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A FAMILY OF ASYMPTOTIC MODELS FOR INTERNAL WAVES PROPAGATING IN INTERMEDIATE/DEEP WATER*

DANIEL G. ALFARO VIGO† AND GLADYS CALLE CARDEÑA‡

Abstract. In this paper, we obtain a family of approximate systems of two partial differential equations for the modeling of weakly nonlinear long internal waves propagating at the interface between two immiscible and irrotational fluids in a channel of intermediate/infinite depth. These systems are approximations of the system of Euler equations that share the same asymptotic order.

The analysis of the corresponding linearized systems leads to the identification of several subfamilies (associated with different subsets in the space of parameters) for which the solutions of the linearized models are physically compatible with the solutions of the linearized system of Euler equations. Finally, for the class of weakly dispersive nonlinear systems which is formed by some of those subfamilies, we establish the existence and uniqueness of local in time solutions.

Keywords. internal waves models; nonlinear waves equations; dispersive waves equations; local well-posedness.

AMS subject classifications. 35Q35; 35E15; 35S30.

1. Introduction

The mathematical theory of internal waves, propagating under the influence of gravity, at the interface between two layers of irrotational and immiscible fluids with different densities attracts a great interest in the scientific community [4–6]. The theory is developed from the Euler equations associated with each fluid layer [14], but in laboratory studies and in engineering applications the full Euler equations appear more complex than is necessary for the modeling situation at hand. Consequently there have appeared many approximate models applying to restricted physical regimes. Nevertheless, these approximate models pose interesting challenges with respect to the mathematical analysis of the involved partial differential equations and their numerical solutions [4, 5]. These challenges are closely related to other problems on dispersive waves (see for instance [2, 4]).

We study a system of two fluids with different densities contained in a long channel with a flat bottom and a rigid lid, in a two-dimensional setting, i.e. considering only one horizontal direction (see Figure 1.1). We focus in the asymptotic regime where the depth of the upper layer is much smaller than the characteristic wavelength of the interface fluctuations (shallow water regime) while the depth of the lower layer is comparable (intermediate water regime) or much larger than that characteristic wavelength (deep water regime). We consider a stable configuration, more specifically, we assume that the lower layer is filled by the denser fluid.

Approximate models for internal wave propagation in the regime of our interest have been developed by various authors. In [5, 6] were introduced several reduced models of different orders of approximation for the propagation of weakly and strongly nonlinear internal waves in a two-fluid system and their solitary waves solutions were studied.

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†Departamento de Ciência da Computação, Universidade Federal do Rio de Janeiro, Caixa Postal 68.530, CEP 21941-590, Rio de Janeiro, RJ, Brazil (dgalfaro@dcc.ufrj.br).

‡Instituto de Ciência, Engenharia e Tecnologia, Universidade Federal dos Vales de Jequitinhonha e Mucuri, Campus do Mucuri, CEP 39803-371, Teófilo Otoni, Minas Gerais, MG, Brazil (gladyscallecardena@gmail.com).

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The models in [6] corresponding to the *shallow water/intermediate water* regime were generalized and extended for the case of rapidly-varying and steep topography in [12,13]. These models consist of system of nonlocal wave equations that describe the evolution of the interface between the two layers of fluids and the average horizontal velocity of the fluid in the upper layer.

In [8] the technique presented in [11] was applied in order to obtain new approximate models for strongly nonlinear waves in the three-dimensional setting considering the *shallow water/deep water* regime. The corresponding equations involve the displacement of the interface and the horizontal velocity at an arbitrary depth in the upper layer, generating a one parameter family of models. All their models are asymptotically consistent with the corresponding models in [5, 13]. They also obtained the equations for the solitary waves propagating in one horizontal direction and numerically studied their profiles.

More recently, in [9] a similar one parameter family of models in the two-dimensional setting (parametrized by the depth at which the horizontal velocity in the upper layer is measured) was introduced. There was established the existence and uniqueness of local in time solutions for one specific member of this family. Additionally, a numerical scheme to approximate the solutions of this models was introduced and a complete error analysis of the semidiscrete scheme was presented.

Other approximate models for weakly and strongly nonlinear internal waves in different asymptotic regimes have been presented and studied by other authors. In [7], a Hamiltonian perturbation theory was used to obtain asymptotic models considering classical scaling regimes and also a novel class of scaling. Moreover, besides the case of an upper fluid with a rigid lid also the situation where it is bounded by a free surface was analyzed. However, they did not introduce any parameterized family of models.

In [4], under the rigid lid assumption and using the expansion of the nonlocal operators involved in the formulation of the Euler equations with respect to small parameters, different models were rigorously derived and analyzed. Their models include classical systems and other model systems that appear to be new. For the regime we are interested in, they obtained a family of models depending on one parameter which is different from the families obtained in [8, 9]. In [15], the local and large time well-posedness for this one parameter family of models was obtained and also a rigorous justification for considering the deep water regime as a limiting case of the intermediate depth regime was presented.

In this paper, we introduce a new family of reduced models for weakly nonlinear internal waves in the *shallow/intermediate or deep waters* regime, applying the techniques used in [2, 11] to a well-known model from [5]. This family depends on seven parameters but only four of them are independent. The advantages of a model with several parameters will be discussed in the finals remarks. In the regime of *shallow/deep waters* we analyze the well-posedness of the initial-value problems corresponding to the linearized models by identifying several subsets of the parameters space where the corresponding solutions are compatible with the linearized Euler equations. Furthermore, for the subclass of nonlinear weakly dispersive models we establish the local well-posedness of the initial-value problems.

The remainder of this paper is organized as follows. In Section 2, we introduce a new family of approximate weakly nonlinear models. In Section 3, we present the well-posedness results of the initial value problem associated with these models and in Section 4, we present the concluding remarks.

