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ON A MULTI-SPECIES CAHN–HILLIARD–DARCY TUMOR GROWTH MODEL WITH SINGULAR POTENTIALS*

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Abstract. We consider a model describing the evolution of a tumor inside a host tissue in terms of the parameters φ_p, φ_d (proliferating and necrotic cells, respectively), $\boldsymbol{u}$ (cell velocity) and n (nutrient concentration). The variables φ_p, φ_d satisfy a vectorial Cahn–Hilliard-type system with nonzero forcing term (implying that their spatial means are not conserved in time), whereas $\boldsymbol{u}$ obeys a variant of Darcy's law and n satisfies a quasi-static diffusion equation. The main novelty of the present work stands in the fact that we are able to consider a configuration potential of singular type implying that the concentration vector (φ_p,φ_d) is constrained to remain in the range of physically admissible values. On the other hand, in the presence of nonzero forcing terms, this choice gives rise to a number of mathematical difficulties, especially related to the control of the mean values of φ_p and φ_d. For the resulting mathematical problem, by imposing suitable initial-boundary conditions, our main result concerns the existence of weak solutions in a proper regularity class.

Keywords. Tumor growth; nonlinear evolutionary system; Cahn–Hilliard–Darcy system; existence of weak solutions; logarithmic potentials.

AMS subject classifications. 35D30; 35Q35; 35Q92; 35K57; 76S05; 92C17; 92B05.

1. Introduction

Tumor growth remains an active area of scientific research due to the impact on the quality of life for those diagnosed with cancer. Starting with the seminal work of Burton [8] and Greenspan [33], many mathematical models have been proposed to emulate the complex biological and chemical processes that occur in tumor growth with the aim of better understanding and ultimately controlling the behavior of cancer cells. In recent years, there has been a growing interest in the mathematical modelling of cancer, see for example [1,3,7,19,21,23]. Mathematical models for tumor growth may have different analytical features: in the present work we are focusing on the subclass of continuum models, namely diffuse interface models. In this framework, the tumor and surrounding host tissue occupy regions of a domain and are subject to various balance laws mimicking the biological processes one would like to model. While it is intuitive to represent the interfaces between the tumor and healthy tissues as idealized surfaces of zero thickness, leading to a sharp interface description that differentiates the tumor and the surrounding host tissue cell-by-cell, these kinds of sharp interface models are often difficult to analyze mathematically, and may break down when the interface undergoes a topological change. Metastasis, which is the spreading of cancer to other parts of the body, is one important example of a change of topology. In such an event, the interface can no longer be represented as a mathematical surface, and thus the sharp interface models are not valid when the tumor exhibits metastasis.

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On the other hand, diffuse interface models consider the interface between the tumor and the healthy tissues as a narrow layer in which tumor and healthy cells are mixed. This alternative representation of the interface gives rise to model equations that are better amenable to mathematical analysis, and the mathematical description remains valid even when the tumor undergoes topological changes. Hence, the recent efforts in the mathematical modeling of tumor growth have been mostly focused on diffuse interface models, see for example [18,19,22,29,31,34,40,44], and their numerical simulations demonstrating complex changes in tumor morphologies due to mechanical stresses and interactions with chemical species such as nutrients or toxic agents.

The interaction of multiple tumor cell species can be described by using multiphase mixture models (see, e.g., [1,20,22,29,42,44]). Indeed, using multiphase porous media mechanics, the authors of [42] represented a growing tumor as a multiphase medium containing an extracellular matrix, tumor and host cells, and interstitial liquid. Numerical simulations were also performed that characterized the process of cancer growth in terms of the initial tumor-to-healthy cell density ratio, nutrient concentration, mechanical strain, cell adhesion, and geometry. The interactions of a growing tumor and a basement membrane were studied in [5], which has been adapted to the multiphase case [9], with additional biophysical details given in [19].

In terms of the theoretical analysis of diffuse interface models, most of the recent literature is restricted to the two-phase variant, i.e., to models that only account for the evolution of a tumor surrounded by healthy tissue. In this setting, there is no differentiation among the tumor cells that exhibit heterogeneous growth behavior, and consequently this kind of two-phase models are just able to describe the growth of a young tumor before the onset of quiescence and necrosis. Analytical results related to well-posedness, asymptotic limits and long-time behavior have been established in [11,13,15,24–28,41] for tumor growth models based on the coupling of Cahn–Hilliard (for the tumor density) and reaction–diffusion (for the nutrient or other chemical factors) equations, and in [27,32,35,37,38] for models of Cahn–Hilliard–Darcy type. There have also been some studies involving the optimal control and sliding modes for diffuse interface tumor growth, see, e.g., [12,14,30].

Comparatively, there have been fewer analytical results for the multi-phase variants, which distinguish between the proliferating and necrotic tumor cells. In [20] a simplification of the tumor model introduced in [9] is studied. In contrast to the original model of [9], which consists of a Cahn–Hilliard–Darcy system coupling transport-type equations with high order source terms, and the natural energy identity of the model appears not to provide sufficient a priori estimates, the authors in [20] analyzed the case of constant and identical mobilities for all tumor species, which allows them to express the simplified model as a Cahn–Hilliard–Darcy system coupled with a transport-type equation without the high order source terms, and establish the existence of a weak solution. Meanwhile, existence of a solution to the original model of [9] remains an open problem due to the high order source terms and a lack of useful a priori estimates. In [29], instead, a vectorial Cahn–Hilliard–Darcy model has been proposed to describe multi-phase tumor growth. A new feature of the said model is the use of a volume-average velocity, which dramatically simplifies the resulting equation for the mixture velocity. Furthermore, differently from the system studied in [20] that consists of one Cahn–Hilliard equation for the total tumor volume fraction coupled with transport-type equations for the individual tumor species, which is the main source of analytical difficulties in the search for a priori estimates, in the model of [29] each tumor species is governed by a Cahn–Hilliard-type equation, and the corresponding natural energy identity yields better a priori estimates for an existence proof.

In this paper, we consider a multi-species tumor model posed in a smooth bounded domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, and over a reference time interval $(0,T)$ with no restriction on the magnitude of T. Our model describes the evolution of proliferating tumor cells, necrotic tumor cells, and healthy host cells. We denote the corresponding volume fractions as $\varphi_p, \varphi_d, \varphi_h \in [0,1]$, respectively, so that $\varphi_p + \varphi_d + \varphi_h = 1$ almost everywhere in $\Omega \times (0,T)$. By this relation, it suffices to track the evolution of φ_p and φ_d in order to deduce the evolution of φ_h; for this reason it is also natural to assume that the vector $\boldsymbol{\varphi} := (\varphi_p, \varphi_d)^\top$ lies in the simplex $\Delta := \{\boldsymbol{y} \in \mathbb{R}^2 : \, 0 \le y_1, y_2, \ y_1 + y_2 \le 1\} \subset \mathbb{R}^2$. This *constraint* will be one of the key points in our approach and we will explain below how it is enforced by the equations. The multi-species tumor model analyzed in this work is given by

$$\partial_t \varphi_p = M_p \Delta \mu_p - \operatorname{div}(\varphi_p \boldsymbol{u}) + S_p, \quad \mu_p = F_{,p} - \Delta \varphi_p \text{ in } Q := \Omega \times (0,T), \tag{1.1a}$$

$$\partial_t \varphi_d = M_d \Delta \mu_d - \operatorname{div}(\varphi_d \boldsymbol{u}) + S_d, \quad \mu_d = F_{,d} - \Delta \varphi_d \text{ in } Q, \tag{1.1b}$$

$$S_p = \Sigma_p(n, \varphi_p, \varphi_d) + m_{pp}\varphi_p + m_{pd}\varphi_d \quad \text{in } Q, \tag{1.1c}$$

$$S_d = \Sigma_d(n, \varphi_p, \varphi_d) + m_{dp}\varphi_p + m_{dd}\varphi_d \quad \text{in } Q, \tag{1.1d}$$

$$\operatorname{div} \boldsymbol{u} = S_p + S_d \quad \text{in } Q, \tag{1.1e}$$

$$\boldsymbol{u} = -\nabla q - \varphi_p \nabla \mu_p - \varphi_d \nabla \mu_d \quad \text{in } Q, \tag{1.1f}$$

$$0 = -\Delta n + \varphi_p n - B(n_C - n) \quad \text{in } Q, \tag{1.1g}$$

$$M_i \partial_{\boldsymbol{n}} \mu_i - \varphi_i \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \partial_{\boldsymbol{n}} \varphi_i = 0, \quad q = 0, \quad n = 1 \quad \text{on } \Gamma := (\partial\Omega) \times (0,T), \tag{1.1h}$$

$$\varphi_p(x,0) = \varphi_{p,0}(x), \quad \varphi_d(x,0) = \varphi_{d,0}(x) \quad \text{in } \Omega. \tag{1.1i}$$

Note that, φ_h can be determined from the relation $1 - \varphi_p - \varphi_d$, and implicitly, we are also assuming that $\varphi_h(x,0) = 1 - \varphi_{p,0}(x) - \varphi_{d,0}(x)$.

Equations (1.1a) and (1.1b) are convective Cahn–Hilliard-type equations (with nonzero forcing terms) that encode the evolution of φ_p and φ_d. The variables μ_p and μ_d are the associated chemical potentials. The constants M_p and M_d denote the mobilities of φ_p and φ_d, and S_p and S_d are the corresponding source terms that account for biological mechanisms experienced by the tumor cells. Furthermore, we have assumed that the cells are tightly packed and move together, leading to the appearance of a cell velocity $\boldsymbol{u}$ governed by Darcy's law (1.1f) with cell pressure q. The subsequent terms $\varphi_p \nabla \mu_p$ and $\varphi_d \nabla \mu_d$ in (1.1f) have the meaning of Korteweg forces. Meanwhile, the gain or loss of volume due to the source terms S_p and S_d (modeled by (1.1c) and (1.1d) with constants m_{pp}, m_{pd}, m_{dp} and m_{dd}) and the change of mass balance is summarized in the relation (1.1e). Lastly, we assumed that a chemical species is present in the domain Ω that serves as a nutrient for tumor proliferation, whose concentration we denote as n. Equation (1.1g) accounts for the diffusion of the nutrient (which is much faster compared to the rate of cell proliferation, resulting in a *quasi-static* evolution), its consumption by the proliferating cells modeled by the term $\varphi_p n$ (host cell nutrient intake is small compared to tumor cell intake), and the transfer of nutrients to and from nearby capillaries modeled by the term $B(n_C - n)$, where $n_C \in (0,1)$ is the level of nutrients in the capillaries and $B \ge 0$ is a constant supply rate.

In (1.1h), $\partial_{\boldsymbol{n}}$ denotes the outer normal derivative to $\partial\Omega$ with unit normal $\boldsymbol{n}$, while in (1.1a) and (1.1b), $F_{,p}$ and $F_{,d}$ denote the partial derivatives of a function $F(\varphi_p, \varphi_d)$ with respect to φ_p and φ_d, respectively, i.e.,

$$F_{,p} := \frac{\partial F}{\partial \varphi_p}, \quad F_{,d} := \frac{\partial F}{\partial \varphi_d}.$$

Associated to (1.1) is the free energy functional $E(\varphi_p,\varphi_d)$ of Ginzburg–Landau type:

$$E(\varphi_p,\varphi_d):=\int_\Omega F(\varphi_p,\varphi_d)+\frac{1}{2}|\nabla\varphi_p|^2+\frac{1}{2}|\nabla\varphi_d|^2\,dx, \tag{1.2}$$

where the function F is a multi-well configuration potential for the variables φ_p and φ_d, which we take to be the sum of a smooth non-convex part F_1 and of a non-smooth *singular* convex part F_0, i.e., $F=F_0+F_1$ with F_0 set to $+\infty$ outside the set $\overline{\Delta}=\{(s,r)\in\mathbb{R}^2:\ s\ge 0,r\ge 0,s+r\le 1\}$ of the "physically admissible" configurations. In other words, if $\int_\Omega F_0(\varphi_p,\varphi_d)\,dx$ is finite, then we necessarily have $\varphi_p,\varphi_d\ge 0$ and $\varphi_p+\varphi_d\le 1$. Moreover, as a further consequence, due to the fact that $\varphi_h=1-\varphi_p-\varphi_d$, we also have $0\le\varphi_h\le 1$. This means, the finiteness of the "configuration energy" $\int_\Omega F(\varphi_p,\varphi_d)\,dx$ automatically ensures the natural bounds

$$0\le\varphi_p,\varphi_d,\varphi_h\le 1 \text{ a.e. in } \Omega. \tag{1.3}$$

An example of singular potential F_0 that we can include in our analysis is

$$F_0(\varphi_p,\varphi_d):=\varphi_p\log\varphi_p+\varphi_d\log\varphi_d+(1-\varphi_p-\varphi_d)\log(1-\varphi_p-\varphi_d), \tag{1.4}$$

which can be seen as a generalization of the standard logarithmic potential commonly used in the framework of Cahn–Hilliard equations (cf. for example [10, 39]). Let us also comment that, in light of the above considerations, the pure phase consisting of proliferating tumor cells is characterized by the region $\{\varphi_p=1,\ \varphi_d=\varphi_h=0\}$, whereas the pure phase corresponding to the necrotic cells is the region $\{\varphi_d=1,\ \varphi_p=\varphi_h=0\}$.

Mathematically speaking, the main novelty of our model, and also its main difficulty from the analytical point of view, comes from the singular component F_0 of the configuration potential coupled with the nonzero source terms in the Cahn–Hilliard relations (1.1a)-(1.1b). Indeed, integrating the first relations in (1.1a), (1.1b) we obtain an evolution law for the spatial mean values $y_i:=\frac{1}{|\Omega|}\int_\Omega\varphi_i\,dx$ of φ_i for $i=p,d$ (cf. (4.2) below) which is satisfied by any hypothetical solution to the system. Such a relation, however, does not involve directly the singular part F_0. Hence, the evolution of y_p,y_d are not automatically compatible with the physical constraint (1.3) and this compatibility (i.e., the fact that y_p, y_d remain well inside the set of meaningful values) has to be carefully proved (see Subsection 4.1) by assuming proper conditions on coefficients and making a careful choice of the boundary conditions. In particular, the first condition in (1.1h) linking the boundary values of $\boldsymbol{u}$, φ_i and μ_i seems to be necessary in order for our arguments to work. It is worth noting that the mathematical literature on multi-component Cahn–Hilliard systems is very poor; we can actually just mention the papers [4, 16, 17, 36] (see also the references cited therein).

Concerning the approach we employ to prove existence of weak solutions to system (1.1a)–(1.1g), the first step is to consider a regularization, which is obtained by replacing the singular potential F_0 with its Moreau–Yosida approximation F_ε depending on an approximation parameter $\varepsilon>0$, and also introducing some suitable truncation functions. The latter choice is due to the fact that F_ε is no longer a singular function, and consequently the uniform boundedness properties $0\le\varphi_p$, $0\le\varphi_d$, $\varphi_p+\varphi_d\le 1$ are not expected to hold in the approximation level. In addition, we remark that in taking the divergence of (1.1f) and using (1.1e) leads to an elliptic equation for the pressure q. Similar to the situation encountered in [27, 35], the presence of source terms S_p and S_d necessarily requires a priori bounds on the pressure q in order to derive useful estimates. Therefore, in the regularization we use the Darcy law (1.1f) to remove explicit dependence on the cell velocity $\boldsymbol{u}$ in the equation, and rewrite the transport terms in (1.1a), (1.1b) directly in terms of the pressure q. Then, we also include a number of regularizing terms depending on a further parameter $\delta>0$ which is intended to go to 0 in the limit.

Introducing the cutoff operator

$$\mathcal{T}(r) := \max\{0, \min\{1, r\}\}, \tag{1.5}$$

our regularized system takes the form

$$\begin{aligned}
\partial_t \varphi_p^{\delta,\varepsilon} &= M_p \Delta \mu_p^{\delta,\varepsilon} + \operatorname{div}\big(\mathcal{T}(\varphi_p^{\delta,\varepsilon})^2 \nabla \mu_p^{\delta,\varepsilon} + \mathcal{T}(\varphi_p^{\delta,\varepsilon})\mathcal{T}(\varphi_d^{\delta,\varepsilon})\nabla \mu_d^{\delta,\varepsilon}\big) && \text{(1.6a)}\\
&\quad + \operatorname{div}(\mathcal{T}(\varphi_p^{\delta,\varepsilon})\nabla q^{\delta,\varepsilon}) + S_p, \\
\mu_p^{\delta,\varepsilon} &= -\delta \Delta \partial_t \varphi_p^{\delta,\varepsilon} + F_{\varepsilon,p}(\varphi_p^{\delta,\varepsilon}, \varphi_d^{\delta,\varepsilon}) + F_{1,p}(\varphi_p^{\delta,\varepsilon}, \varphi_d^{\delta,\varepsilon}) - \Delta \varphi_p^{\delta,\varepsilon}, && \text{(1.6b)}\\
\partial_t \varphi_d^{\delta,\varepsilon} &= M_d \Delta \mu_d^{\delta,\varepsilon} + \operatorname{div}\big(\mathcal{T}(\varphi_d^{\delta,\varepsilon})^2 \nabla \mu_d^{\delta,\varepsilon} + \mathcal{T}(\varphi_p^{\delta,\varepsilon})\mathcal{T}(\varphi_d^{\delta,\varepsilon})\nabla \mu_p^{\delta,\varepsilon}\big) && \text{(1.6c)}\\
&\quad + \operatorname{div}(\mathcal{T}(\varphi_d^{\delta,\varepsilon})\nabla q^{\delta,\varepsilon}) + S_d, \\
\mu_d^{\delta,\varepsilon} &= -\delta \Delta \partial_t \varphi_d^{\delta,\varepsilon} + F_{\varepsilon,d}(\varphi_p^{\delta,\varepsilon}, \varphi_d^{\delta,\varepsilon}) + F_{1,d}(\varphi_p^{\delta,\varepsilon}, \varphi_d^{\delta,\varepsilon}) - \Delta \varphi_d^{\delta,\varepsilon}, && \text{(1.6d)}\\
S_p &= \Sigma_p(n^{\delta,\varepsilon}, \varphi_p^{\delta,\varepsilon}, \varphi_d^{\delta,\varepsilon}) + m_{pp}\varphi_p^{\delta,\varepsilon} + m_{pd}\varphi_d^{\delta,\varepsilon}, && \text{(1.6e)}\\
S_d &= \Sigma_d(n^{\delta,\varepsilon}, \varphi_p^{\delta,\varepsilon}, \varphi_d^{\delta,\varepsilon}) + m_{dp}\varphi_p^{\delta,\varepsilon} + m_{dd}\varphi_d^{\delta,\varepsilon}, && \text{(1.6f)}\\
\delta \partial_t q^{\delta,\varepsilon} &= \Delta q^{\delta,\varepsilon} - \delta \Delta^2 q^{\delta,\varepsilon} + \operatorname{div}\big(\mathcal{T}(\varphi_p^{\delta,\varepsilon})\nabla \mu_p^{\delta,\varepsilon} + \mathcal{T}(\varphi_d^{\delta,\varepsilon})\nabla \mu_d^{\delta,\varepsilon}\big) + S_p + S_d, && \text{(1.6g)}\\
0 &= -\Delta n^{\delta,\varepsilon} + \mathcal{T}(\varphi_p^{\delta,\varepsilon}) n^{\delta,\varepsilon} - B(n_C - n^{\delta,\varepsilon}), && \text{(1.6h)}
\end{aligned}$$

furnished with the initial and boundary conditions

$$\begin{aligned}
\varphi_p^{\delta,\varepsilon}(0) = \varphi_{p,0,\delta}, \quad \varphi_d^{\delta,\varepsilon}(0) = \varphi_{d,0,\delta}, \quad q^{\delta,\varepsilon}(0) = 0 \text{ in } \Omega, && \text{(1.7a)}\\
M_i \partial_{\boldsymbol{n}} \mu_i^{\delta,\varepsilon} + \mathcal{T}(\varphi_i^{\delta,\varepsilon})(\nabla q^{\delta,\varepsilon} + \mathcal{T}(\varphi_p^{\delta,\varepsilon})\nabla \mu_p^{\delta,\varepsilon} + \mathcal{T}(\varphi_d^{\delta,\varepsilon})\nabla \mu_d^{\delta,\varepsilon}) \cdot \boldsymbol{n} = 0 \text{ on } \Gamma, && \text{(1.7b)}\\
\partial_{\boldsymbol{n}} \varphi_i^{\delta,\varepsilon} = 0, \quad n^{\delta,\varepsilon} = 1, \quad q^{\delta,\varepsilon} = 0, \quad \Delta q^{\delta,\varepsilon} = 0 \text{ on } \Gamma, && \text{(1.7c)}
\end{aligned}$$

where the regularized initial data $\{\varphi_{p,0,\delta}, \varphi_{d,0,\delta}\}$ are chosen in such a way that they converge strongly to the original initial data $\{\varphi_{p,0}, \varphi_{d,0}\}$ as $\delta \to 0$. Let us mention that in (1.6b) and (1.6d), we use the notation

$$F_{\varepsilon,p} := \frac{\partial F_\varepsilon}{\partial \varphi_p}, \quad F_{1,p} := \frac{\partial F_1}{\partial \varphi_p}, \quad F_{\varepsilon,d} := \frac{\partial F_\varepsilon}{\partial \varphi_d}, \quad F_{1,d} := \frac{\partial F_1}{\partial \varphi_d}.$$

For fixed $\delta, \varepsilon > 0$, we show the existence of a weak solution to (1.6)-(1.7) by means of a Schauder fixed point argument. It turns out that eliminating the velocity in the equations allows us to decouple the complicated model (1.6) into two subsystems. For given functions $(\bar{q}, \bar{n})$ we first show the well-posedness of (1.6a)-(1.6f) with $q^{\delta,\varepsilon}$ and $n^{\delta,\varepsilon}$ replaced by $\bar{q}$ and $\bar{n}$. Then, using $(\varphi_p^{\delta,\varepsilon}, \mu_p^{\delta,\varepsilon}, \varphi_d^{\delta,\varepsilon}, \mu_d^{\delta,\varepsilon})$ as data, we show the well-posedness of (1.6g)-(1.6h). This allows us to define a compact mapping $\mathcal{K} : (\bar{q}, \bar{n}) \mapsto (q, n)$, and a fixed point of this map $\mathcal{K}$ is a weak solution to the regularized system (1.6)-(1.7). We then derive uniform estimates in δ and ε, and then pass to the limit in the order $\delta \to 0$ followed by $\varepsilon \to 0$ to deduce the existence result for (1.1).

