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RESETTING OF A PARTICLE SYSTEM FOR THE NAVIER–STOKES EQUATIONS*

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Abstract. This work is based on a formulation of the incompressible Navier–Stokes equations developed by P. Constantin and G. Iyer, where the velocity field of a viscous incompressible fluid is written as the expected value of a stochastic process. If we take N copies of the above process (each based on independent Wiener processes), and replace the expected value with the empirical mean, then it was shown that the particle system for the Navier–Stokes equations does not dissipate all its energy as time goes to infinity. This is in contrast to the true (unforced) Navier–Stokes equations, which dissipate all of their energy as time goes to infinity. The objective of this short note is to describe a resetting procedure that removes this deficiency. We prove that if we repeat this resetting procedure often enough, then the new particle system for the Navier–Stokes equations dissipates all its energy.

Keywords. stochastic Lagrangian formulation; particle system; resetting.

AMS subject classifications. 60F10; 60J75; 62P10; 92C37.

1. Introduction

The Navier–Stokes equations:

$$\partial_t u + (u\cdot\nabla)u - \nu\Delta u + \nabla p = 0, \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

are fundamental partial differential equations in fluid dynamics which describe the evolution of the velocity field u of an incompressible fluid. Where $\nu > 0$ denotes the kinematic viscosity, and p denotes the pressure. When the viscosity vanishes, we end up with the incompressible Euler equations:

$$\partial_t u + (u\cdot\nabla)u + \nabla p = 0, \tag{1.3}$$

$$\nabla\cdot u = 0, \tag{1.4}$$

which describe the motion of an ideal incompressible fluid. The mathematical theory of these equations have been extensively studied and the existence of regular solutions is still an open problem in PDE theory [5,9]. We are interested in developing probabilistic techniques, that could help solve this problem.

Probabilistic representations of solutions of partial differential equations as the expected value of functionals of stochastic processes date back to the work of Einstein, Feynman, Kac, and Kolmogorov in physics and mathematics. The Feynman–Kac formula is the most well-known example, which has provided a link between linear parabolic partial differential equations and probability theory [13,15]. These stochastic representation methods have provided in some cases tools to show existence and uniqueness of solutions to partial differential equations. For nonlinear partial differential equations the earliest work was done by McKean [14], where a probabilistic representation of the

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solution for the nonlinear Kolmogorov–Petrovsky–Piskunov equation was given. The theory for nonlinear partial differential equations, however, is far less understood.

The questions studied in this work are motivated by the development of a probabilistic formulation of (1.1)-(1.2) proposed by P. Constantin and G. Iyer in [6]. There, the Navier–Stokes equation is interpreted as the average of a stochastic perturbation of the Euler equation. More specifically, a Weber formula is used to represent the velocity of the inviscid equation in terms of the particle trajectories of the inviscid equation without including time derivatives, then the classical Lagrangian trajectories are replaced by stochastic flows. Averaging these stochastic trajectories gives us the solution of (1.1)-(1.2).

In [11] G. Iyer and J. Mattingly used a Monte–Carlo method to approximate the described probabilistic formulation. They took N independent copies of the Wiener process and replaced the expected value in the above formalism with its empirical mean, $\frac{1}{N}$ times the sum over these N independent copies (we review the details of this method in Section 2). By the law of the large numbers, it is natural to expect that any average could be replaced by its empirical mean: $1/N\sum_{i=1}^{N}X_i\approx\mathbb{E}(X)$, where X and X_i, $i=1,\dots,N$ are i.i.d. It turns out that a straightforward approximation of this average by its empirical mean is not adequate here. It was shown in [11] that in two dimensions the N-particle system for the Navier–Stokes equations does not dissipate all its energy as $t\to\infty$. In contrast, the solution of the corresponding Navier–Stokes equations does dissipate all of its energy as $t\to\infty$. The goal of this paper is to alleviate this deficiency of the particle system developed in [11] by modifying it, so that the modified particle system dissipates all of its energy.