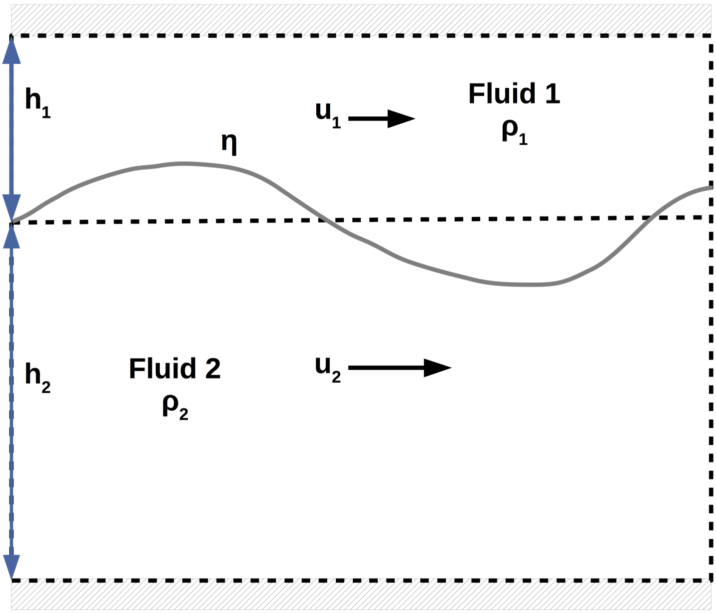

FIG. 1.1. *Physical configuration of the two fluids system.*

2. Family of weakly nonlinear models

In this section, we introduce a family of reduced models that approximate the system of Euler equations for internal waves, see for instance [14]. As a starting point we consider the weakly nonlinear reduced model of Choi and Camassa [5] that corresponds to the asymptotic regime we are interested in. In dimensionless variables, it is given by the following system of partial differential equations

$$\begin{cases} \eta_t - [(1-\alpha\eta)\bar{u}]_x = 0, \\ \bar{u}_t + \alpha\bar{u}\bar{u}_x - \eta_x = \gamma\sqrt{\beta}\,\mathcal{T}_\delta[\bar{u}_{tx}] + \dfrac{\beta}{3}\bar{u}_{txx}, \end{cases} \tag{2.1}$$

where the unknowns η and $\bar{u}$ represent the interface fluctuation and average horizontal velocity of the fluid in the upper layer, respectively, and the variables t, x represent the time and horizontal coordinate, respectively. We also introduce the dimensionless parameters $\alpha = \frac{a}{h_1}$, $\beta = \frac{h_1^2}{\lambda^2}$, $\delta = \frac{h_2}{\lambda}$ and $\gamma = \frac{\varrho_2}{\varrho_1}$ where a, λ represent the amplitude and characteristic wavelength of the interface fluctuations, respectively, and, h_1, ϱ_1 and h_2, ϱ_2 correspond to the depth and density of the fluids in the upper and lower layers, respectively. The parameter α is related to the intensity of the nonlinearity, while β and δ are associated with the dispersive effects of the upper and lower fluid layers, respectively.

The conditions

$$\beta \ll 1, \quad \alpha = O(\beta) \ll 1 \quad \text{and} \quad \delta = O(1),$$

characterize a regime of weakly nonlinear long waves that propagates at the interface between a shallow upper layer and an intermediate lower layer of fluids. Moreover, this model represents an approximation of order $O(\beta^{3/2})$ for the Euler equations. Additionally, we assume that $\gamma > 1$, enforcing the stable configuration with the denser fluid in the lower layer.

The operator $\mathcal{T}_\delta$ represents the Hilbert transform in a layer of height δ (see [12,13]) and is defined by

$$\mathcal{T}_\delta[f](x) = \frac{1}{2\delta} PV \int_{-\infty}^{\infty} f(\tilde{x}) \coth\left(\frac{\pi}{2\delta}(\tilde{x}-x)\right) d\tilde{x},$$

where the integral is calculated in the sense of Cauchy principal value (PV). In the limit of deep water, i.e. the lower layer has infinite depth ($\delta \to \infty$), the operator $\mathcal{T}_\delta$ gives rise to the Hilbert transform (see [10,12,13])

$$\mathcal{H}[f](x) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} f(\tilde{x}) \frac{d\tilde{x}}{\tilde{x}-x}.$$

In the next step, following [11], we use the asymptotic relation between the average horizontal velocity $\bar{u}$ and the horizontal velocity at height $\zeta_0 \in [0,1]$ in the upper layer to transform system (2.1). More specifically, we substitute

$$\bar{u} = u_0 + \frac{\beta}{2}(\zeta_0^2 - \frac{1}{3}) u_{0,xx} + O(\beta^{3/2}),$$

in (2.1) and neglect terms of order $O(\beta^{3/2})$. Thus, we obtain the following family of equivalent models parameterized by height ζ_0

$$\begin{cases} \eta_t = u_{0,x} + \frac{\beta}{2}\left(\zeta_0^2 - \frac{1}{3}\right) u_{0,xxx} - \alpha(\eta u_0)_x, \\ u_{0,t} + \alpha u_0 u_{0,x} = \eta_x + \gamma\sqrt{\beta}\,\mathcal{T}_\delta[u_{0,tx}] + \frac{\beta}{2}\left(1 - \zeta_0^2\right) u_{0,txx}. \end{cases} \tag{2.2}$$

These equations are consistent with the model obtained in [8]. Moreover, for the case when the lower layer is infinitely deep the corresponding one-parameter family is obtained by replacing the operator $\mathcal{T}_\delta$ by the Hilbert transform $\mathcal{H}$. Those models were obtained in [9].

