

QUANTITATIVE ESTIMATES ON LOCALIZED FINITE DIFFERENCES FOR THE FRACTIONAL POISSON PROBLEM, AND APPLICATIONS TO REGULARITY AND SPECTRAL STABILITY*

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Dedicated to Gianni Gilardi on the occasion of his 70th anniversary, with friendship and admiration

Abstract. We establish new quantitative estimates for localized finite differences of solutions to the Poisson problem for the fractional Laplace operator with homogeneous Dirichlet conditions of solid type settled in bounded domains satisfying the Lipschitz cone regularity condition. We then apply these estimates to obtain (i) regularity results for solutions of fractional Poisson problems in Besov spaces; (ii) quantitative stability estimates for solutions of fractional Poisson problems with respect to domain perturbations; (iii) quantitative stability estimates for eigenvalues and eigenfunctions of fractional Laplace operators with respect to domain perturbations.

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1. Introduction

We focus on the Poisson problem for the fractional Laplacian operator, namely on the system

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (1.1)$$

In the above expression, $\Omega \subset \mathbb{R}^N$ is an open and bounded set, s is an index belonging to the interval $(0,1)$, and the regularity of the function f is discussed below. The symbol $(-\Delta)^s$ denotes the s -fractional Laplacian operator: in § 2.1 we provide both the definition of $(-\Delta)^s$ and the rigorous (distributional) formulation of problem (1.1) by following the approach provided, e.g., in [1, 22]. Here, we just mention that we are concerned with solutions u belonging to the space

$$\mathcal{X}_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) : u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega\} \quad (1.2)$$

and that in the following we denote by $\|u\|_s$ the so-called *Gagliardo semi-norm* of u , which is again defined in § 2.1. Note, furthermore, that the so-called *solid* boundary conditions at the second line of (1.1) are consistent with the fact that the fractional Laplacian is a *nonlocal* operator. Also, the fractional Laplacian operator coupled with the solid boundary conditions is usually termed *restricted fractional Laplacian*.

Problem (1.1) can be addressed by relying on variational techniques. For instance, a straightforward application of the Lax-Milgram lemma gives existence and uniqueness of a solution $u \in \mathcal{X}_0^s(\Omega)$, provided that f belongs to the dual space $\mathcal{X}_0^s(\Omega)'$. It is therefore

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natural to investigate whether or not the condition $f \in L^2$ implies additional regularity of u . This is the main goal of the present paper.

In order to state our results in a precise way, we start by introducing some further notation. Let $h \in \mathbb{R}^N$ be a vector, $|h| < 1$. We fix a function $u: \mathbb{R}^N \rightarrow \mathbb{R}$ and a smooth cut-off function $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$, and we define the functions u_h and $T_h u$ by setting

$$u_h(x) := u(x+h), \quad (T_h u)(x) := \phi(x)u_h(x) + [1 - \phi(x)]u(x), \quad \text{for every } x \in \mathbb{R}^N. \tag{1.3}$$

Note that the quantity $T_h u - u = \phi[u_h - u]$ can be viewed as a localized version of a finite difference. In the following we will mostly focus on the case when the domain Ω satisfies a so-called (ρ, θ) -Lipschitz cone condition. The precise definition is provided in § 4.3 below. Very loosely speaking, this condition is a sort of quantified Lipschitz condition imposed on the boundary $\partial\Omega$. Also, $\rho \in (0, +\infty)$ and $\theta \in (0, \pi/2)$ are regularity parameters: the bigger the ρ and the θ , the more regular the domain.

Our main result establishes a precise quantitative control on $T_h u - u$ in the case when u is a weak solution of (1.1).

THEOREM 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set and $f \in L^2(\mathbb{R}^N)$. Assume that ϕ is a smooth cut-off function, namely*

$$\phi \in W^{1,\infty}(\mathbb{R}^N), \quad 0 \leq \phi(x) \leq 1 \tag{1.4}$$

and

$$\text{supp } \phi \subseteq B_1(0). \tag{1.5}$$

Assume also that $u \in \mathcal{X}_0^s(\Omega)$ is a weak solution of (1.1) and that the product

$$\phi u_h \in \mathcal{X}_0^s(\Omega). \tag{1.6}$$

Then there is a constant C , which only depends on $N, s, \text{Lip } \phi$ and $\text{diam } \Omega$, such that

$$\|T_h u - u\|_{L^2(\mathbb{R}^N)}^2 \leq \|u_h - u\|_{L^2(\mathbb{R}^N)}^2 \leq C|h|^{2s} \|f\|_{\mathcal{X}_0^s(\Omega)}^2. \tag{1.7}$$

Moreover, if Ω satisfies a (ρ, θ) -Lipschitz cone condition for some $\rho \in (0, +\infty)$, $\theta \in (0, \pi/2)$ and

$$|h| \leq \frac{\rho \sin \theta}{4},$$

then there is a constant \tilde{C} , which only depends on $N, s, \text{Lip } \phi, \text{diam } \Omega, \rho$ and θ , such that

$$\|T_h u - u\|_s^2 \leq \tilde{C}|h|^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{H^{-s}(\mathbb{R}^N)}. \tag{1.8}$$

The following remarks are in order:

- in the statement of the theorem, $B_1(0)$ is the open unit ball centered at 0, $\text{Lip } \phi$ denotes the Lipschitz constant of ϕ , and $\text{diam } \Omega$ is the diameter of Ω , namely

$$\text{diam } \Omega := \sup_{x,y \in \Omega} |x - y|. \tag{1.9}$$

- A relevant feature of Theorem 1.1 is that by following the proof one can reconstruct the value of the constants C and \tilde{C} .

- The most interesting estimate is (1.8), whereas establishing (1.7) is quite easy. Also, note that (1.7) holds for any open and bounded set Ω , whereas to obtain (1.8) we have to assume the Lipschitz cone condition. Indeed, in the general case we can only establish a weaker version of (1.8), see Lemma 3.1 in § 3 below.
- Let D be a sufficiently large ball containing both Ω and $\Omega + h$, for every $|h| < 1$. By recalling (1.1), we infer that the solution u is only affected by the values attained by f on D . Hence, u does not change if we replace f with its truncation to 0 outside D . This implies that in the right hand side of (1.8) (and in the Besov regularity estimate (1.15) in Theorem 1.3 below) one could for instance use $\|f\|_{L^2(D)}$, instead of $\|f\|_{L^2(\mathbb{R}^N)}$. The replacement of $\|f\|_{H^{-s}(\mathbb{R}^N)}$ with $\|f\|_{H^{-s}(D)}$ when $f \equiv 0$ in $\mathbb{R}^N \setminus D$ is more delicate. As a matter of fact, we can use $\|f\|_{H^{-s}(D)}$ instead of $\|f\|_{H^{-s}(\mathbb{R}^N)}$ only when $s \in (0, 1/2)$. In the other cases there are obstructions (see [5]).
However, to simplify the notation here and in the following we always compute the norms on the whole \mathbb{R}^N .

In the following, we discuss some possible applications of Theorem 1.1. First, we formulate Theorem 1.2, which provides precise quantitative estimates on how the solution of the Poisson problem (1.1) depends on the domain Ω . Note that in the statement of Theorem 1.2 the quantity $\mathfrak{d}(\Omega_b, \Omega_a)$ is a way of measuring the “distance” between the sets Ω_a and Ω_b : the precise definition is provided in § 4.1.

THEOREM 1.2 (Domain perturbations). *Let $\Omega_a, \Omega_b \subset \mathbb{R}^N$ be bounded open sets contained in a sufficiently large open ball D of \mathbb{R}^N . Let us assume that Ω_a satisfies the (ρ, θ) -Lipschitz cone condition. Let $f \in L^2(\mathbb{R}^N)$ and let $u_a \in \mathcal{X}_0^s(\Omega_a)$ and $u_b \in \mathcal{X}_0^s(\Omega_b)$ denote the weak solutions to (1.1) in $\Omega = \Omega_a$ and $\Omega = \Omega_b$, respectively. There is a positive constant C , which only depends on N, s, ρ, θ and $\text{diam} D$, such that, if*

$$\mathfrak{d}(\Omega_b, \Omega_a) < \frac{\rho \sin \theta}{2}, \tag{1.10}$$

then

$$\|u_a - u_b\|_s \leq C \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \mathfrak{d}(\Omega_b, \Omega_a)^{s/2}. \tag{1.11}$$

We point out that, as in the case of Theorem 1.1, by following the proof of Theorem 1.2 one can reconstruct the precise value of the constant C in (1.11). Moreover, we observe that the proof of Theorem 1.2 combines Theorem 1.1 with a localization argument due to Savaré and Schimperna [21]. Finally, in the statement of Theorem 1.2 we impose a regularity assumption on Ω_a only, while Ω_b may be any open and bounded domain satisfying (1.10). This lack of symmetry is consistent with the fact that the quantity $\mathfrak{d}(\Omega_b, \Omega_a)$ is not symmetric in Ω_a and Ω_b , namely in general $\mathfrak{d}(\Omega_b, \Omega_a) \neq \mathfrak{d}(\Omega_a, \Omega_b)$.

In the case of the standard Laplacian, both the optimal regularity properties of the solution u to the Poisson problem corresponding to (1.1) and the relation between the regularity of u and the regularity properties of Ω and of f are well known. In particular, in the case when Ω is a Lipschitz domain, the results in [21] state that, if $f \in L^2$, then u belongs to the Besov space $u \in B_{2,\infty}^{3/2}$ (see § 2.2 below for the definition of $B_{2,\infty}^{3/2}$). In particular, in general one cannot achieve the higher regularity $u \in H^{3/2}$, as one can see by considering the one-dimensional example $u(x) = (1 - x^2)^+$, which solves (1.1) with $s = 1$ and $f = 2\chi_{(-1,1)}$ (cf. also [21, Rem. 2.4]). In this case the regularity $u \in H^{3/2}(\mathbb{R})$ is not attained because u has jump discontinuities at ± 1 .

On the other hand, the regularity theory for the fractional Poisson problem for (1.1) is far less established. In this paper we are interested in possible extensions of the following result [19, Prop. 1.4 (ii)-(iii)] (see also [8, Th. 2.3]):

PROPOSITION 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$ -domain. If $s \in (0, \frac{N}{4}) \cap (0, 1)$, then the solution u to (1.1) satisfies*

$$\|u\|_{L^q(\Omega)} \leq C \|f\|_{L^2(\Omega)} \quad \text{for } q = \frac{2N}{N-4s}, \tag{1.12}$$

while for $s \in (\frac{N}{4}, 1) \cap (0, 1)$ we have

$$\|u\|_{C^\alpha(\bar{\Omega})} \leq C \|f\|_{L^2(\Omega)} \quad \text{for } \alpha = \min \left\{ s, 2s - \frac{N}{2} \right\}. \tag{1.13}$$

In both cases the constant $C > 0$ only depends on s , $|\Omega|$, and q (or α).

We now state our regularity result. Note that the main novelties of Theorem 1.3 compared to Proposition 1.1 are the following. First of all we require low regularity properties on the domain Ω as we assume that Ω has only Lipschitz regularity (see also [9]). Second, we establish Sobolev and Besov-type regularity, more precisely we show that, if $f \in L^2$, then the solution u belongs to the Besov space $B_{2,\infty}^{3s/2}(\mathbb{R}^N)$ (we refer again to § 2.2 for the precise definition).

THEOREM 1.3. *Let Ω be a bounded domain satisfying a (ρ, θ) -Lipschitz cone condition for some values $\rho \in (0, +\infty)$ and $\theta \in (0, \pi/2)$. Assume $f \in L^2(\mathbb{R}^N)$ and let u be the weak solution of (1.1). Then*

$$u \in B_{2,\infty}^{3s/2}(\mathbb{R}^N). \tag{1.14}$$

Moreover, we have the explicit regularity estimate

$$\|u\|_{B_{2,\infty}^{3s/2}(\mathbb{R}^N)} \leq C(N, s, \text{diam } \Omega, \rho, \theta) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2}. \tag{1.15}$$

As a Corollary of the Theorem above (see the embedding 2.20) we obtain the following Sobolev regularity result

COROLLARY 1.1. *Let Ω be a bounded domain satisfying a (ρ, θ) -Lipschitz cone condition for some values $\rho \in (0, +\infty)$ and $\theta \in (0, \pi/2)$. Assume $f \in L^2(\mathbb{R}^N)$ and let u be the weak solution of (1.1). Then*

$$u \in H^{3s/2-\varepsilon}(\mathbb{R}^N). \tag{1.16}$$

Moreover, we have the explicit regularity estimate

$$\|u\|_{H^{3s/2-\varepsilon}(\mathbb{R}^N)} \leq C(N, s, \text{diam } \Omega, \rho, \theta, \varepsilon) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2}. \tag{1.17}$$

We make the following remarks:

- The optimality of (1.15) and of (1.17) is unclear when Ω has only Lipschitz regularity. We refer to the recent papers [12] and [13] where the issue of regularity is investigated for more general non local operators on smooth domains. In § 8.4 we discuss an explicit example in one dimension where the solution has stronger regularity. However, we do not know whether or not this is a general fact.

- The interior regularity in the scale of Nikol'skii spaces has been recently investigated in [6] where it is also proved that weak solutions to (1.1) are in $H^{2s-\varepsilon}(\Omega)$.
- By proceeding as in [21, Corollary 3], one can see that the above result extends to the case when f belongs to the interpolation space $B_{2,1}^{-s/2} = (L^2, H^{-s})_{1/2,1}$. In particular, in that case, estimate (1.15) is replaced by

$$\|u\|_{B_{2,\infty}^{3s/2}(\mathbb{R}^N)} \leq C(N, s, \text{diam}\Omega, \rho, \theta) \|f\|_{B_{2,1}^{-s/2}(\mathbb{R}^N)}. \tag{1.18}$$

We conclude by discussing some new spectral stability estimate for the Poisson problem (1.1). To this aim, we first introduce some notation. We say that (u, λ) is an eigen-couple for the operator $(-\Delta)^s$ in Ω if the eigenfunction $u \in \mathcal{X}_0^s(\Omega)$, $u \neq 0$, the eigenvalue $\lambda \in \mathbb{R}$ and the following holds

$$\begin{cases} (-\Delta)^s u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.19}$$

Owing to classical functional analytic results, the operator $(-\Delta)^s$ admits a diverging sequence of positive eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \nearrow +\infty, \tag{1.20}$$

provided Ω is an open and bounded set. We refer to § 9.1 for a more extended discussion and we point out that here and in the following we count each eigenvalue according to its multiplicity, namely according to the dimension of the associated eigenspace.

By combining Theorem 1.1 with an argument in [16] we establish the stability of the eigenvalues of the operator $(-\Delta)^s$ with respect to domain perturbations.

THEOREM 1.4 (Spectral stability). *Let $\Omega_a, \Omega_b \subset \mathbb{R}^N$ be two open, bounded sets satisfying the following conditions:*

- i) Ω_a and Ω_b both satisfy a (ρ, θ) -Lipschitz cone condition.*
- ii) Ω_a and Ω_b are both contained in some open ball $D \subset \mathbb{R}^N$.*
- iii) There is a ball B_r with radius r such that $B_r \subseteq \Omega_a \cap \Omega_b$.*

Then for every $n \in \mathbb{N}$ there are constants $\nu > 0$ and $C > 0$, which only depend on $N, s, \rho, \theta, \text{diam } D, r$ and n , such that, if

$$d_H^c(\Omega_a, \Omega_b) < \nu, \tag{1.21}$$

then

$$|\lambda_n^a - \lambda_n^b| \leq C d_H^c(\Omega_a, \Omega_b)^s. \tag{1.22}$$

In the previous expression, λ_n^a and λ_n^b denote the n -th eigenvalue of $(-\Delta)^s$ in Ω_a and Ω_b , respectively.

Some remarks are here in order. First, in the statement of the above theorem $d_H^c(\Omega_a, \Omega_b)$ denotes the so-called *complementary Hausdorff distance*, i.e. the Hausdorff distance between the sets $\mathbb{R}^N \setminus \Omega_a$ and $\mathbb{R}^N \setminus \Omega_b$, see § 4.1 for the precise definition. Second, the only reason why we assume hypothesis iii) is because we need an upper bound on $\max\{\lambda_n^a, \lambda_n^b\}$. Indeed, by combining condition iii) with the monotonicity of eigenvalues with respect to set inclusion we obtain an upper bound which only depends

on N , s , r and n . Also, note that as in the case of Theorems 1.1 and 1.2 by following the proof of Theorem 1.4 one can reconstruct the values of the constants ν and C .

REMARK 1.1. As one can infer from the statements of Theorems 1.2 and 1.4, we have the following dichotomy:

- to control the difference between the *eigenvalues*, we need to control the complementary Hausdorff distance $d_H^c(\Omega_a, \Omega_b)$ and we only use property i) in Definition 4.1. However, we have to require that both Ω_a and Ω_b satisfy the Lipschitz regularity condition.
- On the other hand, to control the difference between the *solutions* of the Poisson problem, we need a control on a different type of set distance, namely $\mathfrak{d}(\Omega_b, \Omega_a)$ (cf. (4.9)). This forces us to use both properties i) and ii) in Definition 4.1. On the other hand, we only require Lipschitz regularity of Ω_a (whereas Ω_b can be any domain satisfying (6.4)).

This dichotomy is basically due to the fact that we, a-priori, have some additional information on the behavior of the eigenvalues. Namely, they behave monotonically with respect to domain inclusion, cf. § 9.1, whereas this property is not available for the solutions of the Poisson problem.

We eventually discuss the stability of eigenfunctions for $(-\Delta)^s$ under domain perturbations. We first state the following simple property, which holds for a very large class of domains:

PROPOSITION 1.2. *Fix $s \in (0, 1)$ and let $\Omega \subset \mathbb{R}^N$ be an open bounded set with negligible boundary, namely $\mathcal{L}^N(\partial\Omega) = 0$. Assume that $\{\Omega_j\}_{j \in \mathbb{N}}$ is a sequence of open and bounded sets in \mathbb{R}^N such that*

$$\mathfrak{d}(\Omega_j, \Omega) < \frac{1}{j}. \tag{1.23}$$

Let (w^j, λ^j) be a sequence of eigencouples for the operator $(-\Delta)^s$ on Ω_j such that

$$\|w^j\|_{L^2(\mathbb{R}^N)} = 1. \tag{1.24}$$

Assume furthermore that

$$\lambda^j \rightarrow \lambda \quad \text{as } j \rightarrow +\infty \tag{1.25}$$

for some $\lambda > 0$. Then there are a subsequence $\{j_k\}$ and $u \in \mathcal{X}_0^s(\Omega)$ such that

$$w^{j_k} \rightarrow u \quad \text{strongly in } H^s(\mathbb{R}^N) \tag{1.26}$$

and (u, λ) is an eigencouple for $(-\Delta)^s$ on Ω .

We make the following remarks:

- if the multiplicity of λ is bigger than one, then it might happen that different subsequences converge to different, linearly independent, eigenfunctions associated to λ .
- If the multiplicity of λ is bigger than one, then there is no hope of establishing a convergence rate (see Remark 10.1 in § 10 for a counterexample).
- If the multiplicity of λ is 1 (i.e., if λ is a *simple* eigenvalue), then we can establish quantitative estimates, as the next result shows.

The following result provides an estimate for the convergence rate of (normalized) principal eigenfunctions, which are always simple. In the statement of Theorem 1.5, $\lambda_1(D)$ and $\lambda_1(B_r)$ denote the first eigenvalue of $(-\Delta)^s$ on D and B_r , respectively.

THEOREM 1.5. *Fix $s \in (0, 1)$ and let Ω_a, Ω_b be two bounded, open sets satisfying the conditions i), ii) and iii) in the statement of Theorem 1.4 and*

$$\max\{\mathfrak{d}(\Omega_b, \Omega_a), \mathfrak{d}(\Omega_a, \Omega_b)\} < \frac{\rho \sin \theta}{2}. \tag{1.27}$$

Let λ_1^a, λ_1^b denote the first eigenvalue of the operator $(-\Delta)^s$ in Ω_a and Ω_b , respectively, and let e^a and e^b be the corresponding eigenfunctions satisfying

$$\|e^a\|_{L^2(\mathbb{R}^N)} = \|e^b\|_{L^2(\mathbb{R}^N)} = 1, \quad \int_{\mathbb{R}^N} (-\Delta)^{s/2} e^a(x) (-\Delta)^{s/2} e^b(x) dx \geq 0. \tag{1.28}$$

Define δ by setting

$$\delta := \frac{1}{2} \left(\frac{1}{\lambda_1^a} - \frac{1}{\lambda_2^a} \right) > 0.$$

Then there is a positive constant $\nu > 0$, which only depends on $N, s, \rho, \theta, \text{diam} D, r, \lambda_1(D)$ and δ , such that the following holds:

$$\left\| \frac{e^a}{\sqrt{\lambda_1^a}} - \frac{e^b}{\sqrt{\lambda_1^b}} \right\|_s \leq \frac{C}{\sqrt{\lambda_1(D)}} \max\{\delta^{-1}, \lambda_1(B_r)\} \min\{\mathfrak{d}(\Omega_a, \Omega_b), \mathfrak{d}(\Omega_b, \Omega_a)\}^{s/2}, \tag{1.29}$$

for some constant $C = C(N, s, \rho, \theta, \text{diam} D) \geq 0$, provided that $d_H^c(\Omega_a, \Omega_b) \leq \nu$.

The proof of Theorem 1.5 mainly relies on the abstract theory developed by Feleqi [10]. Also, note that one can also obtain similar results for eigenfunctions associated to other simple eigenvalues. Moreover, in the case of non-simple eigenvalues, we can control a suitable notion of “distance between eigenspaces”. We refer the reader to Theorem 10.1 in § 10 for more details.

We conclude the introduction by outlining the plan of the paper: in the next section we introduce some functional-analytic background. In § 3, we establish the main estimates on localized finite differences of solutions. In § 4 we discuss the regularity conditions on domains and state some related geometrical properties. In § 5 we apply these results to control the difference of two solutions to (1.1) supported in different domains. In § 6 we establish some domain perturbation estimates that constitute a weaker version of Theorem 1.2. These estimates are improved in the subsequent § 7 and § 8 by means of a bootstrap argument. In this way we complete the proofs of Theorem 1.1 and of Theorem 1.2. Next, in § 9 we discuss the behavior of eigenvalues under domain perturbations and establish Theorem 1.4; finally, in § 10 we establish Theorem 1.5 on the behavior of eigenfunctions.