Let us now compare (1.1) and the model of [9] which was analyzed by [20]:

- In [9], the effect of a basement membrane on the growing tumor is also considered, which leads to additional coupling of the model with a Cahn–Hilliard equation transported by the velocity $\boldsymbol{u}$. In this work we do not consider such effects.

- The key distinction is that in our choice of a multi-well potential F in (1.2), we included interfacial energy for the proliferating-necrotic tumor interface and also for the tumor-host interfaces. On the other hand, in [9] the free energy depends only on the total tumor volume fraction $\varphi_T = \varphi_p + \varphi_d$, i.e., $E(\varphi_T) = \int_\Omega f(\varphi_T) + \frac{1}{2}|\nabla\varphi_T|^2\,dx$ for scalar double-well potential f with minima at 0 and 1. This reduction to the total tumor volume fraction implies that the proliferating-necrotic tumor interface in [9] is not energetic.
- More precisely, in [9], like in the multiphase models studied in [22, 44, 45], the differentiation between proliferating and necrotic tumor cells is done a posteriori based on the local density of nutrients after computing φ_T. In contrast, our model (1.1) follows a similar approach to [29] in which φ_p and φ_d are computed without any post processing.
- Moreover, we consider here different boundary conditions with respect to [9], where a zero Dirichlet boundary datum was taken for the chemical potentials, while here we consider a coupled condition for μ_i and $\boldsymbol{u}$ (the first of (1.1h)). It is worth noting that the (easier) case of Dirichlet boundary conditions for μ_i could also be treated, but we preferred to handle (1.1h) which seems to be more reasonable from the modeling point of view. On the contrary, the case of no-flux conditions for μ_i (which would also be meaningful) seems not easy to be treated mathematically.
- The source term in the Cahn–Hilliard equation [9, (2)] have been modified and differs from the earlier model of Wise et al. [44]. The effect of such a modification is that in the corresponding energy identity, the right-hand side can be controlled more easily (cf. [20, Lem. 3.4]). However, in our present setting we encounter terms of the form $S_p\mu_p + S_d\mu_d$ in the energy identity (see (4.4)), which are more difficult to control and require a priori bounds on the mean values of φ_p and φ_d.
- Finally, differently from [20], we allow possibly different mobility coefficients, which would have given rise to a number of mathematical complications in the setting of [20]. On the other hand, we infer stronger spatial and temporal regularities for the individual volume fractions φ_p and φ_d compared to just boundedness in the setting of [20].

Plan of the paper. The assumptions and main results are stated in Section 2. The proof is carried out in the remainder of the paper and is subdivided into several steps: namely, in Section 3 the existence of weak solutions to the regularized system (1.6)-(1.7) is outlined. Then, in Section 4 uniform estimates that are independent of the regularization parameters δ and ε are derived in order to pass to the limit to obtain weak solutions for the original system (1.1).

2. Assumptions and main result

For the remainder of the paper, we denote the mean of a function f over Ω as $\overline{f}$. We start by presenting our assumptions:

ASSUMPTION 2.1.

(A1) *M_p, M_d are strictly positive constants, B is a non-negative constant and $n_C \in (0,1)$.*

(A2) *We set $\Sigma := (\Sigma_p, \Sigma_d)$ and denote as $\underline{\underline{M}} = \begin{pmatrix} m_{pp} & m_{pd} \\ m_{dp} & m_{dd} \end{pmatrix}$ the matrix of the coeffi-*

cients in (1.1c), (1.1d), so that

$$\begin{pmatrix} S_p \\ S_d \end{pmatrix} = \Sigma(n,\varphi_p,\varphi_d) + \underline{\underline{M}} \begin{pmatrix} \varphi_p \\ \varphi_d \end{pmatrix}.$$

Then, we assume that $\Sigma \in C^{0,1}(\mathbb{R}^3;\mathbb{R}^2)$ *and there exist a closed subset* Δ_0 *with* C^1*-boundary contained in the open simplex* Δ *and constants* $K_{p,-}$, $K_{p,+}$, $K_{d,-}$, $K_{d,+} \in \mathbb{R}$ *and* $c>0$, *with* $K_{p,-} \le K_{p,+}$ *and* $K_{d,-} \le K_{d,+}$, *such that*

$$\Sigma(\mathbb{R}^3) \subset [K_{p,-},K_{p,+}] \times [K_{d,-},K_{d,+}], \quad \|D\Sigma\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^6)} \le c, \tag{2.1}$$

and for $\boldsymbol{x} = (\overline{\Sigma_p},\overline{\Sigma_d})^\top \in [K_{p,-},K_{p,+}] \times [K_{d,-},K_{d,+}]$, *it holds that*

$$(\underline{\underline{M}}\boldsymbol{y} + \boldsymbol{x}) \cdot \boldsymbol{n} < 0 \text{ for all } \boldsymbol{y} \in \partial\Delta_0, \tag{2.2}$$

where $\boldsymbol{n}$ *denotes the outer unit normal vector to* Δ_0.

(A3) *The potential* F *is the sum of a convex part* F_0 *and of a (possibly nonconvex) perturbation* F_1. *We assume that* $F_0 : \mathbb{R}^2 \to [0,+\infty]$, *with the effective domain of* F_0 *(i.e., the set where* F_0 *assumes finite values) being given either by* Δ *or by the closure* $\overline{\Delta}$. *Furthermore,* $F_0 \in C^1(\Delta;[0,\infty))$, *i.e.,* F_0 *is smooth once restricted to the simplex* Δ *and there exists constants* $c_1, c_3 > 0$ *and* $c_2, c_4 \ge 0$ *such that*

$$F_0(s,r) \ge c_1(|s|^2 + |r|^2) - c_2 \quad \forall (s,r) \in \overline{\Delta}, \tag{2.3}$$

and for all $(s,r) \neq (S,R) \in \Delta$, *it holds*

$$\nabla F_0(s,r) \cdot (s-S, r-R)^\top \ge c_3 |\nabla F_0(s,r)| - c_4, \tag{2.4}$$

where $\nabla F_0(s,r) = (\frac{\partial F_0(s,r)}{\partial s}, \frac{\partial F_0(s,r)}{\partial r})^\top$. *Meanwhile, we assume* $F_1 \in C^{1,1}(\mathbb{R}^2)$ *with*

$$\left| \frac{\partial F_1(s,r)}{\partial s} \right| + \left| \frac{\partial F_1(s,r)}{\partial r} \right| \le C(1 + |s| + |r|) \quad \forall r,s \in \mathbb{R}. \tag{2.5}$$

(A4) *The initial conditions satisfy* $\varphi_{p,0}, \varphi_{d,0} \in H^1(\Omega)$ *with*

$$\varphi_{p,0} \ge 0, \quad \varphi_{d,0} \ge 0, \quad \varphi_{p,0} + \varphi_{d,0} \le 1 \text{ a.e. in } \Omega, \quad F_0(\varphi_{p,0},\varphi_{d,0}) \in L^1(\Omega). \tag{2.6}$$

Moreover, the mean values $y_{i,0} := \frac{1}{|\Omega|}\int_\Omega \varphi_{i,0}(x)\,dx$ *for* $i = p,d$ *satisfy*

$$(y_{p,0}, y_{d,0}) \in \operatorname{int} \Delta_0. \tag{2.7}$$

Examples. In order to clarify the above assumptions, and particularly those regarding the singular potential F, we introduce one example which is particularly significant and will be considered as a model case in the sequel. Namely, we consider the multi-phase logarithmic potential

$$\begin{aligned} F_0(s,r) &:= s\log s + r\log r + (1-s-r)\log(1-s-r), \\ F_1(s,r) &:= \frac{\chi}{2}\big(r(1-r) + s(1-s) + (1-r-s)(r+s)\big), \end{aligned} \tag{2.8}$$

for a fixed positive constant χ. Then, the Assumption (2.3) is fulfilled by (2.8) due to the boundedness of the simplex Δ, and the Assumption (2.4) is also fulfilled as we prove in Lemma 2.1 below.

LEMMA 2.1. *Let F_0 be defined as*

$$F_0(s,r) = s\log s + r\log r + (1-s-r)\log(1-s-r)$$

and let Δ_0 be a compact subset of Δ. Then, there exist positive constants c_*, C_* depending only on Δ_0 such that (2.4) holds.

Proof. For any $(S,R) \in \Delta_0$, direct computation of ∇F_0 leads to

$$\begin{aligned}\nabla F_0(s,r)\cdot(s-S,r-R)^\top = {} & (s-S)\log s + (r-R)\log r \\ & + ((R+S)-(r+s))\log(1-(r+s)).\end{aligned}$$

Observe for $s, S \in (0,1)$,

$$(s-S)\log s = \begin{cases} >0 & \text{if } s<S, \\ <0 & \text{if } s>S, \\ =0 & \text{if } s=S, \end{cases} \quad \text{and} \quad (s-S)\log s \to \begin{cases} 0 & \text{as } s\to 1, \\ \infty & \text{as } s \to 0. \end{cases}$$

In particular, the function $(s-S)\log(s)$ is bounded from below by some negative constant. Hence, there exists a constant $d_1 \geq 0$ such that

$$(s-S)\log s \geq \frac{S}{2}|\log s| - d_1,$$

and as $S \in (0,1)$ we can choose d_1 independent of S. In a similar fashion, there exists a constant $d_2 \geq 0$ (that can be chosen independent of R) such that

$$(r-R)\log r \geq \frac{R}{2}|\log r| - d_2.$$

Lastly, $(s,r) \in \Delta$ implies that $r+s \in (0,1)$, and consequently there exists a constant $d_3 \geq 0$ independent of R, S such that

$$((R+S)-(r+s))\log(1-(r+s)) \geq \frac{1-(R+S)}{2}|\log(1-(r+s))| - d_3.$$

Summing the above then yields

$$\begin{aligned} & \nabla F_0(s,r)\cdot(s-S,r-R)^\top \\ & \geq \frac{1}{2}\min(R,S,1-(R+S))\big(|\log r| + |\log s| + |\log(1-(r+s))|\big) - C(d_1,d_2,d_3) \\ & \geq \frac{1}{4}\min(R,S,1-(R+S))|\nabla F_0(s,r)| - C(d_1,d_2,d_3). \end{aligned}$$

Now for $(R,S) \in \Delta_0$, we see that

$$\min(R,S,1-(R+S)) \geq c_* > 0$$

for some constant $c_* > 0$ depending only on Δ_0. This concludes the proof of Lemma 2.1. ∎

Moreover, as a consequence of (2.4) we obtain by interchanging the roles of (s,r) and (S,R) an analogous inequality

$$\nabla F_0(S,R)\cdot(S-s,R-r)^\top \geq c_3|\nabla F_0(S,R)| - c_4.$$

Then, a short computation shows that

$$\begin{aligned}
&\big(\nabla F_0(s,r)-\nabla F_0(S,R)\big)\cdot(s-S,r-R)^\top \\
=&\nabla F_0(s,r)\cdot(s-S,r-R)^\top+\nabla F_0(S,R)\cdot(S-s,R-r)^\top \\
\geq& c_3|\nabla F_0(s,r)|+c_3|\nabla F_0(S,R)|-2c_4 \\
\geq& c_3|\nabla F_0(s,r)-\nabla F_0(S,R)|-2c_4.
\end{aligned} \tag{2.9}$$

In particular, the inequality (2.9) together with (2.3) shows that F_0 fulfills the hypotheses of [16, Prop. 2.10]. This property would be important later when we consider the Moreau–Yosida approximation of F_0 for the derivation of uniform estimates.

Let us also sketch a couple of examples of functions Σ_p,Σ_d and $\underline{\underline{M}}$ that fulfill the Assumption (2.2).

- First, we consider the source terms S_p, S_d defined as

$$S_p=\lambda_M g(n)-\lambda_A\varphi_p, \tag{2.10}$$
$$S_d=\lambda_A\varphi_p-\lambda_L\varphi_d, \tag{2.11}$$

for positive constants $\lambda_M,\lambda_A,\lambda_L$ and a bounded positive function g such that $0<g(s)\leq 1$. The archetypal example is $g(s)=\max(n_C,\min(s,1))$ for the constant $n_C\in(0,1)$ in (1.1g). The biological effects that we model here are: the proliferation of tumor cells due to nutrient consumption at a constant rate λ_M, the apoptosis of tumor cells at a constant rate λ_A, which leads to a source term for the necrotic cells, and the lysing/disintegration of necrotic cells at a constant rate λ_L.

With the choice (2.10)-(2.11) one can take $K_{p,-}=0$, $K_{p,+}=\lambda_M$, $K_{d,-}=K_{d,+}=0$. Then, integrating (1.1a) and (1.1b) over Ω, and applying the boundary conditions (1.1h) leads to the following ODE system for the mean values $y_p:=\overline{\varphi_p}$ and $y_d:=\overline{\varphi_d}$:

$$\frac{d}{dt}\begin{pmatrix}y_p\\ y_d\end{pmatrix}=\begin{pmatrix}-\lambda_A & 0\\ \lambda_A & -\lambda_L\end{pmatrix}\begin{pmatrix}y_p\\ y_d\end{pmatrix}+\begin{pmatrix}\lambda_M\overline{g(n)}\\ 0\end{pmatrix}=\underline{\underline{M}}\begin{pmatrix}y_p\\ y_d\end{pmatrix}+\begin{pmatrix}\lambda_M\overline{g(n)}\\ 0\end{pmatrix}.$$

The matrix $\underline{\underline{M}}$ is invertible with eigenvalues $\{-\lambda_A,-\lambda_L\}$, hence the fixed point

$$\begin{pmatrix}y_p^*\\ y_d^*\end{pmatrix}=-\underline{\underline{M}}^{-1}\begin{pmatrix}\lambda_M\overline{g(n)}\\ 0\end{pmatrix}=\begin{pmatrix}\frac{\lambda_M}{\lambda_A}\overline{g(n)}\\ \frac{\lambda_M}{\lambda_L}\overline{g(n)}\end{pmatrix}$$

is asymptotically stable. Under the following constraints on the rates:

$$\lambda_M(\lambda_A+\lambda_L)<\lambda_A\lambda_L, \quad \lambda_A<2\lambda_L,$$

we can show that (y_p^*,y_d^*) lies in the interior of the simplex Δ, and (2.2) holds when we take Δ_0 to be a ball centered at (y_p^*,y_d^*) with sufficiently small radius $\eta>0$. Indeed, thanks to $n_C>0$ we see that $y_p^*,y_d^*>0$. Meanwhile, using $g\leq 1$ shows that

$$y_p^*+y_d^*\leq\lambda_M\left(\frac{1}{\lambda_A}+\frac{1}{\lambda_L}\right)<1$$

when we assume the hypothesis $\lambda_M(\lambda_A+\lambda_L)<\lambda_A\lambda_L$. Furthermore, taking a parameterization of the circle $\partial\Delta_0$ as $(\eta\cos\theta+y_p^*,\eta\sin\theta+y_d^*)$ for $\theta\in[0,2\pi]$

with normal $\boldsymbol{n}=(\cos\theta,\sin\theta)$, a short computation shows that (recalling $\boldsymbol{x}=(\lambda_M\overline{g(n)},0)^\top$)

$$\begin{aligned}&\left[\underline{\underline{M}}\begin{pmatrix}\eta\cos\theta+y_p^*\\ \eta\sin\theta+y_d^*\end{pmatrix}+\begin{pmatrix}\lambda_M\overline{g(n)}\\ 0\end{pmatrix}\right]\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\\ &=-\lambda_A\eta\cos^2\theta+\lambda_A\eta\cos\theta\sin\theta-\lambda_L\eta\sin^2\theta\\ &\le-\frac{\lambda_A}{2}\eta\cos^2\theta-\left(\lambda_L-\frac{\lambda_A}{2}\right)\eta\sin^2\theta\le-\frac{1}{2}\min(\lambda_A,2\lambda_L-\lambda_A)\eta<0\end{aligned}$$

under the assumption $2\lambda_L>\lambda_A$.

- As a second model case, for $\lambda>0$ we take $\underline{\underline{M}}$ as $-\lambda$ times the identity matrix (a more general negative definite diagonal matrix could also be considered) and

$$\Sigma(n,\varphi_p,\varphi_d)=\begin{pmatrix}\lambda/3\\ \lambda/3\end{pmatrix}+\Sigma_0(n,\varphi_p,\varphi_d),$$

where Σ_0 is a $C^1(\mathbb{R}^3;\mathbb{R}^2)$ function such that $\|\Sigma_0\|_{L^\infty(\mathbb{R}^3;\mathbb{R}^2)}\le K$ for some $K>0$. In fact, this choice of Σ allows us to write

$$\begin{pmatrix}S_p\\ S_d\end{pmatrix}=\underline{\underline{M}}\begin{pmatrix}\varphi_p-1/3\\ \varphi_d-1/3\end{pmatrix}+\Sigma_0(n,\varphi_p,\varphi_d)$$

and, in particular, we can take $K_{p,-}=K_{d,-}=\lambda/3-K$ and $K_{p,+}=K_{d,+}=\lambda/3+K$. Note also that the point $(1/3,1/3)$ can be seen as the "center" of the simplex (indeed, it represents the configuration where all the species have the same volume fraction). Hence, we can decompose $(S_p,S_d)^\top$ as the sum of an affine part that tends to keep the configuration close to the center of the simplex and the perturbation part Σ_0.
With this choice, we now check that, at least if λ is large enough (depending on K), then there exists a small constant $\varepsilon>0$ such that

$$y_i=\varepsilon\ \Rightarrow\ y_i'>0,\quad (1-y_p-y_d)=\varepsilon\ \Rightarrow\ y_p'+y_d'<0.$$

Indeed, let $a\in[-K,K]$. Then, for $y_i=\varepsilon$ we have

$$-\lambda(y_i-1/3)+a=\lambda(1/3-\varepsilon)+a\ge\lambda(1/3-\varepsilon)-K$$

which, for $\varepsilon<1/3$, is greater than 0 if λ is large enough compared to K. Analogously, for $a,b\in[-K,K]$ and $y_p+y_d=1-\varepsilon$,

$$-\lambda(y_p+y_d-2/3)+a+b=-\lambda(1/3-\varepsilon)+a+b\le-\lambda(1/3-\varepsilon)+2K$$

which, for $\varepsilon<1/3$, is negative if λ is chosen large enough compared to K. Consequently, one can take Δ_0 as a subset of $\{y_p\ge\varepsilon,\ y_d\ge\varepsilon,y_p+y_d\le1-\varepsilon\}$ with C^1-boundary.