We introduce new parameters in our model using the technique applied in [2], which modifies only the linear terms of the equations. Since $\eta_t = u_{0,x} + O(\beta)$, we get that $u_{0,xxx} = \tilde{a}\eta_{txx} + (1-\tilde{a})u_{0,xxx} + O(\beta)$, where $\tilde{a}$ is an arbitrary number. By plugging this in the first equation of (2.2), we get

$$\eta_t - a\beta\eta_{txx} = u_{0,x} + b\beta u_{0,xxx} - \alpha(\eta u_0)_x + O(\beta^{3/2}), \tag{2.3}$$

where $a = \tilde{a}\left(\frac{\zeta_0^2 - 1/3}{2}\right)$, $b = (1-\tilde{a})\left(\frac{\zeta_0^2 - 1/3}{2}\right)$, and consequently $a + b = \frac{\zeta_0^2 - 1/3}{2}$.

Analogously, we use that $u_{0,t} = \eta_x + \gamma\sqrt{\beta}\,\mathcal{T}_\delta[u_{0,tx}] + O(\beta)$ to transform the second equation of (2.2) and obtain that

$$\begin{aligned} &u_{0,t} - c\gamma\sqrt{\beta}\,\mathcal{T}_\delta[u_{0,tx}] - e\beta u_{0,txx} - g\gamma^2\beta\,\mathcal{T}_\delta^2[u_{0,txx}] \\ &= \eta_x + d\gamma\sqrt{\beta}\,\mathcal{T}_\delta[\eta_{xx}] + f\beta\eta_{xxx} + h\gamma^2\beta\,\mathcal{T}_\delta^2[\eta_{xxx}] - \alpha u_0 u_{0,x} + O(\beta^{3/2}), \end{aligned} \tag{2.4}$$

where c, d, e, f, g, h are arbitrary numbers satisfying $c + d = 1$, $e + f = \frac{1-\zeta_0^2}{2}$ and $g + h = d$.

Therefore, neglecting the order $O(\beta^{3/2})$ terms in (2.3) and (2.4), we obtain the new family of asymptotically equivalent systems

$$\begin{cases} \eta_t - a\beta\eta_{txx} = u_{0,x} + b\beta u_{0,xxx} - \alpha(\eta u_0)_x, \\ u_{0,t} - c\gamma\sqrt{\beta}\mathcal{T}_\delta[u_{0,tx}] - e\beta u_{0,txx} - g\gamma^2\beta\mathcal{T}_\delta^2[u_{0,txx}] \\ = \eta_x + d\gamma\sqrt{\beta}\mathcal{T}_\delta[\eta_{xx}] + f\beta\eta_{xxx} + h\gamma^2\beta\mathcal{T}_\delta^2[\eta_{xxx}] - \alpha u_0 u_{0,x}, \end{cases} \tag{2.5}$$

where the parameters ζ_0, a, b, c, d, e, f, g and h satisfy the relationships

$$\begin{cases} 0 \le \zeta_0 \le 1, \\ a + b = \frac{1}{2}(\zeta_0^2 - \frac{1}{3}), c + d = 1, \\ e + f = \frac{1}{2}(1 - \zeta_0^2), g + h = d. \end{cases} \tag{2.6}$$

Notice that only five of the nine parameters are independent.

In the limit of deep water ($\delta \to \infty$), using that $-\mathcal{H}^2$ corresponds to the identity operator, the family of systems (2.5) reduces to

$$\begin{cases} \eta_t - a\beta\eta_{txx} = u_{0,x} + b\beta u_{0,xxx} - \alpha(\eta u_0)_x, \\ u_{0,t} - c\gamma\sqrt{\beta}\mathcal{H}[u_{0,tx}] - e\beta u_{0,txx} = \eta_x + d\gamma\sqrt{\beta}\mathcal{H}[\eta_{xx}] + f\beta\eta_{xxx} - \alpha u_0 u_{0,x}, \end{cases} \tag{2.7}$$

where the parameters satisfy the relationships

$$\begin{cases} 0 \le \zeta_0 \le 1, \\ a + b = \frac{1}{2}(\zeta_0^2 - \frac{1}{3}), c + d = 1, \\ \gamma^2 d + e + f = \frac{1}{2}(1 - \zeta_0^2). \end{cases} \tag{2.8}$$

In this case, among the seven parameters ζ_0, a, b, c, d, e and f, only four are independent, for instance we can choose ζ_0, a, c and e.

Using the change of variables $t = \sqrt{\beta}\tilde{t}$ and $x = \sqrt{\beta}\tilde{x}$ one can readily eliminate the parameter β, and after suppressing the tilde for the sake of visual clarity the corresponding systems take the form

$$\begin{cases} \eta_t - a\eta_{txx} = u_{0,x} + b u_{0,xxx} - \alpha(\eta u_0)_x, \\ u_{0,t} - c\gamma\mathcal{H}[u_{0,tx}] - e u_{0,txx} = \eta_x + d\gamma\mathcal{H}[\eta_{xx}] + f\eta_{xxx} - \alpha u_0 u_{0,x}. \end{cases} \tag{2.9}$$

3. Analysis of the systems in the deep water regime

From now on, the analysis shall focus on the family of systems (2.9), more specifically we present a study of the well-posedness of this family of systems under certain useful conditions.

3.1. Well-posedness of the linearized systems.

We begin with a study of the well-posedness of the family of systems corresponding to the linearization of (2.9) around the equilibrium state ($\eta = u_0 = 0$), i.e. in the case when $\alpha = 0$. We emphasize that the well-posedness of the linearized system is an important criterion for determining which specific systems of this family of models can be useful in applications (see e.g. [2]). As a result of this analysis, we will determine several subfamilies of models whose solutions are compatible with the solutions of the linearized Euler equations.

