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THE GENERALIZED RIEMANN PROBLEM AND INSTABILITY OF DELTA SHOCK TO THE CHROMATOGRAPHY EQUATIONS*

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Abstract. The generalized Riemann problem for the nonlinear chromatography equations in a neighborhood of the origin $(t>0)$ on the (x,t) plane is considered. The problem is quite different from the previous generalized Riemann problems which have no delta shock wave in the corresponding Riemann solutions. With the method of characteristic analysis and the local existence and uniqueness theorem proposed by Li Ta-tsien and Yu Wen-ci [T.T. Li and W.C. Yu, Duke Univ. Math. Ser. V, Durham, NC, 1985], we constructively solve the generalized Riemann problem and prove the existence and uniqueness of the solutions. It is proved that the generalized Riemann solutions possess a structure similar to the solution of the corresponding Riemann problem for most cases. In case that there is a delta shock wave in the corresponding Riemann solution, we discover that the generalized Riemann solution may turn into a combination of a shock wave and a contact discontinuity, which shows the instability and the internal mechanisms of a delta shock wave.

Keywords. Chromatography equations, generalized Riemann problem, delta shock wave, instability, entropy condition.

AMS subject classifications. 35L65, 35L67, 76L05, 76N10.

1. Introduction

The nonlinear chromatography equations can be expressed as

$$\begin{cases} \dfrac{\partial u}{\partial t}+\dfrac{\partial}{\partial x}\left\{\left(1+\dfrac{1}{1-u+v}\right)u\right\}=0, \\ \dfrac{\partial v}{\partial t}+\dfrac{\partial}{\partial x}\left\{\left(1+\dfrac{1}{1-u+v}\right)v\right\}=0, \end{cases} \tag{1.1}$$

where u and v are non-negative functions of variables $(x,t)\in R\times R^+$, which express the concentrations of the two absorbing species, and $1-u+v>0$. The equations are not only a common analytical tool but are also used to study the preparative separations in the pharmaceutical, food, and agrochemical industries. Yang and Zhang [19], Cheng and Yang [6] studied the Riemann problem of Equation (1.1) and proved the existence and uniqueness of the solution. With the method of the splitting delta function, Guo, Pan and Yin [9] discussed the perturbed Riemann problem for the nonlinear chromatography equations (1.1). Equations (1.1) can be derived from a more general nonlinear chromatography system

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial}{\partial t}\left\{\left(1+\dfrac{a_1}{1-u+v}\right)u\right\}=0, \\ \dfrac{\partial v}{\partial x}+\dfrac{\partial}{\partial t}\left\{\left(1+\dfrac{a_2}{1-u+v}\right)v\right\}=0, \end{cases} \tag{1.2}$$

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where a_1 and a_2 are constants with $a_2 > a_1 > 0$; u and v are the non-negative functions of the variables $(x,t) \in R \times R^+$, and $1-u+v>0$.

A distinctive [1,5,16] feature for Equations (1.1) and (1.2) is that the delta shock wave with Dirac delta function in both u and v will appear in solutions [6]. This fact was also captured numerically and experimentally by Mazzotti et al. [13,14] for (1.2). This delta shock phenomenon originates in the synergistic-competitive behavior of the two species as described in [6].

Another system of nonlinear chromatography equations was introduced in [1,5,16]. The model reads

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x}\left(u + \dfrac{u}{1+u+v}\right) = 0, \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}\left(v + \dfrac{v}{1+u+v}\right) = 0, \end{cases} \tag{1.3}$$

where $u(x,t) \geq 0$, $v(x,t) \geq 0$ express transformations of the concentrations of two solutes. System (1.3) is widely used by chemists and engineers to study the separation of two chemical components in a fluid phase. Unlike in systems (1.1) and (1.2), the delta shock wave does not develop in the solutions of system (1.3) when $u(x,t) \geq 0$, $v(x,t) \geq 0$, see [1,5,16].

Recently, Ambrosio et al. [1] introduced the change of variables

$$\theta = u - v, \qquad \eta = u + v, \tag{1.4}$$

then system (1.3) can be changed to

$$\begin{cases} \dfrac{\partial \theta}{\partial t} + \dfrac{\partial}{\partial x}\left(\theta + \dfrac{\theta}{1+\eta}\right) = 0, \\ \dfrac{\partial \eta}{\partial t} + \dfrac{\partial}{\partial x}\left(\eta + \dfrac{\eta}{1+\eta}\right) = 0, \end{cases} \tag{1.5}$$

where $\eta \geq 0$. Because of the conditions $u \geq 0$ and $v \geq 0$, the change of variables (1.4) is not one-on-one, which implies that system (1.3) and system (1.5) are not equivalent. The existence and uniqueness of solutions to (1.5) were proved by employing the self-similar viscosity vanishing approach in [18]. The delta shock wave appears in the Riemann solution of (1.5). However, for this kind of delta shock waves, only one state variable θ contains the Dirac delta function and the other state variable η has bounded variation. Han and Pan discussed the formation, transition and instability of delta shock waves to the chromatography equations (1.5) in [10].