Leaving the examples behind, we can now define a suitable notion of weak solution to the initial-boundary value problem for system (1.1a)-(1.1g):

DEFINITION 2.1. *We say that a multiple* $(\varphi_p,\mu_p,\eta_p,\varphi_d,\mu_d,\eta_d,\boldsymbol{u},q,n)$ *is a weak solution to the multi-species tumor model* (1.1) *over the interval* $(0,T)$ *if*

(1) *the following regularity properties hold:*

$$\varphi_i \in H^1(0,T;H^1(\Omega)') \cap L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)), \tag{2.12a}$$
$$\text{with } 0 \le \varphi_i \le 1, \quad \varphi_p + \varphi_d \le 1 \text{ a.e. in } Q,$$
$$\mu_i \in L^2(0,T;H^1(\Omega)), \tag{2.12b}$$
$$\eta_i \in L^2(Q), \tag{2.12c}$$
$$\boldsymbol{u} \in L^2(Q) \text{ with } \operatorname{div} \boldsymbol{u} \in L^2(Q), \tag{2.12d}$$
$$q \in L^2(0,T;H^1_0(\Omega)), \tag{2.12e}$$
$$n \in (1 + L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega))), \quad 0 \le n \le 1 \text{ a.e. in } Q, \tag{2.12f}$$

for $i = p,d$.

(2) *Equations* (1.1a)-(1.1g) *hold, for a.e.* $t \in (0,T)$ *and for* $i = p,d$*, in the following weak sense:*

$$\langle \partial_t \varphi_i, \zeta \rangle + \int_\Omega M_i \nabla \mu_i \cdot \nabla \zeta - \varphi_i \boldsymbol{u} \cdot \nabla \zeta \, dx = \int_\Omega S_i \zeta \, dx \quad \forall \zeta \in H^1(\Omega), \tag{2.13a}$$
$$\int_\Omega \mu_i \zeta \, dx = \int_\Omega \nabla \varphi_i \cdot \nabla \zeta + \eta_i \zeta + F_{1,i}(\varphi_p,\varphi_d) \zeta \, dx \quad \forall \zeta \in H^1(\Omega), \tag{2.13b}$$
$$\int_\Omega \boldsymbol{u} \cdot \nabla \xi \, dx = -\int_\Omega (S_p + S_d) \xi \, dx \quad \forall \xi \in H^1_0(\Omega), \tag{2.13c}$$
$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\zeta} \, dx = \int_\Omega -\nabla q \cdot \boldsymbol{\zeta} - \varphi_p \nabla \mu_p \cdot \boldsymbol{\zeta} - \varphi_d \nabla \mu_d \cdot \boldsymbol{\zeta} \, dx \quad \forall \boldsymbol{\zeta} \in (L^2(\Omega))^d, \tag{2.13d}$$
$$0 = -\Delta n + \varphi_p n + B(n_C - n) \quad \text{a.e. in } \Omega, \tag{2.13e}$$
$$\eta_i = F_{0,i}(\varphi_p,\varphi_d) \quad \text{a.e. in } \Omega, \tag{2.13f}$$
$$S_p = \Sigma_p(n,\varphi_p,\varphi_d) + m_{pp}\varphi_p + m_{pd}\varphi_d \quad \text{a.e. in } \Omega, \tag{2.13g}$$
$$S_d = \Sigma_d(n,\varphi_p,\varphi_d) + m_{dp}\varphi_p + m_{dd}\varphi_d \quad \text{a.e. in } \Omega, \tag{2.13h}$$

where $\langle \cdot,\cdot \rangle$ *denotes the duality pairing between* $H^1(\Omega)$ *and its dual* $H^1(\Omega)'$*. Moreover,*

$$\varphi_p(x,0) = \varphi_{p,0}(x), \quad \varphi_d(x,0) = \varphi_{d,0}(x) \quad \text{a.e. in } \Omega. \tag{2.13i}$$

It is worth noting that now the first two boundary conditions in (1.1h) have been incorporated in the weak formulations (2.13a), (2.13b). Moreover, the boundary conditions $q = 0$ and $n = 1$ a.e. on Σ are built into the function spaces in (2.12e) and (2.12f). Furthermore, the attainment of the initial conditions (2.13i) is due to the continuous embedding

$$H^1(0,T;H^1(\Omega)') \cap L^\infty(0,T;H^1(\Omega)) \subset C^0([0,T];L^2(\Omega)),$$

and thus the initial conditions (2.13i) makes sense as equalities in the space $L^2(\Omega)$. Finally, it is worth saying some words about the auxiliary variables η_p and η_d. Using the language of convex analysis, relations (2.13f) for $i = p,d$ may be equivalently stated by saying that the vector $\boldsymbol{\eta} = (\eta_p,\eta_d)$ belongs at almost every point $(x,t) \in Q$ to the subdifferential $\partial F_0(\varphi_p,\varphi_d)$ which is a maximal monotone graph in $\mathbb{R}^2 \times \mathbb{R}^2$. In principle such an object may be a multivalued mapping. Here, however, in view of the fact that F_0 is assumed to be smooth in Δ (cf. (A3)), ∂F_0 may be simply identified with the gradient ∇F_0. On the other hand, the use of some techniques from convex analysis and monotone operators will be required in the last part of the proof.

We are now ready to state the main result of this paper.

THEOREM 2.1. *Let the hypotheses stated in Assumption 2.1 hold. Then, there exists at least one weak solution $(\varphi_p,\mu_p,\eta_p,\varphi_d,\mu_d,\eta_d,\boldsymbol{u},q,n)$ to the multi-species tumor model (1.1) in the sense of Definition 2.1.*

3. Approximation scheme

For $\varepsilon\in(0,1)$ intended to go to 0 in the limit, we consider the Moreau–Yosida approximation of F_0 (cf. [6]) defined as

$$F_\varepsilon(s,r):=\min_{(p,q)\in\mathbb{R}^2}\left(\frac{1}{2\varepsilon}|(p-s,q-r)|^2+F_0(p,q)\right)\quad \text{for } \varepsilon\in(0,1). \tag{3.1}$$

It is well-known that F_ε is convex and differentiable with derivative ∇F_ε that is globally Lipschitz continuous with Lipschitz constant scaling with ε^{-1}. More importantly, thanks to the fact that F_0 satisfies (2.9), it turns out that F_0 fulfills the hypothesis of [16, Prop. 2.10], whence, by [16, Prop. 2.13], there exist positive constants c_*,C_* such that

$$c_*|\nabla F_\varepsilon(s,r)-\nabla F_\varepsilon(S,R)|\le(\nabla F_\varepsilon(s,r)-\nabla F_\varepsilon(S,R))\cdot(s-S,r-R)^\top+C_* \tag{3.2}$$

for all $(s,r)\neq(S,R)\in\mathbb{R}^2$. In particular an analogue of (2.9) also holds for F_ε with constants c_*,C_* independent of $\varepsilon\in(0,1)$.

Concerning the regularized initial data appearing in (1.7a), for each $\delta\in(0,1)$, $i=p,d$, we take $\varphi_{i,0,\delta}\in H^2_{\boldsymbol{n}}(\Omega)$ as the solution f_i to

$$-\delta\Delta f_i+f_i=\varphi_{i,0} \text{ in } \Omega,\quad \partial_{\boldsymbol{n}}f_i=0 \text{ on } \partial\Omega, \tag{3.3}$$

where $\{\varphi_{p,0},\varphi_{d,0}\}$ is the initial data prescribed for (1.1). Note that the parameter ε does not appear and hence the regularized data is only indexed by δ. We have used here the notation $H^2_{\boldsymbol{n}}(\Omega)$ for the space of $H^2(\Omega)$-functions satisfying homogeneous Neumann boundary condition on $\partial\Omega$. Then, it is well-known that, for each $\delta\in(0,1)$, $f_i\in H^2_{\boldsymbol{n}}(\Omega)$. More precisely, testing (3.3) by f_i and $-\Delta f_i$, respectively, one obtains

$$\begin{aligned}2\delta\|\nabla f_i\|^2_{L^2(\Omega)}+\|f_i\|^2_{L^2(\Omega)}&\le\|\varphi_{i,0}\|^2_{L^2(\Omega)},\\ 2\delta\|\Delta f_i\|^2_{L^2(\Omega)}+\|\nabla f_i\|^2_{L^2(\Omega)}&\le\|\nabla\varphi_{i,0}\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.4}$$

Furthermore, elliptic regularity arguments yield the additional estimate

$$\|f_i\|_{H^2(\Omega)}\le C\big(\|\Delta f_i\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}\big)\le C\big(1+\delta^{-\frac12}\big)\|\varphi_{i,0}\|_{H^1(\Omega)}. \tag{3.5}$$

3.1. Auxiliary Cahn–Hilliard problem. Fix now $\bar q\in L^2(0,T;H^1(\Omega))$ and $\bar n\in L^2(Q)$ with $0\le\bar n\le1$ almost everywhere in Q. Then, we first consider the auxiliary problem

$$\partial_t\varphi_p=M_p\Delta\mu_p+\operatorname{div}(\mathcal{T}(\varphi_p)\nabla\bar q)+\operatorname{div}\big(\mathcal{T}(\varphi_p)^2\nabla\mu_p+\mathcal{T}(\varphi_p)\mathcal{T}(\varphi_d)\nabla\mu_d\big)+S_p, \tag{3.6a}$$
$$\mu_p=-\delta\Delta\partial_t\varphi_p+F_{\varepsilon,p}(\varphi_p,\varphi_d)+F_{1,p}(\varphi_p,\varphi_d)-\Delta\varphi_p, \tag{3.6b}$$
$$\partial_t\varphi_d=M_d\Delta\mu_d+\operatorname{div}(\mathcal{T}(\varphi_d)\nabla\bar q)+\operatorname{div}\big(\mathcal{T}(\varphi_p)\mathcal{T}(\varphi_d)\nabla\mu_p+\mathcal{T}(\varphi_d)^2\nabla\mu_d\big)+S_d, \tag{3.6c}$$
$$\mu_d=-\delta\Delta\partial_t\varphi_d+F_{\varepsilon,d}(\varphi_p,\varphi_d)+F_{1,d}(\varphi_p,\varphi_d)-\Delta\varphi_d, \tag{3.6d}$$
$$S_p=\Sigma_p(\bar n,\varphi_p,\varphi_d)+m_{pp}\varphi_p+m_{pd}\varphi_d, \tag{3.6e}$$

$$S_d = \Sigma_d(\bar{n}, \varphi_p, \varphi_d) + m_{dp}\varphi_p + m_{dd}\varphi_d, \tag{3.6f}$$

complemented with the initial and boundary conditions (1.7a)-(1.7c). Recall the cutoff operator $\mathcal{T}$ defined in (1.5). The above is a Cahn–Hilliard system with source term. Note that $\bar{q}$ and $\bar{n}$ are given. Existence of a solution can be proved, for instance, via a Galerkin approximation. We will only derive the necessary a priori estimates.

LEMMA 3.1. *For each $\varepsilon \in (0,1)$, $\delta \in (0,1)$, suppose (2.1) holds, and $F_\varepsilon : \mathbb{R}^2 \to [0,+\infty)$ and $F_1 : \mathbb{R}^2 \to \mathbb{R}$ are given such that ∇F_ε, ∇F_1 are globally Lipschitz continuous. Then, for given $\bar{q} \in L^2(0,T;H^1(\Omega))$ and $\bar{n} \in L^2(Q)$, there exists a* unique *weak solution $(\varphi_p, \mu_p, \varphi_d, \mu_d)$ to (3.6) in the following sense:*

(1) *the functions have the following regularity properties:*

$$\varphi_i \in H^1(0,T;H^2(\Omega)), \quad \mu_i \in L^2(0,T;H^1(\Omega)),$$

with

$$\varphi_i(0) = \varphi_{0,i,\delta} \ \text{a.e. in } \Omega.$$

(2) *Equations (3.6b), (3.6d), (3.6e) and (3.6f) hold a.e. in Q, and Equations (3.6a) and (3.6c) hold for a.e. $t \in (0,T)$ in the following weak sense:*

$$0 = \int_\Omega (\partial_t \varphi_p - S_p)\zeta + \big(M_p \nabla \mu_p + \mathcal{T}(\varphi_p)\big(\nabla \bar{q} + \mathcal{T}(\varphi_p)\nabla \mu_p + \mathcal{T}(\varphi_d)\nabla \mu_d\big)\big) \cdot \nabla \zeta \, dx,$$

$$0 = \int_\Omega (\partial_t \varphi_d - S_d)\zeta + \big(M_d \nabla \mu_d + \mathcal{T}(\varphi_d)\big(\nabla \bar{q} + \mathcal{T}(\varphi_d)\nabla \mu_d + \mathcal{T}(\varphi_p)\nabla \mu_p\big)\big) \cdot \nabla \zeta \, dx,$$

for all $\zeta \in H^1(\Omega)$.

Proof. In the following, the symbol C denotes positive constants that are independent of $(\varphi_p,\ \mu_p,\ \varphi_d,\ \mu_d)$.

First estimate. Testing (3.6a) with μ_p and (3.6b) with $\partial_t \varphi_p$ yields

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\|\nabla \varphi_p\|^2_{L^2(\Omega)} + \int_\Omega (F_{\varepsilon,p} + F_{1,p})(\varphi_p, \varphi_d)\partial_t \varphi_p \, dx \\ &\qquad + \delta \|\nabla \partial_t \varphi_p\|^2_{L^2(\Omega)} + M_p \|\nabla \mu_p\|^2_{L^2(\Omega)} + \|\mathcal{T}(\varphi_p)\nabla \mu_p\|^2_{L^2(\Omega)} \\ &= \int_\Omega S_p \mu_p - \mathcal{T}(\varphi_p)(\nabla \bar{q} + \mathcal{T}(\varphi_d)\nabla \mu_d) \cdot \nabla \mu_p \, dx. \end{aligned} \tag{3.7}$$

An analogous identity is derived similarly by testing (3.6c) with μ_d and (3.6d) with $\partial_t \varphi_d$. Then, adding the two resulting equalities leads to

$$\begin{aligned} &\frac{d}{dt} E_\varepsilon(\varphi_p, \varphi_d) + M_p \|\nabla \mu_p\|^2_{L^2(\Omega)} + M_d \|\nabla \mu_d\|^2_{L^2(\Omega)} \\ &\qquad + \delta \|\nabla \partial_t \varphi_p\|^2_{L^2(\Omega)} + \delta \|\nabla \partial_t \varphi_d\|^2_{L^2(\Omega)} + \int_\Omega |\mathcal{T}(\varphi_p)\nabla \mu_p + \mathcal{T}(\varphi_d)\nabla \mu_d|^2 \, dx \\ &= \int_\Omega (S_p \mu_p + S_d \mu_d)\, dx - \int_\Omega \big(\mathcal{T}(\varphi_p)\nabla \bar{q} \cdot \nabla \mu_p + \mathcal{T}(\varphi_d)\nabla \bar{q} \cdot \nabla \mu_d\big)\, dx, \end{aligned} \tag{3.8}$$

where E_ε is the approximate energy given by

$$E_\varepsilon = \int_\Omega F_\varepsilon(\varphi_p, \varphi_d) + F_1(\varphi_p, \varphi_d) + \frac{1}{2}\Big(|\nabla \varphi_p|^2 + |\nabla \varphi_d|^2\Big)\, dx$$

in light of the Moreau–Yosida approximation F_ε appearing in the system (3.6). Note that the additional (nonnegative) contribution $\int_\Omega |\mathcal{T}(\varphi_d)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d|^2\,dx$ comes from the fact that the equations have been restated in terms of the pressure q.

The approximate energy E_ε may be no longer be coercive with respect to the $H^1(\Omega)$-norms of φ_p and φ_d. For this reason, we test (3.6a) with φ_p to obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi_p\|^2_{L^2(\Omega)} = \int_\Omega S_p\varphi_p - \nabla\varphi_p\cdot(M_p\nabla\mu_p + \mathcal{T}(\varphi_p)(\nabla\bar{q} + \mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d))\,dx,$$

and obtain an analogous identity by testing (3.6c) with φ_d. Then, upon summing the two resulting equalities and using Young's inequality leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big(\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\big)\\
&\le \sigma\big(\|\nabla\mu_p\|^2_{L^2(\Omega)} + \|\nabla\mu_d\|^2_{L^2(\Omega)}\big) + c_\sigma\big(\|\nabla\varphi_p\|^2_{L^2(\Omega)} + \|\nabla\varphi_d\|^2_{L^2(\Omega)}\big)\\
&\quad + C\big(1 + \|\nabla\bar{q}\|^2_{L^2(\Omega)} + \|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\big)
\end{aligned}\tag{3.9}$$

for small constant $\sigma > 0$ and correspondingly large constant c_σ. In (3.9) we employed the boundedness of the cutoff operator as well as the assumption (A2). In particular, (A2) implies

$$\|S_p\|_{L^2(\Omega)} \le c|\Omega| + |m_{pp}|\,\|\varphi_p\|_{L^2(\Omega)} + |m_{pd}|\,\|\varphi_d\|_{L^2(\Omega)},\tag{3.10}$$

and a similar bound holds for $\|S_d\|_{L^2(\Omega)}$. Let $K > 0$ be a positive constant yet to be determined, then upon adding (3.8) and K times (3.9) we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(E_\varepsilon(\varphi_p,\varphi_d) + \frac{K}{2}\left(\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\right)\right) + \delta\|\nabla\partial_t\varphi_d\|^2_{L^2(\Omega)}\\
&\quad + \delta\|\nabla\partial_t\varphi_p\|^2_{L^2(\Omega)} + (M_p - K\sigma)\|\nabla\mu_p\|^2_{L^2(\Omega)} + (M_d - K\sigma)\|\nabla\mu_d\|^2_{L^2(\Omega)}\\
&\le \int_\Omega S_p\mu_p + S_d\mu_d - \nabla\bar{q}\cdot(\mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d)\,dx\\
&\quad + Kc_\sigma\left(\|\nabla\varphi_p\|^2_{L^2(\Omega)} + \|\nabla\varphi_d\|^2_{L^2(\Omega)}\right)\\
&\quad + KC\left(1 + \|\nabla\bar{q}\|^2_{L^2(\Omega)} + \|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\right).
\end{aligned}\tag{3.11}$$

Notice, the Moreau–Yosida approximation F_ε is nonnegative (see (3.1)), and by the growth condition (2.5) of the smooth nonconvex part F_1, we can find positive constants d_1, d_2 such that

$$\int_\Omega F_1(\varphi_p,\varphi_d)\,dx \ge -d_1\left(\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\right) - d_2.$$

Then, taking $K > 2d_1$ shows that

$$E_\varepsilon + \frac{K}{2}\big(\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\big) \ge k\big(\|\varphi_p\|^2_{H^1(\Omega)} + \|\varphi_d\|^2_{H^1(\Omega)}\big) - C,\tag{3.12}$$

for some $k > 0$ independent of ε. Then, after K is fixed, we also take σ sufficiently small so that $\min\{M_p - K\sigma, M_d - K\sigma\} \ge \kappa$ for some constant $\kappa > 0$. It remains to control the integral terms on the right-hand side of (3.11). Observe that, by Young's inequality,

$$\left|\int_\Omega \nabla\bar{q}\cdot(\mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d)\,dx\right|$$

$$\leq \frac{\kappa}{4}\left(\|\nabla\mu_p\|^2_{L^2(\Omega)}+\|\nabla\mu_d\|^2_{L^2(\Omega)}\right)+C\|\nabla\bar{q}\|^2_{L^2(\Omega)}.$$

Furthermore, recalling the notation $\overline{f}$ for the mean value of f over Ω, we have by the Lipschitz continuity of $F_{\varepsilon,p}$ with Lipschitz constant $1/\varepsilon$ together with Assumption (2.5)

$$\begin{aligned}
|\overline{\mu_p}| &\leq \frac{1}{|\Omega|}\int_\Omega |F_{\varepsilon,p}(\varphi_p,\varphi_d)-F_{\varepsilon,p}(0,0)|+|F_{1,p}(\varphi_p,\varphi_d)|\,dx \\
&\quad + \frac{1}{|\Omega|}\int_\Omega |F_{\varepsilon,p}(0,0)|\,dx \\
&\leq C\left(\varepsilon^{-1}+1\right)\left(\|\varphi_p\|_{L^1(\Omega)}+\|\varphi_d\|_{L^1(\Omega)}+1\right)+C_\varepsilon \\
&\leq C_\varepsilon\left(1+\|\varphi_p\|_{L^2(\Omega)}+\|\varphi_d\|_{L^2(\Omega)}\right).
\end{aligned}$$