Let us consider the initial value problem

$$\begin{cases} \eta_t - a\eta_{txx} = u_{0,x} + b u_{0,xxx}, \\ u_{0,t} - c\gamma\mathcal{H}[u_{0,tx}] - e u_{0,txx} = \eta_x + d\gamma\mathcal{H}[\eta_{xx}] + f\eta_{xxx}, \\ \eta(0,x) = \eta^0(x), \quad u_0(0,x) = u_0^0(x), \end{cases} \tag{3.1}$$

in (real) Sobolev spaces $H^s = H^s(\mathbb{R})$ with $s \geq 0$. The standard H^s norm is represented by $\|\cdot\|_s$ (see [10]).

The solution of problem (3.1) can be readily obtained by the Fourier method [10]. Recall that the Fourier transform and its inverse in $L^2(\mathbb{R})$ are defined by

$$\hat{g}(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(x)e^{-ikx}\,dx, \quad g(x) = \int_{-\infty}^{\infty} \hat{g}(k)e^{ikx}\,dk, \quad \forall k \in \mathbb{R}.$$

This leads to the representation of (3.1) as the system of ODEs: $\hat{\mathbf{v}}_t(k) = \widehat{\mathcal{L}\mathbf{v}}(k)$, where

$$\mathbf{v} = \begin{pmatrix} \eta \\ u_0 \end{pmatrix}, \qquad \widehat{\mathcal{L}\mathbf{v}}(k) = ik\begin{bmatrix} 0 & \frac{w_3(k)}{w_1(k)} \\ \frac{w_4(k)}{w_2(k)} & 0 \end{bmatrix}\begin{pmatrix} \hat{\eta} \\ \hat{u}_0 \end{pmatrix}(k), \tag{3.2}$$

with $w_1(k) = 1 + ak^2$, $w_2(k) = 1 + c\gamma|k| + ek^2$, $w_3(k) = 1 - bk^2$, $w_4(k) = 1 - d\gamma|k| - fk^2$. Therefore, after a straightforward computation the solutions are given as

$$\begin{pmatrix} \eta \\ u_0 \end{pmatrix}(t,x) = \int_{-\infty}^{\infty}\left\{ e^{i(kx+\omega(k)t)}\begin{pmatrix} A^+ \\ B^+ \end{pmatrix}(k) + e^{i(kx-\omega(k)t)}\begin{pmatrix} A^- \\ B^- \end{pmatrix}(k)\right\} dk, \tag{3.3}$$

where

$$\begin{pmatrix} A^\pm \\ B^\pm \end{pmatrix}(k) = \frac{1}{2}\begin{bmatrix} 1 & \pm\theta^{-1}(k) \\ \pm\theta(k) & 1 \end{bmatrix}\begin{pmatrix} \hat{\eta}^0 \\ \hat{u}_0^0 \end{pmatrix}(k), \qquad k \in \mathbb{R},$$

and the functions $\omega(k)$, $\theta(k)$ are defined by

$$\omega(k) = k\sqrt{\frac{w_3(k)w_4(k)}{w_1(k)w_2(k)}}, \qquad \theta(k) = \sqrt{\frac{w_1(k)w_4(k)}{w_2(k)w_3(k)}}. \tag{3.4}$$

Remark 3.1. It is important to observe that under conditions (2.8), as $k \to 0$, $\omega(k)$ given by (3.4) represents an approximation of order $O(k^4)$ to the wave frequency $\omega_E(k) = k/\sqrt{k\coth(k) + \gamma|k|}$ associated with the linearized Euler equations. Moreover, if for any $k \in \mathbb{R}$, $\omega(k)$ is real-valued and $\theta(k) \neq 0$, then Equation (3.3) represents a superposition of dispersive waves traveling with speeds $v(k) = \mp\frac{\omega(k)}{k}$. Consequently, under these conditions, the solutions (3.3) are physically compatible with the solutions of the linearized Euler equations.

The following theorem, puts on solid ground the representation (3.3).

Theorem 3.1 (Well-posedness of system (3.1)). *If parameters $(\zeta_0, a, b, c, d, e, f)$ lie in one of the subsets R_i $(i = 1, \ldots, 22)$ defined below, then there exist Δ_i and Γ_i such that for any real $s \geq \max\{0, \Delta_i\}$, the linearized initial value problem (3.1) is well-posed in $H^s \times H^{s-\Delta_i}$, and its solution is given by (3.3). Moreover, for any integer $m \geq 0$ such that $m\Gamma_i \leq \min\{s, s-\Delta_i\}$ we have that*

$$(\partial_t^m \eta(t,x), \partial_t^m u_0(t,x)) \in C\left([0,+\infty); H^{s-m\Gamma_i} \times H^{s-\Delta_i - m\Gamma_i}\right),$$

and for some constants $C_m > 0$

$$\|\partial_t^m \eta(t)\|_{s-m\Gamma_i} + \|\partial_t^m u_0(t)\|_{s-\Delta_i-m\Gamma_i} \leq C_m\left(\|\eta^0\|_s + \|u_0^0\|_{s-\Delta_i}\right), \quad \text{for all } t \geq 0.$$

The definitions of the subsets R_i are given below, and the corresponding values of Δ_i and Γ_i are presented in Table 3.1.