1.1. Notation. For the reader’s convenience, we collect the main notation used in the sequel. In the rest of the paper, we denote by $C(a_1, \dots, a_k)$ a (generic) constant that may only depend on the quantities a_1, \dots, a_k . Its precise value may vary on occurrence. Also, we use the following notation:

- $\mathcal{L}^N(E)$: the Lebesgue measure of the (measurable) set $E \subseteq \mathbb{R}^N$.
- a.e. x : \mathcal{L}^N –almost every x .

- $x \cdot y$: the Euclidean scalar product between the vectors $x, y \in \mathbb{R}^N$.
- $|x|$: the Euclidean norm of $x \in \mathbb{R}^N$.
- $B_r(x)$: the open ball of radius r centered at x in \mathbb{R}^N .
- $d(x, E)$: the distance from the point $x \in \mathbb{R}^N$ to the set $E \subseteq \mathbb{R}^N$, namely

$$d(x, E) := \inf_{y \in E} |x - y|.$$

- $e(E, F)$: the *excess* of the set $E \subseteq \mathbb{R}^N$ with respect to the set $F \subseteq \mathbb{R}^N$, i.e.,

$$e(E, F) := \sup_{y \in E} d(y, F).$$

- $d_H(E, F)$: the *Hausdorff distance* between the sets $E, F \subseteq \mathbb{R}^N$, given by

$$d_H(E, F) = e(E, F) + e(F, E).$$

- $d_H^c(E, F)$: the *complementary Hausdorff distance* defined by formula (4.4) below.
- $\mathfrak{d}(E, F)$: the distance defined by formula (4.9) below. Roughly speaking, it provides a measure of the excess of the boundary ∂E with respect to the boundary ∂F .
- $H^s(\mathbb{R}^N)$: the fractional Sobolev space $W^{s,2}$. We usually focus on the case $s \in (0, 1)$.
- $H^{-s}(\mathbb{R}^N)$: the dual space of $H^s(\mathbb{R}^N)$.
- (\cdot, \cdot) : the standard scalar product in $L^2(\mathbb{R}^N)$, namely

$$(u, v) := \int_{\mathbb{R}^N} u(x)v(x) \, dx.$$

- $\mathcal{X}_0^s(\Omega)$: the functional space

$$\mathcal{X}_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) : u(x) = 0 \text{ for a.e. } x \in \mathbb{R}^N \setminus \Omega\}.$$

Here $\Omega \subseteq \mathbb{R}^N$ is a given open set.

- $B_{2,\infty}^r(\mathbb{R}^N)$: the Besov space defined as in § 2.2.
- $[\cdot, \cdot]_s$: the bilinear form

$$[u, v]_s := ((-\Delta)^{s/2}u, (-\Delta)^{s/2}v) = \int_{\mathbb{R}^N} [(-\Delta)^{s/2}u](x)[(-\Delta)^{s/2}v](x) \, dx,$$

which is defined on $H^s(\mathbb{R}^N)$ and is a scalar product on $\mathcal{X}_0^s(\Omega)$ if Ω is bounded.

- $\|\cdot\|_s$: the Gagliardo semi-norm defined as in (2.6).
- $\mathcal{X}_0^s(\Omega)'$: the dual space of $\mathcal{X}_0^s(\Omega)$.
- $\langle \cdot, \cdot \rangle$: the duality product between $\mathcal{X}_0^s(\Omega)'$ and $\mathcal{X}_0^s(\Omega)$.
- $u_h, T_h u$: the functions defined as in (1.3).
- \hat{v} or $\mathfrak{F}v$: the Fourier transform of the function v (whenever it makes sense).
- $C_c^\infty(\Omega)$: the set of smooth, compactly supported functions, defined on the set Ω .

2. Functional analytic background material: fractional Laplacian and Besov spaces

For the reader’s convenience, in this section we discuss some functional analytic results that are pivotal to our analysis. In particular, in § 2.1 we provide the rigorous formulation of the Poisson problem (1.1). In § 2.2 we introduce the definition and some important property of a particular class of Besov spaces.

2.1. The Poisson problem for the fractional Laplace operator.

2.1.1. The fractional Laplace operator. Given $s \in (0,1)$ and u in the Schwartz class \mathcal{S} of the rapidly decaying functions at infinity, $(-\Delta)^s u$ is defined as

$$(-\Delta)^s u(x) := C(N,s) \text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \tag{2.1}$$

where the notation p.v. means that the integral is taken in the *Cauchy principal value* sense, namely

$$\text{p.v.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy \tag{2.2}$$

and in (2.1) $C(N,s)$ is the normalization constant (cf., e.g., [7])

$$\left(\int_{\mathbb{R}^N} \frac{1 - \cos(x_1)}{|x|^{N+2s}} dx \right)^{-1}.$$

For any $s \in (0,1)$ and any $x, y \in \mathbb{R}^N$ we will also use the shorthand notation $K_s(x - y) = |x - y|^{-N-2s}$ to denote the singular kernel in (2.1). The operator $(-\Delta)^s$ can be equivalently introduced by means of the Fourier transform, which we define for general function $v \in \mathcal{S}$ as follows:

$$\mathfrak{F}v(\xi) = \hat{v}(\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-ix \cdot \xi} u(x) dx.$$

Moreover, \mathfrak{F}^{-1} stands for the inverse transform of \mathfrak{F} . As usual, the above definition can be extended to tempered distributions.

We can then introduce $(-\Delta)^s$ as the pseudo-differential operator with symbol $|\xi|^{2s}$, namely

$$(-\Delta)^s v = \mathfrak{F}^{-1}(|\xi|^{2s} \mathfrak{F}(v)), \quad \forall v \in \mathcal{S}. \tag{2.3}$$

In particular, when $v \in H^s(\mathbb{R}^N)$, then $(-\Delta)^{s/2} v \in L^2(\mathbb{R}^N)$ owing to the above characterization.

2.1.2. Functional framework. Even if the equations we are going to study are settled only in Ω , the behavior of $(-\Delta)^s u$ depends on the interplay between the values of u inside and outside Ω . This is related to the non locality of $(-\Delta)^s$, which implies that, even when u has compact support, $(-\Delta)^s u$ does not necessarily have the same property. For this reason, when we consider a solution u to (1.1), u will be always thought as a function defined on the whole space \mathbb{R}^N that identically vanishes outside Ω . In particular, the global regularity of u will be influenced by this fact (cf. Theorem 1.3 and the examples discussed in the Introduction).

We now proceed along the lines of [22] (see also [1] and [18]). Given $s \in (0,1)$, we introduce the Sobolev type spaces

$$\mathcal{X}_0^s(\Omega) := \{v \in H^s(\mathbb{R}^N) \text{ such that } v=0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}. \tag{2.4}$$

In particular, for $s \in (0,1)$ the extension operator of $u \in \mathcal{X}_0^s(\Omega)$ to 0 outside Ω is a continuous mapping of $H^s(\Omega) \rightarrow H^s(\mathbb{R}^N)$. Then (see [17, Theorem 11.4, Chapter 1]), if $s \in (1/2,1)$, the functions in $\mathcal{X}_0^s(\Omega)$ are equal to zero in the sense of traces on $\partial\Omega$. Hence, $\mathcal{X}_0^s(\Omega)$ can be identified with $H_0^s(\Omega)$ in that case, whereas $\mathcal{X}_0^s \sim H_0^s(\Omega) = H^s(\Omega)$ for $s \in (0,1/2)$. In the limit case $s=1/2$, it turns out that $\mathcal{X}_0^{1/2}(\Omega) \sim H_{00}^{1/2}(\Omega)$ (again, see [17] for more details).

We denote by (\cdot, \cdot) the scalar product in $L^2(\mathbb{R}^N)$, by $\|\cdot\|_{L^2(\mathbb{R}^N)}$ the induced norm and we endow the Hilbert space $\mathcal{X}_0^s(\Omega)$ with the norm

$$\|v\|_{\mathcal{X}_0^s(\Omega)}^2 := \|v\|_{L^2(\mathbb{R}^N)}^2 + \|v\|_s^2. \tag{2.5}$$

Here, $\|\cdot\|_s$ denotes the so-called *Gagliardo-seminorm*

$$\|v\|_s^2 := \iint_{\mathbb{R}^{2N}} K_s(x-y)|v(x)-v(y)|^2 dx dy, \tag{2.6}$$

which is well defined for $v \in H^s(\mathbb{R}^N)$. We also recall the fractional Poincaré inequality

$$\|v\|_{L^2(\mathbb{R}^N)} \leq C_P(\Omega, s)\|v\|_s, \text{ for every } v \in \mathcal{X}_0^s(\Omega), \tag{2.7}$$

where the constant C_P depends, in principle, on Ω and on s . Note that, since the set Ω is bounded, its diameter (which is defined by (1.9)) is finite and, also, Ω is contained in a suitable ball D with radius equal to $\text{diam}\Omega$. In view of the fact that (2.4) implies $\mathcal{X}_0^s(\Omega) \subseteq \mathcal{X}_0^s(D)$, it follows $C_P(\Omega, s) \leq C_P(D, s)$ and, consequently, we can choose the constant C_P in (2.7) depending only on N, s and $\text{diam}\Omega$. Namely, we have

$$\|v\|_{L^2(\mathbb{R}^N)} \leq C(N, s, \text{diam}\Omega)\|v\|_s, \text{ for every } v \in \mathcal{X}_0^s(\Omega). \tag{2.8}$$

As a consequence of (2.8), the Gagliardo seminorm is actually an equivalent norm on $\mathcal{X}_0^s(\Omega)$. Hence, we will generally use $\|\cdot\|_s$ in place of $\|\cdot\|_{\mathcal{X}_0^s(\Omega)}$. Because of this, it is also important to express the Gagliardo-seminorm by using the Fourier transform. We have that (cf. [7, Proposition 3.4 & Proposition 3.6]):

$$C(N, s)\|v\|_s^2 = \|(-\Delta)^{s/2}v\|_{L^2(\mathbb{R}^N)}^2 = \|\left|\xi\right|^s \hat{v}\|_{L^2(\mathbb{R}^N)}^2 \text{ for } v \in H^s(\mathbb{R}^N) \text{ and } s \in (0,1). \tag{2.9}$$

In the following, we will also use the notation

$$[u, v]_s := \int_{\mathbb{R}^N} [(-\Delta)^{s/2}u](x)[(-\Delta)^{s/2}v](x) dx = ((-\Delta)^{s/2}u, (-\Delta)^{s/2}v). \tag{2.10}$$

Note that, owing to (2.8) and (2.9), the above bilinear form is actually a scalar product on $\mathcal{X}_0^s(\Omega)$.

In view of the fact that we will deal with domain variations, it will be generally convenient to view the elements of $\mathcal{X}_0^s(\Omega)$ as functions defined on the whole space \mathbb{R}^N that vanish a.e. outside Ω . In particular, we can continuously embed $\mathcal{X}_0^s(\Omega)$ into $L^2(\mathbb{R}^N)$. This embedding is not dense, since the closure of $\mathcal{X}_0^s(\Omega)$ in $L^2(\mathbb{R}^N)$ coincides with the subspace H_0 of $L^2(\mathbb{R}^N)$ containing those functions that vanish a.e. outside Ω .

In particular, if we denote by $H^{-s}(\mathbb{R}^N)$ the dual space of $H^s(\mathbb{R}^N)$, we have the chain of embeddings

$$L^2(\mathbb{R}^N) \hookrightarrow H^{-s}(\mathbb{R}^N) \hookrightarrow \mathcal{X}_0^s(\Omega)'$$

and both the above embeddings are continuous, namely

$$\|f\|_{\mathcal{X}_0^s(\Omega)'} \leq \|f\|_{H^{-s}(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)} \quad \text{for every } f \in L^2(\mathbb{R}^N). \tag{2.11}$$

2.1.3. The Poisson problem for the fractional Laplacian. With the above functional framework at our disposal, we can make precise the notion of weak solution.

Given $f \in \mathcal{X}_0^s(\Omega)'$, we say that $u : \mathbb{R}^N \rightarrow \mathbb{R}$ is a weak solution to (1.1) if

$$\begin{cases} u \in \mathcal{X}_0^s(\Omega), \\ C(N, s) \iint_{\mathbb{R}^{2N}} K_s(x-y)(u(x) - u(y))(\varphi(x) - \varphi(y)) dx dy = \langle f, \varphi \rangle, \forall \varphi \in \mathcal{X}_0^s(\Omega). \end{cases} \tag{2.12}$$

It is worth noting that (2.12) may be equivalently reformulated as

$$\begin{cases} u \in \mathcal{X}_0^s(\Omega), \\ ((-\Delta)^{s/2} u, (-\Delta)^{s/2} \varphi) = \langle f, \varphi \rangle, \forall \varphi \in \mathcal{X}_0^s(\Omega). \end{cases} \tag{2.13}$$

In what follows, when we speak of a weak (or variational) solution u to (1.1), we will mean a function $u \in \mathcal{X}_0^s(\Omega)$ satisfying (2.12) or, equivalently, (2.13).

By using the Lax-Milgram Lemma one can show that, if $f \in \mathcal{X}_0^s(\Omega)'$ (and hence, in particular, if $f \in L^2(\mathbb{R}^N)$), then there is a unique solution $u \in \mathcal{X}_0^s(\Omega)$. In particular, we have the stability estimate

$$\|u\|_s \leq \|u\|_{\mathcal{X}_0^s(\Omega)} \leq C(N, s, \text{diam } \Omega) \|f\|_{\mathcal{X}_0^s(\Omega)'}. \tag{2.14}$$

2.2. Besov spaces and interpolation. In this paragraph we recall the definition and some properties of the Besov space $B_{2,\infty}^r(\mathbb{R}^N)$. We refer to the books by Triebel [24] and by Stein [23, §V.5] for extended discussions.

If $r \in (0, 1]$, then

$$B_{2,\infty}^r(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u_{2h} - 2u_h + u\|_{L^2(\mathbb{R}^N)}}{|h|^r} < +\infty \right\}, \tag{2.15}$$

where the function u_h is defined as in (1.3) and $u_{2h}(x) := u(x + 2h)$. The space $B_{2,\infty}^r(\mathbb{R}^N)$ is a Banach space with norm

$$\|u\|_{B_{2,\infty}^r(\mathbb{R}^N)} := \|u\|_{L^2(\mathbb{R}^N)} + \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u_{2h} - 2u_h + u\|_{L^2(\mathbb{R}^N)}}{|h|^r}. \tag{2.16}$$

Note that, if $r \in (0, 1)$, then $u \in B_{2,\infty}^r(\mathbb{R}^N)$ if and only if

$$\|u\|_{L^2(\mathbb{R}^N)} + \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u_h - u\|_{L^2(\mathbb{R}^N)}}{|h|^r} < +\infty$$

and, also, the expression on the left hand side is equivalent to the norm $\|\cdot\|_{B_{2,\infty}^r(\mathbb{R}^N)}$, namely

$$\sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u_h - u\|_{L^2(\mathbb{R}^N)}}{|h|^r} \leq C(N, r) \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u_{2h} - 2u_h + u\|_{L^2(\mathbb{R}^N)}}{|h|^r}, \tag{2.17}$$

$$\sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u_{2h} - 2u_h + u\|_{L^2(\mathbb{R}^N)}}{|h|^r} \leq C(N, r) \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u_h - u\|_{L^2(\mathbb{R}^N)}}{|h|^r}. \tag{2.18}$$

If $r > 1$, then

$$B_{2,\infty}^r(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in B_{2,\infty}^{r-1}(\mathbb{R}^N) \text{ for every } i = 1, \dots, N \right\}.$$

and

$$\|u\|_{B_{2,\infty}^r(\mathbb{R}^N)} := \|u\|_{L^2(\mathbb{R}^N)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{B_{2,\infty}^{r-1}(\mathbb{R}^N)}. \tag{2.19}$$

In the above expressions, $\partial u / \partial x_i$ denote the distributional partial derivatives of u . We now recall some properties of $B_{2,\infty}^r(\mathbb{R}^N)$ that will be used in the following. First,

$$B_{2,\infty}^r(\mathbb{R}^N) \subset H^{r-\varepsilon}(\mathbb{R}^N), \quad \text{for every } r \in (0, +\infty), \varepsilon \in (0, r) \tag{2.20}$$

and

$$H^r(\mathbb{R}^N) \subset B_{2,\infty}^r(\mathbb{R}^N), \quad \text{for every } r \in (0, +\infty). \tag{2.21}$$

Also, all the above inclusions are continuous. Second, we follow [23, p. 131] and we consider the equation

$$(I - \Delta)^s u = f \quad \text{in } \mathbb{R}^N, \tag{2.22}$$

where I denotes the identity operator and as usual $s \in (0, 1)$. A rigorous formulation of (2.22) can be provided by using the so-called *Bessel potential*. For our purposes here it is enough to say that, if $f \in L^2(\mathbb{R}^N)$, then $u \in L^2(\mathbb{R}^N)$ satisfies (2.22) if and only if

$$(1 + |\xi|^2)^s \hat{u}(\xi) = \hat{f}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^N. \tag{2.23}$$

As a particular case of [23, Theorem 4', p. 153] we obtain the following

LEMMA 2.1. *Assume that $f \in B_{2,\infty}^r(\mathbb{R}^N)$ and that u satisfies (2.22). Then $u \in B_{2,\infty}^{r+2s}(\mathbb{R}^N)$ and*

$$\|u\|_{B_{2,\infty}^{r+2s}(\mathbb{R}^N)} \leq C(N, s, r) \|f\|_{B_{2,\infty}^r(\mathbb{R}^N)}. \tag{2.24}$$

3. Estimates on localized finite differences

This section aims at establishing estimate (1.7) and Lemma 3.1 below.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set, and let $f \in L^2(\mathbb{R}^N)$. Assume that u and ϕ satisfy the same assumptions as in the statement of Theorem 1.1. Then for every $\sigma \in (0, 1)$ we have*

$$\|T_h u - u\|_s^2 \leq C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) |h|^{\sigma s} \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{\mathcal{X}_0^s(\Omega)} \tag{3.1}$$

Note that in (3.1) the constant C depends on σ and can in principle deteriorate when $\sigma \rightarrow 1^-$. For this reason (3.1) can be regarded as a weaker version of (1.8).

3.1. Proof of the estimate (1.7). We first recall that $0 \leq \phi \leq 1$ owing to (1.4) and we point out that

$$\|T_h u - u\|_{L^2(\mathbb{R}^N)} = \|\phi(u_h - u)\|_{L^2(\mathbb{R}^N)} \leq \|\phi\|_{L^\infty(\mathbb{R}^N)} \|u_h - u\|_{L^2(\mathbb{R}^N)} \leq \|u_h - u\|_{L^2(\mathbb{R}^N)}. \tag{3.2}$$

Hence, to establish (1.7), it is sufficient to control $\|u_h - u\|_{L^2(\mathbb{R}^N)}$. We recall that $\hat{u}_h = e^{i\xi \cdot h} \hat{u}$. Then, the Plancherel theorem and (2.14) give

$$\begin{aligned} \|u_h - u\|_{L^2(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} |e^{i\xi \cdot h} - 1|^{2s} |e^{i\xi \cdot h} - 1|^{2-2s} |\hat{u}(\xi)|^2 d\xi \\ &\stackrel{(3.4), (3.5)}{\leq} 8|h|^{2s} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi \stackrel{(2.9)}{=} C(N, s) |h|^{2s} \|u\|_s^2 \\ &\stackrel{(2.14)}{\leq} C(N, s, \text{diam}\Omega) |h|^{2s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \end{aligned} \tag{3.3}$$

In the previous formula we have used the following elementary inequalities: first, since $s \in (0, 1)$, we have

$$|e^{i\xi \cdot h} - 1|^{2-2s} \leq (|e^{i\xi \cdot h}| + |1|)^{2-2s} \leq 2^{2-2s} \leq 4. \tag{3.4}$$

Second, a direct check shows that

$$|e^{i\xi \cdot h} - 1| = |\cos(\xi \cdot h) - 1 + i \sin(\xi \cdot h)| \leq 2|\xi \cdot h| \leq 2|\xi||h|. \tag{3.5}$$

3.2. Preliminary results.

LEMMA 3.2. *If ϕ is a cut-off function satisfying (1.4) and $s \in (0, 1)$, then*

$$\|(-\Delta)^{s/2} \phi\|_{L^\infty(\mathbb{R}^N)} \leq C(N, s, \text{Lip}\phi). \tag{3.6}$$

Proof. We first observe that, since $s/2 \in (0, 1/2)$, then

$$(-\Delta)^{s/2} \phi(x) = C(N, s) \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N+s}} dy \tag{3.7}$$

because the above integral converges. Now, notice that

$$\begin{aligned} |(-\Delta)^{s/2} \phi(x)| &\leq C(N, s) \int_{\mathbb{R}^N} \frac{|\phi(x) - \phi(y)|}{|x - y|^{N+s}} dy \\ &= C(N, s) \int_{|x-y| \leq 1} \frac{|\phi(x) - \phi(y)|}{|x - y|^{N+s}} dy + C(N, s) \int_{|x-y| > 1} \frac{|\phi(x) - \phi(y)|}{|x - y|^{N+s}} dy \\ &\leq C(N, s, \text{Lip}\phi) \int_{|x-y| \leq 1} \frac{1}{|x - y|^{N-1+s}} dy + C(N, s) \int_{|x-y| > 1} \frac{1}{|x - y|^{N+s}} dy \\ &= C(N, s, \text{Lip}\phi) \int_0^1 \frac{1}{\rho^s} d\rho + C(N, s) \int_1^{+\infty} \frac{1}{\rho^{1+s}} d\rho = C(N, s, \text{Lip}\phi), \end{aligned} \tag{3.8}$$

which establishes (3.6). □

LEMMA 3.3. *If $u \in \mathcal{X}_0^s(\Omega)$ and ϕ is a cut-off function satisfying (1.4), then the product $u\phi \in \mathcal{X}_0^s(\Omega)$.*

Proof. First, we recall Definition (2.4) of $\mathcal{X}_0^s(\Omega)$ and we infer that $\phi(x)u(x) = 0$ for a.e. $x \in \mathbb{R}^N \setminus \Omega$. Hence, to prove that $\phi u \in \mathcal{X}_0^s(\Omega)$ we are left to show that $\phi u \in H^s(\mathbb{R}^N)$.

