\widehat{C}_{n+1}^L - r\widehat{C}_{n+1}^L (H_n^L)^T (\sigma_n^L + H_n^L \widehat{R}_n^L (H_n^L)^T)^{-1} H_n^L \widehat{C}_{n+1}^L$	$q^3 + p^2 q$
total	$q^3 + p^2 q + (d^2)$

Table A.2: *Complexity estimate of the DRKF* (1.5).

A.2. RKF. In the implementation of RKF, we need to exploit the fact that C_n is nonzero only for the upper $p \times p$ sub-block. Therefore its Cholesky decomposition involves a cost of $O(p^3)$. Also one would like see $\widehat{C}_n$ as the sum of $A_n C_n A_n^T$, which is a rank p matrix, and a sparse matrix $A_n D_S A_n^T + \Sigma_n$, instead of a generic $d \times d$ matrix. The Woodbury matrix identity is also useful for gaining computational advantage. For example, when doing the matrix inversion

$$[\sigma_n + H_n \widehat{C}_n H_n^T]^{-1} = [Q_n + H_n A_n C_n A_n^T H_n^T]^{-1},$$

where $Q_n := [\sigma_n + H_n \Sigma'_n H_n^T]$ with $\Sigma'_n = \Sigma_n + D_S$, note that inverting Q_n costs $O(q^3)$. The Woodbury identity indicates that:

$$[Q_n + H_n A_n C_n A_n^T H_n^T]^{-1}$$
$$= Q_n^{-1} - Q_n^{-1} H_n A_n C_n^{1/2} [I + C_n^{1/2} A_n^T H_n^T H_n A_n C_n^{1/2}]^{-1} C_n^{1/2} H_n^T A_n^T Q_n^{-1}.$$

Note that $C_n^{1/2}[I + C_n^{1/2} A_n^T H_n^T H_n A_n C_n^{1/2}]^{-1} C_n^{1/2}$ has only the upper $p \times p$ sub-block being nonzero, so its computation costs only $O(p^3 + pqd + p^2q)$. So the overall cost of computing $[\sigma_n + H_n \widehat{C}_n H_n^T]^{-1}$ is $O(pqd + p^3 + q^3)$, while in the Kalman filter, it is qd^2. The estimate of each step is given below in Table A.3.

Operation	Complexity order
$\widehat{C}_n = A_n C_n A_n^T + A_n D_S A_n^T + \Sigma_n$	d^2
$(\sigma_n + H_n \widehat{C}_n H_n^T)^{-1}$	$pqd + q^3 + p^3$
$\widehat{K}_{n+1} = A_n C_n A_n^T H_n^T (\sigma_n + H_n \widehat{C}_n H_n^T)^{-1}$	$p^2 d + pqd$
$+ \Sigma'_n H_n^T (\sigma_n + H_n \widehat{C}_n H_n^T)^{-1}$	$dq^2 + d^2$
$\mu_{n+1} = A_n \mu_n + B_n - \widehat{K}_{n+1}(Y_n - H_n(A_n \mu_n + B_n))$	$d + dq$
$C_{n+1} = r\mathbf{P}_L \widehat{C}_n \mathbf{P}_L - r\mathbf{P}_L \widehat{C}_n H_n^T (\sigma_n + H_n \widehat{C}_n H_n^T)^{-1} H_n \widehat{C}_n \mathbf{P}_L$	$dp^2 + dq^2$
total	$d^2 + dp^2 + dq^2$

Table A.3: *Complexity estimate of the RKF* (1.6).

Appendix B. Matrix inequalities. The following lemma has been mentioned in [43] for dimension one.

LEMMA B.1.   *The prior-posterior Kalman covariance update mapping $\mathcal{K}$ in* (1.2), *can also be defined as*

$$\mathcal{K}(C) = (I - KH_n)C(I - KH_n)^T + K\sigma K^T$$

where $K := CH_n(\sigma + H_n C H_n^T)^{-1}$ is the corresponding Kalman gain. $\mathcal{K}$ is a concave monotone operator from PD to itself.