R_i	Δ_i	Γ_i	s_i^*	R_i	Δ_i	Γ_i	s_i^*	R_i	Δ_i	Γ_i	s_i^*	R_i	Δ_i	Γ_i
1	-1	2		7	1	2		12	1	0	1	18	0	1
2	0	3		8	$\frac{1}{2}$	$\frac{3}{2}$	1	13	2	1		19	1	2
3	$-\frac{1}{2}$	$\frac{5}{2}$		9	0	1	0	14	$\frac{3}{2}$	$\frac{1}{2}$		20	$\frac{1}{2}$	$\frac{3}{2}$
4	-2	1		10	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{4}$	15	0	-1	0	21	-1	0
5	$-\frac{3}{2}$	$\frac{3}{2}$		11	$-\frac{1}{2}$	$-\frac{1}{2}$	0	16	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	22	$-\frac{1}{2}$	$\frac{1}{2}$
6	-1	0	$-\frac{1}{2}$					17	$\frac{1}{2}$	$-\frac{3}{2}$	1			

TABLE 3.1. *Values of Δ_i, Γ_i and s_i^* (see Prop. 3.1) corresponding to the different sets R_i.*

$$(R_1)\begin{cases} 0\le\zeta_0<\frac{1}{\sqrt{3}}, a=0, b=\frac{1}{2}(\zeta_0^2-\frac{1}{3}),\\ c\in\mathbb{R}, d=1-c,\\ e>\max\left\{\frac{1}{2}(1-\zeta_0^2)+\gamma^2\left((\frac{c+1}{2})^2-1\right), \gamma^2\frac{c^2}{4}\right\},\\ f=\frac{1}{2}(1-\zeta_0^2)-e+(c-1)\gamma^2.\end{cases}\quad (R_2)\begin{cases} 0\le\zeta_0<\frac{1}{\sqrt{3}}, a=0,\\ b=\frac{1}{2}(\zeta_0^2-\frac{1}{3}), c=e=0, d=1\\ f=\frac{1}{2}(1-\zeta_0^2)-\gamma^2.\end{cases}$$

$$(R_3)\begin{cases} 0\le\zeta_0<\frac{1}{\sqrt{3}}, a=0, b=\frac{1}{2}(\zeta_0^2-\frac{1}{3}),\\ 0<c<2\sqrt{1-\frac{1}{2}(1-\zeta_0^2)\frac{1}{\gamma^2}}-1, d=1-c,\\ e=0, f=\frac{1}{2}(1-\zeta_0^2)+(c-1)\gamma^2.\end{cases}\quad (R_4)\begin{cases} 0\le\zeta_0<\min\left\{\frac{1}{\sqrt{3}}, \sqrt{1-\frac{1}{2}\gamma^2}\right\},\\ a=0, b=\frac{1}{2}(\zeta_0^2-\frac{1}{3}),\\ c=1, e=\frac{1}{2}(1-\zeta_0^2), f=d=0.\end{cases}$$

The subset (R_4) is not empty when $\gamma<\sqrt{2}$.

$$(R_5)\begin{cases} 0\le\zeta_0<\frac{1}{\sqrt{3}}, a=0, b=\frac{1}{2}(\zeta_0^2-\frac{1}{3}),\\ c_*<c<2\left(1+\frac{1}{\gamma}\sqrt{\frac{1}{2}(1-\zeta_0^2)}\right), d=1-c,\\ \text{where } c_*=\max\left\{1, 2\left(1-\frac{1}{\gamma}\sqrt{\frac{1}{2}(1-\zeta_0^2)}\right)\right\},\\ e=\frac{1}{2}(1-\zeta_0^2)+(c-1)\gamma^2, f=0.\end{cases}\quad (R_6)\begin{cases} 0\le\zeta_0<\sqrt{1-\frac{1}{2}\gamma^2},\\ a>\max\left\{\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right), 0\right\},\\ b=\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right)-a,\\ c=1, e=\frac{1}{2}(1-\zeta_0^2),\\ d=f=0.\end{cases}$$

The subset (R_6) is not empty when $\gamma<\sqrt{2}$.

$$(R_7)\begin{cases} 0\le\zeta_0\le1, a>\max\left\{\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right), 0\right\},\\ b=\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right)-a, c=e=0, d=1,\\ f=\frac{1}{2}(1-\zeta_0^2)-\gamma^2.\end{cases}\quad (R_8)\begin{cases} 0\le\zeta_0\le1, a>\max\left\{\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right), 0\right\},\\ b=\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right)-a,\\ 0<c<2\sqrt{1-\frac{1-\zeta_0^2}{2\gamma^2}}-1, d=1-c,\\ e=0, f=\frac{1}{2}(1-\zeta_0^2)+(c-1)\gamma^2.\end{cases}$$

$$(R_9)\begin{cases} 0\le\zeta_0\le1, a>\max\left\{\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right), 0\right\},\\ b=\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right)-a, c\in\mathbb{R}, d=1-c,\\ e>\max\left\{\frac{1}{2}(1-\zeta_0^2)+\gamma^2\left((\frac{c+1}{2})^2-1\right), \left(\frac{\gamma c}{2}\right)^2\right\},\\ f=\frac{1}{2}(1-\zeta_0^2)-e+(c-1)\gamma^2.\end{cases}$$

$$(R_{10})\begin{cases} 0\le\zeta_0\le1, a>\max\left\{\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right), 0\right\},\\ b=\frac{1}{2}\left(\zeta_0^2-\frac{1}{3}\right)-a,\\ c_*<c<2\left(1+\frac{1}{\gamma}\sqrt{\frac{1}{2}(1-\zeta_0^2)}\right)\\ \text{where } c_*=\max\left\{1, 2\left(1-\frac{1}{\gamma}\sqrt{\frac{1}{2}(1-\zeta_0^2)}\right)\right\},\\ e=\frac{1}{2}(1-\zeta_0^2)+(c-1)\gamma^2,\\ f=0, d=1-c.\end{cases}\quad (R_{11})\begin{cases} \zeta_0=1, a>\frac{1}{3}, b=\frac{1}{3}-a,\\ c=1, d=e=f=0.\end{cases}$$