Then, using (3.10) and the Poincaré inequality, we see that

$$\begin{aligned}
\int_\Omega S_p\mu_p\,dx &= \int_\Omega S_p(\mu_p-\overline{\mu_p})\,dx+\overline{\mu_p}\int_\Omega S_p\,dx \\
&\leq C\|S_p\|_{L^2(\Omega)}\|\nabla\mu_p\|_{L^2(\Omega)}+C\,|\overline{\mu_p}|\,\|S_p\|_{L^2(\Omega)} \\
&\leq \frac{\kappa}{4}\|\nabla\mu_p\|^2_{L^2(\Omega)}+C_\varepsilon\left(1+\|\varphi_p\|^2_{L^2(\Omega)}+\|\varphi_d\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{3.13}$$

The term $S_d\mu_d$ is controlled analogously. Then, collecting the above computations, (3.11) becomes

$$\begin{aligned}
&\frac{d}{dt}\left(E_\varepsilon(\varphi_p,\varphi_d)+\frac{K}{2}\left(\|\varphi_p\|^2_{L^2(\Omega)}+\|\varphi_d\|^2_{L^2(\Omega)}\right)\right) \\
&\qquad +\frac{\kappa}{2}\left(\|\nabla\mu_p\|^2_{L^2(\Omega)}+\|\nabla\mu_d\|^2_{L^2(\Omega)}\right)+\delta\|\nabla\partial_t\varphi_p\|^2_{L^2(\Omega)}+\delta\|\nabla\partial_t\varphi_d\|^2_{L^2(\Omega)} \\
&\leq C_\varepsilon\left(1+\|\varphi_p\|^2_{H^1(\Omega)}+\|\varphi_d\|^2_{H^1(\Omega)}\right)+C\|\nabla\bar{q}\|^2_{L^2(\Omega)},
\end{aligned} \tag{3.14}$$

for some positive constants C, C_ε that are independent of δ. By virtue of the coercivity property (3.12), a Gronwall argument then yields

$$\|\varphi_p\|_{L^\infty(0,T;H^1(\Omega))}+\|\varphi_d\|_{L^\infty(0,T;H^1(\Omega))}\leq C_\varepsilon, \tag{3.15a}$$

$$\|\mu_p\|_{L^2(0,T;H^1(\Omega))}+\|\mu_d\|_{L^2(0,T;H^1(\Omega))}\leq C_\varepsilon, \tag{3.15b}$$

$$\|\nabla\partial_t\varphi_p\|_{L^2(0,T;L^2(\Omega))}+\|\nabla\partial_t\varphi_d\|_{L^2(0,T;L^2(\Omega))}\leq C_{\varepsilon,\delta}. \tag{3.15c}$$

Second estimate. Testing (3.6b) with $-\Delta\partial_t\varphi_p$ leads to

$$\frac{1}{2}\frac{d}{dt}\|\Delta\varphi_p\|^2_{L^2(\Omega)}+\delta\|\Delta\partial_t\varphi_p\|^2_{L^2(\Omega)}=\int_\Omega(F_{\varepsilon,p}(\varphi_p,\varphi_d)+F_{1,p}(\varphi_p,\varphi_d)-\mu_p)\Delta\partial_t\varphi_p\,dx.$$

One can control the right-hand side with Young's inequality, the linear growth of $F_{\varepsilon,p},F_{1,p}$ and the estimates (3.15a)-(3.15b). Together with the initial condition $\Delta\varphi_{p,0,\delta}\in L^2(\Omega)$, we infer

$$\|\Delta\varphi_p\|_{H^1(0,T;L^2(\Omega))}+\|\Delta\varphi_d\|_{H^1(0,T;L^2(\Omega))}\leq C_{\varepsilon,\delta},$$

where the bound for $\Delta\varphi_d$ follows along a similar argument from testing (3.6d) with $-\Delta\partial_t\varphi_d$. By elliptic regularity we get the following additional estimate:

$$\|\varphi_p\|_{H^1(0,T;H^2(\Omega))}+\|\varphi_d\|_{H^1(0,T;H^2(\Omega))}\leq C_{\varepsilon,\delta}. \tag{3.16}$$

The estimates (3.15a)-(3.16) are sufficient to pass to the limit in a Galerkin approximation to deduce the existence of a quadruple $(\varphi_p, \mu_p, \varphi_d, \mu_d)$ that satisfies the assertions of Lemma 3.1. We now establish the uniqueness of solutions for the auxiliary problem (3.6).

Uniqueness. Let us denote by $\hat{\varphi}_p$, $\hat{\varphi}_d$, $\hat{\mu}_p$ and $\hat{\mu}_d$ the differences $\varphi_{p,1}-\varphi_{p,2}$, $\varphi_{d,1}-\varphi_{d,2}$, $\mu_{p,1}-\mu_{p,2}$ and $\mu_{d,1}-\mu_{d,2}$, respectively. Then, upon testing the difference of the Equations (3.6a) by $\hat{\mu}_p$ and the difference of the Equations (3.6b) by $\partial_t\hat{\varphi}_p - \Delta\hat{\varphi}_p$ leads to

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\left(\|\nabla\hat{\varphi}_p\|^2_{L^2(\Omega)} + \delta\|\Delta\hat{\varphi}_p\|^2_{L^2(\Omega)}\right) \\
&\qquad + \|\Delta\hat{\varphi}_p\|^2_{L^2(\Omega)} + \delta\|\nabla\partial_t\hat{\varphi}_p\|^2_{L^2(\Omega)} + \int_\Omega \left(M_p + (\mathcal{T}_{p,2})^2\right)|\nabla\hat{\mu}_p|^2\,dx \\
&= \int_\Omega \hat{S}_p\hat{\mu}_p - \hat{\mu}_p\Delta\hat{\varphi}_p - \hat{\mathcal{T}}_p\nabla\bar{q}\cdot\nabla\hat{\mu}_p - \left(\hat{F}_{\varepsilon,p} + \hat{F}_{1,p}\right)(\partial_t\hat{\varphi}_p - \Delta\hat{\varphi}_p)\,dx \\
&\qquad - \int_\Omega \nabla\hat{\mu}_p\cdot\left(\widehat{(\mathcal{T}_p)^2}\nabla\mu_{p,1} + \left(\hat{\mathcal{T}}_p\mathcal{T}_{d,1} + \mathcal{T}_{p,2}\hat{\mathcal{T}}_d\right)\nabla\mu_{d,1} + \mathcal{T}_{p,2}\mathcal{T}_{d,2}\nabla\hat{\mu}_d\right)dx \\
&=: J_1 + J_2,
\end{aligned} \tag{3.17}$$

where we used the notation

$$\begin{aligned}
&\mathcal{T}_{p,1} = \mathcal{T}(\varphi_{p,1}), \quad \hat{\mathcal{T}}_p = \mathcal{T}_{p,1} - \mathcal{T}_{p,2}, \quad \hat{F}_{\varepsilon,p} = F_{\varepsilon,p}(\varphi_{p,1},\varphi_{d,1}) - F_{\varepsilon,p}(\varphi_{p,2},\varphi_{d,2}), \\
&\hat{F}_{1,p} = F_{1,p}(\varphi_{p,1},\varphi_{d,1}) - F_{1,p}(\varphi_{p,2},\varphi_{d,2}), \quad \widehat{(\mathcal{T}_p)^2} = (\mathcal{T}_{p,1})^2 - (\mathcal{T}_{p,2})^2 = \hat{\mathcal{T}}_p(\mathcal{T}_{p,1} + \mathcal{T}_{p,2}), \\
&\hat{S}_p = \Sigma_p(\bar{n},\varphi_{p,1},\varphi_{d,1}) - \Sigma_p(\bar{n},\varphi_{p,2},\varphi_{d,2}) + m_{pp}\hat{\varphi}_p + m_{pd}\hat{\varphi}_d.
\end{aligned}$$

Using the Lipschitz continuity of $F_{\varepsilon,p}$, $F_{1,p}$, $\mathcal{T}(\cdot)$, Σ_i and the boundedness of $\mathcal{T}(\cdot)$ and Σ_i, we deduce

$$\begin{aligned}
J_1 &\leq C\left(\|\hat{\varphi}_p\|_{L^2(\Omega)} + \|\hat{\varphi}_d\|_{L^2(\Omega)} + \|\Delta\hat{\varphi}_p\|_{L^2(\Omega)}\right)\left(\|\hat{\mu}_p - \overline{\hat{\mu}_p}\|_{L^2(\Omega)} + |\overline{\hat{\mu}_p}|\right) \\
&\quad + C\left(\|\hat{\varphi}_p\|_{L^2(\Omega)} + \|\hat{\varphi}_d\|_{L^2(\Omega)}\right)\left(\|\Delta\hat{\varphi}_p\|_{L^2(\Omega)} + \|\partial_t\hat{\varphi}_p - \overline{\partial_t\hat{\varphi}_p}\|_{L^2(\Omega)} + |\overline{\partial_t\hat{\varphi}_p}|\right), \\
&\quad + C\|\hat{\varphi}_p\|_{L^\infty(\Omega)}\|\nabla\hat{\mu}_p\|_{L^2(\Omega)}\|\nabla\bar{q}\|_{L^2(\Omega)}, \\
J_2 &\leq C\|\nabla\hat{\mu}_p\|_{L^2(\Omega)}\left(\|\hat{\varphi}_p\|_{L^\infty(\Omega)} + \|\hat{\varphi}_d\|_{L^\infty(\Omega)}\right)\left(\|\nabla\mu_{p,1}\|_{L^2(\Omega)} + \|\nabla\mu_{d,1}\|_{L^2(\Omega)}\right) \\
&\quad + \frac{1}{2}\int_\Omega (\mathcal{T}_{p,2})^2|\nabla\hat{\mu}_p|^2 + (\mathcal{T}_{d,2})^2|\nabla\hat{\mu}_d|^2\,dx.
\end{aligned}$$

Note that by the Lipschitz property of Σ_i, $F_{\varepsilon,p}$ and $F_{1,p}$,

$$\begin{aligned}
|\overline{\partial_t\hat{\varphi}_p}| &= \left|\overline{\hat{S}_p}\right| \leq C\left(\|\hat{\varphi}_p\|_{L^2(\Omega)} + \|\hat{\varphi}_d\|_{L^2(\Omega)}\right), \\
|\overline{\hat{\mu}_p}| &= \left|\overline{\hat{F}_{\varepsilon,p} + \hat{F}_{1,p}}\right| \leq C_\varepsilon\left(\|\hat{\varphi}_p\|_{L^2(\Omega)} + \|\hat{\varphi}_d\|_{L^2(\Omega)}\right),
\end{aligned}$$

and so, upon adding (3.17) to the corresponding equation for $\hat{\varphi}_d$, and applying the estimates for the right-hand sides leads to

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\sum_{i=p,d}\left(\|\nabla\hat{\varphi}_i\|^2_{L^2(\Omega)} + \delta\|\Delta\hat{\varphi}_i\|^2_{L^2(\Omega)}\right) \\
&\qquad + \frac{1}{2}\sum_{i=p,d}\left(\|\Delta\hat{\varphi}_i\|^2_{L^2(\Omega)} + \delta\|\nabla\partial_t\hat{\varphi}_i\|^2_{L^2(\Omega)} + M_i\|\nabla\hat{\mu}_i\|^2_{L^2(\Omega)}\right)
\end{aligned}$$

$$\le C_\varepsilon\Big[1+\|\nabla\bar{q}\|^2_{L^2(\Omega)}+\sum_{\substack{i=p,d\\ j=1,2}}\|\nabla\mu_{i,j}\|^2_{L^2(\Omega)}\Big]\sum_{i=p,d}\Big(\|\hat{\varphi}_i\|^2_{L^2(\Omega)}+\|\Delta\hat{\varphi}_i\|^2_{L^2(\Omega)}\Big), \tag{3.18}$$

where in the above we have used the elliptic estimate and the Sobolev embedding $H^2(\Omega)\subset L^\infty(\Omega)$:

$$\|f\|_{L^\infty(\Omega)}\le C\|f\|_{H^2(\Omega)}\le C\left(\|\Delta f\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}\right), \tag{3.19}$$

for $\hat{\varphi}_p$ and $\hat{\varphi}_d$ as they satisfy no-flux boundary conditions.

Next, we test the difference of the Equations (3.6a) with $\hat{\varphi}_p$ which yields

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\|\hat{\varphi}_p\|^2_{L^2(\Omega)}=&-\int_\Omega\nabla\hat{\varphi}_p\cdot\Big(M_p\nabla\hat{\mu}_p+\hat{T}_p\nabla\bar{q}+\widehat{(T_p)^2}\nabla\mu_{p,1}+(T_{p,2})^2\nabla\hat{\mu}_p\Big)\,dx\\
&-\int_\Omega\nabla\hat{\varphi}_p\cdot\Big(\Big(\hat{T}_pT_{d,1}+T_{p,2}\hat{T}_d\Big)\nabla\mu_{d,1}+T_{p,2}T_{d,2}\nabla\hat{\mu}_d\Big)-\hat{S}_p\hat{\varphi}_p\,dx\\
\le& C\Big[1+\|\nabla\bar{q}\|^2_{L^2(\Omega)}+\sum_{\substack{i=p,d\\ j=1,2}}\|\nabla\mu_{i,j}\|^2_{L^2(\Omega)}\Big]\Bigg(\|\nabla\hat{\varphi}_p\|^2_{L^2(\Omega)}+\sum_{i=p,d}\|\hat{\varphi}_i\|^2_{L^\infty(\Omega)}\Bigg)\\
&+\frac{M_p}{4}\|\nabla\hat{\mu}_p\|^2_{L^2(\Omega)}+\frac{M_d}{4}\|\nabla\hat{\mu}_d\|^2_{L^2(\Omega)},
\end{aligned}$$

and upon adding the analogous estimate obtained from testing (3.6c) with $\hat{\varphi}_d$ and then adding to (3.18), after applying the elliptic estimate (3.19), we arrive at the following differential inequality

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\sum_{i=p,d}\Big(\|\hat{\varphi}_i\|^2_{L^2(\Omega)}+\|\nabla\hat{\varphi}_i\|^2_{L^2(\Omega)}+\delta\|\Delta\hat{\varphi}_i\|^2_{L^2(\Omega)}\Big)\\
&\qquad+\frac{1}{2}\sum_{i=p,d}\Bigg(\|\Delta\hat{\varphi}_i\|^2_{L^2(\Omega)}+\delta\|\nabla\partial_t\hat{\varphi}_i\|^2_{L^2(\Omega)}+\frac{1}{2}M_i\|\nabla\hat{\mu}_i\|^2_{L^2(\Omega)}\Bigg)\\
&\le C_\varepsilon\Big[1+\|\nabla\bar{q}\|^2_{L^2(\Omega)}+\sum_{\substack{i=p,d\\ j=1,2}}\|\nabla\mu_{i,j}\|^2_{L^2(\Omega)}\Big]\sum_{i=p,d}\Big(\|\hat{\varphi}_i\|^2_{H^1(\Omega)}+\|\Delta\hat{\varphi}_i\|^2_{L^2(\Omega)}\Big).
\end{aligned}$$

Applying a Gronwall argument entails uniqueness. ∎

3.2. Auxiliary pressure and nutrient equations. We now consider, for $(\varphi_p,\mu_p,\varphi_d,\mu_d)$ obtained from Lemma 3.1, the following system:

$$\delta\big(\partial_t q+\Delta^2 q\big)-\Delta q=\operatorname{div}\big(\mathcal{T}(\varphi_p)\nabla\mu_p+\mathcal{T}(\varphi_d)\nabla\mu_d\big)+(S_p+S_d)(n,\varphi_p,\varphi_d), \tag{3.20a}$$

$$0=-\Delta n+\mathcal{T}(\varphi_p)n-B(n_C-n), \tag{3.20b}$$

furnished with the initial-boundary conditions resulting from (1.7a)-(1.7c).

LEMMA 3.2. *Let $(\varphi_p,\mu_p,\varphi_d,\mu_d)$ denote a weak solution obtained from Lemma 3.1. Then, there exists a* unique *pair (q,n) of solutions to (3.20) in the following sense:*

(1) *the functions have the following regularity properties:*

$$\begin{aligned}
&q\in L^2(0,T;H^3(\Omega))\cap L^\infty(0,T;H^1_0(\Omega))\cap H^1(0,T;H^{-1}(\Omega)),\\
&n\in L^\infty(0,T;W^{2,r}(\Omega))\ \textit{for any}\ r<\infty\ \textit{and}\ 0\le n\le 1\ \textit{a.e. in}\ Q,
\end{aligned}$$

with

$$q(0)=0\ \textit{in}\ \Omega,\quad q=\Delta q=0,\, n=1\ \textit{on}\ \Gamma.$$

(2) *Equation* (3.20b) *holds a.e. in* Q *and Equation* (3.20a) *holds for a.e.* $t\in(0,T)$ *in the following weak sense:*

$$0=\delta\langle\partial_t q,\zeta\rangle_{H_0^1}+\int_\Omega(\nabla q-\delta\nabla\Delta q+\mathcal{T}(\varphi_p)\nabla\mu_p+\mathcal{T}(\varphi_d)\nabla\mu_d)\cdot\nabla\zeta\,dx \\ -\int_\Omega(S_p+S_d)(n,\varphi_p,\varphi_d)\zeta\,dx,$$

for all $\zeta\in H_0^1(\Omega)$*, where* $\langle\cdot,\cdot\rangle_{H_0^1}$ *denotes the duality pairing between* $H_0^1(\Omega)$ *and* $H^{-1}(\Omega)$*.*

Proof. We investigate the nutrient and pressure equations separately.

Nutrient equation. Since $\mathcal{T}(\cdot)$ is bounded and non-negative, we may first consider a parabolic regularization to (3.20b), namely, we add $\gamma\partial_t n$ on the right-hand side (for $\gamma\in(0,1)$) and we complement the resulting parabolic equation (for example) with the initial condition $n_\gamma(0):=1$ (which is consistent with the boundary datum). Then, applying the standard parabolic theory and the weak comparison principle it is easy to show that there exists a unique function $n_\gamma\in L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))$ with $0\le n_\gamma\le 1$ a.e. in Q (see for example [30, Lem. 3.1]). It turns out that n_γ is uniformly bounded in $L^2(0,T;H^1(\Omega))$ and, passing to the limit $\gamma\to0$, we deduce the existence of a weak solution $n\in L^2(0,T;H^1(\Omega))$ to (3.20b) with $0\le n\le 1$ a.e. in Q. Then, as $\mathcal{T}(\varphi_p)n-B(n_C-n)\in L^\infty(0,T;L^\infty(\Omega))$, applying elliptic regularity we infer $n\in L^\infty(0,T;W^{2,r}(\Omega))$ for any $r<\infty$.