Proof.   The first matrix identity is straightforward to verify, and can be found in many references of Kalman filters [4]. In order to simplify the notations, we let $H = H_n$ and $J(X) = (\sigma + HXH^T)^{-1}$. Then with any symmetric matrix A, the perturbation in direction A is given by

$$D_A J(X) := \frac{d}{dt} J(X + At)\big|_{t=0} = -JHAH^T J.$$

Therefore

$$D_A \mathcal{K} = A - AH^T JHX - XH^T JHA + XH^T JHAH^T JHX$$
$$= (I - H^T JHX)^T A(I - H^T JHX).$$

The Hessian is

$$D_A^2 \mathcal{K} = -2AH^T JHA + 2AH^T JHAH^T JHX + 2XH^T JHAH^T JHA$$
$$- 2XH^T JHAH^T JHAH^T JHX$$
$$= -2(AH^T J^{1/2} - XH^T HAH^T J^{1/2}) \cdot (AH^T J^{1/2} - XH^T HAH^T J^{1/2})^T \preceq 0.$$

Therefore, as long as $X, X + A \succeq 0$, then the concavity holds:

$$\mathcal{K}(X) + \mathcal{K}(X + A) \preceq 2\mathcal{K}(X + \tfrac{1}{2}A).$$

When we require A to be PSD, $D_A \mathcal{K} \succeq 0$ implies the monotonicity of $\mathcal{K}$. □

LEMMA B.2. *Suppose that A, C, D are PSD matrices, C is invertible, while $A \preceq [BCB^T + D]^{-1}$, then*

$$B^T A B \preceq C^{-1}, \quad A^{1/2} D A^{1/2} \preceq I_d.$$

Proof. From the condition, we have $A^{1/2}[BCB^T + D]A^{1/2} \preceq I_d$. Therefore our second claim holds. Moreover,

$$(B^T A B) C (B^T A B) \preceq B^T A^{1/2} A^{1/2} [BCB^T + D] A^{1/2} A^{1/2} B \preceq B^T A B.$$

This leads to our first claim by the next lemma. □

LEMMA B.3. *Let A and B be PSD matrices, if*
- *$A \succeq I_d$, then $ABA \succeq B$.*
- *$A \preceq I_d$, then $ABA \preceq B$. And for any real symmetric matrix C, $CAC \preceq C^2$.*

Proof. If the null subspace of B is D and $\mathbf{P}$ is the projection onto the complementary subspace $D^\perp$, then it suffices to show that $(\mathbf{P}A\mathbf{P})(\mathbf{P}B\mathbf{P})(\mathbf{P}A\mathbf{P}) \succeq \mathbf{P}B\mathbf{P}$. Therefore, without loss of generality, we can assume B is invertible, so it suffices to show

$$(B^{-1/2} A B^{1/2})(B^{-1/2} A B^{1/2})^T \succeq I.$$

But this is equivalent to checking that the singular values of $B^{-1/2} A B^{1/2}$ are greater than 1, which are the same as the eigenvalues of A.

If A and C are invertible, then the second claim follows as the direct inverse of the first claim. Else, it suffice to show the claim on the subspace where A and C are invertible. □

LEMMA B.4. *Let A and B be two PSD matrices, and A is invertible, then*

$$\|AB\| = \|A^{1/2} B A^{1/2}\| = \inf\{\lambda : B \preceq \lambda A^{-1}\}.$$

Proof. $\|AB\| = \|A^{1/2} B A^{1/2}\|$ comes as conjugacy preserves eigenvalues, and $\|A^{1/2} B A^{1/2}\| = \inf\{\lambda : B \preceq \lambda A^{-1}\}$ is obvious. □

LEMMA B.5. *Let $B \in PSD$, then $tr(AB) \leq \|A\| tr(B)$.*

Proof. Suppose the eigenvalue decomposition of B is $\Psi D \Psi^T$. Then we note that

$$\mathrm{tr}(AB) = \mathrm{tr}(A\Psi D \Psi^T) = \mathrm{tr}(\Psi^T A \Psi D), \quad \|A\| \mathrm{tr}(B) = \|\Psi^T A \Psi\| \mathrm{tr}(D).$$

So we can assume B is a diagonal matrix. Then

$$\mathrm{tr}(AB) = \sum_{i=1}^{d} A_{i,i} B_{i,i} \leq \|A\| \sum B_{i,i} = \|A\| \mathrm{tr}(B).$$

□

Appendix C. Convergence to the unique stationary solution. One of the remarkable property of Kalman filter covariance is that it converges to a unique stationary solution to the associated Riccati equation, assuming the system coefficients are stationary, and weak observability and controllability.