From the above discussion, one can observe that the essential difference among the nonlinear chromatography equations (1.1), (1.3) and (1.5) is the coefficients of the absorbing species. Consequently, the structures of solutions for these nonlinear chromatography equations are quite different, in which the delta shock wave plays an important role.

The theory of the delta shock wave has been intensively developed in the last twenty years. The delta shock wave solution and the corresponding Rankine–Hugoniot condition were presented by Zeldovich and Myshkis [20] in the case of the continuity equation. In 1999, Sheng and Zhang [17] discussed the Riemann problem for the zero-pressure gas dynamics, in which a delta shocks appear. The previous investigations mostly focused on the case that only one state variable develops the Dirac delta function and the others have bounded variations. In 2012, Yang and Zhang [19] established a new theory of delta shock waves with Dirac delta functions developing in two state variables for a class of nonstrictly hyperbolic systems of conservation laws. There are numerous excellent papers on delta shock waves, for the related references one can see [17–19] and the references cited therein. We specially mention that, in a series of papers [2–4, 7, 8], Bouchut et al. studied the zero-pressure gas dynamics in different techniques.

In the delta shock wave theories, there are still many open and complicated problems. Research of this area gives a new perspective in the theory of conservation law systems. The aim of this study in this paper is to analyze the internal mechanism and instability of a delta shock wave. We study the nonlinear chromatography equations (1.1) with the following initial data

$$(u,v)\Big|_{t=0}=(u_0(x),v_0(x))=\begin{cases}(u_0^-(x),v_0^-(x)),\ x<0,\\(u_0^+(x),v_0^+(x)),\ x>0,\end{cases}\tag{1.6}$$

where $u_0^\pm(x)$ and $v_0^\pm(x)$ are all bounded C^1 functions with the following property

$$(u_0^\pm(0\pm),v_0^\pm(0\pm))=(\hat{u}^\pm,\hat{v}^\pm).\tag{1.7}$$

Here $\hat{u}^\pm$ and $\hat{v}^\pm$ are constants with $(\hat{u}^-,\hat{v}^-)\neq(\hat{u}^+,\hat{v}^+)$. The initial value (1.6) is a perturbation of Riemann initial value (2.1) at the neighborhood of the origin in the $x-t$ plane. So we call the initial value problem (1.1) with (1.6) a generalized Riemann problem.

It is natural and important to study the Cauchy problem (1.1) with initial data (1.6). For example, error is unavoidable in computation and the error forms a perturbation of the initial data. The interesting question, in this paper, is to discuss whether the generalized Riemann solutions of problem (1.1),(1.6) possess a structure similar to the corresponding Riemann solutions of problem (1.1),(2.1).

Our results show that in a neighborhood of the origin, the Riemann solutions are able to retain their forms after the perturbation of Riemann initial data, if there are only classical elementary waves in the corresponding Riemann solutions. However, when a delta shock wave appears in the corresponding Riemann solution, the perturbation may bring essential change. A distinctive feature of this problem is that the Riemann solution has no local structure stability with respect to the above perturbation. We pay more attention to the differences between Riemann solution and generalized Riemann solution. However, the previous works [11,12] are about the stability of the corresponding Riemann solution.

It is difficult to solve the generalized Riemann problem (1.1),(1.6) because there is a delta shock wave in the corresponding Riemann solutions. However, in the previous work [11, 12] on the generalized Riemann problem, no delta shock wave appears in the corresponding Riemann solutions. Using the method of characteristic analysis and the local existence and uniqueness theorem proposed by Li Ta-tsien and Yu Wen-ci [12], we derive a condition which is used as a criterion to detect the instability of the delta shock wave. If the generalized Riemann initial datum (1.6) satisfies the above condition, the Riemann initial perturbation has no essential influence on the delta shock wave. Furthermore, we analyze some properties of the delta shock wave curve. Conversely, if the condition fails to hold, we prove that a delta shock wave turns into a shock wave and a contact discontinuity, which allows us to better investigate the internal mechanism of a delta shock wave. Finally, we analyze the instability property of the solution of the nonlinear chromatography equation due to the reasonable perturbation on the Riemann initial data.