To this aim, we first observe that, since $0 \leq \phi \leq 1$ and $u \in L^2(\mathbb{R}^N)$, then $\phi u \in L^2(\mathbb{R}^N)$. Hence, it remains to prove that $\|\phi u\|_s < +\infty$. Owing to (2.9), this is equivalent to showing

$$(-\Delta)^{s/2}(\phi u) \in L^2(\mathbb{R}^N). \tag{3.9}$$

The above relation follows from general results related to the so-called *Kato-Ponce* inequality. For instance, we can apply [14, formula (2)] with the choices $f = \phi$, $g = u$, $p_1 = p_2 = \infty$, $q_1 = q_2 = r = 2$, and with $s/2$ in place of s . By recalling (3.6), we then obtain (3.9). Note that, strictly speaking, [14, formula (2)] is only stated for smooth functions in the Schwartz class, but by relying on a standard density argument one can extend it to the (fractional) Sobolev setting. \square

LEMMA 3.4. *Assume that $w \in H^s(\mathbb{R}^N)$ and that ϕ is a cut-off function satisfying (1.4) and (1.5). Let $\mathcal{C}(\phi, w)$ be the commutator defined by setting*

$$\mathcal{C}(\phi, w) := (-\Delta)^{s/2}(\phi w) - w(-\Delta)^{s/2}\phi - \phi(-\Delta)^{s/2}w. \tag{3.10}$$

For every $\sigma \in (0, 1)$, there is a constant $C(N, s, \text{Lip}\phi, \sigma)$ such that

$$\|\mathcal{C}(\phi, w)\|_{L^2(\mathbb{R}^N)} \leq C(N, s, \text{Lip}\phi, \sigma) \|w\|_{L^2(\mathbb{R}^N)}^\sigma \|w\|_s^{1-\sigma}. \tag{3.11}$$

Proof. First we point out that, if $N \leq 2s$, we can choose $s' < s$ such that $N > 2s'$. Now, owing to [7, Proposition 2.1], if $w \in \mathcal{X}_0^s(\Omega)$, then $w \in \mathcal{X}_0^{s'}(\Omega)$ and

$$\|w\|_{s'} \leq C\|w\|_s.$$

Hence, it is enough to establish (3.11) for $w \in \mathcal{X}_0^{s'}(\Omega)$. For this reason we will always assume in the following, with no loss of generality, that $N > 2s$.

We recall the fractional Sobolev embedding inequality, and we refer to [7, Theorem 6.5] for an extended discussion. If $p^* = 2N/(N - 2s)$ and $w \in H^s(\mathbb{R}^N)$, then $w \in L^{p^*}(\mathbb{R}^N)$ and

$$\|w\|_{L^{p^*}(\mathbb{R}^N)} \leq C(N, s)\|w\|_s. \tag{3.12}$$

Next, we fix $\sigma \in (0, 1)$ and choose $r \in (2, p^*)$ in such a way that

$$\frac{1}{r} = \frac{\sigma}{2} + \frac{1-\sigma}{p^*} = C(N, s, \sigma). \tag{3.13}$$

We also recall the elementary interpolation inequality

$$\|w\|_{L^r(\mathbb{R}^N)} \leq \left(\|w\|_{L^2(\mathbb{R}^N)}\right)^\sigma \left(\|w\|_{L^{p^*}(\mathbb{R}^N)}\right)^{1-\sigma}. \tag{3.14}$$

Finally, we use Theorem A.8 in the paper by Kenig, Ponce and Vega [15], which states that

$$\|\mathcal{C}(\phi, w)\|_{L^2(\mathbb{R}^N)} \leq C(N, s, p_1, p_2) \|(-\Delta)^{s_1/2}\phi\|_{L^{p_1}(\mathbb{R}^N)} \|(-\Delta)^{s_2/2}w\|_{L^{p_2}(\mathbb{R}^N)} \tag{3.15}$$

provided that $s_1, s_2 \in [0, s]$, $s = s_1 + s_2$ and $p_1, p_2 \in (1, +\infty)$ satisfy

$$\frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2}.$$

We apply (3.15) with $s_1 = s$, $s_2 = 0$ and $p_2 = r$ and combine the result with (3.12) and (3.14). We infer

$$\|C(\phi, w)\|_{L^2(\mathbb{R}^N)} \leq C(N, s, \sigma) \|(-\Delta)^{s/2} \phi\|_{L^{p_1}(\mathbb{R}^N)} \left(\|w\|_{L^2(\mathbb{R}^N)} \right)^\sigma \left(\|w\|_{L^{p^*}(\mathbb{R}^N)} \right)^{1-\sigma}, \tag{3.16}$$

provided that

$$\frac{1}{p_1} = \frac{1}{2} - \frac{1}{r} = (1 - \sigma) \left(\frac{1}{2} - \frac{1}{p^*} \right) = (1 - \sigma) \frac{s}{N}. \tag{3.17}$$

To obtain the previous expression, we have used the explicit expression (3.13) of r . Also, note that $p_1 \in (2, +\infty)$ since $N > 2s$. To control the term $\|(-\Delta)^{s/2} \phi\|_{L^{p_1}(\mathbb{R}^N)}$ we use an argument similar to (but easier than) the one that gives the proof of [4, Lemma 1]. Namely, we write

$$\|(-\Delta)^{s/2} \phi\|_{L^{p_1}(\mathbb{R}^N)}^{p_1} = \int_{|x| \leq 2} |(-\Delta)^{s/2} \phi(x)|^{p_1} dx + \int_{|x| > 2} |(-\Delta)^{s/2} \phi(x)|^{p_1} dx =: J_1 + J_2. \tag{3.18}$$

To control J_1 , we simply recall Lemma 3.2 and we obtain

$$J_1 \leq \|(-\Delta)^{s/2} \phi\|_{L^\infty(\mathbb{R}^N)}^{p_1} 2^N \omega_N \leq C(N, s, \text{Lip } \phi, p_1) = C(N, s, \text{Lip } \phi, \sigma). \tag{3.19}$$

Here, ω_N denotes the measure of the unit ball in \mathbb{R}^N and we have used the explicit expression of p_1 , namely (3.17), to establish the last equality.

To control J_2 , we first recall the equality (3.7) and the fact that $\text{supp } \phi \subseteq B_1(0)$, owing to (1.5). This implies that, if $|x| > 2$, then

$$\begin{aligned} |(-\Delta)^{s/2} \phi(x)| &= C(N, s) \left| \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N+s}} dy \right| = C(N, s) \int_{\mathbb{R}^N} \frac{\phi(y)}{|x - y|^{N+s}} dy \\ &= C(N, s) \int_{B_1(0)} \frac{\phi(y)}{|x - y|^{N+s}} dy \leq C(N, s) \int_{B_1(0)} \frac{1}{|x - y|^{N+s}} dy \\ &\leq C(N, s) \frac{1}{|x|^{N+s}}. \end{aligned} \tag{3.20}$$

To establish the last inequality we have used the fact that, if $|x| > 2$ and $|y| \leq 1$, then

$$|x - y| \geq |x| - |y| \geq \frac{|x|}{2}.$$

Using (3.20) and recalling that p_1 is given by (3.17), we obtain

$$J_2 = \int_{|x| > 2} |(-\Delta)^{s/2} \phi(x)|^{p_1} dx \leq C(N, s) \int_2^{+\infty} \frac{1}{\rho^{N(p_1-1)+1+sp_1}} d\rho = C(N, s, \sigma). \tag{3.21}$$

Note that the above integral converges since $p_1 > 2$ owing to (3.17). By combining (3.16), (3.18), (3.19) and (3.21) we eventually establish (3.11). \square

3.3. Proof of Lemma 3.1. First of all, we decompose $\|T_h u - u\|_s$ as

$$\|T_h u - u\|_s^2 = \underbrace{\|T_h u\|_s^2 - \|u\|_s^2}_A - \underbrace{C(N, s) \left((-\Delta)^{s/2} u, (-\Delta)^{s/2} [(T_h u) - u] \right)}_B. \tag{3.22}$$

We now separately control the terms A and B by proceeding according to the following steps.

STEP 1: we control the term B . By combining (1.6) with Lemma 3.3 we infer that

$$v = T_h u - u = \phi(u_h - u) \in \mathcal{X}_0^s(\Omega)$$

and hence we can use it as a test function in (2.13). Consequently, by using (1.7) we obtain

$$\begin{aligned} |B| &= \left| \left\langle (-\Delta)^{s/2} u, (-\Delta)^{s/2} [(T_h u) - u] \right\rangle \right| \stackrel{(2.13)}{=} |\langle f, (T_h u) - u \rangle| \leq \|f\|_{L^2(\mathbb{R}^N)} \|(T_h u) - u\|_{L^2(\mathbb{R}^N)} \\ &\stackrel{(1.7)}{\leq} C(N, s, \text{diam} \Omega) |h|^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{\mathcal{X}_0^s(\Omega)}. \end{aligned} \quad (3.23)$$

STEP 2: we rewrite the term A in (3.22) in a more convenient form. We observe that

$$\begin{aligned} \|T_h u\|_s^2 &= C(N, s) \|(-\Delta)^{s/2} (T_h u)\|_{L^2(\mathbb{R}^N)}^2 \\ &= C(N, s) \|(-\Delta)^{s/2} [\phi u_h] + (-\Delta)^{s/2} [(1 - \phi)u]\|_{L^2(\mathbb{R}^N)}^2 \\ &= C(N, s) \|(-\Delta)^{s/2} u + (-\Delta)^{s/2} [\phi(u_h - u)]\|_{L^2(\mathbb{R}^N)}^2 \\ &= C(N, s) \|(-\Delta)^{s/2} u + \phi(-\Delta)^{s/2} [u_h - u] + (u_h - u)(-\Delta)^{s/2} \phi + \mathcal{C}(\phi, u_h - u)\|_{L^2(\mathbb{R}^N)}^2 \\ &= C(N, s) \|T_h((-\Delta)^{s/2} u) + (u_h - u)(-\Delta)^{s/2} \phi + \mathcal{C}(\phi, u_h - u)\|_{L^2(\mathbb{R}^N)}^2. \end{aligned} \quad (3.24)$$

Here we used the fact that $(-\Delta)^{s/2} u_h = [(-\Delta)^{s/2} u]_h$ to ensure the last equality. Indeed, this equation holds true for every $u \in H^s(\mathbb{R}^N)$, since the definition of the fractional Laplacian in terms of the Fourier transform implies

$$\mathfrak{F}((-\Delta)^{s/2} u_h) = |\xi|^s \widehat{u_h} = |\xi|^s e^{ih \cdot \xi} \widehat{u} = e^{ih \cdot \xi} \mathfrak{F}((-\Delta)^{s/2} u) = \mathfrak{F}([(-\Delta)^{s/2} u]_h)$$

for $u \in H^s(\mathbb{R}^N)$. We have

$$\begin{aligned} A &= \|T_h u\|_s^2 - \|u\|_s^2 \\ &= C(N, s) \|T_h((-\Delta)^{s/2} u) + (u_h - u)(-\Delta)^{s/2} \phi + \mathcal{C}(\phi, u_h - u)\|_{L^2(\mathbb{R}^N)}^2 - \|u\|_s^2 \\ &= C(N, s) \left[\|T_h((-\Delta)^{s/2} u) + (u_h - u)(-\Delta)^{s/2} \phi\|_{L^2(\mathbb{R}^N)}^2 - \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2 \right. \\ &\quad \left. + \|\mathcal{C}(\phi, u_h - u)\|_{L^2(\mathbb{R}^N)}^2 + 2 \left(T_h((-\Delta)^{s/2} u) + (u_h - u)(-\Delta)^{s/2} \phi, \mathcal{C}(\phi, u_h - u) \right) \right] \\ &= C(N, s) [I_1 + I_2 + I_3], \end{aligned} \quad (3.25)$$

provided that

$$I_1 := \|T_h((-\Delta)^{s/2} u) + (u_h - u)(-\Delta)^{s/2} \phi\|_{L^2(\mathbb{R}^N)}^2 - \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2 \quad (3.26)$$

$$I_2 := \|\mathcal{C}(\phi, u_h - u)\|_{L^2(\mathbb{R}^N)}^2 \quad (3.27)$$

$$I_3 := 2 \left(T_h((-\Delta)^{s/2} u) + (u_h - u)(-\Delta)^{s/2} \phi, \mathcal{C}(\phi, u_h - u) \right). \quad (3.28)$$

STEP 3: we control the term I_1 . First, we rewrite it as

$$I_1 = \underbrace{\|T_h((-\Delta)^{s/2} u) + (u_h - u)(-\Delta)^{s/2} \phi\|_{L^2(\mathbb{R}^N)}^2 - \|T_h((-\Delta)^{s/2} u)\|_{L^2(\mathbb{R}^N)}^2}_{I_{1,1}}$$

$$+ \underbrace{\|T_h((-\Delta)^{s/2}u)\|_{L^2(\mathbb{R}^N)}^2 - \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)}^2}_{I_{1,2}}. \tag{3.29}$$

To control $I_{1,1}$, we use the elementary identity $|a + b|^2 - |a|^2 = b \cdot (b + 2a)$, which gives

$$\begin{aligned} I_{1,1} &= \int_{\mathbb{R}^N} \left[\left| T_h((-\Delta)^{s/2}u)(x) + (u_h(x) - u(x))((-\Delta)^{s/2}\phi)(x) \right|^2 - |T_h((-\Delta)^{s/2}u)(x)|^2 \right] dx \\ &= \int_{\mathbb{R}^N} \left[u_h(x) - u(x) \right] ((-\Delta)^{s/2}\phi)(x) \\ &\quad \cdot \left[\left[u_h(x) - u(x) \right] ((-\Delta)^{s/2}\phi)(x) + 2T_h((-\Delta)^{s/2}u)(x) \right] dx \\ &\leq \|(-\Delta)^{s/2}\phi\|_{L^\infty(\mathbb{R}^N)} \|u_h - u\|_{L^2(\mathbb{R}^N)} \\ &\quad \left[\|(-\Delta)^{s/2}\phi\|_{L^\infty(\mathbb{R}^N)} \|u_h - u\|_{L^2(\mathbb{R}^N)} + 2\|T_h((-\Delta)^{s/2}u)\|_{L^2(\mathbb{R}^N)} \right]. \end{aligned} \tag{3.30}$$

Next, we use the convexity of the real valued function $y \mapsto |y|^2$. More precisely, recalling that $0 \leq \phi \leq 1$, we have that for almost every $x \in \mathbb{R}^N$

$$\begin{aligned} |T_h((-\Delta)^{s/2}u)(x)|^2 &= \left| \phi(x)((-\Delta)^{s/2}u)_h(x) + [1 - \phi(x)](-\Delta)^{s/2}u(x) \right|^2 \\ &\leq \phi(x) \left| ((-\Delta)^{s/2}u)_h(x) \right|^2 + (1 - \phi(x)) \left| (-\Delta)^{s/2}u(x) \right|^2. \end{aligned} \tag{3.31}$$

Owing to (2.14), the above inequality implies

$$\|T_h((-\Delta)^{s/2}u)\|_{L^2(\mathbb{R}^N)} \leq 2\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)} \stackrel{(2.14)}{\leq} C(N, s, \text{diam}\Omega) \|f\|_{\mathcal{X}_0^s(\Omega)'},$$

whence, recalling (3.6), (3.3) and (3.30), we obtain

$$I_{1,1} \leq C(N, s, \text{Lip}\phi, \text{diam}\Omega) |h|^s \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \tag{3.32}$$

To control $I_{1,2}$ we use (2.14) and (3.31) and we infer

$$\begin{aligned} I_{1,2} &= \int_{\mathbb{R}^N} \left(|T_h((-\Delta)^{s/2}u)(x)|^2 - \|(-\Delta)^{s/2}u(x)\|^2 \right) dx \\ &\leq \int_{\mathbb{R}^N} \phi(x) \left(\left| ((-\Delta)^{s/2}u)_h(x) \right|^2 - \|(-\Delta)^{s/2}u(x)\|^2 \right) dx \\ &= \int_{\mathbb{R}^N} |\phi(x-h) - \phi(x)| \|(-\Delta)^{s/2}u(x)\|^2 dx \leq \text{Lip}\phi |h| \int_{\mathbb{R}^N} \|(-\Delta)^{s/2}u(x)\|^2 dx \\ &\stackrel{(2.14)}{\leq} C(N, s, \text{Lip}\phi, \text{diam}\Omega) |h| \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \end{aligned}$$

Combining the above inequality with (3.32) and recalling that $s \in (0, 1)$ and $|h| \leq 1$, we arrive at

$$\begin{aligned} I_1 &\leq |I_{1,1}| + |I_{1,2}| \leq C(N, s, \text{Lip}\phi, \text{diam}\Omega) (|h|^s + |h|) \|f\|_{\mathcal{X}_0^s(\Omega)'}^2 \\ &\leq C(N, s, \text{Lip}\phi, \text{diam}\Omega) |h|^s \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \end{aligned} \tag{3.33}$$

STEP 4: we control the term I_2 . We combine (2.14), (3.3) and Lemma 3.4. Now, we fix $\sigma \in (0, 1)$ and we obtain

$$I_2 = \|\mathcal{C}(\phi, u_h - u)\|_{L^2(\mathbb{R}^N)}^2 \stackrel{(3.11)}{\leq} C(N, s, \text{Lip}\phi, \sigma) \|u_h - u\|_{L^2(\mathbb{R}^N)}^{2\sigma} \|u - u_h\|_s^{2-2\sigma}$$

$$\begin{aligned}
 &\leq C(N, s, \text{Lip } \phi, \sigma) \|u_h - u\|_{L^2(\mathbb{R}^N)}^{2\sigma} \|u\|_s^{2-2\sigma} \\
 &\stackrel{(2.14)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) \|u_h - u\|_{L^2(\mathbb{R}^N)}^{2\sigma} \|f\|_{\mathcal{X}_0^s(\Omega)'}^{2-2\sigma} \\
 &\stackrel{(1.7)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) |h|^{2\sigma s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^{2\sigma} \|f\|_{\mathcal{X}_0^s(\Omega)'}^{2-2\sigma} \\
 &= C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) |h|^{2\sigma s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \tag{3.34}
 \end{aligned}$$

STEP 5: we control the term I_3 . First, we point out that

$$|I_3| \leq 2\sqrt{I_2(I_1 + \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)}^2)}. \tag{3.35}$$

By combining (3.33) and (3.34), and recalling that $|h| < 1$, we get

$$\begin{aligned}
 \sqrt{I_2 I_1} &\leq C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) \sqrt{|h|^{2\sigma s + s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^4} \\
 &\leq C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) |h|^{\sigma s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \tag{3.36}
 \end{aligned}$$

Next, by combining (2.14) and (3.34) we infer

$$\sqrt{I_2} \|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^N)}^2 \leq C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) |h|^{\sigma s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \tag{3.37}$$

Finally, we plug (3.36) and (3.37) into (3.35) and we arrive at

$$I_3 \leq C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma) |h|^{\sigma s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^2. \tag{3.38}$$

STEP 6: we conclude the proof of Lemma 3.1. We plug (3.34), (3.33) and (3.38) into (3.25) and, by recalling that $\sigma \in (0, 1)$, we obtain

$$A = \|T_h u\|_s^2 - \|u\|_s^2 \leq C(N, s, \text{Lip } \phi, \sigma, \text{diam } \Omega) |h|^{\sigma s} \|f\|_{\mathcal{X}_0^s(\Omega)'}^2.$$

Recalling (3.22) and using (3.23) we then deduce

$$\|T_h u - u\|_s^2 \leq C(N, s, \text{Lip } \phi, \sigma, \text{diam } \Omega) |h|^{\sigma s} \left(\|f\|_{\mathcal{X}_0^s(\Omega)'}^2 + \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{\mathcal{X}_0^s(\Omega)'} \right),$$

whence, using the inequality $\|f\|_{\mathcal{X}_0^s(\Omega)'} \leq \|f\|_{L^2(\mathbb{R}^N)}$, we eventually arrive at (3.1).

4. Geometric background material: Hausdorff distance, Lipschitz cone condition and cut-off functions

In this section we introduce some preliminary notions related to distance between sets in \mathbb{R}^N , regularity properties of open sets, covering lemmas and cut-off functions.

4.1. Hausdorff and related distances between sets in \mathbb{R}^N . We start by recalling some definitions and we refer to [21, § 2.3] for more details. Let $E, F \subseteq \mathbb{R}^N$ be two sets. The *excess* or *unilateral Hausdorff distance* of E with respect to F is defined as

$$e(E, F) := \sup_{x \in E} d(x, F) = \sup_{x \in E \setminus F} d(x, F) = \sup_{x \in E \setminus F} d(x, \partial F). \tag{4.1}$$

The (bilateral) *Hausdorff distance* between E and F is then given by

$$d_H(E, F) := e(E, F) + e(F, E) = \sup_{x \in E} d(x, F) + \sup_{y \in F} d(y, E). \tag{4.2}$$

We also introduce the notions of *internal excess*:

$$e^c(F, E) := e(\mathbb{R}^N \setminus E, \mathbb{R}^N \setminus F) = \sup_{x \in F \setminus E} d(x, \mathbb{R}^N \setminus F) = \sup_{x \in F \setminus E} d(x, \partial F) \tag{4.3}$$

and of *complementary Hausdorff distance*:

$$d_H^c(E, F) := e^c(F, E) + e^c(E, F) = d_H(\mathbb{R}^N \setminus E, \mathbb{R}^N \setminus F). \tag{4.4}$$

Next, given a set $E \subseteq \mathbb{R}^N$ and a real number $\varepsilon > 0$, we term $E^{-\varepsilon}$ the (possibly empty) set

$$E^{-\varepsilon} := \{x \in E : B_\varepsilon(x) \subseteq E\}. \tag{4.5}$$

Moreover, we denote by E^ε the set

$$E^\varepsilon := \{x \in \mathbb{R}^N : d(x, E) < \varepsilon\}. \tag{4.6}$$

The following result is well known. We provide a proof just for the sake of completeness.

LEMMA 4.1. *Let E, F be subsets of \mathbb{R}^N and let $\varepsilon > 0$. Then we have the following:*

i) *if*

$$e^c(F, E) < \varepsilon, \tag{4.7}$$

then

$$F^{-\varepsilon} \subseteq E;$$

ii) *if*

$$e(E, F) < \varepsilon, \tag{4.8}$$

then

$$E \subseteq F^\varepsilon.$$

Proof. Let us first prove i). By contradiction, let us assume there is $x \in F$ such that $B_\varepsilon(x) \subseteq F$ and $x \notin E$. Since $B_\varepsilon(x) \subseteq F$, then

$$d(x, \mathbb{R}^N \setminus F) \geq \varepsilon.$$

Moreover, since $x \notin E$, then $x \in F \setminus E$. Hence, we have

$$e^c(F, E) = \sup_{y \in F \setminus E} d(y, \mathbb{R}^N \setminus F) \geq d(x, \mathbb{R}^N \setminus F) \geq \varepsilon,$$

which contradicts (4.7) and proves i).