Pressure equation. As (3.20a) is a linear fourth-order parabolic equation with given right-hand side (recall we treat $(\varphi_p,\mu_p,\varphi_d,\mu_d)$ as given data), the existence of a solution can be obtained via a Galerkin approximation once we establish the necessary a priori estimates below. Given n, φ_p, μ_p, φ_d and μ_d, we test (3.20a) with $q-\Delta q$. Using the boundary conditions $q=\Delta q=0$ on Γ, we then obtain

$$\begin{aligned}&\frac{d}{dt}\frac{\delta}{2}\left(\|q\|^2_{L^2(\Omega)}+\|\nabla q\|^2_{L^2(\Omega)}\right)+(1+\delta)\|\Delta q\|^2_{L^2(\Omega)}+\delta\|\nabla\Delta q\|^2_{L^2(\Omega)}+\|\nabla q\|^2_{L^2(\Omega)}\\ &=\int_\Omega(T_p\nabla\mu_p+T_d\nabla\mu_d)\cdot\nabla(\Delta q-q)+(S_p+S_d)(q-\Delta q)\,dx\\ &\le\frac{C}{\delta}\left(\|\nabla\mu_p\|^2_{L^2(\Omega)}+\|\nabla\mu_d\|^2_{L^2(\Omega)}\right)+\frac{\delta}{2}\left(\|\nabla\Delta q\|^2_{L^2(\Omega)}+\|\nabla q\|^2_{L^2(\Omega)}\right)\\ &\quad+\frac{C}{\delta}\left(1+\|\varphi_p\|^2_{L^2(\Omega)}+\|\varphi_d\|^2_{L^2(\Omega)}\right)+\frac{\delta}{2}\left(\|q\|^2_{L^2(\Omega)}+\|\Delta q\|^2_{L^2(\Omega)}\right).\end{aligned}\tag{3.21}$$

Integrating in time, using $q(0)=0$, and applying first a Gronwall argument and then elliptic regularity leads to

$$\|q\|_{L^\infty(0,T;H^1(\Omega))}+\|q\|_{L^2(0,T;H^3(\Omega))}\le C_\delta.\tag{3.22}$$

Then, consider testing (3.20a) with an arbitrary test function $\zeta\in H_0^1(\Omega)$ and integrating by parts; then, employing the estimates (3.22) allows us to infer that

$$\|\partial_t q\|_{L^2(0,T;H^{-1}(\Omega))}\le C_\delta.\tag{3.23}$$

Uniqueness. Let $\hat n:=n_1-n_2$ and $\hat q:=q_1-q_2$ denote the difference between two solution pairs (q_1,n_1) and (q_2,n_2) corresponding to the same data $(\varphi_p,\mu_p,\varphi_d,\mu_d)$. Then it is straightforward to see that

$$0=-\Delta\hat n+\mathcal{T}(\varphi_p)\hat n+B\hat n,\quad \delta(\partial_t\hat q+\Delta^2\hat q)-\Delta\hat q=\hat\Sigma_p+\hat\Sigma_d,\tag{3.24}$$

where for $i=p,d$,

$$\hat{\Sigma}_i := \Sigma_i(n_1,\varphi_p,\varphi_d) - \Sigma_i(n_2,\varphi_p,\varphi_d).$$

By testing the first equation of (3.24) with $\hat{n}$ we easily deduce that $\hat{n}=0$ by the Poincaré inequality. Then, testing the second equation of (3.24) with $\hat{q}$ and noting that $\hat{\Sigma}_i=0$ due to $n_1=n_2$, the uniqueness of solutions is clear. ◻

3.3. Fixed point argument. We will now apply a fixed point argument locally in time, and consider for some $T_0\in(0,T]$ the pair $(\bar{q},\bar{n})\in L^2(0,T_0;H^1(\Omega))\times L^2(0,T_0;L^2(\Omega))$ with $0\le\bar{n}\le 1$ a.e. in $\Omega\times(0,T_0)$. Let us introduce the mapping $\mathcal{K}:(\bar{q},\bar{n})\to(q,n)$, where (q,n) is the unique solution pair to (3.20) with $(\varphi_p,\mu_p,\varphi_d,\mu_d)$ as the unique solution quadruple to (3.6). To specify the domain of $\mathcal{K}$ we define

$$X := \big\{(q,n): \|q\|_{L^2(0,T_0;H^1(\Omega))} + \|n\|_{L^2(0,T_0;L^2(\Omega))} \le R,\ 0\le n\le 1 \text{ a.e. in } \Omega\times(0,T_0)\big\},$$

where $R>0$ is arbitrary but otherwise fixed. For example, one can take $R=1$. Let us mention that applying Gronwall's inequality to (3.14) on the interval $[0,t]$ yields analogous bounds to (3.15a)-(3.15b) with T replaced by t and constants $C_{\varepsilon,R}$, $C_{\varepsilon,\delta,R}$ now also depending on R due to the term $\|\nabla\bar{q}\|_{L^2(0,t;L^2(\Omega))}$ that will appear on the right-hand side. Then, applying Gronwall's inequality to (3.21) on $[0,t]$ leads to the estimate $\|q\|_{L^\infty(0,t;H^1(\Omega))}\le C_{\varepsilon,\delta,R}$. Hence, one obtains the estimate

$$\|q\|^2_{L^2(0,t;H^1(\Omega))} \le t\|q\|^2_{L^\infty(0,T;H^1(\Omega))} \le tC_{\varepsilon,\delta,R}$$

for any $t\in(0,T]$. On the other hand, since $0\le n\le 1$ a.e. in Q, we get

$$\|n\|^2_{L^2(0,t;L^2(\Omega))} \le t|\Omega|.$$

Consequently, for T_0 sufficiently small (in a way that possibly depends on ε, δ and R), we have

$$\|q\|_{L^2(0,T_0;H^1(\Omega))} + \|n\|_{L^2(0,T_0;L^2(\Omega))} \le C_{\varepsilon,\delta,R}T_0^{\frac{1}{2}} \le R. \tag{3.25}$$

This implies that for such a choice of T_0, the operator $\mathcal{K}$ maps X (which is a convex closed subset of the product Banach space $L^2(0,T_0;H^1(\Omega))\times L^2(0,T_0;L^2(\Omega))$) into itself.

Continuity. We now aim to show that $\mathcal{K}:X\to X$ is continuous with respect to the norm of $L^2(0,T_0;H^1(\Omega))\times L^2(0,T_0;L^2(\Omega))$, keeping in mind that thanks to the uniqueness results for the auxiliary problems (3.6) and (3.20), $\mathcal{K}$ is a single-valued mapping. Let $(\bar{q}_k,\bar{n}_k)_{k\in\mathbb{N}}\subset X$ be a sequence that converges strongly to a limit $(\bar{q},\bar{n})$ in X. We denote $(q_k,n_k)=\mathcal{K}(\bar{q}_k,\bar{n}_k)$ and $(q,n):=\mathcal{K}(\bar{q},\bar{n})$. Then, it is easy to see that from Lemma 3.1 (more precisely (3.15a)-(3.16)) there exists a corresponding sequence $(\varphi_{p,k},\mu_{p,k},\varphi_{d,k},\mu_{d,k})_{k\in\mathbb{N}}$ such that

$$\|\varphi_{i,k}\|_{H^1(0,T_0;H^2(\Omega))} + \|\mu_{i,k}\|_{L^2(0,T_0;H^1(\Omega))} \le C_{\varepsilon,\delta,R}$$

for $i=p,d$ and some constant $C=C_{\varepsilon,\delta,R}$ independent of k. Then, standard compactness results [43, § 8, Cor. 4] yield

$$\begin{aligned}
&\varphi_{i,k}\to\varphi_i \text{ strongly in } C^0([0,T_0];W^{1,r}(\Omega))\cap C^0(\overline{\Omega}\times[0,T_0]),\\
&\mu_{i,k}\to\mu_i \text{ weakly in } L^2(0,T_0;H^1(\Omega)),
\end{aligned}$$

along a non-relabelled subsequence for $i=p,d$, and any $r\in[1,\infty)$ in two dimensions and $r\in[1,6)$ in three dimensions. Hence, along a non-relabelled subsequence, $\varphi_{p,k}\to\varphi_p$ uniformly in $\overline{\Omega}\times[0,T_0]$ and thus $\mathcal{T}(\varphi_{p,k})\to\mathcal{T}(\varphi_p)$ uniformly in $\overline{\Omega}\times[0,T_0]$. Moreover, one can easily check that the limit functions φ_i, μ_i solve (3.6) with q,n in place of $\bar{q},\bar{n}$. Next, taking the difference of (3.20b) for two indices a and b leads to

$$-\Delta(n_a-n_b)+(\mathcal{T}(\varphi_{p,a})-\mathcal{T}(\varphi_{p,b}))n_a+\mathcal{T}(\varphi_{p,b})(n_a-n_b)+B(n_a-n_b)=0,$$

and by testing with n_a-n_b we obtain by the Poincaré inequality

$$\begin{aligned}&\|\nabla(n_a-n_b)\|^2_{L^2(0,T_0;L^2(\Omega))}\\ &\le\|n_a-n_b\|_{L^2(0,T_0;L^2(\Omega))}\|\mathcal{T}(\varphi_{p,a})-\mathcal{T}(\varphi_{p,b})\|_{L^2(0,T_0;L^2(\Omega))}\\ &\le C\|\nabla(n_a-n_b)\|_{L^2(0,T_0;L^2(\Omega))}\|\mathcal{T}(\varphi_{p,a})-\mathcal{T}(\varphi_{p,b})\|_{L^2(0,T_0;L^2(\Omega))}\end{aligned}\tag{3.26}$$

after neglecting the non-negative term $(\mathcal{T}(\varphi_{p,b})+B)|n_a-n_b|^2$. Applying the uniform convergence of $\mathcal{T}(\varphi_{p,k})$ we see that $\{n_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(0,T_0;H^1(\Omega))$ and thus $n_k\to n_*$ strongly in $L^2(0,T_0;L^2(\Omega))$ for some limit function n_*. Meanwhile, from the a priori estimates (3.22)-(3.23) and standard compactness results, along a non-relabelled subsequence it holds that

$$q_k\to q_* \text{ strongly in } L^2(0,T;H^1(\Omega)).$$

Let us mention here that thanks to the strong convergence of $n_k\to n_*$ in $L^2(0,T_0;L^2(\Omega))$, along a further subsequence we have a.e. convergence in $\Omega\times(0,T_0)$. Continuity of Σ_i, $i=p,d$, and boundedness are sufficient to ensure that the source terms $\Sigma_i(n_k,\varphi_{p,k},\varphi_{d,k})$, $i=p,d$, converge to $\Sigma_i(n,\varphi_p,\varphi_d)$ strongly in $L^2(0,T_0;L^2(\Omega))$.

Hence, along a non-relabelled subsequence $\mathcal{K}(\bar{q}_k,\bar{n}_k)\to(q_*,n_*)$. On the other hand, it is easy to check that (q_*,n_*) solve (3.20) (with the limit φ_i,μ_i). Then, thanks to the uniqueness of the solutions for the auxiliary Equations (3.20), one infers that, necessarily, $(q_*,n_*)=(q,n)=\mathcal{K}(\bar{q},\bar{n})$ and the whole sequence converges. This shows the required continuity of the map $\mathcal{K}$.

Compactness. To apply Schauder's fixed point theorem to $\mathcal{K}$, it remains to show that $\mathcal{K}:X\to X$ is a compact mapping. This amounts to prove for any sequence $(\bar{q}_k,\bar{n}_k)_{k\in\mathbb{N}}\subset X$, there exists a subsequence $(\bar{q}_{k_l},\bar{n}_{k_l})_{l\in\mathbb{N}}$ such that $(q_{k_l},n_{k_l}):=\mathcal{K}(\bar{q}_{k_l},\bar{n}_{k_l})$ converges strongly to some limit (q,n) in $L^2(0,T_0;H^1(\Omega))\times L^2(0,T_0;L^2(\Omega))$. Note that by the definition of X we have

$$\|q_k\|_{L^2(0,T_0;H^1(\Omega))}+\|n_k\|_{L^2(0,T_0;L^2(\Omega))}\le R$$

and $0\le n_k\le 1$ a.e. in $\Omega\times(0,T_0)$. This boundedness, and a similar argument to the proof of the continuity of $\mathcal{K}$, permit us to conclude the proof. Indeed, by repeating the a priori estimates given above, one can prove that the sequence (q_k,n_k) is uniformly bounded in a better space, whence follows the desired compactness assertion.

We now state the main result of this section.

THEOREM 3.1 (Local existence). *Let Assumption 2.1 hold. Moreover, for each $\varepsilon\in(0,1)$, $\delta\in(0,1)$ let us assume that $F_\varepsilon:\mathbb{R}^2\to[0,+\infty)$ and $F_1:\mathbb{R}^2\to\mathbb{R}$ are given such that ∇F_ε, ∇F_1 are globally Lipschitz continuous. Then, there exist a time $T_0\in(0,T]$ and functions $(\varphi_p,\mu_p,\varphi_d,\mu_d,q,n)$ such that*

(1) *the following regularity properties*

$$\begin{aligned}
&\varphi_i \in H^1(0,T_0;H^2(\Omega)) \quad \text{for } i=p,d,\\
&\mu_i \in L^2(0,T_0;H^1(\Omega)) \quad \text{for } i=p,d,\\
&q \in L^2(0,T_0;H^3(\Omega)) \cap L^\infty(0,T_0;H^1_0(\Omega)) \cap H^1(0,T_0;H^{-1}(\Omega)),\\
&n \in L^\infty(0,T_0;W^{2,r}(\Omega)) \text{ for any } r<\infty \text{ and } 0 \le n \le 1 \text{ a.e. in } \Omega\times(0,T_0),
\end{aligned}$$

hold together with

$$\varphi_i(0)=\varphi_{0,i,\delta},\, q(0)=0 \text{ in } \Omega, \quad \Delta q=0,\, n=1 \text{ on } \partial\Omega\times(0,T_0),$$

for $i=p,d$.

(2) *Equations* (1.6b), (1.6d), (1.6e), (1.6f) *and* (1.6h) *hold a.e. in* $\Omega\times(0,T_0)$, *and Equations* (1.6a), (1.6c) *and* (1.6g) *hold for a.e.* $t\in(0,T_0)$ *in the following weak sense:*

$$\begin{aligned}
0&=\int_\Omega (\partial_t\varphi_p - S_p)\zeta + \big(M_p\nabla\mu_p + \mathcal{T}(\varphi_p)\big(\nabla q + \mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d\big)\big)\cdot\nabla\zeta\,dx,\\
0&=\int_\Omega (\partial_t\varphi_d - S_d)\zeta + \big(M_d\nabla\mu_d + \mathcal{T}(\varphi_d)\big(\nabla q + \mathcal{T}(\varphi_d)\nabla\mu_d + \mathcal{T}(\varphi_p)\nabla\mu_p\big)\big)\cdot\nabla\zeta\,dx,\\
0&=\delta\langle\partial_t q,\xi\rangle_{H^1_0} + \int_\Omega \big(\nabla q - \delta\nabla\Delta q + \mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d\big)\cdot\nabla\xi - (S_p+S_d)\xi\,dx
\end{aligned}$$

for all $\zeta\in H^1(\Omega)$ *and* $\xi\in H^1_0(\Omega)$.

3.4. A priori estimates. We now derive some a priori estimates for the solution $(\varphi_p,\mu_p,\varphi_d,\mu_d,q,n)$ to (1.6) obtained from Theorem 3.1. All these estimates will be independent of T_0, which will allow us to extend the solution up to the full time interval $[0,T]$. For this reason, although with some abuse of notation, we shall directly work on the original time interval $[0,T]$ and postpone the details of the extension argument to the next subsection. Below the symbol C denotes constants that are independent of δ and ε.

First estimate. Testing the nutrient Equation (1.6h) with $n-1\in H^1_0(\Omega)$, we obtain from the boundedness of the cut-off operator $\mathcal{T}$ and of n the estimate

$$\begin{aligned}
&\|\nabla n\|^2_{L^2(\Omega)} + \int_\Omega \underbrace{(\mathcal{T}(\varphi_p)+B)|n-1|^2}_{\ge 0}\,dx\\
&= \int_\Omega \mathcal{T}(\varphi_p)(1-n) + B(1-n_C)(1-n)\,dx \le C.
\end{aligned}$$

Hence, integrating in time and applying the Poincaré inequality yields

$$\|n\|_{L^2(0,T;H^1(\Omega))}\le C.$$

The weak comparison principle then yields that $0\le n\le 1$ a.e. in $\Omega\times(0,T)$. Hence, by elliptic regularity, we arrive at

$$\|n\|_{L^\infty(0,T;W^{2,r}(\Omega))}\le C \quad \forall r<\infty. \tag{3.27}$$

Second estimate. Testing (1.6a) with μ_p, (1.6b) with $\partial_t\varphi_p$ and comparing leads to an analogous identity to (3.7) but with $\bar q$ replaced by q. Combining this with the identity obtained from testing (1.6c) with μ_d and (1.6d) with $\partial_t\varphi_d$ yields an analogous identity to (3.8) but with $\bar{q}$ replaced by q. Then, adding the resulting identity to that obtained from testing (1.6g) with q leads to the equality

$$\begin{aligned}&\frac{d}{dt}\Big(E_\varepsilon(\varphi_p,\varphi_d)+\frac{\delta}{2}\|q\|_{L^2(\Omega)}^2\Big)+\sum_{i=p,d}\Big(M_i\|\nabla\mu_i\|_{L^2(\Omega)}^2+\delta\|\nabla\partial_t\varphi_i\|_{L^2(\Omega)}^2\Big)\\&\quad+\delta\|\Delta q\|_{L^2(\Omega)}^2+\|\nabla q+\mathcal{T}(\varphi_p)\nabla\mu_p+\mathcal{T}(\varphi_d)\nabla\mu_d\|_{L^2(\Omega)}^2\\&=\int_\Omega(S_p\mu_p+S_d\mu_d)+(S_p+S_d)q\,dx.\end{aligned}\tag{3.28}$$

Testing now (1.6a) with φ_p, (1.6c) with φ_d and summing the obtained relations yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\varphi_p\|_{L^2(\Omega)}^2+\|\varphi_d\|_{L^2(\Omega)}^2\Big)\\&=-\sum_{i=p,d}\int_\Omega M_i\nabla\mu_i\cdot\nabla\varphi_i-S_i\varphi_i\,dx\\&\quad-\int_\Omega\big(\nabla q+\mathcal{T}(\varphi_p)\nabla\mu_p+\mathcal{T}(\varphi_d)\nabla\mu_d\big)\cdot\big(\mathcal{T}(\varphi_p)\nabla\varphi_p+\mathcal{T}(\varphi_d)\nabla\varphi_d\big)\,dx.\end{aligned}\tag{3.29}$$

Summing (3.28) and (3.29) then gives

$$\begin{aligned}&\frac{d}{dt}\left(\int_\Omega(F_\varepsilon+F_1)(\varphi_p,\varphi_d)\,dx+\sum_{i=p,d}\frac{1}{2}\|\varphi_i\|_{H^1(\Omega)}^2+\frac{\delta}{2}\|q\|_{L^2(\Omega)}^2\right)\\&\quad+\delta\|\Delta q\|_{L^2(\Omega)}^2+\sum_{i=p,d}\Big(\frac{1}{2}M_i\|\nabla\mu_i\|_{L^2(\Omega)}^2+\delta\|\nabla\partial_t\varphi_i\|_{L^2(\Omega)}^2\Big)\\&\quad+\frac{1}{2}\|\nabla q+\mathcal{T}(\varphi_p)\nabla\mu_p+\mathcal{T}(\varphi_d)\nabla\mu_d\|_{L^2(\Omega)}^2\\&\le C+C\sum_{i=p,d}\Big(\|\varphi_i\|_{L^2(\Omega)}^2+\|\nabla\varphi_i\|_{L^2(\Omega)}^2\Big)+\int_\Omega S_p\mu_p+S_d\mu_d+(S_p+S_d)q\,dx.\end{aligned}\tag{3.30}$$

It remains to control the integral on the right-hand side of (3.30). To handle the pressure term we consider, for a.e. $t\in(0,T)$, the function $N_q(t)\in H^2(\Omega)\cap H_0^1(\Omega)$ as the unique solution to the Poisson problem

$$-\Delta N_q(t)=q(t)\text{ in }\Omega,\quad N_q(t)=0\text{ on }\Gamma.$$

As $q(t)\in L^2(\Omega)$, elliptic regularity shows that $\|N_q\|_{H^2(\Omega)}\le C_*\|q\|_{L^2(\Omega)}$ for a positive constant C_* depending only on Ω. Furthermore, it can be shown that (see for example [26, §2.2])

$$\langle\partial_t q,N_q\rangle_{H_0^1}=\frac{1}{2}\frac{d}{dt}\|\nabla N_q\|_{L^2(\Omega)}^2.$$

We additionally claim that $N_q(0)=0$. Indeed, as $q(0)=0$ from (1.7a), the only solution to the Laplace equation with zero Dirichlet condition is zero. Then, upon testing (1.6g) with N_q leads to

$$\frac{\delta}{2}\frac{d}{dt}\|\nabla N_q\|_{L^2(\Omega)}^2+\|q\|_{L^2(\Omega)}^2+\delta\|\nabla q\|_{L^2(\Omega)}^2$$

$$
\begin{aligned}
&= \int_\Omega (S_p+S_d)N_q - (\mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d)\cdot\nabla N_q\,dx \\
&\le C\Big[1+\sum_{i=p,d}\|\varphi_i\|_{L^2(\Omega)}\Big]\|N_q\|_{L^2(\Omega)} + \sum_{i=p,d}\|\nabla\mu_i\|_{L^2(\Omega)}\|\nabla N_q\|_{L^2(\Omega)},
\end{aligned}
$$

where we have also used that

$$
-\int_\Omega \delta\Delta q\,\Delta N_q\,dx = \int_\Omega \delta q\Delta q\,dx = -\delta\|\nabla q\|^2_{L^2(\Omega)}.
$$

Therefore, by Young's inequality, Poincaré's inequality and the estimate $\|N_q\|_{H^2(\Omega)} \le C_*\|q\|_{L^2(\Omega)}$, we arrive at

$$
\begin{aligned}
&\frac{\delta}{2}\frac{d}{dt}\|\nabla N_q\|^2_{L^2(\Omega)} + \frac{1}{2}\|q\|^2_{L^2(\Omega)} + \delta\|\nabla q\|^2_{L^2(\Omega)} \\
&\le C\Big[1+\sum_{i=p,d}\Big(\|\nabla\mu_i\|^2_{L^2(\Omega)} + \|\varphi_i\|^2_{L^2(\Omega)}\Big)\Big].
\end{aligned} \tag{3.31}
$$