The paper is organized as follows. In Section 2, we present some preliminary knowledge about the nonlinear chromatography equations (1.1). The construction and proof of the generalized Riemann solutions to problem (1.1),(1.6) are presented in Section 3.

2. Preliminaries

Consider system (1.1) with Riemann initial data

$$(u,v)(x,0)=(\hat{u}^{\pm},\hat{v}^{\pm}) \qquad \pm x>0, \tag{2.1}$$

where $\hat{u}^{\pm}$ and $\hat{v}^{\pm}$ are constants with $(\hat{u}^{-},\hat{v}^{-})\neq(\hat{u}^{+},\hat{v}^{+})$, see [6, 19] for a more detailed study of the model.

The eigenvalues of the chromatography equations (1.1) are

$$\lambda_1(u,v)=1+\frac{1}{1-u+v}, \qquad \lambda_2(u,v)=1+\frac{1}{(1-u+v)^2}. \tag{2.2}$$

The corresponding left eigenvectors are

$$l_1(u,v)=(v,-u), \qquad l_2(u,v)=(-1,1), \tag{2.3}$$

and the right eigenvectors are

$$r_1(u,v)=(1,1)^T, \qquad r_2(u,v)=(u,v)^T. \tag{2.4}$$

From Equations (2.2) and (2.4), we have

$$\nabla\lambda_1\cdot r_1=0, \qquad \nabla\lambda_2\cdot r_2=\frac{-2(-u+v)}{(1-u+v)^3}. \tag{2.5}$$

Therefore λ_1 is always linearly degenerate, λ_2 is genuinely nonlinear if $u\neq v$, and linearly degenerate if $u=v$. From Equation (2.2), we notice that system (1.1) is no longer strictly hyperbolic in the region of the (u,v) plane where $u=v$.

For the chromatography equations (1.1), the Riemann invariants along the characteristic fields are

$$\zeta(u,v)=-u+v, \qquad \varsigma(u,v)=\frac{v}{u}. \tag{2.6}$$

Definition 2.1 ([15, 19]). *A pair of (u,v) is called a generalized delta shock wave solution to (1.1) with the initial data (1.6) on $[0,T)$, if there exists a smooth curve $l=\{(x_\delta(t),t):0\le t<T\}$ and a weight $\omega(x,t)$ such that u and v are represented in the following forms*

$$u=U(x,t)+\omega(x,t)\delta(l), \qquad v=V(x,t)+\omega(x,t)\delta(l), \tag{2.7}$$

in which $\delta(x)$ is the delta function, $\omega\in C^1(l)$, $U,V\in L^\infty(R\times[0,T);R)$ and satisfies

$$\begin{aligned}\int_0^T\int_{-\infty}^{+\infty}&\left(U\phi_t+\left(U+\frac{U}{1-U+V}\right)\phi_x\right)dxdt\\ &+\int_0^T\omega(x_\delta(t),t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt+\int_{-\infty}^{+\infty}u_0(x)\phi(x,0)dx=0,\end{aligned} \tag{2.8}$$

$$\int_0^T\int_{-\infty}^{+\infty}\left(V\phi_t+\left(V+\frac{V}{1-U+V}\right)\phi_x\right)dxdt$$

$$+\int_0^T \omega(x_\delta(t),t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt+\int_{-\infty}^{+\infty} v_0(x)\phi(x,0)dx=0, \tag{2.9}$$

for all the test functions $\phi\in C_0^\infty((-\infty,+\infty)\times[0,T))$. *Here* σ_δ *is the tangential derivative of the curve* l, *and* $\dfrac{\partial\phi(x,t)}{\partial l}$ *stands for the tangential derivative of the function* ϕ *on the curve* l.

The Riemann solutions for problem (1.1),(2.1) are divided into the following cases through the following conditions [6]:

(1) When $0<-\hat{u}^-+\hat{v}^-<-\hat{u}^++\hat{v}^+$, the solution is $\overleftarrow{S}+J$;

(2) When $-\hat{u}^-+\hat{v}^-\le 0\le -\hat{u}^++\hat{v}^+$, the solution is delta shock wave δS;

(3) When $-\hat{u}^-+\hat{v}^-<-\hat{u}^++\hat{v}^+<0$, the solution is $J+\overrightarrow{S}$;

(4) When $-\hat{u}^++\hat{v}^+<0<-\hat{u}^-+\hat{v}^-$, the solution is $\overleftarrow{R}_1+\overrightarrow{R}_2$;

(5) When $0\le -\hat{u}^++\hat{v}^+<-\hat{u}^-+\hat{v}^-$, the solution is $\overleftarrow{R}+J$;

(6) When $-\hat{u}^++\hat{v}^+<-\hat{u}^-+\hat{v}^-\le 0$, the solution is $J+\overrightarrow{R}$.