To prove ii), we assume again by contradiction that there is $x \in E$ such that $d(x, F) \geq \varepsilon$. This implies that

$$e(E, F) = \sup_{y \in E} d(y, F) \geq d(x, F) \geq \varepsilon,$$

which contradicts (4.8). The lemma is proved. □

It is worth noting that, if $d_H^c(E, F) < \varepsilon$, then we have both $E^{-\varepsilon} \subseteq F$ and $F^{-\varepsilon} \subseteq E$. Analogously, if $d_H(E, F) < \varepsilon$, then $E \subseteq F^\varepsilon$ and $F \subseteq E^\varepsilon$.

In addition, we observe that, if we define the distance

$$\mathfrak{d}(E, F) := e(E, F) + e^c(F, E), \tag{4.9}$$

then it turns out that

$$e(F \triangle E, \partial F) \leq \mathfrak{d}(E, F) \leq 2e(F \triangle E, \partial F),$$

where $F \triangle E = (E \setminus F) \cup (F \setminus E)$. Also,

$$\mathfrak{d}(E, F) < \varepsilon \implies F^{-\varepsilon} \subseteq E \subseteq F^\varepsilon. \tag{4.10}$$

In other words, if the distance $\mathfrak{d}(E, F)$ is smaller than ε , then the boundary of E is included in the ε -neighbourhood of the boundary of F .

4.2. Construction of cut-off functions. The following covering lemma is classical (cf., e.g., [2, p. 49]). Also in this case, we provide a proof for completeness:

LEMMA 4.2. *Let $E \subseteq \mathbb{R}^N$ satisfy $E \subseteq B_R(y)$ for some $R > 0$, $y \in \mathbb{R}^N$. Fix $r > 0$. Then there are $x_1, \dots, x_k \in E$ such that:*

i) the cardinality k satisfies

$$k \leq \left(\frac{2R+r}{r} \right)^N; \tag{4.11}$$

ii) the balls $B_r(x_i)$ cover E , namely

$$E \subseteq \bigcup_{i=1}^k B_r(x_i); \tag{4.12}$$

iii) the balls $B_{r/2}(x_i)$ are pairwise disjoint, namely

$$B_{r/2}(x_i) \cap B_{r/2}(x_j) = \emptyset \quad \text{if } i \neq j. \tag{4.13}$$

Proof. We choose $x_1 \in E$ and we set $E_1 := E \setminus B_r(x_1)$. Next, we choose $x_2 \in E_1$ and we set $E_2 := E_1 \setminus B_r(x_2)$. We iterate this procedure: since E is bounded, after some finite number k of steps we obtain $E_{k+1} = \emptyset$. Then, by construction, (4.12) is satisfied. To establish (4.13), we point out that $|x_i - x_j| \geq r$ if $i \neq j$. To establish (4.11), we observe that the balls $B_{r/2}(x_1), \dots, B_{r/2}(x_k)$ are also contained in the ball $B_{R+r/2}(y)$. Hence, we deduce

$$\begin{aligned} k \omega_N \left(\frac{r}{2} \right)^N &= \sum_{i=1}^k \mathcal{L}^N \left(B_{r/2}(x_i) \right) = \mathcal{L}^N \left(\bigcup_{i=1}^k B_{r/2}(x_i) \right) \\ &\leq \mathcal{L}^N \left(B_{R+r/2}(y) \right) = \omega_N (R+r/2)^N, \end{aligned}$$

which implies (4.11). In the above expression, ω_N denotes the measure of the unit ball in \mathbb{R}^N . □

We conclude this subsection with a “kind of partition of unity” result that gives the existence of a suitable family of cut-off functions. The proof is very standard (see for instance the proof of [16, Lemma 9]), so we omit it.

LEMMA 4.3. *Let x_1, \dots, x_k belong to \mathbb{R}^N . Let $r > 0$ and let $B_r(x_1), \dots, B_r(x_k)$ be open balls such that (4.13) holds. Then there are $(k + 1)$ Lipschitz continuous functions $\phi_0, \dots, \phi_k : \mathbb{R}^N \rightarrow \mathbb{R}$ satisfying the following requirements:*

$$0 \leq \phi_i(x) \leq 1, \quad |\nabla \phi_i(x)| \leq \frac{C(N)}{r} \text{ for a.e. } x \in \mathbb{R}^N \text{ and every } i = 1, \dots, k; \tag{4.14}$$

$$\phi_0(x) = 0 \text{ if } x \in \bigcup_{i=1}^k B_r(x_i), \quad \phi_0(x) = 1 \text{ if } x \in \mathbb{R}^N \setminus \bigcup_{i=1}^k B_{2r}(x_i); \tag{4.15}$$

$$\phi_i(x) = 0 \text{ if } x \in \mathbb{R}^N \setminus B_{2r}(x_i) \text{ for } i = 1, \dots, k; \tag{4.16}$$

$$\sum_{i=0}^k \phi_i(x) = 1 \text{ for every } x \in \mathbb{R}^N. \tag{4.17}$$

4.3. The Lipschitz cone condition. We first introduce the regularity assumption we use. We fix $\theta \in (0, \pi/2)$, $\rho > 0$ and \mathbf{n} a unitary vector in \mathbb{R}^N . We term $C_{\rho, \theta}(\mathbf{n})$ the open cone

$$C_{\rho, \theta}(\mathbf{n}) := \{ \mathbf{v} \in \mathbb{R}^N : 0 < |\mathbf{v}| < \rho, \mathbf{v} \cdot \mathbf{n} > |\mathbf{v}| \cos \theta \}.$$

DEFINITION 4.1. *Assume $\Omega \subseteq \mathbb{R}^N$ is an open set and fix $x_0 \in \mathbb{R}^N$, $\theta \in (0, \pi/2)$, $\rho > 0$. We term $\mathcal{N}_{\rho, \theta}(x_0, \Omega)$ the (possibly empty) set of unit vectors $\mathbf{n} \in \mathbb{R}^N$ such that*

- i) for every $\mathbf{v} \in C_{\rho, \theta}(\mathbf{n})$ and every $y \in B_{3\rho}(x_0) \cap \Omega$, we have $(y - \mathbf{v}) \in \Omega$.*
- ii) for every $\mathbf{v} \in C_{\rho, \theta}(\mathbf{n})$ and every $y \in B_{3\rho}(x_0) \setminus \Omega$, we have $(y + \mathbf{v}) \in \mathbb{R}^N \setminus \Omega$.*

We say that Ω satisfies the uniform (ρ, θ) -Lipschitz cone condition if

$$\mathcal{N}_{\rho, \theta}(x_0, \Omega) \neq \emptyset \quad \forall x_0 \in \mathbb{R}^N.$$

We have the following simple result. We refer to Figure 4.1 for a representation.

LEMMA 4.4. *Let \mathbf{n} be a unit vector and let $\rho > 0$ and $\theta \in (0, \pi/2)$. Assume that $0 < t < \rho/2$ and let $\varepsilon = t \sin \theta$. If $x = t\mathbf{n}$, then*

$$B_\varepsilon(x) \subseteq C_{\rho, \theta}(\mathbf{n}).$$

Proof. We can refer to Figure 4.1 and infer that the inclusion holds true. For completeness, we also provide an analytic proof. Assume that $\mathbf{v} \in B_\varepsilon(x)$. Then

$$\mathbf{v} = t\mathbf{n} + \varepsilon \mathbf{e} \tag{4.18}$$

for some \mathbf{e} in the open unit ball centered at the origin. Since

$$0 < t(1 - \sin \theta) \leq |\mathbf{v}| \leq t(1 + \sin \theta) < \rho,$$

because $\varepsilon = t \sin \theta$, we are left to show that

$$\mathbf{v} \cdot \mathbf{n} > |\mathbf{v}| \cos \theta. \tag{4.19}$$

Actually, using (4.18), we infer

$$\mathbf{v} \cdot \mathbf{n} = t + \varepsilon \mathbf{e} \cdot \mathbf{n}, \tag{4.20}$$

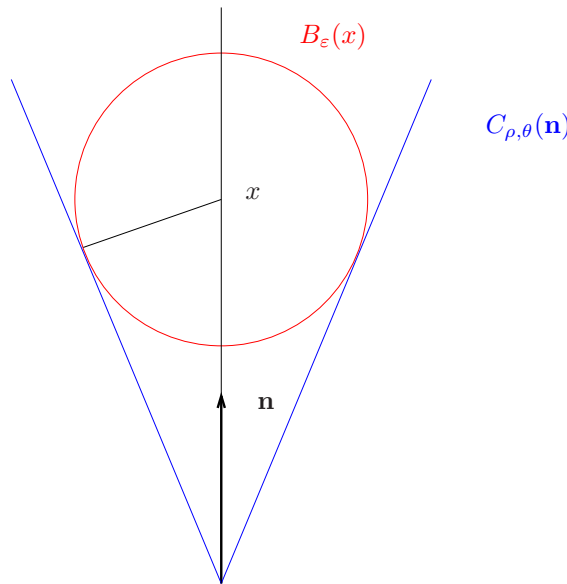


FIG. 4.1.

whence in particular $\mathbf{v} \cdot \mathbf{n} > 0$. Moreover, using that $|\mathbf{n}| = 1$ and $|\mathbf{e}| < 1$, we infer

$$|\mathbf{v}|^2 = t^2 + 2\varepsilon t \mathbf{e} \cdot \mathbf{n} + \varepsilon^2 |\mathbf{e}|^2 < t^2 + 2\varepsilon t \mathbf{e} \cdot \mathbf{n} + \varepsilon^2, \tag{4.21}$$

Then, squaring both sides of (4.19), we are left to check that

$$(\mathbf{v} \cdot \mathbf{n})^2 - |\mathbf{v}|^2 \cos^2 \theta > 0.$$

Using (4.20), (4.21) and, subsequently, that $\varepsilon = t \sin \theta$, we then obtain

$$\begin{aligned} (\mathbf{v} \cdot \mathbf{n})^2 - |\mathbf{v}|^2 \cos^2 \theta &> t^2 + \varepsilon^2 (\mathbf{e} \cdot \mathbf{n})^2 + 2\varepsilon t \mathbf{e} \cdot \mathbf{n} - (t^2 + 2\varepsilon t \mathbf{e} \cdot \mathbf{n} + \varepsilon^2) \cos^2 \theta \\ &= t^2 \sin^2 \theta + t^2 (\mathbf{e} \cdot \mathbf{n})^2 \sin^2 \theta + 2t^2 (\mathbf{e} \cdot \mathbf{n}) \sin^3 \theta - t^2 \sin^2 \theta \cos^2 \theta \\ &= t^2 \sin^2 \theta (\sin \theta + (\mathbf{e} \cdot \mathbf{n}))^2 \geq 0, \end{aligned}$$

whence follows (4.19), as desired. □

5. Projection estimates

In this section we establish two preliminary results that are pivotal to the proof of Theorem 1.2 and of Theorem 1.4.

LEMMA 5.1. *Assume Ω is an open, bounded set satisfying a (ρ, θ) -Lipschitz cone condition for some $\rho \in (0, 1/2]$, $\theta \in (0, \pi/2)$. Assume furthermore that $f \in L^2(\mathbb{R}^N)$ and that $u \in \mathcal{X}_0^s(\Omega)$ is the weak solution of (1.1). If*

$$0 < \varepsilon < \frac{\rho \sin \theta}{2}, \tag{5.1}$$

then there is $\tilde{u} \in \mathcal{X}_0^s(\Omega^{-\varepsilon})$ such that

$$\|\tilde{u} - u\|_{L^2(\mathbb{R}^N)}^2 \leq C(N, s, \text{diam } \Omega, \rho) \left(\frac{\varepsilon}{\sin \theta} \right)^{2s} \|f\|_{\mathcal{X}_0^s(\Omega)}^2, \tag{5.2}$$

and, for every $\sigma \in (0, 1)$,

$$\|\tilde{u} - u\|_s^2 \leq C(N, s, \text{diam}\Omega, \rho, \sigma) \left(\frac{\varepsilon}{\sin\theta}\right)^{\sigma s} \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{\mathcal{X}_0^s(\Omega)}. \tag{5.3}$$

Note that the assumption that $\rho \leq 1/2$ is not restrictive: indeed, Definition 4.1 implies that, if $\rho_1 \leq \rho_2$ and Ω satisfies a (ρ_2, θ) -Lipschitz cone condition, then it also satisfies a (ρ_1, θ) -Lipschitz cone condition.

LEMMA 5.2. *Assume Ω is an open, bounded set satisfying a (ρ, θ) -Lipschitz cone condition for some $\rho \in (0, 1/2]$, $\theta \in (0, \pi/2)$. Let ε satisfy*

$$0 < \varepsilon < \frac{\rho \sin\theta}{2}. \tag{5.4}$$

Fix $f \in L^2(\mathbb{R}^N)$ and let w be the weak solution of

$$\begin{cases} (-\Delta)^s w = f & \text{in } \Omega^\varepsilon \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega^\varepsilon. \end{cases} \tag{5.5}$$

Then there is $\hat{w} \in \mathcal{X}_0^s(\Omega)$ such that

$$\|\hat{w} - w\|_{L^2(\mathbb{R}^N)}^2 \leq C(N, s, \text{diam}\Omega, \rho) \left(\frac{\varepsilon}{\sin\theta}\right)^{2s} \|f\|_{\mathcal{X}_0^s(\Omega^\varepsilon)}^2, \tag{5.6}$$

and, for every $\sigma \in (0, 1)$,

$$\|\hat{w} - w\|_s^2 \leq C(N, s, \text{diam}\Omega, \rho, \sigma) \left(\frac{\varepsilon}{\sin\theta}\right)^{\sigma s} \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{\mathcal{X}_0^s(\Omega^\varepsilon)}. \tag{5.7}$$

5.1. Proof of Lemma 5.1.

5.1.1. Construction of \tilde{u} . We fix Ω , ε and u as in the statement of Lemma 5.1. We also fix a number t such that

$$\frac{\varepsilon}{\sin\theta} < t < \frac{\rho}{2}. \tag{5.8}$$

We proceed according to the following steps.

STEP 1: we introduce the set

$$E := \bigcup_{y \in \partial\Omega} B_\rho(y). \tag{5.9}$$

We apply Lemma 4.2 with $R = \text{diam}\Omega + 1$, $r = \rho$ and we obtain that

$$E \subseteq \bigcup_{i=1}^k B_\rho(x_i), \quad x_1, \dots, x_k \in \mathbb{R}^N, \tag{5.10}$$

where every $x \in \mathbb{R}^N$ belongs to at most 5^N of the balls $B_{2\rho}(x_1), \dots, B_{2\rho}(x_k)$. Owing to (4.11), the cardinality k satisfies

$$k \leq C(N, \text{diam}\Omega, \rho). \tag{5.11}$$

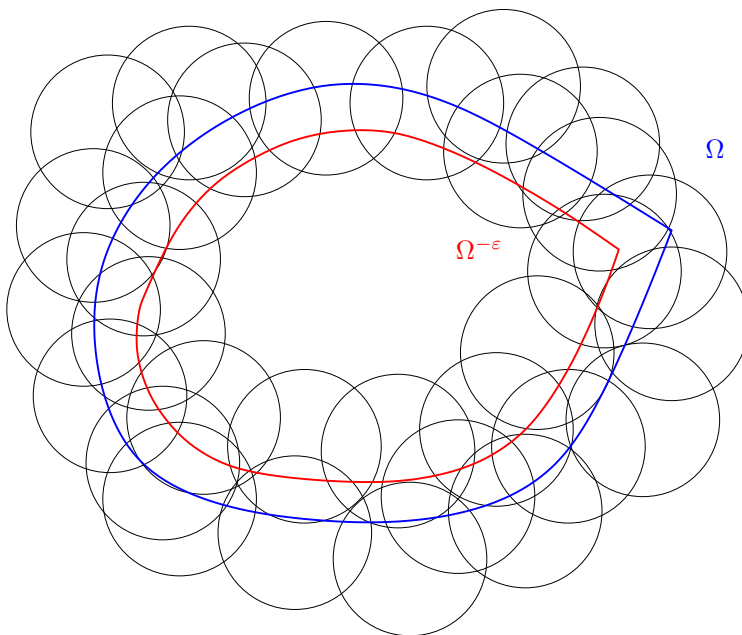


FIG. 5.1. The sets $\Omega^{-\varepsilon}$ (inside, in red) and Ω (outside, in blue) and the balls covering the set E , see (5.9) and (5.10).

STEP 2: we apply Lemma 4.3, again with $r = \rho$, and we consider the functions ϕ_0, \dots, ϕ_k . For every $i = 1, \dots, k$ we fix a vector $\mathbf{n}_i \in \mathcal{N}_{\rho, \theta}(x_i, \Omega)$ and we define the function u_{it} by setting

$$u_{it}(x) := u(x + t\mathbf{n}_i). \tag{5.12}$$

STEP 3: finally, we define the function \tilde{u} by setting

$$\tilde{u}(x) := \phi_0(x)u(x) + \sum_{i=1}^k \phi_i(x)u_{it}(x). \tag{5.13}$$

5.1.2. Proof of the inclusion $\tilde{u} \in \mathcal{X}_0^s(\Omega^{-\varepsilon})$. First, combining Lemma 3.3 with the Definition (5.13) of \tilde{u} , we conclude that $\tilde{u} \in H^s(\mathbb{R}^N)$. Hence, we are left to show that

$$\tilde{u}(x) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N \setminus \Omega^{-\varepsilon}. \tag{5.14}$$

We fix $x \in \mathbb{R}^N \setminus \Omega^{-\varepsilon}$ and we separately consider two cases.

CASE 1: if $d(x, \Omega) \geq t$, then $x \notin \Omega$ and moreover $(x + t\mathbf{n}_i) \notin \Omega$ for all $i = 1, \dots, k$ because \mathbf{n}_i is a unit vector. Since $u \in \mathcal{X}_0^s(\Omega)$, then $u \equiv 0$ in $\mathbb{R}^N \setminus \Omega$, whence

$$0 = u(x) = u(x + t\mathbf{n}_i) = u_{it}(x).$$

This implies that $\tilde{u}(x) = 0$.

CASE 2: we are left to consider the case when $d(x, \Omega) < t$: we have

$$x \in \bigcup_{z \in \partial\Omega} B_{2t}(z) \subseteq \bigcup_{i=1}^k B_\rho(x_i). \tag{5.15}$$

To prove the above inclusion, we have combined the inequality $t < \rho/2$, which follows from (5.8), with STEP 1 in § 5.1.1. By combining (5.15) with (4.15) we get $\phi_0(x) = 0$. Next, we fix $i \in \{1, \dots, k\}$ such that $\phi_i(x) \neq 0$. Owing to (4.16), this implies that $x \in B_{2\rho}(x_i)$. We set $y := x + t\mathbf{n}_i$. We want to show that $y \notin \Omega$.

First, we apply Lemma 4.4 and we conclude that

$$B_\varepsilon(x) \subseteq y - C_{\rho,\theta}(\mathbf{n}_i). \tag{5.16}$$

Now, we point out that $y \in B_{3\rho}(x_i)$ because $x \in B_{2\rho}(x_i)$ and $|t| \leq \rho/2$. Hence, we can use property i) in Definition 4.1: if $y \in \Omega$, then $y - C_{\rho,\theta}(\mathbf{n}_i) \subseteq \Omega$. By recalling (5.16) and the definition of $\Omega^{-\varepsilon}$ we conclude that, if $y \in \Omega$, then $B_\varepsilon(x) \subseteq \Omega$ and hence $x \in \Omega^{-\varepsilon}$. This contradicts our assumption and hence we can conclude that $y \notin \Omega$. This implies

$$0 = u(y) = u(x + t\mathbf{n}_i) = u_{it}(x),$$

whence follows that $\tilde{u}(x) = 0$. The proof of (5.14) is complete.

5.1.3. Proof of (5.2) and (5.3). We first establish (5.2). We combine (4.17) with (5.13) and we conclude that, for every $x \in \mathbb{R}^N$, there holds

$$u(x) - \tilde{u}(x) = \sum_{i=1}^k \phi_i(x) [u(x) - u_{it}(x)]. \tag{5.17}$$

To control $\|u - \tilde{u}\|_{L^2(\mathbb{R}^N)}$ we use (1.7). More precisely, we first recall (5.11) and conclude that

$$\|u - \tilde{u}\|_{L^2(\mathbb{R}^N)} \leq k \max_{i=1, \dots, k} \|\phi_i[u - u_{it}]\|_{L^2(\mathbb{R}^N)} \stackrel{(5.11)}{\leq} C(N, \text{diam}\Omega, \rho) \max_{i=1, \dots, k} \|\phi_i[u - u_{it}]\|_{L^2(\mathbb{R}^N)}. \tag{5.18}$$

Next, we set $h = t\mathbf{n}_i$ (we do not highlight the dependence of h on the index i , for simplicity) and recall the Definition (1.3) of $T_h v$. Then we infer that $T_h v - v = \phi[v_h - v]$.

We now apply Theorem 1.1 to the function $\phi_i[u - u_{it}]$, for every $i = 1, \dots, k$. The hypotheses of Theorem 1.1 are satisfied because ϕ_i is a cut-off function as in the statement of Lemma 4.3 and consequently satisfies (1.4) and also (1.5), since $r = \rho \leq 1/2$. Also, the analysis in § 5.1.2 shows that $\phi_i u_{it} \in \mathcal{X}_0^s(\Omega^{-\varepsilon}) \subseteq \mathcal{X}_0^s(\Omega)$, whence condition (1.6) is also satisfied. By combining (5.18) with (1.7) we arrive at the inequality

$$\|u - \tilde{u}\|_{L^2(\mathbb{R}^N)} \leq C(N, s, \text{diam}\Omega, \rho, \theta) t^s \|f\|_{\mathcal{X}_0^s(\Omega)'} = C(N, s, \text{diam}\Omega, \rho, \theta) |h|^s \|f\|_{\mathcal{X}_0^s(\Omega)'}$$

Finally, we point out that the above inequality holds for every t satisfying (5.8) and we eventually arrive at (5.2).