Our modification is inspired by [12], where G. Iyer and A. Novikov studied a particle system formulation for the Burgers equation. There we can find another example, where the particle system does not fully mimic the properties of the corresponding PDE. Namely, the viscous Burgers equation does not develop shocks, but the corresponding N-particle system shocks almost surely in finite time. In order to remove these shocks, the authors [12] considered a resetting procedure that prevented their formation. In this paper we propose another resetting procedure, such that the particle system for the Navier–Stokes equations dissipate all its energy.

The particle system in [11] does not completely dissipate its energy, because, roughly speaking, the gradients of the velocities for the N particles become decorrelated with time. We reinforce correlation of these velocities and their gradients by resetting, and this allows complete dissipation of energy. When the resetting condition holds, then the particle system dissipates its energy exponentially. Once the resetting condition is not satisfied, we reset the particle system, and restart the procedure again. In addition to the exponential dissipation of energy, the resetting procedure itself adds more dissipation each time we average our data. Our theorem states that if we keep repeating the resetting procedure, the particle system will dissipate all its energy.

We now highlight briefly some other probabilistic formulations for the Navier–Stokes equations and related work. The initial work on probabilistic representation of the Navier–Stokes equations was done by A. Chorin. It was shown in [1] that in two dimensions vorticity evolves according to Fokker–Planck type equation. Using this A. Chorin gave a probabilistic representation for the vorticity using random walks and a particle limit, and then related it to the velocity vector using the Biot-Savart Law. Using a different approach Y. Le Jan and A. Sznitman in [17] developed a probabilistic representation of Navier–Stokes equations, where they used a backward-in-time branching process in Fourier space to express the velocity of the three dimensional viscous fluid as the average of a stochastic process. These works do not allow to develop a self contained well-posedness theory. Here our motivation is to develop new probabilistic techniques that may help us establish regularity theory for partial differential equations of Fluid Dynamics. The stochastic representation in [6] allows such theory, and we, therefore, chose to work with it.

Building on the work done by P. Constantin and G. Iyer in [6], X. Zhang in [18] gave a stochastic representation for the backward incompressible Navier–Stokes equation using stochastic Lagrangian path. Later, linking the work of X. Zhang in [18] with P. Constantin and G. Iyer in [6], F. Delbaen, J. Qiu, and S. Tang in [7] developed a coupled forward-backward stochastic differential system in the space of fields for the incompressible Navier–Stokes equation in the whole space. Using probabilistic tools, they were able to obtain local uniqueness results for the forward-backward stochastic differential system. In addition, they were able to show the existence of global solutions for the case with small Reynolds number or when the dimension is two. We also mention [2], where F. Cipriano and A. Cruzeiro, using the Brownian motions on the group of homeomorphisms on the torus, established a stochastic variational principle for the two dimensional Navier–Stokes equations. Moreover, A. Cruzeiro and E. Shamarova in [3] formulated a connection between the Navier–Stokes equations and a system of forward-backward stochastic differential equations on the group of volume-preserving diffeomorphisms of a flat torus.

This paper is organized as follows. In the next section, Section 2, we describe the stochastic-Lagrangian representation of the Navier–Stokes equations and construct the particle system. In addition, we explain the resetting scheme which will then be used in our main theorem, Theorem 2.1. Finally, in Section 2, we study the energy of the Navier–Stokes's particle system and we use the resetting scheme to show that by repeating it often enough, the particle system for the Navier–Stokes equations dissipates all its energy.

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2. The particle system, resetting, and energy decay

First in this section, we construct the particle system for the Navier–Stokes equations based on stochastic Lagrangian trajectories. After that, we will describe resetting scheme which will be used in Theorem 2.1. Finally, at the end of this section, we will prove Theorem 2.1. We first begin by describing a stochastic Lagrangian formulation of the Navier–Stokes equations [11].