$$(R_{12})\begin{cases}\frac{1}{\sqrt{3}}<\zeta_0\le 1, a=\frac{1}{2}(\zeta_0^2-\frac{1}{3}), b=0,\\ c\in\mathbb{R}, d=1-c,\\ e>\max\left\{\frac{1}{2}(1-\zeta_0^2)+\gamma^2\left((\frac{c+1}{2})^2-1\right),\gamma^2\frac{c^2}{4}\right\},\\ f=\frac{1}{2}(1-\zeta_0^2)-e+(c-1)\gamma^2.\end{cases}\quad (R_{13})\begin{cases}\frac{1}{\sqrt{3}}<\zeta_0\le 1, a=\frac{1}{2}(\zeta_0^2-\frac{1}{3}),\\ b=c=e=0, d=1,\\ f=\frac{1}{2}(1-\zeta_0^2)-\gamma^2.\end{cases}$$

$$(R_{14})\begin{cases}\frac{1}{\sqrt{3}}<\zeta_0\le 1, a=\frac{1}{2}(\zeta_0^2-\frac{1}{3}), b=0,\\ 0<c<2\sqrt{1-\frac{1-\zeta_0^2}{2\gamma^2}}-1, d=1-c,\\ e=0, f=\frac{1}{2}(1-\zeta_0^2)+(c-1)\gamma^2.\end{cases}\quad (R_{15})\begin{cases}\frac{1}{\sqrt{3}}<\zeta_0<\sqrt{1-\frac{1}{2}\gamma^2},\\ a=\frac{1}{2}(\zeta_0^2-\frac{1}{3}), b=0,\\ c=1, d=f=0,\\ e=\frac{1}{2}(1-\zeta_0^2).\end{cases}$$

The subset (R_{15}) is not empty when $\gamma<\frac{2}{\sqrt{3}}$.

$$(R_{16})\begin{cases}\frac{1}{\sqrt{3}}<\zeta_0\le 1, a=\frac{1}{2}(\zeta_0^2-\frac{1}{3}), b=0,\\ c_*<c<2\left(1+\frac{1}{\gamma}\sqrt{\frac{1}{2}(1-\zeta_0^2)}\right)\\ \text{where } c_*=\max\left\{1,2\left(1-\frac{1}{\gamma}\sqrt{\frac{1}{2}(1-\zeta_0^2)}\right)\right\},\\ d=1-c, e=\frac{1}{2}(1-\zeta_0^2)+(c-1)\gamma^2, f=0.\end{cases}\quad (R_{17})\begin{cases}\zeta_0=1, a=\frac{1}{3}, b=0,\\ c=1, d=e=f=0.\end{cases}$$

$$(R_{18})\begin{cases}\zeta_0=\sqrt{\frac{1}{3}}, a=b=0,\\ c\in\mathbb{R}, d=1-c,\\ e>\max\left\{\frac{1}{3}+\gamma^2\left((\frac{c+1}{2})^2-1\right),\gamma^2\frac{c^2}{4}\right\},\\ f=\frac{1}{3}-e+(c-1)\gamma^2.\end{cases}\quad (R_{19})\begin{cases}\zeta_0=\sqrt{\frac{1}{3}}, a=b=0,\\ c=e=0, d=1,\\ f=\frac{1}{3}-\gamma^2.\end{cases}$$

$$(R_{20})\begin{cases}\zeta_0=\sqrt{\frac{1}{3}}, a=b=0,\\ 0<c<2\sqrt{1-\frac{1}{3\gamma^2}}-1, d=1-c,\\ e=0, f=\frac{1}{3}+(c-1)\gamma^2.\end{cases}\quad (R_{21})\begin{cases}\zeta_0=\sqrt{\frac{1}{3}}, a=b=0,\\ c=1, e=\frac{1}{3}, d=f=0.\end{cases}$$

$$(R_{22})\begin{cases}\zeta_0=\sqrt{\frac{1}{3}}, a=b=0,\\ c_*<c<2\left(1+\frac{1}{\gamma}\sqrt{\frac{1}{3}}\right)\text{ where } c_*=\max\left\{1,2\left(1-\frac{1}{\gamma}\sqrt{\frac{1}{3}}\right)\right\},\\ d=1-c, e=\frac{1}{3}+(c-1)\gamma^2, f=0.\end{cases}$$

Remark 3.2. The proof is straightforward and is based on the same arguments presented in [2]. The subsets R_i are obtained after a lengthy calculation imposing the conditions $w_j(k)>0$, $j=1,\ldots,4$, and taking into account restrictions (2.8).

Remark 3.3. It is worth noticing that Theorem 3.1 establishes that when the parameters (ζ_0,a,b,c,d,e,f) lie in the subset R_i, the system (3.1) generates a bounded C^0 semigroup in $H^s\times H^{s-\Delta_i}$ defined by Equation (3.3).

3.2. Well-posedness of the nonlinear systems. In this section, we study the well-posedness of the initial value problem corresponding to the nonlinear systems (2.9) when the parameters belong to several of the subsets R_i presented in the previous section.

It is convenient to write the initial value problem for system (2.9) in the form

$$\mathbf{v}_t = \mathcal{L}\mathbf{v} + \mathbf{F}(\mathbf{v}), \qquad \mathbf{v}(0) = \mathbf{v}_0, \tag{3.5}$$

where $\mathbf{v}$ and $\mathcal{L}$ are defined in (3.2), $\mathbf{v}_0$ represents the initial condition and $\mathbf{F}$ is the nonlinear map given as

$$\mathbf{F}(\mathbf{v}) = \begin{pmatrix} \mathcal{B}_1(\eta u_0) \\ \mathcal{B}_2(u_0^2) \end{pmatrix}, \quad \text{where} \quad \widehat{\mathcal{B}_1 g}(k) = \frac{\alpha i k}{w_1(k)}\hat{g}(k), \quad \widehat{\mathcal{B}_2 g}(k) = \frac{\alpha i k}{2 w_2(k)}\hat{g}(k). \tag{3.6}$$

We say that the nonlinear system (3.5), associated with the subset R_i, is *weakly dispersive*, if there exists a number s_i^* such that for any $s > s_i^*$ the function $\mathbf{F}$ is locally Lipschitz continuous in $H^s \times H^{s-\Delta_i}$ (with respect to the norm $\|(v_1,v_2)\|_{H^s \times H^{s-\Delta_i}} := \|(v_1,v_2)\|_{s,s-\Delta_i} = \|v_1\|_s + \|v_2\|_{s-\Delta_i}$). We borrowed this definition from [3].