The proof of (5.3) relies on (3.1) and is entirely analogous. The only new point is that we have to use the inequality $\text{Lip}\phi \leq C(N, \rho)$, which follows from (4.14). Details are omitted for brevity.

5.2. Proof of Lemma 5.2.

5.2.1. Construction of \hat{w} . We fix Ω , ε and w as in the statement of Lemma 5.2. We also fix a number t satisfying (5.8). We proceed as in STEP 1 and STEP 2 in § 5.1.1 and we define the function w_{it} by setting

$$w_{it}(x) := w(x + t\mathbf{n}_i). \tag{5.19}$$

Finally, we define the function \hat{w} by setting

$$\hat{w}(x) := \phi_0(x)w(x) + \sum_{i=1}^k \phi_i(x)w_{it}(x). \tag{5.20}$$

5.2.2. Proof of the inclusion $\hat{w} \in \mathcal{X}_0^s(\Omega)$. We combine Lemma 3.3 with the Definition (5.20) of \hat{w} and we conclude that $\hat{w} \in H^s(\mathbb{R}^N)$. Hence, we are left to show that

$$\hat{w}(x) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N \setminus \Omega. \tag{5.21}$$

We fix $x \notin \Omega$ and we separately consider two cases:

CASE 1: if $d(x, \Omega^\varepsilon) \geq t$, then $x \notin \Omega^\varepsilon$ and moreover $(x + t\mathbf{n}_i) \notin \Omega^\varepsilon$ because \mathbf{n}_i is a unit vector. Since $w \in \mathcal{X}_0^s(\Omega^\varepsilon)$, then $w \equiv 0$ in $\mathbb{R}^N \setminus \Omega^\varepsilon$, whence

$$0 = w(x) = w(x + t\mathbf{n}_i) = w_{it}(x).$$

This implies that $\hat{w}(x) = 0$.

CASE 2: we are left to consider the case when $d(x, \Omega^\varepsilon) < t$. By recalling Definition (4.6), this implies $d(x, \Omega) < t + \varepsilon < 2t$. Thus, we have

$$x \in \bigcup_{z \in \partial\Omega} B_{2t}(z) \subseteq \bigcup_{i=1}^k B_\rho(x_i).$$

Combining the above formula with (4.15) we deduce that $\phi_0(x) = 0$. Next, we fix $i \in \{1, \dots, k\}$ such that $\phi_i(x) \neq 0$. Owing to (4.16), this implies that $x \in B_{2\rho}(x_i)$. We set $y := x + t\mathbf{n}_i$ and we want to show that $y \notin \Omega^\varepsilon$. Since $x \in B_{2\rho}(x_i)$, we can use property ii) in Definition 4.1: since $x \notin \Omega$, then $x + C_{\rho, \theta}(\mathbf{n}_i) \subseteq \mathbb{R}^N \setminus \Omega$. Next, we apply Lemma 4.4 and we conclude that

$$B_\varepsilon(y) \subseteq x + C_{\rho, \theta}(\mathbf{n}_i) \subseteq \mathbb{R}^N \setminus \Omega.$$

This means that $d(y, \Omega) \geq \varepsilon$ and hence that $y \notin \Omega^\varepsilon$. Consequently we have

$$0 = w(y) = w(x + t\mathbf{n}_i) = w_{it}(x),$$

whence we obtain $\hat{w}(x) = 0$. The proof of (5.21) is complete.

5.2.3. Proof of (5.6) and (5.7)

We proceed as in § 5.1.3 and we apply estimates (1.7) and (3.1) in the domain Ω^ε . The details are omitted.

6. Domain perturbation estimates

This section aims at establishing the following result, which can be regarded as a weaker version of Theorem 1.2:

LEMMA 6.1. *Under the same assumptions as in the statement of Theorem 1.2, for every $\sigma \in (0, 1)$ we have*

$$\|u_a - u_b\|_s \leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{L^2(D)}^{1/2} \|f\|_{\mathcal{X}_0^s(D)}^{1/2} \mathfrak{d}(\Omega_b, \Omega_a)^{s\sigma/2}. \tag{6.1}$$

6.1. Notation and preliminary results. Let D , Ω_a and Ω_b be as in the statement of Theorem 1.2. We recall that we term u_a and u_b the solutions of (1.1) when $\Omega = \Omega_a$ and $\Omega = \Omega_b$, respectively. Also, we recall that the sets $\Omega^{-\varepsilon}$ and Ω^ε are defined as in (4.5) and (4.6), respectively, and we term $u^{-\varepsilon}$ and u^ε the solutions of the Poisson problem (1.1) when $\Omega = \Omega^{-\varepsilon}$ and $\Omega = \Omega^\varepsilon$, respectively.

We introduce some additional notation. Given two bounded subdomains Ω and $\tilde{\Omega}$ of D with $\Omega \subseteq \tilde{\Omega}$, we denote with $P_{\tilde{\Omega} \rightarrow \Omega}$ the orthogonal projection

$$P_{\tilde{\Omega} \rightarrow \Omega} : \mathcal{X}_0^s(\tilde{\Omega}) \rightarrow \mathcal{X}_0^s(\Omega) \tag{6.2}$$

with respect to the scalar product (2.10). Namely, for $u \in \mathcal{X}_0^s(\tilde{\Omega})$, this projection is characterized by

$$[u - P_{\tilde{\Omega} \rightarrow \Omega}(u), v]_s = 0 \quad \forall v \in \mathcal{X}_0^s(\Omega).$$

Recall also that

$$\|w - P_{\tilde{\Omega} \rightarrow \Omega}(w)\|_s = \min_{v \in \mathcal{X}_0^s(\Omega)} \|w - v\|_s$$

for $w \in \mathcal{X}_0^s(\tilde{\Omega})$. We have the following simple, albeit important, property:

LEMMA 6.2. *Assume that $\Omega_a \subseteq \Omega$ and that u_a and u solve (1.1) respectively in Ω_a and in Ω . Then*

$$P_{\Omega \rightarrow \Omega_a}(u) = u_a,$$

and $P_{\Omega \rightarrow \Omega_a}$ is linear.

Proof. Since $\mathcal{X}_0^s(\Omega_a) \subset \mathcal{X}_0^s(\Omega)$ and since u and u_a are weak solutions of (1.1) in Ω and in Ω_a , respectively, we have, for all $v \in \mathcal{X}_0^s(\Omega_a)$,

$$[u, v]_s = \langle f, v \rangle = [u_a, v]_s,$$

whence $[u - u_a, v]_s = 0$ for all $v \in \mathcal{X}_0^s(\Omega_a)$, that is the thesis. □

6.2. Proof of Lemma 6.1: conclusion. First, we fix $\varepsilon > 0$ such that

$$\mathfrak{d}(\Omega_b, \Omega_a) < \varepsilon < \frac{\rho \sin \theta}{2}. \tag{6.3}$$

We recall (4.10) and we conclude that

$$\Omega_a^{-\varepsilon} \subseteq \Omega_b \subseteq \Omega_a^\varepsilon. \tag{6.4}$$

By using Lemma 6.2, we have

$$u_b = P_{\Omega_a^\varepsilon \rightarrow \Omega_b}(u_a^\varepsilon),$$

where u_a^ε denotes the weak solution of (1.1) in Ω_a^ε . Hence we obtain the following chain of inequalities:

$$\|u_a^\varepsilon - u_b\|_s = \min_{v \in \mathcal{X}_0^s(\Omega_b)} \|u_a^\varepsilon - v\|_s \leq \|u_a^\varepsilon - u_a^{-\varepsilon}\|_s \leq \|u_a^\varepsilon - u_a\|_s + \|u_a - u_a^{-\varepsilon}\|_s. \tag{6.5}$$

Note that to establish the first inequality we used the inclusion $\mathcal{X}_0^s(\Omega_a^{-\varepsilon}) \subseteq \mathcal{X}_0^s(\Omega_b)$ following from (6.4). By using (6.5) we infer

$$\|u_a - u_b\|_s \leq \|u_a - u_a^\varepsilon\|_s + \|u_a^\varepsilon - u_b\|_s \leq 2\|u_a - u_a^\varepsilon\|_s + \|u_a - u_a^{-\varepsilon}\|_s. \tag{6.6}$$

Applying again Lemma 6.2, we deduce

$$\|u_a - u_a^{-\varepsilon}\|_s = \min_{v \in \mathcal{X}_0^s(\Omega_a^{-\varepsilon})} \|u_a - v\|_s.$$

By using Lemma 5.1 we conclude that

$$\begin{aligned} \|u_a - u_a^{-\varepsilon}\|_s &\leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \varepsilon^{\sigma s/2} \|f\|_{L^2(D)}^{1/2} \|f\|_{\mathcal{X}_0^s(\Omega_a)}^{1/2} \\ &\leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \varepsilon^{\sigma s/2} \|f\|_{L^2(D)}^{1/2} \|f\|_{\mathcal{X}_0^s(D)}^{1/2}. \end{aligned} \tag{6.7}$$

In the previous estimate we used the inequality $\|f\|_{\mathcal{X}_0^s(\Omega_a)} \leq \|f\|_{\mathcal{X}_0^s(D)}$, which holds for $f \in L^2(\mathbb{R}^N)$ and can be established by arguing as follows: from the inclusion $\Omega_a \subset D$ we infer that, for every $v \in \mathcal{X}_0^s(\Omega_a)$, $\|v\|_{\mathcal{X}_0^s(D)} = \|v\|_{\mathcal{X}_0^s(\Omega_a)}$. This implies

$$\|f\|_{\mathcal{X}_0^s(\Omega_a)} = \sup_{v \in \mathcal{X}_0^s(\Omega_a)} \frac{\int_{\mathbb{R}^N} f(x)v(x) dx}{\|v\|_{\mathcal{X}_0^s(\Omega_a)}} \leq \sup_{v \in \mathcal{X}_0^s(\Omega_a)} \frac{\|f\|_{\mathcal{X}_0^s(D)} \|v\|_{\mathcal{X}_0^s(D)}}{\|v\|_{\mathcal{X}_0^s(\Omega_a)}} = \|f\|_{\mathcal{X}_0^s(D)}.$$

By applying once more Lemma 6.2 we get

$$\|u_a^\varepsilon - u_a\|_s = \min_{v \in \mathcal{X}_0^s(\Omega_a)} \|u_a^\varepsilon - v\|_s,$$

which combined with Lemma 5.2 gives

$$\begin{aligned} \|u_a^\varepsilon - u_a\|_s &\leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \varepsilon^{\sigma s/2} \|f\|_{L^2(D)}^{1/2} \|f\|_{\mathcal{X}_0^s(\Omega_a^\varepsilon)}^{1/2} \\ &\leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \varepsilon^{\sigma s/2} \|f\|_{L^2(D)}^{1/2} \|f\|_{\mathcal{X}_0^s(D)}^{1/2}. \end{aligned} \tag{6.8}$$

By plugging (6.7) and (6.8) into (6.6) we arrive at

$$\|u_a - u_b\|_s \leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \varepsilon^{\sigma s/2} \|f\|_{L^2(D)}^{1/2} \|f\|_{\mathcal{X}_0^s(D)}^{1/2}.$$

We recall that the above inequality holds for every ε satisfying (6.3) and we conclude the proof of (6.1).

7. Regularity estimates

In this section we establish the following result, which can be regarded as a weaker version of Theorem 1.3:

LEMMA 7.1. *Under the same assumptions as in the statement of Theorem 1.3, for every $\sigma \in (0, 1)$ we have*

$$u \in B_{2,\infty}^{3\sigma s/2}(\mathbb{R}^N), \quad \|u\|_{B_{2,\infty}^{3\sigma s/2}(\mathbb{R}^N)} \leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2}. \tag{7.1}$$

The proof is based on an argument similar to that given in the proof of [21, Proposition 2.3] combined with the use of Lemma 7.2 below.

7.1. Preliminary results. The following result is classical, but we provide a proof for the sake of completeness and for future reference.

LEMMA 7.2. *Assume that $v \in L^2(\mathbb{R}^N)$ satisfies*

$$(-\Delta)^s v = g \text{ in } \mathbb{R}^N. \tag{7.2}$$

If $g \in B_{2,\infty}^r(\mathbb{R}^N)$ for some $r > 0$, then

$$v \in B_{2,\infty}^{r+2s}(\mathbb{R}^N) \text{ and } \|v\|_{B_{2,\infty}^{r+2s}(\mathbb{R}^N)} \leq C(N, s, r) [\|v\|_{L^2(\mathbb{R}^N)} + \|g\|_{B_{2,\infty}^r(\mathbb{R}^N)}]. \tag{7.3}$$

Proof. The basic idea of the proof can be outlined as follows: first, we observe that (7.2) implies that

$$v + (-\Delta)^s v = \ell := v + g \text{ in } \mathbb{R}^N. \tag{7.4}$$

Next, by using the Fourier transform, we show that the regularity properties of the above equation are basically the same as those of the Equation (2.22). Finally, we apply Lemma 2.1 and we conclude.

The details of this procedure are organized into a number of steps.

STEP 1: we show that $v \in B_{2,\infty}^r(\mathbb{R}^N)$.

First of all, we show that v has some fractional Sobolev regularity. More precisely, we fix $\varepsilon = \min\{r, s\}$, and we show that $v \in H^{r+2s-\varepsilon}(\mathbb{R}^N)$ and that

$$\|v\|_{H^{r+2s-\varepsilon}(\mathbb{R}^N)} \leq C(N, s, r) [\|v\|_{L^2(\mathbb{R}^N)} + \|g\|_{B_{2,\infty}^r(\mathbb{R}^N)}]. \tag{7.5}$$

To establish (7.5), we first use the inclusion property (2.20) and we conclude that $g \in H^{r-\varepsilon}(\mathbb{R}^N)$ and that

$$\|g\|_{H^{r-\varepsilon}(\mathbb{R}^N)} \stackrel{(2.20)}{\leq} C(N, s, r) \|g\|_{B_{2,\infty}^r(\mathbb{R}^N)}. \tag{7.6}$$

Next, we point out that proving that $v \in H^{r+2s-\varepsilon}(\mathbb{R}^N)$ amounts to show that

$$(1 + |\xi|^2)^{(r+2s-\varepsilon)/2} \hat{v} \in L^2(\mathbb{R}^N).$$

We recall (2.3) and we infer the following chain of equalities:

$$\begin{aligned} & (1 + |\xi|^2)^{(r+2s-\varepsilon)/2} |\hat{v}| \\ &= \frac{(1 + |\xi|^2)^{(r+2s-\varepsilon)/2}}{1 + |\xi|^{r+2s-\varepsilon}} (1 + |\xi|^{r+2s-\varepsilon}) |\hat{v}| \stackrel{(2.3)}{=} \frac{(1 + |\xi|^2)^{(r+2s-\varepsilon)/2}}{1 + |\xi|^{r+2s-\varepsilon}} (|\hat{v}| + |\xi|^{r-\varepsilon} |\hat{g}|) \\ &\leq \frac{(1 + |\xi|^2)^{(r+2s-\varepsilon)/2}}{1 + |\xi|^{r+2s-\varepsilon}} \left[|\hat{v}| + (1 + |\xi|^2)^{(r-\varepsilon)/2} |\hat{g}| \right]. \end{aligned} \tag{7.7}$$

Next, we recall that $\varepsilon = \min\{r, s\}$ and, since

$$\left| \frac{(1 + |\xi|^2)^{(r+2s-\varepsilon)/2}}{1 + |\xi|^{r+2s-\varepsilon}} \right| \leq C(N, s, \varepsilon) = C(N, s, r) \text{ for every } \xi \in \mathbb{R}^N,$$

then by combining (7.6) and (7.7) we conclude that $v \in H^{r+2s-\varepsilon}(\mathbb{R}^N)$ and that the inequality (7.5) is satisfied.

We now turn to the proof of the Besov regularity of v . We recall that $\varepsilon = \min\{r, s\}$ and, owing to (2.21), we conclude that

$$v \in H^{r+2s-\varepsilon}(\mathbb{R}^N) \subset H^r(\mathbb{R}^N) \stackrel{(2.20)}{\subset} B_{2,\infty}^r(\mathbb{R}^N)$$

and, by using (7.5), that

$$\|v\|_{B_{2,\infty}^r(\mathbb{R}^N)} \stackrel{(2.20)}{\leq} C(N, s, r) \|v\|_{H^{r+2s-\varepsilon}(\mathbb{R}^N)} \stackrel{(7.5)}{\leq} C(N, s, r) [\|v\|_{L^2(\mathbb{R}^N)} + \|g\|_{B_{2,\infty}^r(\mathbb{R}^N)}]. \tag{7.8}$$

STEP 2: we conclude the proof of the lemma in the case when $r + 2s \leq 1$. First, we point out that by using (2.3) again we infer from (7.4) the equality

$$\hat{v}(\xi) + |\xi|^{2s} \hat{v}(\xi) = \hat{\ell}(\xi) \quad \text{for a.e. } \xi \in \mathbb{R}^N. \tag{7.9}$$

Note that, owing to (7.8), $\ell = v + g \in B_{2,\infty}^r(\mathbb{R}^N)$ and

$$\|\ell\|_{B_{2,\infty}^r(\mathbb{R}^N)} \leq C(N, s, r) [\|v\|_{L^2(\mathbb{R}^N)} + \|g\|_{B_{2,\infty}^r(\mathbb{R}^N)}]. \tag{7.10}$$

Since by assumption $v \in L^2(\mathbb{R}^N)$, owing to the Plancherel Theorem and to Definition (2.15), proving that $v \in B_{2,\infty}^{r+2s}(\mathbb{R}^N)$ amounts to show that

$$\sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|\hat{v}_{2h} - 2\hat{v}_h + \hat{v}\|_{L^2(\mathbb{R}^N)}}{|h|^{r+2s}} < +\infty.$$

By directly computing \hat{v}_h and using (7.9) we obtain

$$\begin{aligned} & \hat{v}_{2h}(\xi) - 2\hat{v}_h(\xi) + \hat{v}(\xi) \\ &= (e^{i2\xi \cdot h} - 2e^{i\xi \cdot h} + 1)\hat{v}(\xi) \\ \stackrel{(7.9)}{=} & \frac{e^{i2\xi \cdot h} - 2e^{i\xi \cdot h} + 1}{1 + |\xi|^{2s}} \hat{\ell}(\xi) = \frac{(1 + |\xi|^2)^s}{1 + |\xi|^{2s}} \frac{1}{(1 + |\xi|^2)^s} (e^{i2\xi \cdot h} - 2e^{i\xi \cdot h} + 1)\hat{\ell}(\xi). \end{aligned} \tag{7.11}$$

Owing to (2.23),

$$\frac{1}{(1 + |\xi|^2)^s} (e^{i2\xi \cdot h} - 2e^{i\xi \cdot h} + 1)\hat{\ell}(\xi) = \hat{u}_{2h}(\xi) - 2\hat{u}_h(\xi) + \hat{u}(\xi) \tag{7.12}$$

provided that u solves the equation

$$(I - \Delta)^s u = \ell \quad \text{in } \mathbb{R}^N. \tag{7.13}$$

Owing to Lemma 2.1, since $\ell \in B_{2,\infty}^r(\mathbb{R}^N)$, then $u \in B_{2,\infty}^{r+2s}(\mathbb{R}^N)$. Moreover,

$$\begin{aligned} \|\hat{u}\|_{L^2(\mathbb{R}^N)} + \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|\hat{u}_{2h} - 2\hat{u}_h + \hat{u}\|_{L^2(\mathbb{R}^N)}}{|h|^{r+2s}} &= \|u\|_{B_{2,\infty}^{r+2s}(\mathbb{R}^N)} \\ &\stackrel{(2.24)}{\leq} C(N, s, r) \|\ell\|_{B_{2,\infty}^r(\mathbb{R}^N)} \\ &\stackrel{(7.10)}{\leq} C(N, s, r) [\|v\|_{L^2(\mathbb{R}^N)} + \|g\|_{B_{2,\infty}^r(\mathbb{R}^N)}]. \end{aligned} \tag{7.14}$$

To conclude, we point out that

$$\frac{(1+|\xi|^2)^s}{1+|\xi|^{2s}} \leq 1 \quad \text{for every } \xi \in \mathbb{R}^N. \tag{7.15}$$

By combining (2.16), (7.11), (7.12), (7.14) and (7.15) we eventually arrive at (7.3).

STEP 3: we conclude the proof by dealing with the case when $r+2s > 1$. Recall (2.19), we fix $j=1, \dots, N$ and term w the distributional derivative

$$w := \frac{\partial v}{\partial x_j}.$$

Next, we point out that

$$\hat{w}_{2h}(\xi) - 2\hat{w}_h(\xi) + \hat{w}(\xi) = i\xi_j(e^{i2\xi \cdot h} - 2e^{i\xi \cdot h} + 1)\hat{v}(\xi)$$

and by arguing as in (7.11) and (7.12) we conclude that

$$\hat{w}_{2h}(\xi) - 2\hat{w}_h(\xi) + \hat{w}(\xi) = \frac{(1+|\xi|^2)^s}{1+|\xi|^{2s}} (\hat{z}_{2h}(\xi) - 2\hat{z}_h(\xi) + \hat{z}(\xi)),$$

provided that

$$z = \frac{\partial u}{\partial x_j}$$

and u solves (7.13).

If $1 < r+2s \leq 2$, then by following the same argument as in STEP 2 we conclude the proof of the lemma.