Let B_t be a standard 2 or 3-dimensional Brownian motion on a torus $\mathbb{T}$, and let u_0 be some given periodic, and divergence free $C^{k,\alpha}$ initial data. Let $\mathbb{E}$ denote the expected value with respect to the Wiener measure and $\mathbf{P}$ be the Leray–Hodge projection onto divergence free vector fields. Consider the system of equations

$$dX_t(x) = u_t(X_t(x))\,dt + \sqrt{2\nu}\,dB_t, \qquad X_0(x) = x, \tag{2.1}$$

$$u_t = \mathbb{E}\mathbf{P}\left[(\nabla^* Y_t)(u_0 \circ Y_t)\right], \qquad Y_t = X_t^{-1}. \tag{2.2}$$

Above X_t is the stochastic flow of diffeomorphisms on $\mathbb{T}$, and we denote $Y_t = X_t^{-1}$ to be the spatial inverse of X_t for any given time $t \geqslant 0$. We denote $\nabla^* Y_t$ to be the transpose of the Jacobian of Y_t.

It was shown in [6] that if the initial data $u_0 \in C^{k,\alpha}$ is a deterministic divergence free vector field with $k \geqslant 2$ and if we impose periodic boundary conditions on u_t and

$X_t - \mathbb{I}$, then for a short time the system (2.1)-(2.2) is equivalent to the incompressible Navier–Stokes equations, that is, u_t satisfies

$$\partial_t u_t + (u_t \cdot \nabla) u_t - \nu \Delta u_t + \nabla p = 0, \ \nabla \cdot u_t = 0. \tag{2.3}$$

In the case when the viscosity is zero $\nu = 0$, the equations (2.1)-(2.2) are the Lagrangian formulation for the incompressible Euler equation developed in [4]. Note that we need the law of the entire flow X in order to compute u, this is due to the fact that the term $\nabla^* Y$ is present in (2.2). In order to approximate the system (2.1)-(2.2), we replace the flow X_t with N different copies X_t^i where each one is driven independently by a Wiener process B_t^i, $i = 1,2,\ldots,N$. Fix a (sufficiently large) N, and we end up with the following approximate system

$$dX_t^i = u_t\left(X_t^i\right) dt + \sqrt{2\nu}\, dB_t^i, \quad Y_t^i = \left(X_t^i\right)^{-1}, \tag{2.4}$$

$$u_t = \frac{1}{N} \sum_{i=1}^{N} u_t^i, \quad u_t^i = \mathbf{P}[(\nabla^* Y_t^i) u_0 \circ Y_t^i], \tag{2.5}$$

with initial data $X_0(x) = x$. We impose periodic boundary conditions on the initial data u_0, and the displacement $\lambda_t^i(x) = X_t^I(x) - x$.

The following Lemma describes the evolution of the velocity of the particle system (2.5) as SPDE.

LEMMA 2.1 (Lemma 4.2 in [11]). *Let $u_t^i = \mathbf{P}[(\nabla^* Y_t^i) u_0 \circ Y_t^i]$ be the i^{th} summand in (2.5). Then u_t^i satisfies the SPDE*

$$du_t^i + \left[(u_t \cdot \nabla) u_t^i - \nu \triangle u_t^i + (\nabla^* u_t) u_t^i + \nabla p_t^i\right] dt + \sqrt{2\nu} \nabla u_t^i dB_t^i = 0, \tag{2.6}$$

and u_t satisfies the SPDE

$$du_t + [(u_t \cdot \nabla) u_t - \nu \triangle u_t + \nabla p_t]\, dt + \frac{\sqrt{2\nu}}{N} \sum_{i=1}^{N} \nabla u_t^i dB_t^i = 0. \tag{2.7}$$

In contrast to the true Navier–Stokes equations (1.1)-(1.2), the particle system (2.4)-(2.5), for any finite N, may not dissipate all of its energy as $t \to \infty$. This was proven for two dimensions in [11]. In this work, we propose to alleviate this deficiency by considering a resetting scheme, in which we start by solving the system (2.4)-(2.5) on the interval $(t_0, t_1]$, where the resetting time t_1 is specified later according to our proposed resetting condition (2.13) below. Next, we average our data, replace the original initial data with $u_{t_1}^N$, and restart the system (2.4)-(2.5) for the next time interval using this new initial data. We keep repeating this procedure on each interval $(t_m, t_{m+1}]$ for $m \in \mathbb{N}$.