By virtue of the computations performed in (3.13) we infer that

$$
\begin{aligned}
&\int_\Omega S_p\mu_p + S_d\mu_d\,dx \\
&\le \frac{M_p}{4}\|\nabla\mu_p\|^2_{L^2} + \frac{M_d}{4}\|\nabla\mu_d\|^2_{L^2(\Omega)} + C_\varepsilon\Big(1+\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\Big).
\end{aligned} \tag{3.32}
$$

Then, letting κ be a sufficiently small constant such that $\kappa C \le \frac{1}{4}\min(M_p,M_d)$, where C is the constant on the right-hand side of (3.31), and adding κ times (3.31) to (3.30) yields

$$
\begin{aligned}
&\frac{d}{dt}\left(\int_\Omega (F_\varepsilon+F_1)(\varphi_p,\varphi_d)\,dx + \sum_{i=p,d}\frac{1}{2}\|\varphi_i\|^2_{H^1(\Omega)} + \frac{\delta}{2}\Big(\|q\|^2_{L^2(\Omega)} + \kappa\|\nabla N_q\|^2_{L^2(\Omega)}\Big)\right) \\
&\quad + \sum_{i=p,d}\Big(\tfrac{1}{4}M_i\|\nabla\mu_i\|^2_{L^2(\Omega)} + \delta\|\nabla\partial_t\varphi_i\|^2_{L^2(\Omega)}\Big) + \frac{\kappa}{4}\|q\|^2_{L^2(\Omega)} + \delta\kappa\|\nabla q\|^2_{L^2(\Omega)} \\
&\qquad + \delta\|\Delta q\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla q + \mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d\|^2_{L^2(\Omega)} \\
&\le C_\varepsilon + C_\varepsilon\sum_{i=p,d}\|\varphi_i\|^2_{H^1(\Omega)},
\end{aligned} \tag{3.33}
$$

where we have estimated the last term on the right-hand side of (3.30) as follows:

$$
\int_\Omega (S_p+S_d)q\,dx \le \frac{\kappa}{4}\|q\|^2_{L^2(\Omega)} + C\big(1+\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\big).
$$

Then, applying Gronwall's inequality to (3.33) yields the following estimates uniform in δ:

$$
\begin{aligned}
\|(F_\varepsilon+F_1)(\varphi_p,\varphi_d)\|_{L^\infty(0,T;L^1(\Omega))} + \|\varphi_p\|_{L^\infty(0,T;H^1(\Omega))} + \|\varphi_d\|_{L^\infty(0,T;H^1(\Omega))} &\le C_\varepsilon, \\
\|\nabla\mu_p\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\mu_d\|_{L^2(0,T;L^2(\Omega))} + \|q\|_{L^2(0,T;L^2(\Omega))} &\le C_\varepsilon, \\
\sqrt{\delta}\big(\|q\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla\partial_t\varphi_p\|_{L^2(0,T;L^2(\Omega))} + \|\nabla\partial_t\varphi_d\|_{L^2(0,T;L^2(\Omega))}\big) &\le C_\varepsilon,
\end{aligned} \tag{3.34}
$$

notice that the above is true thanks to the fact that $q(0)=\Delta q(0)=N_q(0)=0$ and that

$$\|\varphi_{i,0,\delta}\|_{H^1(\Omega)} \le C\|\varphi_{i,0}\|_{H^1(\Omega)}$$

from (3.4). Then, testing (1.6g) with q and estimating the right-hand side gives

$$\begin{aligned}&\frac{\delta}{2}\frac{d}{dt}\|q\|^2_{L^2(\Omega)} + \|\nabla q\|^2_{L^2(\Omega)} + \delta\|\nabla\Delta q\|^2_{L^2(\Omega)}\\ &\le C\Big[1+\|q\|^2_{L^2(\Omega)} + \sum_{i=p,d}\Big(\|\varphi_i\|^2_{L^2(\Omega)} + \|\nabla\mu_i\|^2_{L^2(\Omega)}\Big)\Big] + \frac{1}{2}\|\nabla q\|^2_{L^2(\Omega)}.\end{aligned}$$

In light of (3.34), and recalling the initial condition $q(0)=0$, we find that

$$\|\nabla q\|_{L^2(0,T;L^2(\Omega))} \le C_\varepsilon. \tag{3.35}$$

Third estimate. Thanks to the Lipschitz regularity of $F_{\varepsilon,i}$ and $F_{1,i}$ for $i=p,d$, it is easy to see that by (3.34)

$$|\overline{\mu_i}|^2 \le C_\varepsilon\left(1+\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\right) \in L^\infty(0,T).$$

Hence, by Poincaré's inequality and (3.34), we deduce

$$\|\mu_p\|_{L^2(0,T;H^1(\Omega))} + \|\mu_d\|_{L^2(0,T;H^1(\Omega))} \le C_\varepsilon. \tag{3.36}$$

Fourth estimate. Testing (1.6b) with $\Delta\varphi_p$, and in light of (3.36) and the Lipschitz regularity of $F_{\varepsilon,p}$ and $F_{1,p}$, we have

$$\frac{1}{2}\|\Delta\varphi_p\|^2_{L^2(\Omega)} + \frac{d}{dt}\frac{\delta}{2}\|\Delta\varphi_p\|^2_{L^2(\Omega)} \le C_\varepsilon\left(1+\|\mu_p\|^2_{L^2(\Omega)} + \|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\right). \tag{3.37}$$

Recalling (3.5) we see that

$$\delta\|\Delta\varphi_{p,0,\delta}\|^2_{L^2(\Omega)} \le C\delta(1+\delta^{-1})\|\varphi_{p,0}\|^2_{H^1(\Omega)} \le C. \tag{3.38}$$

Thus, integrating (3.37) in time and applying the elliptic estimate

$$\|v\|_{H^2(\Omega)} \le C\big(\|\Delta v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big)$$

(holding as v satisfies no-flux boundary conditions), we obtain

$$\begin{aligned}\|\varphi_p\|_{L^2(0,T;H^2(\Omega))} + \|\varphi_d\|_{L^2(0,T;H^2(\Omega))} &\le C_\varepsilon,\\ \sqrt{\delta}\big(\|\varphi_p\|_{L^\infty(0,T;H^2(\Omega))} + \|\varphi_d\|_{L^\infty(0,T;H^2(\Omega))}\big) &\le C_\varepsilon.\end{aligned} \tag{3.39}$$

Then, by inspection of (1.6a) we find that

$$\|\partial_t\varphi_p\|_{H^1(\Omega)'} \le C\left(\|\nabla q\|_{L^2(\Omega)} + \|\nabla\mu_p\|_{L^2(\Omega)} + \|\nabla\mu_d\|_{L^2(\Omega)} + \|S_p\|_{L^2(\Omega)}\right),$$

with a similar relation holding for φ_d. Hence, we infer that

$$\|\partial_t\varphi_p\|_{L^2(0,T;H^1(\Omega)')} + \|\partial_t\varphi_d\|_{L^2(0,T;H^1(\Omega)')} \le C_\varepsilon. \tag{3.40}$$

Fifth estimate. Testing (1.6g) with $q-\Delta q \in H^1_0(\Omega)$ and performing standard computations leads to the analogue of (3.21). Then, multiplying both sides of (3.21) by δ and using a Gronwall argument yields

$$\delta\|q\|_{L^\infty(0,T;H^1(\Omega))} + \delta\|\nabla\Delta q\|_{L^2(0,T;L^2(\Omega))} + \sqrt{\delta}\|\Delta q\|_{L^2(0,T;L^2(\Omega))} \le C_\varepsilon. \tag{3.41}$$

Then, by inspection of (1.6g), and recalling (3.34), (3.35) and (3.41), we infer

$$\begin{aligned}\delta\|\partial_t q\|_{L^2(0,T;H^{-1}(\Omega))} \leq & C\sum_{i=p,d}\left(1+\|\nabla\mu_i\|_{L^2(0,T;L^2(\Omega))}+\|\varphi_i\|_{L^2(0,T;L^2(\Omega))}\right)\\ &+C\|\nabla q\|_{L^2(0,T;L^2(\Omega))}+C\delta\|\nabla\Delta q\|_{L^2(0,T;L^2(\Omega))}+C\\ \leq & C_\varepsilon. \end{aligned} \tag{3.42}$$

Note that by the Lipschitz continuity of ∇F_ε and ∇F_1, and the boundedness of φ_p,φ_d in $L^2(0,T;L^2(\Omega))$ from (3.34), we can easily infer that

$$\|F_{\varepsilon,i}(\varphi_p,\varphi_d)\|_{L^2(0,T;L^2(\Omega))}+\|F_{1,i}(\varphi_p,\varphi_d)\|_{L^2(0,T;L^2(\Omega))}\leq C_\varepsilon \text{ for } i=p,d.$$

Then, by testing (1.6b) with $-\delta\Delta\partial_t\varphi_p$ and (1.6d) with $-\delta\Delta\partial_t\varphi_d$ we obtain using the boundedness of $\mu_i - F_{\varepsilon,i}(\varphi_p,\varphi_d)-F_{1,i}(\varphi_p,\varphi_d)-\Delta\varphi_i$ in $L^2(0,T;L^2(\Omega))$

$$\delta\left(\|\Delta\partial_t\varphi_p\|_{L^2(0,T;L^2(\Omega))}+\|\Delta\partial_t\varphi_d\|_{L^2(0,T;L^2(\Omega))}\right)\leq C_\varepsilon. \tag{3.43}$$

On the other hand, testing (1.6a) with $\sqrt{\delta}\partial_t\varphi_p$, we obtain

$$\begin{aligned}\sqrt{\delta}\|\partial_t\varphi_p\|^2_{L^2(\Omega)} \leq & C\Big(1+\sum_{i=,p,d}\left(\|\nabla\mu_i\|^2_{L^2(\Omega)}+\|\varphi_i\|^2_{L^2(\Omega)}\right)+\|\nabla q\|^2_{L^2(\Omega)}+\delta\|\nabla\partial_t\varphi_p\|^2_{L^2(\Omega)}\Big)\\ &+\frac{1}{2}\sqrt{\delta}\|\partial_t\varphi_p\|^2_{L^2(\Omega)}.\end{aligned}$$

Recalling (3.34) and (3.35), we then deduce that

$$\sqrt{\delta}\|\partial_t\varphi_p\|^2_{L^2(0,T;L^2(\Omega))}+\sqrt{\delta}\|\partial_t\varphi_d\|^2_{L^2(0,T;L^2(\Omega))}\leq C_\varepsilon,$$

whence, by repeating the same argument on φ_d and by applying elliptic regularity, (3.43) yields

$$\delta\|\varphi_p\|_{H^1(0,T;H^2(\Omega))}+\delta\|\varphi_d\|_{H^1(0,T;H^2(\Omega))}\leq C_\varepsilon. \tag{3.44}$$

3.5. Extension to $[0,T]$. Thanks to the a priori estimates (3.27), (3.34), (3.36), (3.39), (3.40), (3.41), (3.42), (3.43) and (3.44), which have a uniform character with respect to the time variable, we can extend the local solution obtained from Theorem 3.1 up to the full reference interval $[0,T]$. This can be achieved by means of a standard contradiction argument which we now outline. Suppose there exists a maximal time of existence $T_m\in(0,T]$ for the weak solution $(\varphi_p,\mu_p,\varphi_d,\mu_d,q,n)$ to (1.6). To be precise, T_m is defined as the largest time such that $(\varphi_p,\mu_p,\varphi_d,\mu_d,q,n)$ exists with the regularity properties specified in the statement of Theorem 3.1. We want to prove that, in fact, $T_m=T$. If this is not the case, repeating the a priori estimates mentioned above (but now working on the maximal time interval $[0,T_m]$), we deduce in particular that

$$\|\varphi_p\|_{C^0([0,T_m];H^2(\Omega))}+\|\varphi_d\|_{C^0([0,T_m];H^2(\Omega))}+\|q\|_{C^0([0,T_m];H^1_0(\Omega))}\leq C_{\varepsilon,\delta},$$

where $C_{\varepsilon,\delta}$ is independent of T_m. Note that, to obtain the above bound, we used in particular (3.43) with the continuous embedding $H^1(0,T_m)\subset C^0([0,T_m])$ and (3.41)-(3.42) with the continuous embedding

$$L^2(0,T_m;H^3(\Omega)\cap H^1_0(\Omega))\cap H^1(0,T_m;H^{-1}(\Omega))\subset C^0([0,T_m];H^1_0(\Omega)).$$

In particular, the triple $(\varphi_p(t),\varphi_d(t),q(t))$ remains bounded in $H^2(\Omega)\times H^2(\Omega)\times H^1_0(\Omega)$, and actually (strongly) converges in the same space to a limit $(\varphi_p(T_m),\varphi_d(T_m),q(T_m))$, as $t\nearrow T_m$. This allows us to restart the system by taking $\varphi_p(T_m)$, $\varphi_d(T_m)$ and $q(T_m)$ as new "initial" data (note that the other equations of the system have a quasi-static nature; hence they do not involve any initial data). To be precise, we should observe that we performed the fixed point argument by assuming the initial condition $q(0)=0$, while we are restarting the argument from $q(T_m)\neq 0$. On the other hand, it is easy to realize that the choice $q(0)=0$ was taken just for convenience (indeed, that initial datum will disappear when taking the limit $\delta\to 0$) and the argument still works for any datum in $H^1_0(\Omega)$ (as is $q(T_m)$). Hence, restarting from T_m we get a new local solution which is defined on an interval of the form $(T_m,T_m+\epsilon)$ for some $\epsilon>0$ and still enjoys the regularity properties detailed in Theorem 3.1. This contradicts the maximality of T_m. Hence $T_m=T$.

3.6. Passing to the limit $\delta\to 0$. We now pass to the limit $\delta\to 0$ to obtain a weak solution $(\varphi_p^\varepsilon,\mu_p^\varepsilon,\varphi_d^\varepsilon,\mu_d^\varepsilon,q^\varepsilon,n^\varepsilon)$ defined over $(0,T)$ to the following problem:

$$\partial_t\varphi_p = M_p\Delta\mu_p + \operatorname{div}(\mathcal{T}(\varphi_p)\nabla q) + \operatorname{div}\big(\mathcal{T}(\varphi_p)^2\nabla\mu_p + \mathcal{T}(\varphi_p)\mathcal{T}(\varphi_d)\nabla\mu_d\big) + S_p, \tag{3.45a}$$
$$\mu_p = F_{\varepsilon,p}(\varphi_p,\varphi_d) + F_{1,p}(\varphi_p,\varphi_d) - \Delta\varphi_p, \tag{3.45b}$$
$$\partial_t\varphi_d = M_d\Delta\mu_d + \operatorname{div}(\mathcal{T}(\varphi_d)\nabla q) + \operatorname{div}\big(\mathcal{T}(\varphi_p)\mathcal{T}(\varphi_d)\nabla\mu_p + \mathcal{T}(\varphi_d)^2\nabla\mu_d\big) + S_d, \tag{3.45c}$$
$$\mu_d = F_{\varepsilon,d}(\varphi_p,\varphi_d) + F_{1,d}(\varphi_p,\varphi_d) - \Delta\varphi_d, \tag{3.45d}$$
$$S_p = \Sigma_p(n,\varphi_p,\varphi_d) + m_{pp}\varphi_p + m_{pd}\varphi_d, \tag{3.45e}$$
$$S_d = \Sigma_d(n,\varphi_p,\varphi_d) + m_{dp}\varphi_p + m_{dd}\varphi_d, \tag{3.45f}$$
$$0 = \Delta q + \operatorname{div}\big(\mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d\big) + S_p + S_d, \tag{3.45g}$$
$$0 = -\Delta n + \mathcal{T}(\varphi_p)n - B(n_C - n), \tag{3.45h}$$

furnished with the initial and boundary conditions

$$\varphi_p(0)=\varphi_{p,0},\quad \varphi_d(0)=\varphi_{d,0} \text{ in } \Omega, \tag{3.46a}$$
$$M_i\partial_{\boldsymbol{n}}\mu_i + \mathcal{T}(\varphi_i)(\nabla q + \mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d)\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \tag{3.46b}$$
$$n=1,\quad q=0,\quad \partial_{\boldsymbol{n}}\varphi_i = 0 \text{ on } \Gamma. \tag{3.46c}$$

Note that in (3.45) the regularized convex part F_ε of the potential F is still present.

THEOREM 3.2. *Let Assumption 2.1 hold. For $\varepsilon\in(0,1)$ and $\delta\in(0,1)$, let $F_\varepsilon:\mathbb{R}^2\to[0,+\infty)$ be the Moreau-Yosida approximation of F_0 as detailed in Sec. 3. Let also $\varphi_{i,0,\delta}\in H^2_n(\Omega)$ be the unique solution to (3.3). Then, there exists $\delta_0>0$ such that for all $\delta<\delta_0$, the weak solution $(\varphi_p^{\delta,\varepsilon},\mu_p^{\delta,\varepsilon},\varphi_d^{\delta,\varepsilon},\mu_d^{\delta,\varepsilon},q^{\delta,\varepsilon},n^{\delta,\varepsilon})$ to (1.6) defined on $[0,T]$ and obtained from Theorem 3.1 satisfies the following properties:*

(1) *there exist functions $(\varphi_p^\varepsilon,\mu_p^\varepsilon,\varphi_d^\varepsilon,\mu_d^\varepsilon,q^\varepsilon,n^\varepsilon)$ such that for $i=p,d$ and any $s<\infty$ in two dimensions and any $s\in[1,6)$ in three dimensions, and any $r<\infty$,*

$$\begin{aligned}
&\varphi_i^{\delta,\varepsilon}\to\varphi_i^\varepsilon \ \text{weakly* in } L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega)'),\\
&\varphi_i^{\delta,\varepsilon}\to\varphi_i^\varepsilon \ \text{strongly in } C^0([0,T];L^s(\Omega))\cap L^2(0,T;W^{1,s}(\Omega)) \text{ and a.e. in } Q,\\
&\mu_i^{\delta,\varepsilon}\to\mu_i^\varepsilon \ \text{weakly in } L^2(0,T;H^1(\Omega)),\\
&q^{\delta,\varepsilon}\to q^\varepsilon \ \text{weakly in } L^2(0,T;H^1(\Omega)),\\
&n^{\delta,\varepsilon}\to n^\varepsilon \ \text{weakly* in } L^\infty(0,T;W^{2,r}(\Omega)) \text{ and strongly in } L^2(0,T;H^1(\Omega)).
\end{aligned}$$

(2) *The tuple* $(\varphi_p^\varepsilon,\mu_p^\varepsilon,\varphi_d^\varepsilon,\mu_d^\varepsilon,q^\varepsilon,n^\varepsilon)$ *satisfies Equations* (3.45b), (3.45d), (3.45e), (3.45f), (3.45h) *a.e. in* Q*, whereas Equations* (3.45a), (3.45c) *and* (3.45g) *hold for a.e.* $t\in(0,T)$ *in the following weak sense:*

$$0=\langle\partial_t\varphi_p^\varepsilon,\zeta\rangle+\int_\Omega\big(M_p\nabla\mu_p^\varepsilon+\mathcal{T}(\varphi_p^\varepsilon)\big(\nabla q^\varepsilon+\mathcal{T}(\varphi_p^\varepsilon)\nabla\mu_p^\varepsilon+\mathcal{T}(\varphi_d^\varepsilon)\nabla\mu_d^\varepsilon\big)\big)\cdot\nabla\zeta-S_p\zeta\,dx,$$
$$0=\langle\partial_t\varphi_d^\varepsilon,\zeta\rangle+\int_\Omega\big(M_d\nabla\mu_d^\varepsilon+\mathcal{T}(\varphi_d^\varepsilon)\big(\nabla q^\varepsilon+\mathcal{T}(\varphi_d^\varepsilon)\nabla\mu_d^\varepsilon+\mathcal{T}(\varphi_p^\varepsilon)\nabla\mu_p^\varepsilon\big)\big)\cdot\nabla\zeta-S_d\zeta\,dx,$$
$$0=\int_\Omega\big(\nabla q^\varepsilon+\mathcal{T}(\varphi_p^\varepsilon)\nabla\mu_p^\varepsilon+\mathcal{T}(\varphi_d^\varepsilon)\nabla\mu_d^\varepsilon\big)\cdot\nabla\xi-(S_p+S_d)\xi\,dx \qquad \blacksquare$$

for all $\zeta\in H^1(\Omega)$ *and* $\xi\in H_0^1(\Omega)$*. Moreover,* $0\le n^\varepsilon\le 1$ *a.e. in* Q*, and* $\varphi_i^\varepsilon(0)=\varphi_{0,i}$ *a.e. in* Ω*.*

Proof. Recalling the estimate (3.4), we immediately infer the following properties of the initial data $(\varphi_{p,0,\delta},\varphi_{d,0,\delta})$:

$$\|\varphi_{p,0,\delta}\|_{H^1(\Omega)}+\|\varphi_{d,0,\delta}\|_{H^1(\Omega)}\le C,$$
$$\varphi_{p,0,\delta}\to\varphi_{p,0},\quad \varphi_{d,0,\delta}\to\varphi_{d,0}\text{ weakly in }H^1(\Omega)\text{ and strongly in }L^2(\Omega).$$

Furthermore, this choice of initial data for the regularized system (1.6) implies that the estimate (3.34) is uniform in $\delta\in(0,\delta_0)$.