If $r+2s > 2$ we iterate the above argument and we eventually arrive at (7.3). □

7.2. Proof of Lemma 7.1. We fix $f \in L^2(\mathbb{R}^N)$ and $h \in \mathbb{R}^N$. As usual we term u the weak solution of (1.1) and we define the functions u_h and f_h as in (1.3). We have now all the ingredients required to prove Lemma 7.1. Owing to the translation invariance of the fractional Laplacian, $u_h \in \mathcal{X}_0^s(\Omega - h)$ is the weak solution of

$$\begin{cases} (-\Delta)^s u_h = f_h & \text{in } \Omega - h, \\ u_h = 0 & \text{in } \mathbb{R}^N \setminus (\Omega - h). \end{cases}$$

Here and in the following we use the notation

$$\Omega - h := \{x \in \mathbb{R}^N : x+h \in \Omega\}.$$

Note that, if $|h|$ is sufficiently small (which is not restrictive for our purposes, as it will be clear in the following), then

$$\mathfrak{d}(\Omega - h, \Omega) = e(\Omega - h, \Omega) + e^c(\Omega, \Omega - h) \leq 2|h| \leq \frac{\rho \sin \theta}{2}. \tag{7.16}$$

We term v_h the weak solution of

$$\begin{cases} (-\Delta)^s v_h = f_h & \text{in } \Omega, \\ v_h = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Owing to (7.16), the sets $\Omega_a = \Omega$ and $\Omega_b = \Omega - h$ satisfy (1.10). By applying Lemma 6.1, we conclude that for every $\sigma \in (0, 1)$ we have

$$\|u_h - v_h\|_s \leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f_h\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f_h\|_{L^2(\mathbb{R}^N)}^{1/2} |h|^{\sigma s/2}. \tag{7.17}$$

Next, we consider the function $w := u - v_h$, which satisfies $w \in \mathcal{X}_0^s(\Omega)$. Moreover, by linearity, w is the weak solution of

$$\begin{cases} (-\Delta)^s w = f - f_h & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Using (2.14), we infer

$$\|u - v_h\|_s \stackrel{(2.14)}{\leq} C(N, s, \text{diam}\Omega) \|f - f_h\|_{\mathcal{X}_0^s(\Omega)} \stackrel{(2.11)}{\leq} C(N, s, \text{diam}\Omega) \|f - f_h\|_{H^{-s}(\mathbb{R}^N)}. \tag{7.18}$$

Next, we control $\|f - f_h\|_{H^{-s}(\mathbb{R}^N)}$. We first fix $R > 0$ (to be determined later) and we point out that

$$\begin{aligned} \|f - f_h\|_{H^{-s}(\mathbb{R}^N)}^2 &= C(N) \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} |1 - e^{i\xi \cdot h}|^2 |\hat{f}|^2 d\xi \\ &= C(N) \left(\underbrace{\int_{|\xi| \leq R} (1 + |\xi|^2)^{-s} |1 - e^{i\xi \cdot h}|^2 |\hat{f}|^2 d\xi}_{I_1} + \underbrace{\int_{|\xi| > R} (1 + |\xi|^2)^{-s} |1 - e^{i\xi \cdot h}|^2 |\hat{f}|^2 d\xi}_{I_2} \right). \end{aligned} \tag{7.19}$$

Next, we introduce the decomposition

$$|1 - e^{i\xi \cdot h}|^2 = |1 - e^{i\xi \cdot h}|^{2-s} |1 - e^{i\xi \cdot h}|^s \stackrel{(3.4), (3.5)}{\leq} 8|\xi|^s |h|^s, \tag{7.20}$$

which gives

$$\begin{aligned} I_1 &\stackrel{(7.20)}{\leq} C(N) \int_{|\xi| \leq R} (1 + |\xi|^2)^{-s} |\xi|^s |h|^s |\hat{f}|^2 d\xi \leq C(N) R^s |h|^s \int_{\mathbb{R}^N} (1 + |\xi|^2)^{-s} |\hat{f}|^2 d\xi \\ &\leq C(N) R^s |h|^s \|f\|_{H^{-s}(\mathbb{R}^N)}^2. \end{aligned} \tag{7.21}$$

On the other hand, $(1 + |\xi|^2)^{-s} \leq |\xi|^{-2s}$, whence

$$I_2 \stackrel{(7.20)}{\leq} C(N) \int_{|\xi| > R} |\xi|^{-2s} |\xi|^s |h|^s |\hat{f}|^2 d\xi = C(N) \int_{|\xi| > R} |\xi|^{-s} |h|^s |\hat{f}|^2 d\xi \leq R^{-s} |h|^s \|f\|_{L^2(\mathbb{R}^N)}^2. \tag{7.22}$$

By choosing R in such a way that $R^s = \|f\|_{L^2(\mathbb{R}^N)} / \|f\|_{H^{-s}(\mathbb{R}^N)}$, plugging this equality into (7.21) and (7.22) and by recalling (7.19) we eventually get

$$\|f - f_h\|_{H^{-s}(\mathbb{R}^N)} \leq C(N) \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} |h|^{s/2}. \tag{7.23}$$

By combining (7.17), (7.18) and (7.23) we arrive at

$$\|u - u_h\|_s \leq \|u - v_h\|_s + \|v_h - u_h\|_s$$

$$\begin{aligned}
 &\leq C(N, s, \text{diam}\Omega) \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} |h|^{s/2} \\
 &\quad + C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f_h\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f_h\|_{L^2(\mathbb{R}^N)}^{1/2} |h|^{\sigma s/2} \\
 &\leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2} [|h|^{\sigma s/2} + |h|^{s/2}] \\
 &\leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2} |h|^{\sigma s/2}, \tag{7.24}
 \end{aligned}$$

where to establish the last inequality we used that $|h| \leq 1$ and $\sigma \in (0, 1)$.

We now set $z := (-\Delta)^{s/2}u$ and we point out that, thanks to (2.9), (7.24) implies

$$\|z - z_h\|_{L^2(\mathbb{R}^N)} \leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2} |h|^{\sigma s/2}.$$

We point that $\sigma s/2 \in (0, 1)$ since $s, \sigma \in (0, 1)$ and we recall that in this case the Besov norm can be characterized as in (2.17). We conclude that the above inequality implies

$$z \in B_{2,\infty}^{\sigma s/2}(\mathbb{R}^N), \quad \|z\|_{B_{2,\infty}^{\sigma s/2}(\mathbb{R}^N)} \leq \|z\|_{L^2(\mathbb{R}^N)} + C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2}. \tag{7.25}$$

Note that

$$\begin{aligned}
 \|z\|_{L^2(\mathbb{R}^N)} &= C(N, s) \|u\|_s \stackrel{(2.14)}{\leq} C(N, s, \text{diam}\Omega) \|f\|_{\mathcal{X}_0^s(\Omega)} \stackrel{(2.11)}{\leq} C(N, s, \text{diam}\Omega) \|f\|_{H^{-s}(\mathbb{R}^N)} \\
 &\stackrel{(2.11)}{\leq} C(N, s, \text{diam}\Omega) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2},
 \end{aligned}$$

whence from (7.25) we infer

$$\|z\|_{B_{2,\infty}^{\sigma s/2}(\mathbb{R}^N)} \leq C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2}. \tag{7.26}$$

Finally, we recall that $z := (-\Delta)^{s/2}u$ and we apply Lemma 7.2. We conclude that

$$\begin{aligned}
 u \in B_{2,\infty}^{(\sigma s/2)+s}(\mathbb{R}^N), \quad \|u\|_{B_{2,\infty}^{(\sigma s/2)+s}(\mathbb{R}^N)} &\stackrel{(7.3)}{\leq} C(N, s, \sigma) (\|u\|_{L^2(\mathbb{R}^N)} + \|z\|_{B_{2,\infty}^{\sigma s/2}(\mathbb{R}^N)}) \\
 &\stackrel{(7.26)}{\leq} C(N, s, \text{diam}\Omega, \rho, \theta, \sigma) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2}. \tag{7.27}
 \end{aligned}$$

Since $\sigma \in (0, 1)$, then $B_{2,\infty}^{(\sigma s/2)+s}(\mathbb{R}^N) \subseteq B_{2,\infty}^{3\sigma s/2}(\mathbb{R}^N)$ and the inclusion is continuous. Hence, from (7.27) we infer (7.1) which concludes the proof of the lemma.

8. Conclusion of the bootstrap argument

8.1. Proof of Theorem 1.1. First, we point out that we have already given the proof of (1.7) in § 3.1, so we are left to prove (1.8). To this end, we proceed as in § 3.3 and we point out that

$$\|T_h u - u\|_s^2 \leq |A| + |B|, \tag{8.1}$$

where A and B are as in (3.22). Owing to (3.23),

$$\begin{aligned}
 |B| &\stackrel{(3.23)}{\leq} C(N, s, \text{diam}\Omega) |h|^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{\mathcal{X}_0^s(\Omega)} \stackrel{(2.11)}{\leq} \\
 &\leq C(N, s, \text{diam}\Omega) |h|^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{H^{-s}(\mathbb{R}^N)}. \tag{8.2}
 \end{aligned}$$

Next, we recall (3.25) and we decompose A as

$$A = C(N, s) [I_1 + I_2 + I_3], \tag{8.3}$$

where I_1 , I_2 and I_3 are defined as in (3.26), (3.27) and (3.28), respectively. Owing to (3.33),

$$\begin{aligned} |I_1| &\stackrel{(3.33)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega) |h|^s \|f\|_{\mathcal{X}_0^s(\Omega)}^2 \tag{2.11} \\ &\leq C(N, s, \text{Lip } \phi, \text{diam } \Omega) |h|^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{H^{-s}(\mathbb{R}^N)}. \end{aligned} \tag{8.4}$$

To control I_2 , we first choose $\sigma_1 \in (2/3, 2/(3s)) \neq \emptyset$ so that

$$\frac{3}{2} \sigma_1 s < 1. \tag{8.5}$$

We apply Lemma 7.1 and we recall that, when $r \in (0, 1)$, the $B_{2,\infty}^r$ -norm can be characterized as in (2.17). We conclude that

$$\begin{aligned} \sup_{h \in \mathbb{R}^N \setminus \{0\}} \frac{\|u - u_h\|_{L^2(\mathbb{R}^N)}}{|h|^{3\sigma_1 s/2}} &\stackrel{(2.16), (2.17)}{\leq} C(N, s) \|u\|_{B_{2,\infty}^{3\sigma_1 s/2}(\mathbb{R}^N)} \\ &\stackrel{(7.1)}{\leq} C(N, s, \text{diam } \Omega, \rho, \theta, \sigma_1) \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2}. \end{aligned} \tag{8.6}$$

Next, we choose

$$\sigma_2 = \frac{2}{3\sigma_1} \in (s, 1) \tag{8.7}$$

and by proceeding as in (3.34) we obtain

$$\begin{aligned} I_2 &= \|\mathcal{C}(\phi, u_h - u)\|_{L^2(\mathbb{R}^N)}^2 \stackrel{(3.11)}{\leq} C(N, s, \text{Lip } \phi, \sigma_2) \|u_h - u\|_{L^2(\mathbb{R}^N)}^{2\sigma_2} \|u - u_h\|_s^{2-2\sigma_2} \\ &\leq C(N, s, \text{Lip } \phi, \sigma_2) \|u_h - u\|_{L^2(\mathbb{R}^N)}^{2\sigma_2} \|u\|_s^{2-2\sigma_2} \\ &\stackrel{(2.14)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \sigma_2) \|u_h - u\|_{L^2(\mathbb{R}^N)}^{2\sigma_2} \|f\|_{\mathcal{X}_0^s(\Omega)}^{2-2\sigma_2} \\ &\stackrel{(8.6)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta, \sigma_1, \sigma_2) |h|^{3\sigma_1 \sigma_2 s} \|f\|_{H^{-s}(\mathbb{R}^N)}^{\sigma_2} \|f\|_{L^2(\mathbb{R}^N)}^{\sigma_2} \|f\|_{\mathcal{X}_0^s(\Omega)}^{2-2\sigma_2} \\ &\stackrel{(8.7)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta, \sigma_1, \sigma_2) |h|^{2s} \|f\|_{H^{-s}(\mathbb{R}^N)}^{\sigma_2} \|f\|_{L^2(\mathbb{R}^N)}^{\sigma_2} \|f\|_{\mathcal{X}_0^s(\Omega)}^{2-2\sigma_2} \\ &\stackrel{(2.11)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta, \sigma_1, \sigma_2) |h|^{2s} \|f\|_{H^{-s}(\mathbb{R}^N)}^{\sigma_2} \|f\|_{L^2(\mathbb{R}^N)}^{\sigma_2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1-\sigma_2} \|f\|_{L^2(\mathbb{R}^N)}^{1-\sigma_2} \\ &= C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta, \sigma_1, \sigma_2) |h|^{2s} \|f\|_{H^{-s}(\mathbb{R}^N)} \|f\|_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{8.8}$$

We point out that σ_1 , and consequently σ_2 , can be chosen in such a way that they depend only on s , and we simplify the above estimate to

$$I_2 \leq C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta) |h|^{2s} \|f\|_{H^{-s}(\mathbb{R}^N)} \|f\|_{L^2(\mathbb{R}^N)}. \tag{8.9}$$

To control I_3 , we recall (3.35) and we obtain

$$|I_3| \stackrel{(3.35)}{\leq} 2\sqrt{I_2 (|I_1| + \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2)}$$

$$\begin{aligned}
 &\stackrel{(8.9)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta) |h|^s \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \sqrt{|I_1| + \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^N)}^2} \\
 &\stackrel{(2.14), (8.4)}{\leq} C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta) |h|^s \|f\|_{H^{-s}(\mathbb{R}^N)} \|f\|_{L^2(\mathbb{R}^N)} \sqrt{|h|^s + 1} \\
 &\leq C(N, s, \text{Lip } \phi, \text{diam } \Omega, \rho, \theta) |h|^s \|f\|_{H^{-s}(\mathbb{R}^N)} \|f\|_{L^2(\mathbb{R}^N)}. \tag{8.10}
 \end{aligned}$$

By combining (8.1), (8.2), (8.3), (8.4), (8.9) and (8.10) we eventually arrive at (1.8) and this concludes the proof of Theorem 1.1.

8.2. Proof of Theorem 1.2.

8.2.1. Preliminary results. First, we establish a sharper version of Lemma 5.1.

LEMMA 8.1. *Let $f \in L^2(\mathbb{R}^N)$. Under the same assumptions as in the statement of Lemma 5.1, there is $\tilde{u} \in \mathcal{X}_0^s(\Omega^{-\varepsilon})$ such that (5.2) holds and moreover*

$$\| \tilde{u} - u \|_s^2 \leq C(N, s, \text{diam } \Omega, \rho) \left(\frac{\varepsilon}{\sin \theta} \right)^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{H^{-s}(\mathbb{R}^N)}. \tag{8.11}$$

Proof. We take the same function \tilde{u} as in the proof of Lemma 5.1 (see in particular § 5.1.1). Owing to the analysis in § 5.1.2, $\tilde{u} \in \mathcal{X}_0^s(\Omega^{-\varepsilon})$ and hence we are left to establish (8.11). To this aim, we proceed as in § 5.1.3 and we combine (5.17) and (1.8). We get, for $h = h_i = t\mathbf{n}_i$,

$$\begin{aligned}
 \|u - \tilde{u}\|_s &\stackrel{(5.17)}{\leq} k \max_{i=1, \dots, k} \|\phi_i[u - u_{it}]\|_s \stackrel{(5.11)}{\leq} C(N, \text{diam } \Omega, \rho) \max_{i=1, \dots, k} \|\phi_i[u - u_{it}]\|_s \\
 &\stackrel{(1.3), (5.12)}{=} C(N, \text{diam } \Omega, \rho) \max_{i=1, \dots, k} \|T_{h_i} u - u\|_s \\
 &\stackrel{(1.8), (4.14)}{\leq} C(N, s, \text{diam } \Omega, \rho) \sqrt{|h|^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{H^{-s}(\mathbb{R}^N)}} \\
 &\leq C(N, \text{diam } \Omega, \rho) \sqrt{\left(\frac{\varepsilon}{\sin \theta} \right)^s \|f\|_{L^2(\mathbb{R}^N)} \|f\|_{H^{-s}(\mathbb{R}^N)}},
 \end{aligned}$$

owing to the arbitrariness of $t \in (\varepsilon/\sin \theta, \rho/2)$. This establishes (8.11). □

We now state a sharper version of Lemma 5.2.

LEMMA 8.2. *Let $f \in L^2(\mathbb{R}^N)$. Under the same assumptions as in the statement of Lemma 5.2, there is $\hat{w} \in \mathcal{X}_0^s(\Omega)$ such that (5.6) holds and moreover*

$$\| \hat{w} - w \|_s^2 \leq C(N, s, \text{diam } \Omega, \rho) \left(\frac{\varepsilon}{\sin \theta} \right)^s \|f\|_{L^2(\Omega^\varepsilon)} \|f\|_{H^{-s}(\mathbb{R}^N)}. \tag{8.12}$$

Proof. We take the same function \hat{w} as in the proof of Lemma 5.2, namely we define \hat{w} as in (5.20). By arguing as in the proof of Lemma 8.1 and applying (1.8) we arrive at (8.12). The details are omitted. □

8.2.2. Proof of Theorem 1.2: conclusion. We proceed as in the proof of Lemma 6.1, but we apply Lemma 8.1 and 8.2 instead of Lemma 5.1 and 5.2, respectively.

In particular, in place of (6.7) we get

$$\|u_a - u_a^{-\varepsilon}\|_s \leq C(N, s, \text{diam } \Omega, \rho, \theta) \varepsilon^{s/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2}, \tag{8.13}$$

and, in place of (6.8),

$$\|u_a^\varepsilon - u_a\|_s \leq C(N, s, \text{diam } \Omega, \rho, \theta) \varepsilon^{s/2} \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2}. \tag{8.14}$$

We plug (8.13) and (8.14) into (6.6), we recall that ε can be any number satisfying (6.3) and we eventually arrive at (1.11).

8.3. Proof of Theorem 1.3. We proceed as in the proof of Lemma 7.1, but we apply (1.11) instead of (6.1). In particular, we can replace (7.17) with

$$\|u_h - v_h\|_s \leq C(N, s, \text{diam}\Omega, \rho, \theta) \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} |h|^{s/2} \tag{8.15}$$

and hence we can improve (7.24) to

$$\|u - u_h\|_s \leq C(N, s, \text{diam}\Omega, \rho, \theta) \|f\|_{L^2(\mathbb{R}^N)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} |h|^{s/2}.$$

By arguing as in the proof of Lemma 7.1 from the above inequality we infer that $z := (-\Delta)^{s/2}u$ belongs to the Besov space $B_{2,\infty}^{s/2}$. Thus, by applying Lemma 7.2, we eventually arrive at (1.14).

8.4. An explicit example. In this paragraph we discuss the H^s (and henceforth Besov) regularity of the solution of (1.1) in a specific example. Let us fix $s \in (0, 1)$ and consider the Poisson problem

$$\begin{cases} (-\Delta)^s u = 1 & \text{in } B_1(0), \\ u = 0 & \text{in } \mathbb{R}^N \setminus B_1(0). \end{cases} \tag{8.16}$$

In the above expression, $B_1(0)$ is the unit ball, centered at the origin, of \mathbb{R}^N . The solution u is then given by

$$u(x) = \begin{cases} C(N, s)(1 - |x|^2)^s & \text{if } |x| < 1, \\ 0 & \text{elsewhere.} \end{cases} \tag{8.17}$$

A proof of the above fact is given by Gettoor [11, Theorem 5.2] (cf. also [3] and [20]).

We have the following regularity result:

LEMMA 8.3. *Assume $N = 1$. Let u be the solution of (8.16). Then*

$$u \in H^r(\mathbb{R}) \quad \text{for every } r < s + \frac{1}{2}. \tag{8.18}$$

Note that, owing to (2.21), the above lemma implies, in particular,

$$u \in B_{2,\infty}^r(\mathbb{R}) \quad \text{for every } r < s + \frac{1}{2}. \tag{8.19}$$

We now compare this result with the regularity provided by Theorem 1.3. We set

$$f(x) := \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{elsewhere} \end{cases}$$

and we point out that $f \in L^2(\mathbb{R})$. Theorem 1.3 implies that $u \in B_{2,\infty}^{3s/2}(\mathbb{R})$. Since $s < 1$, then $3s/2 < s + 1/2$, whence, in particular,

$$B_{2,\infty}^r(\mathbb{R}) \subset B_{2,\infty}^{3s/2}(\mathbb{R}) \quad \text{if } 3s/2 < r < s + 1/2.$$

This means that Lemma 8.3 is consistent with Theorem 1.2 since the regularity result established in Lemma 8.3 is stronger than the regularity provided by Theorem 1.2.

Proof. (Proof of Lemma 8.3.) We use the explicit formula (8.17) and we proceed according to the following steps:

STEP 1: we make some preliminary considerations. First, we point out that establishing (8.18) amounts to show that $(1 + |\xi|^2)^{r/2} \hat{u} \in L^2(\mathbb{R})$, or, equivalently,

$$\int_{\mathbb{R}} (1 + |\xi|^2)^r \hat{u}^2(\xi) \, d\xi < +\infty. \tag{8.20}$$

Since

$$\left| \frac{(1 + |\xi|^2)^r}{1 + |\xi|^{2r}} \right| < C(r) \quad \text{for every } \xi \in \mathbb{R},$$

then establishing (8.20) is equivalent to proving

$$\int_{\mathbb{R}} (1 + |\xi|^{2r}) \hat{u}^2(\xi) \, d\xi < +\infty. \tag{8.21}$$

Since $u \in L^2(\mathbb{R})$, then $\hat{u} \in L^2(\mathbb{R})$ and hence (8.21) holds if and only if

$$\int_{\mathbb{R}} |\xi|^{2r} \hat{u}^2(\xi) \, d\xi < +\infty. \tag{8.22}$$

Finally, we point out that

$$\int_{\mathbb{R}} |\xi|^{2r} \hat{u}^2(\xi) \, d\xi = \int_{-2}^2 |\xi|^{2r} \hat{u}^2(\xi) \, d\xi + \int_{|\xi|>2} |\xi|^{2r} \hat{u}^2(\xi) \, d\xi \leq C \|\hat{u}\|_{L^2(\mathbb{R})}^2 + \int_{|\xi|>2} |\xi|^{2r} \hat{u}^2(\xi) \, d\xi \tag{8.23}$$

and this implies that to establish (8.18) it suffices to show that

$$\int_{|\xi|>2} |\xi|^{2r} \hat{u}^2(\xi) \, d\xi < +\infty \quad \text{for every } r < s + \frac{1}{2}. \tag{8.24}$$

STEP 2: we compute the Fourier transform of u . To this end, we note that u is smooth on the interval $(-1, 1)$ and satisfies

$$(1 - x^2)u'(x) + 2sxu(x) = 0 \quad \text{for } x \in (-1, 1). \tag{8.25}$$

It can be shown that then (8.25) holds in fact in the sense of distributions on \mathbb{R} as u is defined by (8.17). Thus, we can take the Fourier transform of both sides of (8.25) and obtain

$$\mathcal{F}((1 - x^2)u'(x) + 2sxu(x)) = 0 \quad \text{in } \mathbb{R}_\xi. \tag{8.26}$$

A straightforward computation ensures that

$$\mathcal{F}(x^2 u') = -\frac{d^2}{d\xi^2} \hat{v}(\xi),$$

provided that $v(x) = u'(x)$. This implies that

$$\mathcal{F}(x^2 u') = -i \left(\xi \frac{d^2}{d\xi^2} \hat{u} + 2 \frac{d}{d\xi} \hat{u} \right).$$

By using the above equality we can re-write (8.25) as

$$\xi \frac{d^2}{d\xi^2} \hat{u} + (2+2s) \frac{d}{d\xi} \hat{u} + \xi \hat{u} = 0 \quad \text{in } \mathbb{R}_\xi. \tag{8.27}$$

Note furthermore that \hat{u} is a smooth function since u is compactly supported.