The resetting criterion comes from the comparison of the rate of change of the energies of the true Navier–Stokes equations and the particle system. Namely, the rate of change of the energy of our particle system (2.4)-(2.5) is (see Theorem 2.1 below) as follows:

$$\frac{1}{2\nu} \partial_t \mathbb{E} \|u_t\|_{L^2}^2 = -\frac{1}{N^2} \sum_{i \neq j}^{N} \mathbb{E}\left[\langle \nabla u_t^j, \nabla u_t^i \rangle\right]. \tag{2.8}$$

Observe that the rate of change of energy depends on the average of inner products of the gradients of N velocities. In contrast, for the true Navier–Stokes equations, the rate of change of the energy is:

$$\frac{1}{2\nu} \partial_t \|u_t\|_{L^2}^2 = -\|\nabla u_t\|_{L^2}^2. \tag{2.9}$$

For large N the right-hand sides of (2.8) and (2.9) are essentially[1] the same, if $\nabla u_t^j = \nabla u_t^i$ for all i and j. This observation motivates our approach. We will use resetting to keep the sum of the expected value of the inner products

$$\sum_{i\neq j}^{N} \mathbb{E}\left[\langle \nabla u_t^j, \nabla u_t^i\rangle\right] \geqslant cN^2\,\mathbb{E}\|u_t\|_{L^2}^2, \tag{2.10}$$

on each interval $(t_m, t_{m+1}]$ where $m \in \mathbb{N}$ and for some constant $c > 0$ that does not depend on N. This will make the inner products in (2.8) to be positive and thus the rate of energy dissipation will be negative on each interval $(t_m, t_{m+1}]$. Therefore we consider the following resetting system

$$dX_t^i = u_t\left(X_t^i\right)dt + \sqrt{2\nu}\,dB_t^i, \quad X_{t_m}^i(x_0) = x_0, \quad Y_t^i = \left(X_t^i\right)^{-1}, \tag{2.11}$$

$$u_t = \frac{1}{N}\sum_{i=1}^{N} u_t^i, \quad u_t^i = \mathbf{P}\left[(\nabla^* Y_t^i)(u_{t_m}\circ Y_t^i)\right], \quad \text{for } \ t \in (t_m, t_{m+1}], \tag{2.12}$$

where $m \in \mathbb{N}$, the set of non-negative integers, $t_0 = 0$, and the resetting times are defined recursively

$$t_m = \inf\left\{t > t_{m-1} : \sum_{i\neq j}^{N} \mathbb{E}\left[\langle \nabla u_t^i, \nabla u_t^j\rangle\right] < (1-\varepsilon)N(N-1)\left\|\nabla u_{t_{m-1}}\right\|_{L^2}^2\right\}, \tag{2.13}$$

for some positive fixed $\varepsilon < 1$. We say we reset the system at every $t = t_m$, because we treat u_{t_m} as initial conditions for each of the intervals $t \in [t_m, t_{m+1})$.

THEOREM 2.1. *Suppose we are in two dimensions. Let the initial condition u_0 be a $\mathcal{F}_0$-measurable, periodic mean zero function such that the norm $\|u_0\|_{1,\alpha}$, $\alpha > 0$ is almost surely bounded. If we let $\{t_m\}_{m=0}^{\infty}$ to be the sequence of resetting times defined in (2.13), then the particle system with resetting (2.11)-(2.12) dissipates all its energy.*

We remark that the particle system with resetting (2.11)-(2.12) dissipates its energy using two mechanisms. It dissipates energy exponentially anytime the inequality (2.10) holds, and it dissipates energy when we average the initial data each time we reset the particle system (2.11)-(2.12). We also want to highlight that the manner in which we defined our resetting times in (2.13) causes the length of time increments $\delta_m = t_m - t_{m-1}$ to vary among resetting intervals. This gives rise to the case that if the sequence of time increments δ_m decays to zero too fast, then the limit of the sequence of resetting times t_m can tend to some finite time T, as $m \to \infty$. Thus, in order to prove our theorem, we have to consider two cases. The first case is when the limit of sequence of resetting times t_m tends to ∞, as $m \to \infty$. In this case, we show that the particle system with resetting (2.11)-(2.12) dissipates its energy mainly exponentially. The second case is when the limit of the sequence of resetting times t_m tends to a time $T < \infty$, as $m \to \infty$. In this case, we show that the system (2.11)-(2.12) dissipates its energy mainly by averaging of the initial data each time we reset.