PROPOSITION 3.1. *Suppose that the parameters (ζ_0,a,b,c,d,e,f) lie in one of the subsets R_i where $i \in \{6,8,9,10,11,12,15,16,17\}$. Then the system (3.5), associated with any of those subsets R_i, is weakly dispersive when we set*

$$s_i^* = \max\left\{ \frac{\Delta_i}{2}, \Delta_i, \Delta_i + \frac{1}{2} + l_{i1}, \Delta_i + \frac{1}{2} + l_{i2} \right\},$$

where l_{i1} and l_{i2} represent the order of the operators $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. The values of s_i^* are presented in Table 3.1.

Proof. We sketch the proof for the subset R_9, the other cases can be handled using similar arguments. In this case, $\Delta_i = 0$ and $l_{i1} = l_{i2} = -1$, so $s_i^* = 0$. Notice that $s - 1 \le s$, also, if $s > 0$ then $s - 1 < 2s - 1/2$. Then, applying theorem C.10 in [1, p. 472], for any $\eta_1,\eta_2,u_{01},u_{02} \in H^s$ we get that

$$\begin{aligned}
\|\mathcal{B}_1(\eta_1 u_{01} - \eta_2 u_{02})\|_s &\le C_1 \|\eta_1 u_{01} - \eta_2 u_{02}\|_{s-1} \\
&\le C_1 \left(\|u_{01}(\eta_1 - \eta_2)\|_{s-1} + \|(u_{01} - u_{02})\eta_2\|_{s-1}\right) \\
&\le C_2 \left(\|u_{01}\|_s \|\eta_1 - \eta_2\|_s + \|u_{01} - u_{02}\|_s \|\eta_2\|_s\right) \\
&\le C_2 \max\{\|u_{01}\|_s, \|\eta_2\|_s\} \left(\|\eta_1 - \eta_2\|_s + \|u_{01} - u_{02}\|_s\right) \\
\|\mathcal{B}_2(u_{01}^2 - u_{02}^2)\|_s &\le C_1 \|u_{01}^2 - u_{02}^2\|_{s-1} \le C_2 \|u_{01} + u_{02}\|_s \|u_{01} - u_{02}\|_s \\
&\le 2C_2 \max\{\|u_{01}\|_s, \|u_{02}\|_s\} \|u_{01} - u_{02}\|_s.
\end{aligned} \tag{3.7}$$

From these estimates, the Lipschitz continuity of $\mathbf{F}$ on the open balls $B_R \subset H^s \times H^s$ of radius $R > 0$ centered at zero immediately follows. □

Now, we can formulate our main result.

THEOREM 3.2 (Well-posedness of weakly dispersive systems). *Suppose that the parameters (ζ_0,a,b,c,d,e,f) lie in one of the subsets R_i considered in proposition 3.1. Then for any $s \ge \max\{0,\Delta_i\}$ satisfying $s > s_i^*$, the initial value problem (3.5) is locally well-posed in $H^s \times H^{s-\Delta_i}$. More specifically, there exist a maximal $T > 0$ and a unique $\mathbf{v} = (\eta, u_0) \in C([0,T]; H^s \times H^{s-\Delta_i})$ that solves problem (3.5).*

Proof. The Duhamel's formulation for Equation (3.5) is given by

$$\mathbf{v}(t) = e^{t\mathcal{L}}\mathbf{v}_0 + \int_0^t e^{(t-t')\mathcal{L}} \mathbf{F}(\mathbf{v}(t'))\, dt', \tag{3.8}$$

where $\mathbf{v}_0=\mathbf{v}(0)$ and $e^{t\mathcal{L}}$ represents the bounded C^0 semigroup defined by (3.3) (see Remark 3.3). Let $C_0\ge 1$ be a bound of the semigroup norm, i.e $\|e^{t\mathcal{L}}\|\le C_0$ for any $t\ge 0$, consider $R=2C_0\|\mathbf{v}_0\|_{s,s-\Delta_i}$ and $T>0$ an arbitrary number. Let us define the map $\mathbf{u}\mapsto \mathbf{A}\mathbf{u}$ as

$$(\mathbf{A}\mathbf{u})(t)=e^{t\mathcal{L}}\mathbf{v}_0+\int_0^t e^{(t-t')\mathcal{L}}\mathbf{F}(\mathbf{u}(t'))\,dt',\quad t\in[0,T],$$

for any $\mathbf{u}\in X_T=C([0,T];\overline{B_R})$.

It is well-known that X_T is a complete metric space with respect to the distance $d(\mathbf{u}_1,\mathbf{u}_2)=\max_{t\in[0,T]}\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{s,s-\Delta_i}$. For $0\le t_1<t_2\le T$ and $\mathbf{u}\in X_T$, we get

$$\begin{aligned}&\|(\mathbf{A}\mathbf{u})(t_1)-(\mathbf{A}\mathbf{u})(t_2)\|_{s,s-\Delta_i}\\ &\le\int_0^{t_1}\|(e^{(t_2-t_1)\mathcal{L}}-\mathbf{I})e^{(t_1-t')\mathcal{L}}\mathbf{F}(\mathbf{u}(t'))\|_{s,s-\Delta_i}\,dt'+\int_{t_1}^{t_2}\|e^{(t_2-t')\mathcal{L}}\mathbf{F}(\mathbf{u}(t'))\|_{s,s-\Delta_i}\,dt'.\end{aligned}$$