Then, most of the weak/weak* convergence properties in the statement are directly deduced from the uniform estimates (3.27), (3.34), (3.36) and (3.39), while the strong convergences follow from using [43, § 8, Cor. 4]. On the other hand, the strong convergence of $n^{\delta,\varepsilon}$ is proved, similarly as before, by a Cauchy argument which we now sketch. Let (a small) $\eta>0$ and (a large) $C_*>0$ be given but otherwise arbitrary. Then, thanks to the a.e. convergence of $\varphi_p^{\delta,\varepsilon}$ to φ_p^ε in Q, by Egorov's theorem there exists a measurable subset $X_\eta\subset Q$ with $C_*|X_\eta|<\frac14\eta$ and $\varphi_p^{\delta,\varepsilon}\to\varphi_p^\varepsilon$ uniformly in the complement $Q\setminus X_\eta$. By this uniform convergence, there exists $\delta_*>0$ such that for any two indices $0<\delta_1,\delta_2<\delta_*$,

$$C_*\int_{Q\setminus X_\eta}|\mathcal{T}(\varphi_p^{\delta_1,\varepsilon})-\mathcal{T}(\varphi_p^{\delta_2,\varepsilon})|^2\,dx\,dt<\frac{\eta}{2}.$$

Then, following the computation in (3.26) and using the boundedness of T, we find that

$$\begin{aligned}&\|n^{\delta_1,\varepsilon}-n^{\delta_2,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))}\le C_*\|\mathcal{T}(\varphi_p^{\delta_1,\varepsilon})-\mathcal{T}(\varphi_p^{\delta_2,\varepsilon})\|^2_{L^2(0,T;L^2(\Omega))}\\ &\le C_*\int_{Q\setminus X_\eta}|\mathcal{T}(\varphi_p^{\delta_1,\varepsilon})-\mathcal{T}(\varphi_p^{\delta_2,\varepsilon})|^2\,dx\,dt+C_*\int_{X_\eta}|\mathcal{T}(\varphi_p^{\delta_1,\varepsilon})-\mathcal{T}(\varphi_p^{\delta_2,\varepsilon})|^2\,dx\,dt\\ &<\frac{\eta}{2}+2C_*|X_\eta|<\eta,\end{aligned}$$

for $0<\delta_1,\delta_2<\delta_*$. Here C_* is exactly the constant C in (3.26). This shows that $\{n^{\delta,\varepsilon}\}_{\delta\in(0,\delta_*)}$ is a Cauchy sequence in $L^2(0,T;H^1(\Omega))$. The property $0\le n^\varepsilon\le 1$ a.e. in Q can be deduced also from a weak comparison principle.

Now passing to the limit $\delta\to0$ in (1.6e), (1.6f), (1.6h) lead to (3.45e), (3.45f) and (3.45h), respectively. Let us fix $\zeta\in L^2(0,T;H^1(\Omega))$ and test (1.6b) with ζ. Then,

$$\int_0^T\int_\Omega\big(\mu_p^{\delta,\varepsilon}+\Delta\varphi_p^{\delta,\varepsilon}-(F_{\varepsilon,p}+F_{1,p})(\varphi_p^{\delta,\varepsilon},\varphi_d^{\delta,\varepsilon})\big)\zeta-\delta\nabla\partial_t\varphi_p^{\delta,\varepsilon}\cdot\nabla\zeta\,dx\,dt=0.$$

Using the weak convergences of $\mu_p^{\delta,\varepsilon}$, $\Delta\varphi_p^{\delta,\varepsilon}$ in $L^2(0,T;L^2(\Omega))$ and the Lipschitz continuity of $F_{\varepsilon,p}$ and $F_{1,p}$, as well as the boundedness $\|\sqrt{\delta}\nabla\partial_t\varphi_p^{\delta,\varepsilon}\|_{L^2(0,T;L^2(\Omega))}\le C_\varepsilon$ resulting from (3.34), passing to the limit $\delta\to 0$ in the above equality leads to

$$\int_0^T\int_\Omega \big(\mu_p^\varepsilon+\Delta\varphi_p^\varepsilon-(F_{\varepsilon,p}+F_{1,p})(\varphi_p^\varepsilon,\varphi_d^\varepsilon)\big)\zeta\,dx\,dt=0.$$

Since the above identity holds for arbitrary $\zeta\in L^2(0,T;H^1(\Omega))$ and all the terms in the integrand belong to $L^2(0,T;L^2(\Omega))$, the fundamental lemma of calculus of variations then yields (3.45b).

In a similar fashion, we infer from testing (1.6a) with an arbitrary test function $\zeta\in L^2(0,T;H^1(\Omega))$ and then passing to the limit $\delta\to 0$ the identity

$$\begin{aligned}0=&\int_0^T\langle\partial_t\varphi_p^\varepsilon,\zeta\rangle\,dt-\int_0^T\int_\Omega S_p(n^\varepsilon,\varphi_p^\varepsilon,\varphi_d^\varepsilon)\zeta\,dx\,dt\\&+\int_0^T\int_\Omega\big(M_p\nabla\mu_p^\varepsilon+\mathcal{T}(\varphi_p^\varepsilon)\big(\nabla q^\varepsilon+\mathcal{T}(\varphi_p^\varepsilon)\nabla\mu_p^\varepsilon+\mathcal{T}(\varphi_d^\varepsilon)\nabla\mu_d^\varepsilon\big)\big)\cdot\nabla\zeta\,dx\,dt.\end{aligned}$$

For this, we used the strong L^2-convergences of $n^{\delta,\varepsilon}$ and $\varphi_i^{\delta,\varepsilon}$ with the generalized Lebesgue dominated convergence theorem and the Assumption (2.1) to deduce that $S_p(n^{\delta,\varepsilon},\varphi_p^{\delta,\varepsilon},\varphi_d^{\delta,\varepsilon})$ converges to $S_p(n^\varepsilon,\varphi_p^\varepsilon,\varphi_d^\varepsilon)$ strongly in $L^2(0,T;L^2(\Omega))$. Furthermore, by the continuity and boundedness of $\mathcal{T}(\cdot)$, it is easy to see that

$$\mathcal{T}(\varphi_p^{\delta,\varepsilon})\to\mathcal{T}(\varphi_p^\varepsilon)\text{ weakly}*\text{ in }L^\infty(Q)\text{ and strongly in }L^p(Q)\text{ for all }p\in[1,\infty).$$

Moreover, the strong convergence of the initial data $\varphi_{p,0,\delta}$ to $\varphi_{p,0}$ in $L^2(\Omega)$ and the strong convergence of $\varphi_p^{\delta,\varepsilon}$ to φ_p^ε in $C^0([0,T];L^2(\Omega))$ yield $\varphi_p^\varepsilon(0)=\varphi_{p,0}$ as an equality in $L^2(\Omega)$.

Lastly, it remains to pass to the limit in (1.6g). Consider testing (1.6g) with the product $\eta(t)\xi(x)$ for arbitrary test functions $\eta\in C^1(0,T)$ with $\eta(T)=0$ and $\xi\in H^2(\Omega)\cap H_0^1(\Omega)$, then we have

$$\begin{aligned}0=&\int_0^T\int_\Omega-\delta q^{\delta,\varepsilon}\xi\partial_t\eta-(S_p+S_d)(n^{\delta,\varepsilon},\varphi_p^{\delta,\varepsilon},\varphi_d^{\delta,\varepsilon})\eta(t)\xi\,dx\,dt\\&+\int_0^T\eta(t)\int_\Omega\big(\nabla q^{\delta,\varepsilon}+\mathcal{T}(\varphi_p^{\delta,\varepsilon})\nabla\mu_p^{\delta,\varepsilon}+\mathcal{T}(\varphi_d^{\delta,\varepsilon})\nabla\mu_d^{\delta,\varepsilon}\big)\cdot\nabla\xi+\delta\Delta q^{\delta,\varepsilon}\cdot\Delta\xi\,dx\,dt.\end{aligned}$$

Thanks to $\|q^{\delta,\varepsilon}\|_{L^2(0,T;L^2(\Omega))}\le C_\varepsilon$ from (3.34) and $\sqrt{\delta}\|\Delta q^{\delta,\varepsilon}\|_{L^2(0,T;L^2(\Omega))}\le C_\varepsilon$ from (3.41), after passing to the limit we obtain

$$0=\int_0^T\eta(t)\int_\Omega\big(\nabla q^\varepsilon+\mathcal{T}(\varphi_p^\varepsilon)\nabla\mu_p^\varepsilon+\mathcal{T}(\varphi_d^\varepsilon)\nabla\mu_d^\varepsilon\big)\cdot\nabla\xi-(S_p+S_d)(n^\varepsilon,\varphi_p^\varepsilon,\varphi_d^\varepsilon)\xi\,dx\,dt,$$

holding for all $\xi\in H^2(\Omega)\cap H_0^1(\Omega)$ and $\eta\in C^1(0,T)$ with $\eta(T)=0$. Using the density of $H^2(\Omega)\cap H_0^1(\Omega)$ in $H_0^1(\Omega)$ and the fundamental lemma of calculus of variations, we obtain the weak formulation of (3.45g) as stated in Theorem 3.2. ∎

4. Passing to the limit $\varepsilon\to 0$

Let $(\varphi_p^\varepsilon,\mu_p^\varepsilon,\varphi_d^\varepsilon,\mu_d^\varepsilon,q^\varepsilon,n^\varepsilon)$ denote a weak solution to (3.45) obtained from Theorem 3.2. Introducing the velocity variable as $\boldsymbol{u}^\varepsilon:=-\nabla q^\varepsilon-\mathcal{T}(\varphi_p^\varepsilon)\nabla\mu_p^\varepsilon-\mathcal{T}(\varphi_d^\varepsilon)\nabla\mu_d^\varepsilon$, we can now rewrite (3.45) as

$$\partial_t\varphi_p^\varepsilon=M_p\Delta\mu_p^\varepsilon-\operatorname{div}(\mathcal{T}(\varphi_p^\varepsilon)\boldsymbol{u}^\varepsilon)+S_p,\tag{4.1a}$$

$$\mu_p^\varepsilon = F_{\varepsilon,p}(\varphi_p^\varepsilon,\varphi_d^\varepsilon) + F_{1,p}(\varphi_p^\varepsilon,\varphi_d^\varepsilon) - \Delta\varphi_p^\varepsilon, \tag{4.1b}$$
$$\partial_t\varphi_d^\varepsilon = M_d\Delta\mu_d^\varepsilon - \operatorname{div}(\mathcal{T}(\varphi_d^\varepsilon)\boldsymbol{u}^\varepsilon) + S_d, \tag{4.1c}$$
$$\mu_d^\varepsilon = F_{\varepsilon,d}(\varphi_p^\varepsilon,\varphi_d^\varepsilon) + F_{1,d}(\varphi_p^\varepsilon,\varphi_d^\varepsilon) - \Delta\varphi_d^\varepsilon, \tag{4.1d}$$
$$S_p = \Sigma_p(n^\varepsilon,\varphi_p^\varepsilon,\varphi_d^\varepsilon) + m_{pp}\varphi_p^\varepsilon + m_{pd}\varphi_d^\varepsilon, \tag{4.1e}$$
$$S_d = \Sigma_d(n^\varepsilon,\varphi_p^\varepsilon,\varphi_d^\varepsilon) + m_{dp}\varphi_p^\varepsilon + m_{dd}\varphi_d^\varepsilon, \tag{4.1f}$$
$$\boldsymbol{u}^\varepsilon = -\nabla q^\varepsilon - \mathcal{T}(\varphi_p^\varepsilon)\nabla\mu_p^\varepsilon - \mathcal{T}(\varphi_d^\varepsilon)\nabla\mu_d^\varepsilon, \tag{4.1g}$$
$$\operatorname{div}\boldsymbol{u}^\varepsilon = S_p + S_d, \tag{4.1h}$$
$$0 = -\Delta n^\varepsilon + \mathcal{T}(\varphi_p^\varepsilon)n^\varepsilon - B(n_C - n^\varepsilon), \tag{4.1i}$$

furnished with the initial-boundary conditions (3.46a)-(3.46c) (in fact, the system is satisfied in the weak form specified in the statement; nevertheless, it is probably clearer to report the equations in their strong formulation).

The aim of this section is to derive uniform a priori estimates in ε and then pass to the limit $\varepsilon \to 0$. Let us point out that the estimate (3.27) involving n^ε is already uniform in ε. For convenience, we will drop the superscript ε in the variables, and denote with the symbol C positive constants that are independent of ε.

4.1. A priori estimates. We will now derive a number of estimates that are uniform with respect to ε. We start controlling the mean values of φ_p and φ_d. Denoting

$$\boldsymbol{y}(t) := (\overline{\varphi_p}(t), \overline{\varphi_d}(t)), \quad \overline{\boldsymbol{\Sigma}}(t) = (\overline{\Sigma_p}(t), \overline{\Sigma_d}(t)),$$

then by testing (4.1a) and (4.1c) with 1 leads to the following system of ordinary differential equations:

$$\frac{d}{dt}\boldsymbol{y}(t) = \overline{\boldsymbol{\Sigma}}(t) + \underline{\underline{M}}\boldsymbol{y}(t) \tag{4.2}$$

for any $0 \le t \le T$. Thanks to (2.1), (2.2) and (2.7) we infer that the vector $\boldsymbol{y}(t) = (\overline{\varphi_p}(t), \overline{\varphi_d}(t))$ belongs to the interior $\operatorname{int}\Delta_0$ for all times $t \in [0,T]$. Indeed, at the time $t = 0$, $\boldsymbol{y}(0) \in \operatorname{int}\Delta_0$ by (2.7). Suppose that there exists a time t_* such that $\boldsymbol{y}(t_*) \in \partial\Delta_0$. Then, taking $t = t_*$ in the above ODE, multiplying with the outer unit normal $\boldsymbol{n}$ to Δ_0 and applying (2.2), we necessarily have that

$$\frac{d}{dt}\boldsymbol{y}(t_*) \cdot \boldsymbol{n} < 0.$$

As a consequence, $\boldsymbol{y}(t) \in \operatorname{int}\Delta_0$ for t in a right neighbourhood of t_*, whence it is apparent that $\boldsymbol{y}(t)$ can never leave Δ_0.

From this we deduce that there exist positive constants $0 < c_1 < c_2 < 1$ independent of ε such that

$$c_1 \le \overline{\varphi_p}(t), \overline{\varphi_d}(t) \le c_2, \quad c_1 \le \overline{(\varphi_p + \varphi_d)}(t) \le c_2 \quad \forall t \in [0,T]. \tag{4.3}$$

Testing now (4.1a) with μ_p, (4.1c) with μ_d, (4.1b) with $\partial_t\varphi_p$, (4.1d) with $\partial_t\varphi_d$, (4.1g) with $\boldsymbol{u}$ and summing leads to

$$\frac{d}{dt}\int_\Omega F_\varepsilon(\varphi_p,\varphi_d) + F_1(\varphi_p,\varphi_d) + \frac{1}{2}\left(|\nabla\varphi_p|^2 + |\nabla\varphi_d|^2\right)dx$$
$$+ M_p\|\nabla\mu_p\|_{L^2(\Omega)}^2 + M_d\|\nabla\mu_d\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}\|_{L^2(\Omega)}^2$$

$$= \int_\Omega S_p\mu_p + S_d\mu_d + q(S_p+S_d)\,dx. \tag{4.4}$$

In the above we used Darcy's law and integration by parts to deduce that

$$\int_\Omega (\mathcal{T}(\varphi_p)\nabla\mu_p + \mathcal{T}(\varphi_d)\nabla\mu_d)\cdot \boldsymbol{u}\,dx = \int_\Omega -\nabla q\cdot\boldsymbol{u} - |\boldsymbol{u}|^2\,dx = \int_\Omega q(S_p+S_d) - |\boldsymbol{u}|^2\,dx.$$

Let us now observe that, by the boundedness of Σ_p, we have

$$\begin{aligned}\int_\Omega S_p\mu_p\,dx &\le C\|\mu_p - \overline{\mu_p}\|_{L^1(\Omega)} + C|\overline{\mu_p}| + \sum_{i=p,d}\int_\Omega m_{pi}\varphi_i(\mu_p - \overline{\mu_p} + \overline{\mu_p})\,dx\\ &= C\|\mu_p - \overline{\mu_p}\|_{L^1(\Omega)} + C|\overline{\mu_p}| + \sum_{i=p,d}\int_\Omega m_{pi}(\varphi_i - \overline{\varphi_i})(\mu_p - \overline{\mu_p})\,dx,\\ &\quad + \overline{\mu_p}\int_\Omega m_{pp}\varphi_p + m_{pd}\varphi_d\,dx\\ &\le C\|\mu_p - \overline{\mu_p}\|_{L^1(\Omega)} + C|\overline{\mu_p}| + C\sum_{i=p,d}\|\nabla\varphi_i\|_{L^2(\Omega)}\|\nabla\mu_p\|_{L^2(\Omega)},\end{aligned}$$

where we have used that $(\overline{\varphi_p},\overline{\varphi_d})$ never leaves the set Δ_0 and so $m_{pp}\overline{\varphi_p} + m_{pd}\overline{\varphi_d}$ is bounded. An analogous estimate holds for $S_d\mu_d$, whence, by the Poincaré and Young inequalities, we obtain

$$\begin{aligned}\left|\int_\Omega S_p\mu_p + S_d\mu_d\,dx\right| &\le C\,(|\overline{\mu_p}| + |\overline{\mu_d}|) + \frac{M_p}{4}\|\nabla\mu_p\|^2_{L^2(\Omega)}\\ &\quad + \frac{M_d}{4}\|\nabla\mu_d\|^2_{L^2(\Omega)} + C\left(1 + \|\nabla\varphi_p\|^2_{L^2(\Omega)} + \|\nabla\varphi_d\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{4.5}$$

For the term involving the pressure q, we have

$$\left|\int_\Omega (S_p+S_d)q\,dx\right| \le C_\eta\left(1 + \|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\right) + \eta\|q\|^2_{L^2(\Omega)}$$

for some positive constant η to be fixed below. To get an L^2-estimate of the pressure, we use the Poincaré inequality for $H^1_0(\Omega)$-functions and Darcy's law to deduce that

$$\|q\|^2_{L^2(\Omega)} \le C\|\nabla q\|^2_{L^2(\Omega)} \le C\left(\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla\mu_p\|^2_{L^2(\Omega)} + \|\nabla\mu_d\|^2_{L^2(\Omega)}\right). \tag{4.6}$$

Take now η sufficiently small so that

$$\begin{aligned}\left|\int_\Omega (S_p+S_d)q\,dx\right| &\le \frac{1}{2}\|\boldsymbol{u}\|^2_{L^2(\Omega)} + \frac{M_p}{4}\|\nabla\mu_p\|^2_{L^2(\Omega)} + \frac{M_d}{4}\|\nabla\mu_d\|^2_{L^2(\Omega)}\\ &\quad + C\left(1 + \|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{4.7}$$

Then, substituting (4.5) and (4.7) into (4.4) yields

$$\begin{aligned}&\frac{d}{dt}\int_\Omega F_\varepsilon(\varphi_p,\varphi_d) + F_1(\varphi_p,\varphi_d) + \frac{1}{2}\left(|\nabla\varphi_p|^2 + |\nabla\varphi_d|^2\right)dx\\ &\quad + \frac{M_p}{2}\|\nabla\mu_p\|^2_{L^2(\Omega)} + \frac{M_d}{2}\|\nabla\mu_d\|^2_{L^2(\Omega)} + \frac{1}{2}\|\boldsymbol{u}\|^2_{L^2(\Omega)}\end{aligned}$$

$$\leq C\left(1+\|\varphi_p\|_{H^1(\Omega)}^2+\|\varphi_d\|_{H^1(\Omega)}^2+|\overline{\mu_p}|+|\overline{\mu_d}|\right). \tag{4.8}$$