Proof. (**Proof of Theorem 2.1.**) By Lemma 2.1, u_t^i satisfies the SPDE

$$du_t^i + \left[(u_t\cdot\nabla)u_t^i - \nu\triangle u_t^i + (\nabla^* u_t)u_t^i + \nabla p_t^i\right]dt + \sqrt{2\nu}\,\nabla u_t^i\,dB_t^i = 0, \tag{2.14}$$

[1] If $\nabla u_t^j = \nabla u_t^i$, then the right-hand side of (2.8) is $(N-1)N^{-1}\|\nabla u_t\|_{L^2}^2 \to \|\nabla u_t\|_{L^2}^2$, as $N \to \infty$.

on each interval $t\in?(t_m,t_{m+1}]$. In two dimensions, the vorticity $\omega_t^i=\nabla\times u_t^i$ solves

$$d\omega_t^i+[(u_t\cdot\nabla)\omega^i-\nu\Delta\omega_t^i]\,dt+\sqrt{2\nu}\nabla\omega_t^i\,dB_t^i=0. \tag{2.15}$$

By Itô's formula, we obtain

$$\frac{1}{2}d\left|\omega_t^i\right|^2+\omega_t^i\cdot\left[(u_t\cdot\nabla)\omega_t^i-\nu\triangle\omega_t^i\right]dt+\sqrt{2\nu}\omega_t^i\cdot(\nabla\omega_t^i\,dB_t^i)-\nu\left|\nabla\omega_t^i\right|^2dt=0. \tag{2.16}$$

Integrating in space and using the fact that u_t^i and u_t are divergences free, we have $d\|\omega_t^i\|_{L^2}^2=0$ for all $t\in[t_m,t_{m+1})$. Thus, the norm of all the vorticities is preserved on such time-intervals. Since $\|\omega_t^i\|_{L^2}^2=\|\nabla u_t^i\|_{L^2}^2$, we have

$$\|\nabla u_{t_m}\|_{L^2}^2=\left\|\nabla u_t^1\right\|_{L^2}^2=\left\|\nabla u_t^2\right\|_{L^2}^2=\cdots=\left\|\nabla u_t^N\right\|_{L^2}^2, \tag{2.17}$$

for $t\in(t_m,t_{m+1}]$.

Using Lemma 2.1 and Itô's formula, we also have

$$\frac{1}{2}d|u_t|^2+u_t\cdot[(u_t\cdot\nabla)u_t-\nu\triangle u_t+\nabla p]\,dt+\frac{\sqrt{2\nu}}{N}\sum_{i=1}^N u_t\cdot(\nabla u_t^i\,dB_t^i)=\frac{\nu}{N^2}\sum_{i=1}^N\left|\nabla u_t^i\right|^2dt. \tag{2.18}$$

Integrating in space and taking expected values, we obtain

$$\begin{aligned}\frac{1}{2\nu}\partial_t\mathbb{E}\|u_t\|_{L^2}^2&=\mathbb{E}\left[\frac{1}{N^2}\sum_{i=1}^N\left\|\nabla u_t^i\right\|_{L^2}^2-\|\nabla u_t\|_{L^2}^2\right]\\&=\mathbb{E}\left[\frac{1}{N^2}\sum_{i=1}^N\left\|\nabla u_t^i\right\|_{L^2}^2-\frac{1}{N^2}\left[\sum_{i=1}^N\|\nabla u_t^i\|_{L^2}^2+\sum_{i\neq j}^N\langle\nabla u_t^j,\nabla u_t^i\rangle\right]\right].\end{aligned} \tag{2.19}$$

This simplifies to

$$\frac{1}{2\nu}\partial_t\mathbb{E}\|u_t\|_{L^2}^2=-\frac{1}{N^2}\sum_{i\neq j}^N\mathbb{E}\left[\langle\nabla u_t^j,\nabla u_t^i\rangle\right]. \tag{2.20}$$