THEOREM C.1 (Bougerol 93).   *Suppose that* $(A_n, B_n, H_n, \Sigma_n)$ *is an ergodic stationary sequence. Define the observability and controllability Gramian as follows:*

$$\mathcal{O}_n = \sum_{k=1}^n A_{k,1}^T H_k^T \sigma_k^{-1} H_k A_{k,1}, \quad \mathcal{C}_n = \sum_{k=1}^n A_{n,k+1}^T \Sigma_k A_{n,k+1}, \quad A_{k,j} = A_{k-1} \cdots A_j.$$

Suppose the system (1.1) *is weakly observable and controllable, that is there is an n such that*

$$\mathbb{P}(\det(\mathcal{O}_n) \neq 0, \det(\mathcal{C}_n) \neq 0) > 0.$$

Suppose also the following random variables are integrable,

$$\log\log^+ \|A_1\|, \quad \log\log^+ \|A_1^{-1}\|, \quad \log\log^+ \|\Sigma_1\|, \quad \log\log^+ \|H_1^T H_1\|.$$

where $\log^+ x = \max\{0, \log x\}$. *Then there is a stationary PD sequence* $\widetilde{R}_n$ *that follows*

$$\widetilde{R}_{n+1} = \mathcal{K}(A_n \widetilde{R}_n A_n^T + \Sigma_n).$$

And for the covariance matrix R_n *of another Kalman filter started with an initial value* R_0, *will converge to* R_n^s *asymptotically:* $\limsup_{n\to\infty} \frac{1}{n}\delta(R_n, \widetilde{R}_n) \leq \alpha$. *Here* α *is a negative constant, and* δ *defines a Riemannian distance on* S_d^+ *by*

$$\delta(P, Q) = \sqrt{\sum_{i=1}^d \log^2 \lambda_i}, \quad \lambda_i \text{ are eigenvalues of } PQ^{-1}.$$

One simple and useful fact is that, if the original system (1.1) meets the requirement of Theorem C.1, then so does the inflated systems (1.9) and (1.10). To see this, one need to write down the corresponding observability and controllability Gramians $\mathcal{O}_n'$ and $\mathcal{C}_n'$.

In the general scenarios, it's straightforward to verify that the Gramians of reference system (1.10) are larger than the ones of (1.1)

$$\mathcal{O}_n' = \sum_{k=1}^n A_{k,1}'^T H_{k-1}^T \sigma_{k-1}^{-1} H_{k-1} A_{k,1}^T \succeq \mathcal{O}_n, \quad \mathcal{C}_n' = \sum_{k=1}^n A_{n,k+1}'^T \Sigma_k' A_{n,k+1}' \succeq \mathcal{C}_n.$$

Since these Gramian matrices are PSD matrices, $\mathcal{O}_n$ and $\mathcal{C}_n$ are nonsingular indicate that $\mathcal{O}_n'$ and $\mathcal{C}_n'$ are nonsingular. Also note

$$\log \|A_1'\| = \frac{1}{2}\log \widetilde{R} + \log \|A_1\|,$$

moreover,

$$\begin{aligned}
\log^+ \|\Sigma_1'\| &\leq \log \widetilde{R} + \log^+ (\|\Sigma_n\| + \|A_n D_S A_n^T\|) \\
&\leq \log \widetilde{R} + \max\{\log^+ (2\|\Sigma_n\|), \log^+ (2\|A_n D_S A_n^T\|)\} \\
&\leq \log \widetilde{R} + \log 2 + \log^+ \|\Sigma_n\| + 2\log^+ \|A_n\| + \log^+ \|D_S\|
\end{aligned}$$

so the integrability condition holds naturally.

When the system had dynamical two-scale decoupling (1.4), it is easy to see that $A_{k,j}$ has a block-diagonal structure, and so do the Gramians $\mathcal{C}_n$ and $\mathcal{O}_n$. It is also easy to verify that the controllability Gramian of (1.9) satisfies

$$\mathcal{C}'_n = \sum_{k=1}^{n} A'^T_{n,k+1} \Sigma^L_k A'_{n,k+1} \succeq \sum_{k=1}^{n} A^T_{n,k+1} \Sigma^L_k A^L_{n,k+1} = \mathbf{P}_L \mathcal{C}_n \mathbf{P}_L.$$

As for the observability Gramian of (1.9), notice that σ_k^{-1} is invertible, so there is a constant D_n such that $D_n \sigma_k^{-1} \preceq [\sigma_k^L]^{-1}$, then it is straightforward to verify that

$$D_n \mathcal{O}'_n = D_n \sum_{k=1}^{n} A'^T_{k,1} H^T_{k-1} \sigma^{-1}_{k-1} H_{k-1} A'^T_{k,1} \succeq \mathbf{P}_L \mathcal{O}_n \mathbf{P}_L.$$

This shows the weak observability and controllability of (1.1) implies the ones of (1.9).

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