The key point is now to derive uniform estimates on the mean values $|\overline{\mu_p}|$ and $|\overline{\mu_d}|$ in order to obtain useful a priori bounds from (4.8). To this aim, we test (4.1b) with $\varphi_p-\overline{\varphi_p}$, leading to

$$\int_\Omega |\nabla\varphi_p|^2+F_{\varepsilon,p}(\varphi_p-\overline{\varphi_p})\,dx=\int_\Omega(\mu_p-\overline{\mu_p}-F_{1,p})(\varphi_p-\overline{\varphi_p})\,dx.$$

Here, we have used that

$$\int_\Omega \overline{\mu_p}(\varphi_p-\overline{\varphi_p})\,dx=\overline{\mu_p}\int_\Omega \varphi_p-\overline{\varphi_p}\,dx=0.$$

Then, using the growth condition (2.5) and the Poincaré inequality, we obtain

$$\int_\Omega |\nabla\varphi_p|^2+F_{\varepsilon,p}(\varphi_p-\overline{\varphi_p})\,dx\leq C\left(1+\|\varphi_p\|_{L^2(\Omega)}+\|\varphi_d\|_{L^2(\Omega)}+\|\nabla\mu_p\|_{L^2(\Omega)}\right)\|\nabla\varphi_p\|_{L^2(\Omega)}.$$

A similar inequality can be obtained by testing (4.1d) with $\varphi_d-\overline{\varphi_d}$. Upon adding these two inequalities and recalling that $\nabla F_\varepsilon=(F_{\varepsilon,p},F_{\varepsilon,d})^\top$, we have

$$\begin{aligned}&\|\nabla\varphi_p\|_{L^2(\Omega)}^2+\|\nabla\varphi_d\|_{L^2(\Omega)}^2+\int_\Omega \nabla F_\varepsilon(\varphi_p,\varphi_d)\cdot(\varphi_p-\overline{\varphi_p},\varphi_d-\overline{\varphi_d})^\top\,dx\\ &\leq C\|\nabla\mu_p\|_{L^2(\Omega)}\|\nabla\varphi_p\|_{L^2(\Omega)}+C\|\nabla\mu_d\|_{L^2(\Omega)}\|\nabla\varphi_d\|_{L^2(\Omega)}\\ &\quad+C\left(1+\|\varphi_p\|_{H^1(\Omega)}^2+\|\varphi_d\|_{H^1(\Omega)}^2\right).\end{aligned} \tag{4.9}$$

At this point we will use the fact that F_ε satisfies (3.2). Consider $s=\varphi_p$, $r=\varphi_d$, $S=\overline{\varphi_p}$, $R=\overline{\varphi_d}$. Then, we find that

$$\begin{aligned}&c_*|\nabla F_\varepsilon(\varphi_p,\varphi_d)-\nabla F_\varepsilon(\overline{\varphi_p},\overline{\varphi_d})|\\ &\leq(\nabla F_\varepsilon(\varphi_p,\varphi_d)-\nabla F_\varepsilon(\overline{\varphi_p},\overline{\varphi_d}))\cdot(\varphi_p-\overline{\varphi_p},\varphi_d-\overline{\varphi_d})^\top+C_*.\end{aligned} \tag{4.10}$$

We recall another property of the derivative of the Moreau–Yosida approximation, namely

$$|\nabla F_\varepsilon(p,q)|\leq|(\partial F_0)^\circ(p,q)|\quad\forall(p,q)\in\Delta,$$

where ∂ denotes here the subdifferential in the sense of convex analysis and $(\partial F_0)^\circ(p,q)$ is the element of minimum norm in the set $\partial F_0(p,q)$, that, at least in principle, could contain more than one element. Here, however, F_0 is assumed to be C^1 in Δ and, consequently, $|(\partial F_0)^\circ(p,q)|=|(\nabla F_0)(p,q)|<\infty$. Then, thanks to the fact that $(\overline{\varphi_p},\overline{\varphi_d})\in\Delta_0$ for all $t\in[0,T]$, we see that $|\nabla F_\varepsilon(\overline{\varphi_p},\overline{\varphi_d})|\leq C$ for all $t\in[0,T]$, and hence integrating (4.10), rearranging and applying the Poincaré inequality leads to

$$\begin{aligned}c_*\|\nabla F_\varepsilon(\varphi_p,\varphi_d)\|_{L^1(\Omega)}&\leq c_*\|\nabla F_\varepsilon(\overline{\varphi_p},\overline{\varphi_d})\|_{L^1(\Omega)}+c_*\|\nabla F_\varepsilon(\varphi_p,\varphi_d)-\nabla F_\varepsilon(\overline{\varphi_p},\overline{\varphi_d})\|_{L^1(\Omega)}\\ &\leq C+C_*|\Omega|+\int_\Omega \nabla F_\varepsilon(\varphi_p,\varphi_d)\cdot(\varphi_p-\overline{\varphi_p},\varphi_d-\overline{\varphi_d})^\top\,dx\\ &\quad+C\left(\|\nabla\varphi_p\|_{L^2(\Omega)}+\|\nabla\varphi_d\|_{L^2(\Omega)}\right).\end{aligned} \tag{4.11}$$

Substituting this inequality into (4.9) and applying Young's inequality then yields

$$\|\nabla\varphi_p\|_{L^2(\Omega)}^2+\|\nabla\varphi_d\|_{L^2(\Omega)}^2+\|\nabla F_\varepsilon(\varphi_p,\varphi_d)\|_{L^1(\Omega)}$$

$$\leq \theta \left(\frac{M_p}{4} \|\nabla \mu_p\|^2_{L^2(\Omega)} + \frac{M_d}{4} \|\nabla \mu_d\|^2_{L^2(\Omega)} \right) + C \left(1 + \|\varphi_p\|^2_{H^1(\Omega)} + \|\varphi_d\|^2_{H^1(\Omega)} \right), \tag{4.12}$$

for some constant $\theta > 0$ yet to be determined. Then, in light of (4.12), observe that, by testing (4.1b) and (4.1d) with ± 1, we obtain that the terms involving the mean values $|\overline{\mu_p}|$ and $|\overline{\mu_d}|$ appearing on the right-hand side of (4.8) can be estimated as follows:

$$\begin{aligned} C(|\overline{\mu_p}| + |\overline{\mu_d}|) &\leq C \left(\|\nabla F_1(\varphi_p, \varphi_d)\|_{L^1(\Omega)} + \|\nabla F_\varepsilon(\varphi_p, \varphi_d)\|_{L^1(\Omega)} \right) \\ &\leq \frac{M_p}{4} \|\nabla \mu_p\|^2_{L^2(\Omega)} + \frac{M_d}{4} \|\nabla \mu_d\|^2_{L^2(\Omega)} + C \left(1 + \|\varphi_p\|^2_{H^1(\Omega)} + \|\varphi_d\|^2_{H^1(\Omega)} \right), \end{aligned} \tag{4.13}$$

by choosing θ appropriately small. Returning to (4.8) and substituting the estimate (4.13), we infer

$$\begin{aligned} &\frac{d}{dt} \int_\Omega F_\varepsilon(\varphi_p, \varphi_d) + F_1(\varphi_p, \varphi_d) + \frac{1}{2} \left(|\nabla \varphi_p|^2 + |\nabla \varphi_d|^2 \right) dx \\ &\quad + \frac{M_p}{4} \|\nabla \mu_p\|^2_{L^2(\Omega)} + \frac{M_d}{4} \|\nabla \mu_d\|^2_{L^2(\Omega)} + \frac{1}{2} \|\boldsymbol{u}\|^2_{L^2(\Omega)} \\ &\leq C \left(1 + \|\varphi_p\|^2_{H^1(\Omega)} + \|\varphi_d\|^2_{H^1(\Omega)} \right). \end{aligned} \tag{4.14}$$

To (4.14) we now add the following inequality obtained from testing (4.1a) with φ_p and (4.1c) with φ_d and summing (cf. (3.29)):

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|\varphi_p\|^2_{L^2(\Omega)} + \|\varphi_d\|^2_{L^2(\Omega)} \right) \leq \frac{M_p}{8} \|\nabla \mu_p\|^2_{L^2(\Omega)} + \frac{M_d}{8} \|\nabla \mu_d\|^2_{L^2(\Omega)} + \frac{1}{4} \|\boldsymbol{u}\|^2_{L^2(\Omega)} \\ + C \left(1 + \|\varphi_p\|^2_{H^1(\Omega)} + \|\varphi_d\|^2_{H^1(\Omega)} \right), \end{aligned}$$

leading to

$$\begin{aligned} &\frac{d}{dt} \int_\Omega F_\varepsilon(\varphi_p, \varphi_d) + F_1(\varphi_p, \varphi_d) \, dx + \frac{d}{dt} \frac{1}{2} \left(\|\varphi_p\|^2_{H^1(\Omega)} + \|\varphi_d\|^2_{H^1(\Omega)} \right) \\ &\quad + \frac{M_p}{8} \|\nabla \mu_p\|^2_{L^2(\Omega)} + \frac{M_d}{8} \|\nabla \mu_d\|^2_{L^2(\Omega)} + \frac{1}{4} \|\boldsymbol{u}\|^2_{L^2(\Omega)} \\ &\leq C \left(1 + \|\varphi_p\|^2_{H^1(\Omega)} + \|\varphi_d\|^2_{H^1(\Omega)} \right). \end{aligned} \tag{4.15}$$

By definition of the Moreau–Yosida approximation, we have

$$F_\varepsilon(s, r) \leq F_0(s, r) \quad \forall (s, r) \in \mathbb{R}^2, \quad \forall \varepsilon \in (0, 1).$$

Hence, recalling (2.6), we arrive at

$$\int_\Omega F_\varepsilon(\varphi_{p,0}, \varphi_{d,0}) + F_1(\varphi_{p,0}, \varphi_{d,0}) \, dx \leq C.$$

Applying Gronwall's inequality to (4.15), we deduce

$$\begin{aligned} &\|F_\varepsilon(\varphi_p, \varphi_d)\|_{L^\infty(0,T;L^1(\Omega))} + \|\varphi_p\|_{L^\infty(0,T;H^1(\Omega))} + \|\varphi_d\|_{L^\infty(0,T;H^1(\Omega))} \\ &\quad + \|\nabla \mu_p\|_{L^2(0,T;L^2(\Omega))} + \|\nabla \mu_d\|_{L^2(0,T;L^2(\Omega))} + \|\boldsymbol{u}\|_{L^2(0,T;L^2(\Omega))} \leq C. \end{aligned} \tag{4.16}$$

Thus, returning to (4.9), using the boundedness of $\|\varphi_p(t)\|_{H^1(\Omega)}$ and $\|\varphi_d(t)\|_{H^1(\Omega)}$ for all $t\in[0,T]$ leads to

$$\int_\Omega \nabla F_\varepsilon(\varphi_p,\varphi_d)\cdot(\varphi_p-\overline{\varphi_p},\varphi_d-\overline{\varphi_d})^\top\,dx \le C\left(1+\|\nabla\mu_p\|_{L^2(\Omega)}+\|\nabla\mu_d\|_{L^2(\Omega)}\right). \tag{4.17}$$

Then, substituting the above inequality into (4.11) yields

$$\|\nabla F_\varepsilon(\varphi_p,\varphi_d)\|_{L^1(\Omega)} \le C\left(1+\|\nabla\mu_p\|_{L^2(\Omega)}+\|\nabla\mu_d\|_{L^2(\Omega)}\right) \tag{4.18}$$

thanks to the estimate (4.16), whence $\nabla F_\varepsilon(\varphi_p,\varphi_d)$ is bounded in $L^2(0,T;L^1(\Omega))$. In turn, by the first line of (4.13) we find that $|\overline{\mu_p}|$ and $|\overline{\mu_d}|$ are bounded in $L^2(0,T)$. Hence, recalling (4.16) and using once more the Poincaré inequality, we get

$$\|\mu_p\|_{L^2(0,T;H^1(\Omega))}+\|\mu_d\|_{L^2(0,T;H^1(\Omega))}\le C. \tag{4.19}$$

Furthermore, recalling (4.6), thanks to (4.16) we now have

$$\|q\|_{L^2(0,T;H^1(\Omega))}\le C. \tag{4.20}$$

Next, we infer estimates on $F_{\varepsilon,p}$ by testing (4.1b) with $F_{\varepsilon,p}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,p}(\varphi_p,\varphi_d)}$, leading to

$$\begin{aligned}
&\|F_{\varepsilon,p}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,p}(\varphi_p,\varphi_d)}\|^2_{L^2(\Omega)}\\
&\quad+\int_\Omega F_{\varepsilon,pp}(\varphi_p,\varphi_d)|\nabla\varphi_p|^2+F_{\varepsilon,pd}(\varphi_p,\varphi_d)\nabla\varphi_p\cdot\nabla\varphi_d\,dx\\
&=\int_\Omega(\mu_p-\overline{\mu_p}-F_{1,p}(\varphi_p,\varphi_d))(F_{\varepsilon,p}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,p}(\varphi_p,\varphi_d)})\,dx,
\end{aligned}$$

where

$$F_{\varepsilon,pp}=\frac{\partial^2F_\varepsilon}{\partial\varphi_p^2},\quad F_{\varepsilon,pd}=\frac{\partial^2F_\varepsilon}{\partial\varphi_p\partial\varphi_d}.$$

Adding the similar identity obtained testing (4.1d) with $F_{\varepsilon,d}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,d}(\varphi_p,\varphi_d)}$ and employing the Poincaré inequality together with the linear growth of ∇F_1, it is not difficult to deduce

$$\begin{aligned}
&\frac12\|F_{\varepsilon,p}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,p}(\varphi_p,\varphi_d)}\|^2_{L^2(\Omega)}+\frac12\|F_{\varepsilon,d}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,d}(\varphi_p,\varphi_d)}\|^2_{L^2(\Omega)}\\
&\quad+\int_\Omega(\nabla\varphi_p,\nabla\varphi_d)\cdot D^2F_\varepsilon(\varphi_p,\varphi_d)(\nabla\varphi_p,\nabla\varphi_d)^\top\,dx\\
&\le C\left(1+\|\nabla\mu_p\|^2_{L^2(\Omega)}+\|\nabla\mu_d\|^2_{L^2(\Omega)}+\|\varphi_p\|^2_{L^2(\Omega)}+\|\varphi_d\|^2_{L^2(\Omega)}\right).
\end{aligned}$$

Since F_ε is convex, the Hessian D^2F_ε is non-negative and consequently we can neglect the integral term on the left-hand side. Then, recalling (4.16) leads to

$$\begin{aligned}
&\|F_{\varepsilon,p}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,p}(\varphi_p,\varphi_d)}\|_{L^2(0,T;L^2(\Omega))}\\
&\quad+\|F_{\varepsilon,d}(\varphi_p,\varphi_d)-\overline{F_{\varepsilon,d}(\varphi_p,\varphi_d)}\|_{L^2(0,T;L^2(\Omega))}\le C.
\end{aligned} \tag{4.21}$$

Upon recalling the boundedness of $\nabla F_\varepsilon(\varphi_p,\varphi_d)$ in $L^2(0,T;L^1(\Omega))$ resulting from (4.18), we deduce a control of the quantities $\|\overline{F_{\varepsilon,p}}\|_{L^2(0,T)}$ and $\|\overline{F_{\varepsilon,d}}\|_{L^2(0,T)}$. Hence, from (4.21) we eventually obtain

$$\|F_{\varepsilon,p}(\varphi_p,\varphi_d)\|_{L^2(0,T;L^2(\Omega))}+\|F_{\varepsilon,d}(\varphi_p,\varphi_d)\|_{L^2(0,T;L^2(\Omega))}\le C. \tag{4.22}$$

Viewing (4.1b) and (4.1d) as elliptic equations for φ_p and φ_d, respectively, with right-hand sides bounded in $L^2(0,T;L^2(\Omega))$ and no-flux boundary conditions, the elliptic regularity theory gives

$$\|\varphi_p\|_{L^2(0,T;H^2(\Omega))}+\|\varphi_d\|_{L^2(0,T;H^2(\Omega))}\leq C. \tag{4.23}$$

Lastly, from inspection of (4.1a) and (4.1c), and thanks to the estimate (4.16) we have

$$\|\partial_t\varphi_p\|_{L^2(0,T;H^1(\Omega)')}+\|\partial_t\varphi_d\|_{L^2(0,T;H^1(\Omega)')}\leq C. \tag{4.24}$$

4.2. Compactness and passing to the limit. Thanks to the uniform estimates (4.16), (4.19), (4.20), (4.22), (4.23) and (4.24), by standard compactness arguments we infer the existence of functions $(\varphi_p,\mu_p,\varphi_d,\mu_d,q,\boldsymbol{u})$ and of a pair (η_p,η_d) such that

$$\varphi_i^\varepsilon\to\varphi_i \text{ weakly}* \text{ in } L^\infty(0,T;H^1(\Omega))\cap L^2(0,T;H^2(\Omega))\cap H^1(0,T;H^1(\Omega)'), \tag{4.25a}$$
$$\varphi_i^\varepsilon\to\varphi_i \text{ strongly in } C^0([0,T];L^s(\Omega))\cap L^2(0,T;W^{1,s}(\Omega)), \tag{4.25b}$$
$$\varphi_i^\varepsilon\to\varphi_i \text{ a.e. in } \Omega\times(0,T), \tag{4.25c}$$
$$\mu_i^\varepsilon\to\mu_i \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{4.25d}$$
$$\boldsymbol{u}^\varepsilon\to\boldsymbol{u} \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{4.25e}$$
$$q^\varepsilon\to q \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{4.25f}$$

and

$$F_{\varepsilon,p}(\varphi_p,\varphi_d)\to\eta_p \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{4.26a}$$
$$F_{\varepsilon,d}(\varphi_p,\varphi_d)\to\eta_d \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{4.26b}$$

for any $s<\infty$ in two dimensions and any $s\in[1,6)$ in three dimensions. Using a similar argument as in the proof of Theorem 3.2, by the a.e. convergence of φ_p^ε to φ_p in $\Omega\times(0,T)$ and Egorov's theorem, we can show that $\{n^\varepsilon\}_{\varepsilon\in(0,1)}$ is a Cauchy family in $L^2(0,T;H^1(\Omega))$. Then, there also exists a function $n\in L^\infty(0,T;W^{2,r}(\Omega))$, for any $r<\infty$, with $0\leq n\leq 1$ a.e. in $\Omega\times(0,T)$, such that

$$n^\varepsilon\to n \text{ strongly in } L^2(0,T;H^1(\Omega)).$$

It now remains to pass to the limit $\varepsilon\to 0$ in (4.1). Actually, in view of the above convergence properties, the argument is very similar to that used before when we pass to the limit $\delta\to 0$. Hence, we just outline the differences which are mainly related to the terms depending on F_ε. Actually, combining (4.25b), (4.26a)-(4.26b) with the standard monotonicity argument in [2, Prop. 1.1, p. 42], we readily deduce that $(\varphi_p,\varphi_d)\in\Delta$ a.e. in $\Omega\times(0,T)$ and that $\eta_p=F_{0,p}(\varphi_p,\varphi_d)$, $\eta_d=F_{0,d}(\varphi_p,\varphi_d)$. This, in particular, implies that the truncation operator $\mathcal{T}(\cdot)$ disappears in the limit formulation of the problem, namely, we have $\mathcal{T}(\varphi_p)=\varphi_p$ and $\mathcal{T}(\varphi_d)=\varphi_d$ a.e. in Q.

Let us also point out that from the structural assumption (2.2) and from the derivation of (4.3) the limit functions φ_p and φ_d satisfy $(\overline{\varphi_p}(t),\overline{\varphi_d}(t))\in\Delta_0$ and

$$0<c_1\leq\overline{\varphi_p}(t),\overline{\varphi_d}(t)\leq c_2<1,\quad c_1\leq\overline{\varphi_p+\varphi_d}(t)\leq c_2$$

for all $t\in[0,T]$. Hence, we have proved that the tuple $(\varphi_p,\mu_p,\varphi_d,\mu_d,\boldsymbol{u},q,n)$ is a weak solution to system (1.1) in the sense of Definition 2.1. This concludes the proof of Theorem 2.1.

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