Case I: $\lim_{m\to\infty}t_m\to\infty$. In this case, the sequence of resetting times goes to infinity as $m\to\infty$. Using resetting, we have

$$\sum_{i\neq j}^N\mathbb{E}\left[\langle\nabla u_t^i,\nabla u_t^j\rangle\right]\geqslant(1-\varepsilon)N(N-1)\,\|\nabla u_{t_m}\|_{L^2}^2, \tag{2.21}$$

for all $t\in[t_m,t_{m+1})$. Thus, using (2.21) we can obtain the following estimate on the rate of change of energy (2.20)

$$\begin{aligned}\frac{1}{2\nu}\partial_t\mathbb{E}\|u_t\|_{L^2}^2&\leqslant-(1-\varepsilon)\frac{N-1}{N}\|\nabla u_{t_m}\|_{L^2}^2\\&\leqslant-C(1-\varepsilon)\frac{N-1}{N}\mathbb{E}\|u_{t_m}\|_{L^2}^2\leqslant-C(1-\varepsilon)\frac{N-1}{N}\mathbb{E}\|u_t\|_{L^2}^2,\end{aligned} \tag{2.22}$$

for some constant $C > 0$ that arises from using Poincaré's inequality. Hence, using Gronwall's inequality, we obtain an exponential dissipation of energy.

Case II: $\lim_{m\to\infty} t_m \to T < \infty$. In this case the sequence of resetting times converges to a finite time T. At every resetting time, we have

$$\omega_{t_m} = \frac{1}{N} \sum_{i=1}^{N} \omega_{t_m}^i. \tag{2.23}$$

Thus,

$$\begin{aligned} \mathbb{E}\|\omega_{t_m}\|_{L^2}^2 &= \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{E}\|\omega_{t_m}^i\|_{L^2}^2 + \frac{1}{N^2}\sum_{i\neq j}^{N}\mathbb{E}\left[\langle \omega_{t_m}^i, \omega_{t_m}^j\rangle\right] \\ &= \frac{1}{N}\|\omega_{t_{m-1}}\|_{L^2}^2 + \frac{1}{N^2}\sum_{i\neq j}^{N}\mathbb{E}\left[\langle \omega_{t_m}^i, \omega_{t_m}^j\rangle\right]. \end{aligned} \tag{2.24}$$

Since $\langle \nabla u_{t_m}^i, \nabla u_{t_m}^j\rangle = \langle \omega_{t_m}^i, \omega_{t_m}^j\rangle$, we have

$$\sum_{i\neq j}^{N} \mathbb{E}\left[\langle \omega_{t_m}^i, \omega_{t_m}^j\rangle\right] \leqslant (1-\varepsilon)N(N-1)\left\|\omega_{t_{m-1}}\right\|_{L^2}^2. \tag{2.25}$$

Thus, we obtain the following estimate on (2.24)

$$\mathbb{E}\|\omega_{t_m}\|_{L^2}^2 \leqslant (1-\alpha)\left\|\omega_{t_{m-1}}\right\|_{L^2}^2, \alpha = 1-\varepsilon+\frac{\varepsilon}{N} < 1. \tag{2.26}$$

Iterating over m, we have

$$\mathbb{E}\|\omega_{t_m}\|_{L^2}^2 \leqslant (1-\alpha)^m \|\omega_0\|_{L^2}^2, \tag{2.27}$$

where $\omega_0 = \nabla \times u_0$. Thus, if $\lim_{m\to\infty} t_m \to T$, for some finite time T, this means we will reset the particle system a countable number of times. Hence,

$$\mathbb{E}\|\omega_{t_m}\|_{L^2}^2 \leqslant (1-\alpha)^m \|\omega_0\|_{L^2}^2 \to 0 \quad \text{as} \quad m \to \infty. \tag{2.28}$$

To show that $u_{t_m} \to 0$ as $m \to \infty$, we apply Poincaré's inequality and obtain

$$\|u_{t_m}\|_{L^2} \leqslant c\|\nabla u_{t_m}\|_{L^2} = c\|\omega_{t_m}\|_{L^2} \to 0 \qquad \text{as} \qquad m \to \infty. \tag{2.29}$$