STEP 3: we only consider the case $\xi \in (2, +\infty)$, since the case $\xi \in (-\infty, -2)$ is analogous. We set $z(\xi) := \xi^{1+s} \hat{u}(\xi)$. Then, noting for simplicity by $v'(\xi)$ the derivative of a generic function $v(\xi)$, by a direct computation we can check that z solves

$$z''(\xi) + \left(1 - \frac{s(1+s)}{\xi^2}\right) z(\xi) = 0 \quad \text{for } \xi \in (2, +\infty). \tag{8.28}$$

By multiplying the above expression times $z'(\xi)$ we then infer

$$\frac{1}{2} \frac{d}{d\xi} \left[(z')^2(\xi) + z^2(\xi) \left(1 - \frac{s(1+s)}{\xi^2}\right) \right] = \frac{s(1+s)z^2(\xi)}{\xi^3} \quad \text{for } \xi \in (2, +\infty). \tag{8.29}$$

Next, we point out that, if $s \in (0, 1)$ and $\xi \in (2, +\infty)$, then

$$\left(1 - \frac{s(1+s)}{\xi^2}\right) \geq \frac{1}{2}. \tag{8.30}$$

We then set

$$m(\xi) := \left[z'(\xi)^2 + z^2(\xi) \left(1 - \frac{s(1+s)}{\xi^2}\right) \right], \tag{8.31}$$

and we obtain the differential inequality

$$m'(\xi) \leq \frac{C}{\xi^3} m(\xi) \quad \text{for } \xi \in (2, +\infty). \tag{8.32}$$

By applying Gronwall’s lemma, we deduce that m , and consequently z , is bounded in the interval $(2, +\infty)$. By performing a similar argument on the interval $(-\infty, -2)$, we then have

$$\int_{|\xi|>2} |\xi|^{2r} \hat{u}^2(\xi) d\xi \leq C \int_{|\xi|>2} |\xi|^{2(r-1-s)} d\xi < +\infty$$

provided that $2(r-1-s) < -1$, namely that $r < s+1/2$. This establishes (8.24) and henceforth (8.18) and concludes the proof of the lemma. \square

9. Proof of Theorem 1.4

9.1. Preliminary results. The following results is well-known. A proof is given, e.g., in [22, Prop. 9] under the additional assumption $2s < N$. Actually, the argument in [22] seems to work for general $s \in (0, 1)$. However, for the reader’s convenience, we provide here a sketch of an alternative proof.

LEMMA 9.1. *Let $\Omega \subset \mathbb{R}^N$ an open and bounded set and let $s \in (0, 1)$. Then the following properties hold:*

- i) The operator $(-\Delta)^s$ admits a diverging sequence of positive eigenvalues*

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \nearrow +\infty \tag{9.1}$$

in Ω . As usual, in (9.1) we count each eigenvalue according to its multiplicity. Note furthermore that the first eigenvalue λ_1 is simple, namely it has multiplicity 1.

ii) The Rayleigh min-max principle holds, namely for every $n \in \mathbb{N}$ we have

$$\lambda_n = \min_{V \in \mathcal{V}(n)} \max_{u \in V \setminus \{0\}} \frac{\|u\|_s^2}{\|u\|_{L^2(\Omega)}^2} = \max_{u \in S_n \setminus \{0\}} \frac{\|u\|_s^2}{\|u\|_{L^2(\mathbb{R}^N)}^2} = \frac{\|u_n\|_s^2}{\|u_n\|_{L^2(\mathbb{R}^N)}^2}. \tag{9.2}$$

In the previous expression, $\mathcal{V}(n)$ is the set of n -dimensional subspaces of $\mathcal{X}_0^s(\Omega)$, u_1, \dots, u_n are the eigenfunctions associated to the eigenvalues $\lambda_1, \dots, \lambda_n$ and S_n is the subspace generated by u_1, \dots, u_n .

Proof. We first establish i). We consider the linear operator $R: L^2(\Omega) \rightarrow \mathcal{X}_0^s(\Omega)$ which maps the function $f \in L^2(\Omega) \subseteq \mathcal{X}_0^s(\Omega)'$ to the weak solution $u = R(f)$ of the Poisson problem (1.1). We start with showing that R is continuous. We recall that the bilinear form $[\cdot, \cdot]_s$ is defined by (2.10) and by plugging u as a test function in (2.13) we get

$$C(N, s) \|u\|_s^2 = [u, u]_s = \langle f, u \rangle \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C(N, s, \text{diam}\Omega) \|f\|_{L^2(\Omega)} \|u\|_s.$$

To establish the last inequality, we have used (2.8). The above inequality implies

$$\|R(f)\|_s \leq C(N, s, \text{diam}\Omega) \|f\|_{L^2(\Omega)}$$

and hence establishes the continuity of R .

Next, we term i the immersion $i: \mathcal{X}_0^s(\Omega) \rightarrow L^2(\Omega)$. Since Ω is bounded, by general results on fractional Sobolev spaces (see [7, Theorem 8.2]) i is a compact map.

Finally, we consider the operator $R \circ i$, which is continuous and compact because it is the composition of a continuous operator with a compact operator. Note furthermore that the operator $R \circ i$ is self-adjoint with respect to the bilinear form $[\cdot, \cdot]_s$, which is a scalar product on $\mathcal{X}_0^s(\Omega)$. Indeed, owing to (2.13), for every $u, v \in \mathcal{X}_0^s(\Omega)$ we have

$$[R(u), v]_s = \langle u, v \rangle = (u, v) = (v, u) = \langle v, u \rangle = [R(v), u]_s = [u, R(v)]_s.$$

We conclude that $R \circ i$ is a compact, self-adjoint operator on a separable Hilbert space and henceforth admits a sequence of eigencouples $\{(\mu_n, u_n)\}_{n \in \mathbb{N}}$ with $\{\mu_n\}$ converging to 0 as $n \rightarrow +\infty$. Namely, for $n \in \mathbb{N}$, we have

$$\mu_n (-\Delta)^s (u_n) = u_n.$$

By using u_n as a test function we then obtain

$$\mu_n C(N, s) \|u_n\|_s^2 = \|u_n\|_{L^2(\Omega)}^2 > 0,$$

where we have also used that $u_n \neq 0$ by definition of eigenvector. The above equality implies that $\mu_n > 0$ for every n . By setting $\lambda_n := 1/\mu_n > 0$ we obtain a sequence of eigenvalues for the operator $(-\Delta)^s$. Note that, for $n \rightarrow +\infty$, λ_n diverges to $+\infty$ because μ_n converges to 0.

Finally, arguing as in the case of the standard Laplace operator, one can prove that the first eigenvalue is simple and that the Rayleigh min-max principle holds. The details are omitted. \square

To state the next result, we have to introduce some notation. First, we fix two open and bounded sets $\Omega_a, \Omega_b \subseteq \mathbb{R}^N$ and an open ball D containing both Ω_a and Ω_b . As in (6.2) we denote by $P_{D \rightarrow \Omega_a}: \mathcal{X}_0^s(D) \rightarrow \mathcal{X}_0^s(\Omega_a)$ the projection operator with respect to the scalar product (2.10). Also, we fix $s \in (0, 1)$ and we term (λ_n^a, u_n^a) , (λ_n^b, u_n^b) the

sequence of eigencouples of the operator $(-\Delta)^s$ in Ω_a and Ω_b , respectively. Finally, we term

$$S_n^b := \text{span} \{u_i^b : i = 1, \dots, n\} \tag{9.3}$$

the subspace generated by the eigenfunctions u_1^b, \dots, u_n^b . Note that $S_n^b \subseteq \mathcal{X}_0^s(\Omega_b) \subseteq \mathcal{X}_0^s(D)$.

The following lemma reduces the problem of controlling the eigenvalues to the problem of controlling the projections of the corresponding eigenfunctions.

LEMMA 9.2. *Fix $n \in \mathbb{N}$ and suppose that there are positive constants $A > 0$ and $0 < B < 1$ such that, for every $u \in S_n^b$,*

$$\|P_{D \rightarrow \Omega_a}(u) - u\|_s^2 \leq A \|u\|_{L^2(\mathbb{R}^N)}^2, \tag{9.4}$$

$$\|P_{D \rightarrow \Omega_a}(u) - u\|_{L^2(\mathbb{R}^N)}^2 \leq B \|u\|_{L^2(\mathbb{R}^N)}^2. \tag{9.5}$$

Then

$$\lambda_n^a - \lambda_n^b \leq \frac{A}{(1 - \sqrt{B})^2}. \tag{9.6}$$

Proof. We simply apply [16, Lemma 15] with

$$H := \mathcal{X}_0^s(D), \quad V_a := \mathcal{X}_0^s(\Omega_a), \quad \mathcal{H}(u, v) := [u, v]_s, \quad h(u, v) := (u, v).$$

□

9.2. Conclusion of the proof of Theorem 1.4. Let n, s and Ω_a, Ω_b be as in the statement of Theorem 1.4. We also fix ε such that

$$0 < d_H^c(\Omega_a, \Omega_b) < \varepsilon < \nu \tag{9.7}$$

and we proceed according to the following steps.

STEP 1: owing to (4.4), condition (9.7) implies $e^c(\Omega_b, \Omega_a) < \varepsilon$. By using Lemma 4.1, we infer

$$\Omega_b^{-\varepsilon} \subseteq \Omega_a. \tag{9.8}$$

We now fix $i = 1, \dots, n$ and consider the i -th eigencouple (λ_i^b, u_i^b) of $(-\Delta)^s$ on Ω_b . We apply Lemma 8.1 and we infer that if ν , and henceforth ε , satisfies (5.1), then there is $\tilde{u} \in \mathcal{X}_0^s(\Omega_b^{-\varepsilon})$ such that

$$\|u_i^b - \tilde{u}\|_s \leq C(N, s, \text{diam } D, \rho, \theta) \varepsilon^{s/2} \lambda_i^b \|u_i^b\|_{L^2(\mathbb{R}^N)}. \tag{9.9}$$

By (9.8), we have $\mathcal{X}_0^s(\Omega_b^{-\varepsilon}) \subseteq \mathcal{X}_0^s(\Omega_a)$. Hence, using (9.9), we get

$$\|P_{D \rightarrow \Omega_a}(u_i^b) - u_i^b\|_s \leq C(N, s, \text{diam } D, \rho, \theta) \varepsilon^{s/2} \lambda_i^b \|u_i^b\|_{L^2(\mathbb{R}^N)}. \tag{9.10}$$

Finally, we recall that by assumption Ω_b contains a ball B_r of radius r . This implies that $\mathcal{X}_0^s(B_r) \subseteq \mathcal{X}_0^s(\Omega_b)$ and thanks to the monotonicity of the eigenvalues with respect to set inclusion (which follows from the Rayleigh min-max principle (9.2)) we conclude that $\lambda_i^b \leq C(N, s, r, i)$. Using (9.10) we finally arrive at

$$\|P_{D \rightarrow \Omega_a}(u_i^b) - u_i^b\|_s \leq C(N, s, \text{diam } D, \rho, \theta, r, i) \varepsilon^{s/2} \|u_i^b\|_{L^2(\mathbb{R}^N)}. \tag{9.11}$$

STEP 2: we fix $u \in S_n^b$ (see (9.3)), namely

$$u = \sum_{i=1}^n z_i u_i^b \tag{9.12}$$

for some $z_1, \dots, z_n \in \mathbb{R}$. We recall that, by construction, the eigenfunctions u_j^b are orthogonal with respect to the scalar product $[\cdot, \cdot]_s$, which implies

$$(u_j^b, u_i^b) = \frac{1}{\lambda_j^b} [u_j^b, u_i^b]_s = 0 \quad \text{if } i \neq j.$$

By using (9.11) we get

$$\begin{aligned} \|P_{D \rightarrow \Omega_a}(u) - u\|_s &= \left\| \sum_{i=1}^n z_i [P_{D \rightarrow \Omega_a}(u_i^b) - u_i^b] \right\|_s \leq \sum_{i=1}^n |z_i| \|P_{D \rightarrow \Omega_a}(u_i^b) - u_i^b\|_s \\ &\leq C(N, s, \text{diam } D, \rho, \theta, r, n) \varepsilon^{s/2} \sum_{j=1}^n |z_j| \|u_j^b\|_{L^2(\mathbb{R}^N)} \\ &\leq C(N, s, \text{diam } D, \rho, \theta, r, n) \varepsilon^{s/2} \|u\|_{L^2(\mathbb{R}^N)}. \end{aligned} \tag{9.13}$$

Owing to (2.8) the above inequality also implies

$$\|P_{D \rightarrow \Omega_a}(u) - u\|_{L^2(\mathbb{R}^N)} \leq C(N, s, \text{diam } D, \rho, \theta, r, n) \varepsilon^{s/2} \|u\|_{L^2(\mathbb{R}^N)}. \tag{9.14}$$

STEP 3: we apply Lemma 9.2. We recall (9.13) and (9.14) and we conclude that the hypotheses are satisfied if we assume that

$$B := C(N, s, \text{diam } D, \rho, \theta, r, n) \varepsilon^s \leq \frac{1}{2} < 1.$$

We can choose the constant ν in the statement of Theorem 1.4 in such a way that the above condition is satisfied for every $\varepsilon < \nu$. By using Lemma 9.2 we conclude that

$$\lambda_n^a - \lambda_n^b \leq \frac{C(N, s, \text{diam } D, \rho, \theta, r, n) \varepsilon^s}{(1 - 1/\sqrt{2})^2} \leq C(N, s, \text{diam } D, \rho, \theta, r, n) \varepsilon^s.$$

Since the above inequality holds for every ε satisfying (9.7), we arrive at

$$\lambda_n^a - \lambda_n^b \leq C(N, s, \text{diam } D, \rho, \theta, r, n) d_H^c(\Omega_a, \Omega_b)^s$$

and by exchanging the roles of Ω_a and Ω_b we eventually conclude the proof of (1.22).

10. Proofs of Proposition 1.2 and Theorem 1.5

In this section, we discuss the stability of the eigenfunctions of $(-\Delta)^s$ with respect to domain perturbation. We first provide the proof of Proposition 1.2.

Proof. (Proof of Proposition 1.2.) We use (1.24) and (1.25) and we infer that $\|u^j\|_s^2 = \lambda^j \rightarrow \lambda$ and $u^j = 0$ in $\mathbb{R}^N \setminus \Omega_j$. This implies that there exists $u \in H^s(\mathbb{R}^N)$ such that, up to subsequences, we have

$$u^j \rightarrow u \quad \text{weakly in } H^s(\mathbb{R}^N) \quad \text{and strongly in } L^2(\mathbb{R}^N).$$

By recalling (1.24), this implies $\|u\|_{L^2(\mathbb{R}^N)} = 1$ and hence $u \neq 0$.

We now show that (u, λ) is an eigencouple for $(-\Delta)^s$ on Ω by proceeding according to the following steps.

STEP 1: we show that $u \in \mathcal{X}_0^s(\Omega)$. Since u belongs to $H^s(\mathbb{R}^N)$, we are left to show $u = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$. To this end, we fix $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \overline{\Omega})$. We claim that

$$\text{supp } \varphi \subset \mathbb{R}^N \setminus \Omega_j \quad \text{for any } j \text{ sufficiently large.}$$

Indeed, $\text{supp } \varphi$ is compactly contained in $\mathbb{R}^N \setminus \overline{\Omega}$. Hence, there exists $\varepsilon_0 > 0$ such that $\text{supp } \varphi \subset \mathbb{R}^N \setminus \overline{\Omega^{\varepsilon_0}}$ (see (4.6) for the definition of Ω^{ε_0}). On the other hand, Assumption (1.23) implies, in particular, that $e(\Omega_j, \Omega) < 1/j$ and hence by using Lemma 4.1 we conclude that $\Omega_j \subseteq \Omega^{1/j}$. In other words, $\text{supp } \varphi \subset \mathbb{R}^N \setminus \overline{\Omega_j}$ for $j > 1/\varepsilon_0$ and this implies

$$\int_{\mathbb{R}^N} u^j(x) \varphi(x) \, dx = 0, \quad \text{for every } j > 1/\varepsilon_0.$$

We let $j \rightarrow \infty$ and we obtain

$$\int_{\mathbb{R}^N} u(x) \varphi(x) \, dx = 0.$$

Owing to the arbitrariness of $\varphi \in C_c^\infty(\mathbb{R}^N \setminus \overline{\Omega})$ and to the fact that $\mathcal{L}^N(\partial\Omega) = 0$, we conclude that u vanishes a.e. in $\mathbb{R}^N \setminus \Omega$ and hence that $u \in \mathcal{X}_0^s(\Omega)$.

STEP 2: we show that $(-\Delta)^s u = \lambda u$ a.e. in Ω . We fix $\varphi \in C_c^\infty(\Omega)$. By using Assumption (1.23), we infer $e^c(\Omega, \Omega_j) < 1/j$ and hence that

$$\text{supp } \varphi \subset \Omega_j \quad \text{for any } j \text{ sufficiently large.}$$

By using φ as a test function in the equation for u^j , we get

$$\int_{\mathbb{R}^N} u^j(x) (-\Delta)^s \varphi(x) \, dx = \lambda^j \int_{\mathbb{R}^N} u^j(x) \varphi(x) \, dx$$

and by passing to the limit as $j \rightarrow \infty$, we arrive at

$$\int_{\mathbb{R}^N} u(x) (-\Delta)^s \varphi(x) \, dx = \lambda \int_{\mathbb{R}^N} u(x) \varphi(x) \, dx.$$

By taking into account the regularity of u and the arbitrariness of $\varphi \in C_c^\infty(\Omega)$, we deduce that $(-\Delta)^s u = \lambda u$ a.e. in Ω , and therefore that (u, λ) is an eigencouple of $(-\Delta)^s$ on Ω . Moreover, u satisfies the relation $\|u\|_s^2 = \lambda \|u\|_{L^2(\mathbb{R}^N)}^2$, which combined with the equality $\|u\|_{L^2(\mathbb{R}^N)}^2 = 1$ gives $\|u\|_s^2 = \lambda$.

Finally, we recall that $u^j \rightarrow u$ weakly in $H^s(\mathbb{R}^N)$ and that $\|u^j\|_s \rightarrow \|u\|_s$, we use the uniform convexity of $H^s(\mathbb{R}^N)$ and we conclude that $u^j \rightarrow u$ strongly in $H^s(\mathbb{R}^N)$. \square

Proposition 1.2 does not provide any information on the rate of the convergence $u^j \rightarrow u$. Indeed, in contrast to the stability result for eigenvalues, the convergence rate for eigenfunctions is not uniquely determined in general. The following remark shows that this happens in particular when the corresponding eigenvalues are not simple.

REMARK 10.1. Let us consider the case when the (geometric) multiplicities of λ^j and λ are two. We denote by u_ℓ^j and u_ℓ ($\ell = 1, 2$) the corresponding eigenfunctions such that $\|u_\ell^j\|_{L^2(\mathbb{R}^N)} = \|u_\ell\|_{L^2(\mathbb{R}^N)} = 1$ and $[u_\ell^j, u_m^j]_s = \lambda^j \delta_{\ell m}$. We furthermore assume

that $u_\ell^j \rightarrow u_\ell$ strongly in $H^s(\mathbb{R}^N)$ for each $\ell=1,2$. Such a situation occurs, e.g., for ball-shaped domains. Note that

$$u^j(x) := (1 - \sigma^j)u_1^j(x) + \sigma^j u_2^j(x) \rightarrow u_1(x) \quad \text{strongly in } H^s(\mathbb{R}^N) \tag{10.1}$$

for every sequence $\{\sigma^j\}$ such that $\sigma^j \rightarrow 0$ as $j \rightarrow \infty$. This implies that u^j is also an eigenfunction corresponding to λ^j over Ω_j . Moreover, one has

$$\|u^j - u_1\|_s \geq \|u^j - u_1^j\|_s - \|u_1^j - u_1\|_s \geq \sigma^j \|u_1^j - u_2^j\|_s - \|u_1^j - u_1\|_s = \sqrt{2\lambda^j} \sigma^j - \|u_1^j - u_1\|_s.$$

If we choose $\{\sigma^j\}$ in such a way that

$$\liminf_{j \rightarrow \infty} \frac{\sigma^j}{\|u_1^j - u_1\|_s} \in (1/\sqrt{2\lambda}, +\infty],$$

then

$$\|u^j - u_1\|_s \geq \kappa \sigma^j \quad \text{for every sufficiently large } j$$

for some constant $\kappa > 0$. This means that we can construct $\{\sigma^j\}$ in such a way that the eigenfunction u^j defined as in (10.1) has an arbitrarily slow rate of convergence to u_1 . Hence, in general, the convergence rate of eigenfunctions is not uniquely determined.

On the other hand, the convergence rate for principal eigenfunctions might be estimated, since the principal eigenvalue is simple and consequently the situation outlined in Remark 10.1 cannot occur. In the general case, we can control the convergence rate of eigenspaces. More precisely, in the rest of this section, we will control a suitable notion of “distance between eigenspaces” by the domain perturbation rate. In particular, we will give a proof of Theorem 1.5.

Throughout the rest of this section, Ω_a, Ω_b are bounded, open sets of \mathbb{R}^N satisfying assumptions i)–iii) in the statement of Theorem 1.4. Let (λ_j^a, e_j^a) denote the j -th eigencouple of $(-\Delta)^s$ on Ω_a (see (1.19)). We can assume that (e_j^a) are an orthonormal basis of $L^2(\Omega_a)$: in particular, $(e_i^a, e_j^a) = \delta_{ij}$ and $[e_i^a, e_j^a]_s = \lambda_j^a \delta_{ij}$ (cf., e.g., [22, Proposition 9]). Moreover, we assume that

$$\lambda_{k-1}^a < \lambda_k^a \leq \lambda_{k+1}^a \leq \dots \leq \lambda_{k+m-1}^a < \lambda_{k+m}^a \tag{10.2}$$

for some $k, m \in \mathbb{N}$ (if $k=1$, we replace λ_{k-1}^a by 0) and we define the m -dimensional space

$$N_{k,m}^a := \text{span}\{e_k^a, e_{k+1}^a, \dots, e_{k+m-1}^a\}.$$

If, for instance, λ_k^a is an eigenvalue with (geometric) multiplicity m , then (10.2) holds true and $N_{k,m}^a$ is the corresponding eigenspace. Note that we equip both $N_{k,m}^a$ and $N_{k,m}^b$ with the norm $\|\cdot\|_s$.