From the above estimate, using basic properties of the semigroup, the boundedness of $e^{(t_i-t')\mathcal{L}}\mathbf{F}(\mathbf{u}(t'))$ and the dominated convergence theorem, we immediately get the continuity of the function $(\mathbf{A}\mathbf{u})(t)$. Moreover, we have

$$\begin{aligned}\|(\mathbf{A}\mathbf{u})(t)\|_{s,s-\Delta_i}&\le\|e^{t\mathcal{L}}\mathbf{v}_0\|_{s,s-\Delta_i}+\int_0^t\|e^{(t-t')\mathcal{L}}\mathbf{F}(\mathbf{u}(t'))\|_{s,s-\Delta_i}\,dt'\\ &\le C_0\|\mathbf{v}_0\|_{s,s-\Delta_i}+C_0C_F(R)RT=\frac{R}{2}+C_0C_F(R)RT,\end{aligned}$$

where $C_F(R)$ is a Lipschitz constant for $\mathbf{F}$ on the ball B_R. Consequently, for $T<(2C_0C_F(R))^{-1}$ one has that $\mathbf{A}$ maps X_T into itself.

Take $T\le(2C_0C_F(R))^{-1}$, then for any $\mathbf{u}_1,\mathbf{u}_2\in X_T$ and $t\in[0,T]$ we have

$$\begin{aligned}\|(\mathbf{A}\mathbf{u}_1)(t)-(\mathbf{A}\mathbf{u}_2)(t)\|_{s,s-\Delta_i}&\le\int_0^t\|e^{(t-t')\mathcal{L}}(\mathbf{F}(\mathbf{u}_1(t'))-\mathbf{F}(\mathbf{u}_2(t')))\|_{s,s-\Delta_i}\,dt'\\ &\le C_0C_F(R)T\,d(\mathbf{u}_1,\mathbf{u}_2)<\frac{1}{2}d(\mathbf{u}_1,\mathbf{u}_2).\end{aligned}$$

Therefore, the map $\mathbf{A}$ is a contraction on X_T and from the Banach fixed point theorem follows the local existence of a solution. This solution can be extended to a maximal solution on the interval $[0,T_{\mathbf{v}_0})$. Moreover, using (3.8) and Gronwall's inequality one can readily obtain its uniqueness. □

4. Final remarks

In this work, we presented a new family of reduced equations which is asymptotically equivalent to the system obtained in [5] for weakly nonlinear internal waves. The system of equations depends on four independent parameters. The analysis of the linearized equations allowed us to identify several subfamilies, which correspond to different subsets of the parameters, whose solutions are compatible with the solutions of the system of Euler equations.

Moreover, for the class of weakly dispersive models, we presented a result on the local well-posedness for the nonlinear systems. The study of the well-posedness for other systems not included in this class will be pursued elsewhere.

We remark that one of the main advantages of this new four-parameter family is the freedom to choose well-posed reduced models asymptotically equivalent to the Euler system. Moreover, this freedom can also be exploited to develop appropriate schemes for numerical simulations. Furthermore, the equations are stated in terms of the horizontal fluid velocity corresponding to a fixed (but arbitrarily chosen) depth, which give more flexibility than the depth-averaged velocity employed for instance in [5, 6, 12]. This flexibility represents an advantage regarding possible engineering applications as the former quantity can be easier to measure than the later.

REFERENCES

[1] S. Benzoni-Gavage and D. Serre, *Multidimensional Hyperbolic Partial Differential Equations*, Oxford University Press, New York, 2007.

[2] J.L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I: Derivation and linear Theory*, J. Nonlinear Sci., 12:283–318, 2002.

[3] J.L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media: II. The nonlinear theory*, Nonlinearity, 17:925–952, 2004.

[4] J.L. Bona, D. Lannes, and J.-C. Saut, *Asymptotic models for internal waves*, J. Math. Pures Appl., 89:538–566, 2008.

[5] W. Choi and R. Camassa, *Weakly nonlinear internal waves in a two-fluid system*, J. Fluid Mech., 313:83–103, 1996.

[6] W. Choi and R. Camassa, *Fully nonlinear Internal waves in two-fluid system*, J. Fluid Mech., 396:1–36, 1999.

[7] W. Craig, P. Guyenne, and H. Kalisch, *Hamiltonian long-wave expansions for free surfaces and interfaces*, Comm. Pure Appl. Math., 58:1587–1641, 2005.

[8] S. Debsarma, K.P. Das, and J.T. Kirby, *Fully nonlinear higher-order model equations for long internal waves in a two-fluid system*, J. Fluid Mech., 654:281–303, 2010.

[9] J.M. Grajales, *Existence and numerical approximation of solutions of an improved internal wave model*, Math. Model. Anal., 19:309–333, 2014.

[10] R. Iorio and V. Iorio, *Fourier Analysis and Partial Differential Equations*, Cambridge University Press, New York, 2001.

[11] O. Nwogu, *An alternative form of Boussinesq equations for nearshore wave propagation*, J. Waterway, Port, Coastal, Ocean Eng., 119:618–638, 1993.

[12] A. Ruiz de Zárate and A. Nachbin, *A reduced model for internal waves interacting with topography at intermediate depth*, Commun. Math. Sci., 6:385–396, 2008.

[13] A. Ruiz de Zárate, D.G. Alfaro Vigo, A. Nachbin, and W. Choi, *A higher-order internal wave model accounting for large bathymetric variations*, Stud. Appl. Math., 122:275–294, 2009.

[14] G.B. Whitham, *Linear and nonlinear waves*, John Wiley & Sons, New York, 1999.

[15] L. Xu, *Intermediate long waves systems for internal waves*, Nonlinearity, 25(3):597–640, 2012.