Thus, the particle system dissipates all its energy in a finite time T. □

Remark 2.1. In our Theorem 2.1 we prove that the particle system will dissipate all of its energy. We think this dissipation occurs in finite time as follows. When vorticity is large, inequality (2.10) is more likely to hold for longer time increments δ_m, and the particle system will mainly dissipate its energy at an exponential rate. However, once the vorticity becomes small, the Brownian motion dominates dynamics, and it makes impossible for inequality (2.10) to hold for long time increments δ_m. When that happens, the energy is going to be dissipated through the averaging each time we reset. The latter will likely imply that the sequence of resetting times converges to a finite time.

REFERENCES

[1] A. Chorin, *Numerical study of slightly viscous flow*, J. Fluid Mech., 57:785–796, 1973.

[2] F. Cipriano and A.B. Cruzeiro, *Navier–Stokes equation and Diffusions on the group of homeomorphisms of the torus*, Comm. Math. Phys., 275:255–269, 2007.

[3] A. Cruzeiro and E. Shamarova, *Navier–Stokes equations and forward-backward SDEs on the group of diffeomorphisms of a torus*, Stochastic Processes and their Applications, 119:4034–4060, 2009.

[4] P. Constantin, *An Eulerian–Lagrangian approach for incompressible fluids: local theory*, JAMS, 14:263–278, 2001.

[5] P. Constantin, *Some open problems and research directions in the mathematical study of fluid dynamics*, Mathematics unlimited-2001 and beyond, Springer, Berlin, 353–360, 2001.

[6] P. Constantin and G. Iyer, *A stochastic Lagrangian representation of the three-dimensional incompressible Navier–Stokes equations*, Commun. Pure Appl. Math., 330–345, 2008.

[7] F. Delbaen, J. Qiu, and S. Tang, *Forward-backward stochastic differential systems associated to Navier–Stokes equations in the whole space*, Stochastic Processes and their Applications, 125(7):2516–2561, 2015.

[8] R. Esposito, R. Marra, M. Pulvirenti, and C. Sciarretta, *A stochastic Lagrangian picture for the three dimensional Navier–Stokes equation*, Comm. Part. Diff. Eqs., 13(12):1601–1610, 1988.

[9] C.L. Fefferman, *Existence and smoothness of the Navier–Stokes equation*, The millennium prize problems, Clay Mathematics Institute, Cambridge, 57–67, 2006.

[10] G. Iyer, *A stochastic perturbation of inviscid flows*, Comm. Math. Phys., 266(3):631–645, 2006.

[11] G. Iyer and J. Mattingly, *A stochastic-Lagrangian particle system for the Navier–Stokes equations*, Nonlinearity, 21:2537–2553, 2008.

[12] G. Iyer and A. Novikov, *The regularizing effects of resetting in a particle system for the Burgers equation*, Ann. of Probability, 39(4):1468–1501, 2011.

[13] N.V. Krylov and B.L. Rozovski, *Stochastic partial differential equations and diffusion processes*, Russian Math. Surveys, 37:81–105, 1982.

[14] H.P. McKean, *Application of Brownian motion to the equation of Kolmogorov, Petrovskii, and Piskunov*, Commun. Pure Appl. Math., 28(3):323–331, 1975.

[15] J. Pedlosky, *Geophysical Fluid Dynamics*, Springer-Verlag, 1982.

[16] A.S. Sznitman, *Topics in propagation of chaos*, Lecture Notes in Math. 1464. Springer-Verlag, 165–251, 1991.

[17] Y. Le Jan and A.S. Sznitman, *Stochastic cascades and 3-dimensional Navier–Stokes equations*, Probab. Theory Related Fields, 109(3):343–366, 1997.

[18] X. Zhang, *A stochastic representation for backward incompressible Navier–Stokes equations*, Probab. Theory Related Fields, 148:305–332, 2010.

[19] X. Zhang, *Stochastic Lagrangian particle approach to fractal Navier–Stokes equations*, Comm. Math. Phys., 311(1):133–155, 2012.