Next, we recall some notions of distance between two subspaces M, N of $\mathcal{X}_0^s(D)$. Let $D \subset \mathbb{R}^N$ be an open ball containing both Ω_a and Ω_b as in assumption ii) of Theorem 1.4. Assume moreover that $\mathcal{X}_0^s(D)$ is endowed with the norm $\|\cdot\|_s$, which is equivalent to $\|\cdot\|_{H^s(\mathbb{R}^N)}$. We define the excess of M from N in $\mathcal{X}_0^s(D)$ by setting

$$e_s(M, N) := \sup_{x \in M, \|x\|_s=1} \text{dist}_s(x, N),$$

where $\text{dist}_s(x, N) := \inf_{y \in N} \|x - y\|_s$. Also, we define the Hausdorff distance between M and N by setting

$$d_{H,s}(M, N) := e_s(M, N) + e_s(N, M).$$

Finally, we define the operator $T_a : \mathcal{X}_0^s(\Omega_a) \rightarrow \mathcal{X}_0^s(\Omega_a)'$ by setting

$$T_a u = f \iff [u, \varphi]_s = \langle f, \varphi \rangle_{\mathcal{X}_0^s(\Omega_a)} \quad \text{for all } \varphi \in \mathcal{X}_0^s(\Omega_a).$$

Note that T_a is bounded, linear and bijective (see (2.14)). By the Open Mapping Theorem, the map $T_a^{-1} : \mathcal{X}_0^s(\Omega_a)' \rightarrow \mathcal{X}_0^s(\Omega_a)$ is a well defined and bounded linear operator. One can analogously define $T_b : \mathcal{X}_0^s(\Omega_b) \rightarrow \mathcal{X}_0^s(\Omega_b)'$ and its inverse $T_b^{-1} : \mathcal{X}_0^s(\Omega_b)' \rightarrow \mathcal{X}_0^s(\Omega_b)$. Moreover, every $v \in L^2(D)$ can be regarded as an element f_v of $\mathcal{X}_0^s(\Omega_a)'$ by setting

$$\langle f_v, \varphi \rangle_{\mathcal{X}_0^s(\Omega_a)} := \int_{\Omega_a} v(x)\varphi(x) dx \quad \text{for every } \varphi \in \mathcal{X}_0^s(\Omega_a).$$

To simplify notation, we will directly write v instead of f_v . Note that T_a^{-1} and T_b^{-1} are also well-defined if $v \in L^2(D)$ (see [19]), namely

$$T_a u = v \iff (-\Delta)^s u = v \text{ in } \Omega_a, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega_a.$$

In the following, T_a^{-1} and T_b^{-1} are often regarded as bounded linear operators from $L^2(D)$ into $\mathcal{X}_0^s(D)$.

The proof of the following lemma is based on an abstract theory due to Feleqi [10, Lemma 2.4]. For the reader's convenience we provide a proof, which is specific to our setting.

LEMMA 10.1. *Assume that condition (10.2) holds for some $k, m \in \mathbb{N}$. Define δ by setting*

$$\delta := \begin{cases} \frac{1}{2} \min \left\{ \frac{1}{\lambda_{k-1}^a} - \frac{1}{\lambda_k^a}, \frac{1}{\lambda_{k+m-1}^a} - \frac{1}{\lambda_{k+m}^a} \right\} & \text{if } k \geq 2, \\ \frac{1}{2} \left(\frac{1}{\lambda_1^a} - \frac{1}{\lambda_2^a} \right) & \text{if } k = 1. \end{cases} \tag{10.3}$$

Then

$$e_s(N_{k,m}^a, N_{k,m}^b) \leq m \max \{ \delta^{-1}, \lambda_{k+m-1}^a \} \left\| (T_a^{-1} - T_b^{-1})|_{N_{k,m}^a} \right\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))},$$

provided that

$$\begin{cases} \max \left\{ \left| \frac{1}{\lambda_{k-1}^b} - \frac{1}{\lambda_{k-1}^a} \right|, \left| \frac{1}{\lambda_{k+m}^b} - \frac{1}{\lambda_{k+m}^a} \right| \right\} < \delta & \text{if } k \geq 2, \\ \left| \frac{1}{\lambda_1^b} - \frac{1}{\lambda_1^a} \right| < \delta & \text{if } k = 1. \end{cases} \tag{10.4}$$

In the previous expression $(T_a^{-1} - T_b^{-1})|_{N_{k,m}^a} : N_{k,m}^a \rightarrow \mathcal{X}_0^s(D)$ denotes the restriction of $T_a^{-1} - T_b^{-1}$ to $N_{k,m}^a$, which is bounded and linear. Also, $\|\cdot\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))}$ denotes the (standard) norm of bounded linear operators from $N_{k,m}^a$ to $\mathcal{X}_0^s(D)$ (see (10.9) below for its definition).

Proof. We first point out that, if M, N are closed subspaces of a given Hilbert space X with inner product $(\cdot, \cdot)_X$, then

$$e_X(M, N) = \|(1 - Q) \circ P\|_{\mathcal{L}(X)}. \tag{10.5}$$

In the previous expression, P and Q are the projection maps from X onto M and N , respectively, and $e_X(M, N)$ is the excess of M from N computed with respect to the norm $\|\cdot\| := \sqrt{(\cdot, \cdot)_X}$. To establish (10.5) we point out that

$$\begin{aligned} \|(1-Q) \circ P\|_{\mathcal{L}(X)} &:= \sup_{u \in X, \|u\|=1} \|(1-Q) \circ Pu\| \\ &= \sup_{v \in M, \|v\|=1} \|(1-Q)v\| = \sup_{v \in M, \|v\|=1} \text{dist}_X(v, N) = e_X(M, N). \end{aligned}$$

In the rest of the proof, we will always choose $X = \mathcal{X}_0^s(D)$, $\|\cdot\| = \|\cdot\|_s$, $e_X(\cdot, \cdot) = e_s(\cdot, \cdot)$, $M = N_{k,m}^a$ and $N = N_{k,m}^b$.

Next, we fix $i = 1, 2, \dots, m$ and observe that

$$\begin{aligned} \left\| (T_a^{-1} - T_b^{-1}) \Big|_{N_{k,m}^a} \right\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))} &= \sup_{f \in N_{k,m}^a, \|f\|_s=1} \left\| (T_a^{-1} - T_b^{-1}) f \right\|_s \\ &\geq \left\| (T_a^{-1} - T_b^{-1}) \frac{e_{k+i-1}^a}{\sqrt{\lambda_{k+i-1}^a}} \right\|_s \\ &= \frac{1}{\sqrt{\lambda_{k+i-1}^a}} \left\| \frac{e_{k+i-1}^a}{\lambda_{k+i-1}^a} - T_b^{-1} e_{k+i-1}^a \right\|_s. \end{aligned} \tag{10.6}$$

Let us define the orthogonal projection $P_{D \rightarrow \Omega_b} : \mathcal{X}_0^s(D) \rightarrow \mathcal{X}_0^s(\Omega_b)$ as in (6.2). By using the fact that $\{e_j^b\}_{j \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega_b)$, we get

$$P_{D \rightarrow \Omega_b}(v) = \sum_{j=1}^{\infty} \frac{[v, e_j^b]_s}{\lambda_j^b} e_j^b \quad \text{for } v \in \mathcal{X}_0^s(D). \tag{10.7}$$

Note that the series at the right-hand side of the above equality is convergent, since $P_{D \rightarrow \Omega_b}(v) \in \mathcal{X}_0^s(\Omega_b)$ and $\{e_j^b\}_{j \in \mathbb{N}}$ is a basis.

Since $e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a) \in \mathcal{X}_0^s(\Omega_b)^\perp$ and $T_b^{-1}(e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)) = 0$, then

$$\begin{aligned} &\left\| \frac{e_{k+i-1}^a}{\lambda_{k+i-1}^a} - T_b^{-1} e_{k+i-1}^a \right\|_s^2 \\ &= \left\| \frac{P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)}{\lambda_{k+i-1}^a} - T_b^{-1} \circ P_{D \rightarrow \Omega_b}(e_{k+i-1}^a) \right. \\ &\quad \left. + \frac{e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)}{\lambda_{k+i-1}^a} - T_b^{-1}(e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)) \right\|_s^2 \\ &= \left\| \frac{P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)}{\lambda_{k+i-1}^a} - T_b^{-1} \circ P_{D \rightarrow \Omega_b}(e_{k+i-1}^a) \right\|_s^2 + \left\| \frac{e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)}{\lambda_{k+i-1}^a} \right\|_s^2 \\ &\stackrel{(10.7)}{=} \left\| \frac{1}{\lambda_{k+i-1}^a} \sum_{j=1}^{\infty} \frac{[e_{k+i-1}^a, e_j^b]_s}{\lambda_j^b} e_j^b - \sum_{j=1}^{\infty} \frac{1}{\lambda_j^b} \frac{[e_{k+i-1}^a, e_j^b]_s}{\lambda_j^b} e_j^b \right\|_s^2 + \left\| \frac{e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)}{\lambda_{k+i-1}^a} \right\|_s^2 \\ &= \sum_{j=1}^{\infty} \left(\frac{1}{\lambda_{k+i-1}^a} - \frac{1}{\lambda_j^b} \right)^2 \frac{[e_{k+i-1}^a, e_j^b]_s^2}{\lambda_j^b} + \left\| \frac{e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)}{\lambda_{k+i-1}^a} \right\|_s^2 \\ &\geq \sum_{j \neq k, \dots, k+m-1} \left(\frac{1}{\lambda_{k+i-1}^a} - \frac{1}{\lambda_j^b} \right)^2 \frac{[e_{k+i-1}^a, e_j^b]_s^2}{\lambda_j^b} + \left\| \frac{e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)}{\lambda_{k+i-1}^a} \right\|_s^2. \end{aligned}$$

Note furthermore that by combining (10.3) and (10.4) we get

$$\left| \frac{1}{\lambda_{k+i-1}^a} - \frac{1}{\lambda_j^b} \right| > \delta \quad \text{if } j \leq k-1 \text{ or } j \geq k+m. \tag{10.8}$$

Indeed, for every $j \geq k+m$ we have

$$\begin{aligned} \left| \frac{1}{\lambda_{k+i-1}^a} - \frac{1}{\lambda_j^b} \right| &\geq \frac{1}{\lambda_{k+m-1}^a} - \frac{1}{\lambda_{k+m}^b} \quad (\text{by } \lambda_{k+i-1}^a \leq \lambda_{k+m-1}^a \text{ and } \lambda_j^b \geq \lambda_{k+m}^b) \\ &\geq \frac{1}{\lambda_{k+m-1}^a} - \frac{1}{\lambda_{k+m}^a} - \left| \frac{1}{\lambda_{k+m}^b} - \frac{1}{\lambda_{k+m}^a} \right| \\ &\stackrel{(10.3), (10.4)}{>} 2\delta - \delta = \delta. \end{aligned}$$

By using an analogous argument we can establish (10.8) when $j \leq k-1$. Also, we have

$$\begin{aligned} \sum_{j \neq k, \dots, k+m-1} \frac{[e_{k+i-1}^a, e_j^b]_s^2}{\lambda_j^b} &\stackrel{(10.7)}{=} \sum_{j \neq k, \dots, k+m-1} \frac{[P_{D \rightarrow \Omega_b}(e_{k+i-1}^a), e_j^b]_s^2}{\lambda_j^b} \\ &= \|(1-Q) \circ P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)\|_s^2 \end{aligned}$$

and $e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a) = (1-Q)(e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)) \in \mathcal{X}_0^s(\Omega_b)^\perp$. By going back to (10.6) and using the above relations, we obtain

$$\begin{aligned} &\left\| (T_a^{-1} - T_b^{-1}) \Big|_{N_{k,m}^a} \right\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))}^2 \\ &\geq \frac{1}{\lambda_{k+i-1}^a} \left[\delta^2 \underbrace{\|(1-Q) \circ P_{D \rightarrow \Omega_b}(e_{k+i-1}^a)\|_s^2}_{\in \mathcal{X}_0^s(\Omega_b)} \right. \\ &\quad \left. + \left(\frac{1}{\lambda_{k+i-1}^a} \right)^2 \underbrace{\|(1-Q)(e_{k+i-1}^a - P_{D \rightarrow \Omega_b}(e_{k+i-1}^a))\|_s^2}_{\in \mathcal{X}_0^s(\Omega_b)^\perp} \right] \\ &\geq \frac{1}{\lambda_{k+i-1}^a} \min \left\{ \delta^2, \left(\frac{1}{\lambda_{k+i-1}^a} \right)^2 \right\} \|(1-Q)e_{k+i-1}^a\|_s^2. \end{aligned}$$

By using (10.5), we deduce that

$$\begin{aligned} e_s(N_{k,m}^a, N_{k,m}^b) &= \|(1-Q) \circ P\|_{\mathcal{L}(\mathcal{X}_0^s(D))} \\ &= \sup_{u \in \mathcal{X}_0^s(D), \|u\|_s=1} \|(1-Q) \circ Pu\|_s \\ &= \sup_{u \in \mathcal{X}_0^s(D), \|u\|_s=1} \left\| (1-Q) \sum_{i=1}^m \frac{[u, e_{k+i-1}^a]_s}{\lambda_{k+i-1}^a} e_{k+i-1}^a \right\|_s \\ &\leq \sup_{u \in \mathcal{X}_0^s(D), \|u\|_s=1} \sum_{i=1}^m \frac{|[u, e_{k+i-1}^a]_s|}{\lambda_{k+i-1}^a} \|(1-Q)e_{k+i-1}^a\|_s \\ &\leq \sup_{u \in \mathcal{X}_0^s(D), \|u\|_s=1} \sum_{i=1}^m \|u\|_s \frac{\|e_{k+i-1}^a\|_s}{\lambda_{k+i-1}^a} \|(1-Q)e_{k+i-1}^a\|_s \\ &\leq \sum_{i=1}^m \frac{1}{\sqrt{\lambda_{k+i-1}^a}} \|(1-Q)e_{k+i-1}^a\|_s \\ &\leq m \max \{ \delta^{-1}, \lambda_{k+m-1}^a \} \left\| (T_a^{-1} - T_b^{-1}) \Big|_{N_{k,m}^a} \right\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))} \end{aligned}$$

and this completes the proof. □

In the following we use this lemma:

LEMMA 10.2. *Under the same assumptions as in Theorem 1.2, we have*

$$\left\| (T_a^{-1} - T_b^{-1}) \Big|_{N_{k,m}^a} \right\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))} \leq C(N, s, \rho, \theta, \text{diam } D) \mathfrak{d}(\Omega_b, \Omega_a)^{s/2}.$$

Proof. We observe that

$$\begin{aligned} \left\| (T_a^{-1} - T_b^{-1}) \Big|_{N_{k,m}^a} \right\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))} &:= \sup_{f \in N_{k,m}^a, \|f\|_s=1} \|(T_a^{-1} - T_b^{-1})f\|_s \\ &= \sup_{f \in N_{k,m}^a, \|f\|_s=1} \|u_a - u_b\|_s, \end{aligned} \tag{10.9}$$

where $u_a := T_a^{-1}f$ and $u_b := T_b^{-1}f$. By applying Theorem 1.2, we deduce that

$$\begin{aligned} &\left\| (T_a^{-1} - T_b^{-1}) \Big|_{N_{k,m}^a} \right\|_{\mathcal{L}(N_{k,m}^a, \mathcal{X}_0^s(D))} \\ &\leq C(N, s, \rho, \theta, \text{diam } D) \sup_{f \in N_{k,m}^a, \|f\|_s=1} \|f\|_{L^2(D)}^{1/2} \|f\|_{H^{-s}(\mathbb{R}^N)}^{1/2} \mathfrak{d}(\Omega_b, \Omega_a)^{s/2} \\ &\stackrel{(2.8), (2.11)}{\leq} C(N, s, \rho, \theta, \text{diam } D) \mathfrak{d}(\Omega_b, \Omega_a)^{s/2}, \end{aligned}$$

which is the desired result. □

We can now state our main results concerning the eigenspace stability.

THEOREM 10.1. *Let assumptions i)–iii) in Theorem 1.4 and (1.10) hold. Assume furthermore that (10.2) holds for some $k, m \in \mathbb{N}$ and let $\delta > 0$ be the same as in (10.3). Let $\lambda_1(D) > 0$ denote the principal eigenvalue for $(-\Delta)^s$ on D . Then there is a positive constant ν , only depending on $N, s, \rho, \theta, \text{diam } D, r, k, m, \lambda_1(D)$ and δ , such that, if $d_H^c(\Omega_a, \Omega_b) < \nu$, then*

$$e_s(N_{k,m}^a, N_{k,m}^b) \leq C(N, s, \rho, \theta, \text{diam } D) m \max\{\delta^{-1}, \lambda_{k+m-1}^a\} \mathfrak{d}(\Omega_b, \Omega_a)^{s/2}. \tag{10.10}$$

Also, let $\lambda_j(B_r)$ denote the j -th eigenvalue of $(-\Delta)^s$ on B_r . If, in addition, $\mathfrak{d}(\Omega_a, \Omega_b) < (\rho \sin \theta)/2$, then we also have

$$\begin{aligned} &d_{H,s}(N_{k,m}^a, N_{k,m}^b) \\ &\leq C(N, s, \rho, \theta, \text{diam } D) m \max\{\delta^{-1}, \lambda_{k+m-1}(B_r)\} (\mathfrak{d}(\Omega_b, \Omega_a) + \mathfrak{d}(\Omega_a, \Omega_b))^{s/2}. \end{aligned} \tag{10.11}$$

Proof. By recalling that $\lambda_j^a \geq \lambda_1^a \geq \lambda_1(D) > 0$ we get that

$$\left| \frac{1}{\lambda_j^b} - \frac{1}{\lambda_j^a} \right| = \left| \frac{\lambda_j^a - \lambda_j^b}{\lambda_j^a \lambda_j^b} \right| \leq \frac{1}{\lambda_1(D)^2} |\lambda_j^a - \lambda_j^b| \rightarrow 0 \quad \text{for } j = k-1, k+m \text{ as } d_H^c(\Omega_a, \Omega_b) \rightarrow 0_+.$$

This implies that (10.4) holds true if $\nu > 0$ is small enough (the smallness threshold of ν may also depend on $\lambda_1(D)$ and δ). By applying Lemmas 10.1 and 10.2, we get (10.10).

If, in addition, $\mathfrak{d}(\Omega_a, \Omega_b) < (\rho \sin \theta)/2$, then by switching a and b and by repeating the same argument as before, we obtain

$$e_s(N_{k,m}^b, N_{k,m}^a) \leq C(N, s, \rho, \theta, \text{diam } D) m \max\{\delta^{-1}, \lambda_{k+m-1}^b\} \mathfrak{d}(\Omega_a, \Omega_b)^{s/2}$$

(with same ν). By using the fact that $\lambda_{k+m-1}^a, \lambda_{k+m-1}^b \leq \lambda_{k+m-1}(B_r)$ and adding the above inequality to (10.10), we eventually establish (10.11). \square

We can now give the proof of Theorem 1.5.

Proof. (Proof of Theorem 1.5.) Since the first eigenvalue is simple, assumption (10.4) is satisfied with $k = m = 1$ provided $\nu > 0$ is small enough (the smallness threshold of ν may again depend on $\lambda_1(D)$ and δ). By applying Theorem 10.1 we conclude that

$$e_s(N_{1,1}^a, N_{1,1}^b) \leq C \max \{ \delta^{-1}, \lambda_1^a \} \mathfrak{d}(\Omega_b, \Omega_a)^{s/2}. \tag{10.12}$$

Note furthermore that

$$\begin{aligned} e_s(N_{1,1}^a, N_{1,1}^b) &= \max \left\{ \inf_{c \in \mathbb{R}} \|e_1^a - ce_1^b\|_s, \inf_{c \in \mathbb{R}} \|-e_1^a - ce_1^b\|_s \right\} \\ &= \inf_{c \in \mathbb{R}} \|e_1^a - ce_1^b\|_s = \left\| e_1^a - \frac{[e_1^a, e_1^b]_s}{\lambda_1^b} e_1^b \right\|_s = \sqrt{\lambda_1^a - \frac{[e_1^a, e_1^b]_s^2}{\lambda_1^b}}, \end{aligned}$$

which implies

$$\left\| \frac{e_1^a}{\sqrt{\lambda_1^a}} - \frac{e_1^b}{\sqrt{\lambda_1^b}} \right\|_s^2 = 2 \left(1 - \left[\frac{e_1^a}{\sqrt{\lambda_1^a}}, \frac{e_1^b}{\sqrt{\lambda_1^b}} \right]_s \right) \leq 2 \left(1 - \left[\frac{e_1^a}{\sqrt{\lambda_1^a}}, \frac{e_1^b}{\sqrt{\lambda_1^b}} \right]_s^2 \right) = \frac{2}{\lambda_1^a} e_s(N_{1,1}^a, N_{1,1}^b)^2.$$

By combining the above equality with (10.12) we arrive at

$$\begin{aligned} \left\| \frac{e_1^a}{\sqrt{\lambda_1^a}} - \frac{e_1^b}{\sqrt{\lambda_1^b}} \right\|_s &\leq \frac{C}{\sqrt{\lambda_1^a}} \max \{ \delta^{-1}, \lambda_1^a \} \mathfrak{d}(\Omega_b, \Omega_a)^{s/2} \\ &\leq \frac{C}{\sqrt{\lambda_1(D)}} \max \{ \delta^{-1}, \lambda_1(B_r) \} \mathfrak{d}(\Omega_b, \Omega_a)^{s/2}. \end{aligned}$$

On the other hand, by switching a and b , we obtain

$$\left\| \frac{e_1^b}{\sqrt{\lambda_1^b}} - \frac{e_1^a}{\sqrt{\lambda_1^a}} \right\|_s \leq \frac{C}{\sqrt{\lambda_1(D)}} \max \{ \delta^{-1}, \lambda_1(B_r) \} \mathfrak{d}(\Omega_a, \Omega_b)^{s/2}.$$

Taking the minimum of the right-hand side of both the previous inequalities, we eventually establish (1.29). \square

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