Here "+" means "followed by"; the capitals S, J and R denote shock wave, contact discontinuity and rarefaction wave, respectively.

Definition 2.2. *For an* $n\times n$ *matrix* $H=(a_{ij})$, *define*

$$\| H \| = \underset{i=1,\cdots,n}{\mathrm{Max}} \sum_{i=1}^n |a_{ij}|$$

and

$$\| H \|_{min} = \inf\{\|\gamma H\gamma^{-1}\|;\gamma=diag\{\gamma_i\},\gamma_i\neq 0,i=1,\cdots,n\}.$$

Lemma 2.1 ([12]). *If* A *is a* 2×2 *matrix of form*

$$\begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix},$$

then

$$\| A \|_{min} = \sqrt{|ab|}.$$

Let

$$D(\epsilon)=\{(x,t)\,|\,x_1(t)\le x\le x_2(t),0\le t<\epsilon\}, \tag{2.10}$$

where $x=x_1(t)$ and $x=x_2(t)$ are given or unknown C^2 curves with

$$x_1(0)=x_2(0)=0, \qquad x_1'(0)<x_2'(0).$$

On an angular domain (2.10), consider the free boundary problem (1.1) with boundary conditions as follows:

on $x=x_1(t)$,

$$-u^*(x_1(t),t)+v^*(x_1(t),t)=F(x_1(t),t,\hat{v}^*u^*(x_1(t),t)-\hat{u}^*v^*(x_1(t),t)), \tag{2.11}$$

and

$$\frac{dx_1(t)}{dt} = f(x_1(t),t,u^*(x_1(t),t),v^*(x_1(t),t)); \tag{2.12}$$

on $x = x_2(t)$,

$$\hat{v}^* u^*(x_2(t),t) - \hat{u}^* v^*(x_2(t),t) = G(x_2(t),t,-u^*(x_2(t),t)+v^*(x_2(t),t)), \tag{2.13}$$

and

$$\frac{dx_2(t)}{dt} = g(x_2(t),t,u^*(x_2(t),t),v^*(x_2(t),t)), \tag{2.14}$$

where F, f, G and g are known C^1 functions, $u^*(x,t)$, $v^*(x,t)$ are unknown functions on $D(\epsilon)$. $\hat{u}^* = u^*(0,0)$ and $\hat{v}^* = v^*(0,0)$ can be determined uniquely from boundary conditions (2.11) and (2.13). We assume that $\lambda_1 < \lambda_2$ (the similar discussion can be done for $\lambda_1 > \lambda_2$) in this section. Set

$$\begin{pmatrix} \mathcal{U}(x,t) \\ \mathcal{V}(x,t) \end{pmatrix} = \begin{pmatrix} l_1(\hat{u}^*,\hat{v}^*) \\ l_2(\hat{u}^*,\hat{v}^*) \end{pmatrix} \begin{pmatrix} u^*(x,t) \\ v^*(x,t) \end{pmatrix} = \begin{pmatrix} \hat{v}^* & -\hat{u}^* \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u^*(x,t) \\ v^*(x,t) \end{pmatrix}. \tag{2.15}$$

The boundary conditions (2.11) on $x = x_1(t)$ and (2.13) on $x = x_2(t)$ can be rewritten, respectively, as on $x = x_1(t)$,

$$\mathcal{V}(x_1(t),t) = \mathcal{F}(x_1(t),t,\mathcal{U}(x_1(t),t)), \tag{2.16}$$

and on $x = x_2(t)$,

$$\mathcal{U}(x_2(t),t) = \mathcal{G}(x_2(t),t,\mathcal{V}(x_2(t),t)). \tag{2.17}$$

The characterizing matrix A of this free boundary problem is of the form

$$A = \begin{pmatrix} 0 & \mathcal{G}_3' \\ \mathcal{F}_3' & 0 \end{pmatrix}, \tag{2.18}$$

where $\mathcal{F}_3'$ and $\mathcal{G}_3'$ represent the values at the origin of the first derivatives of $\mathcal{F}$ and $\mathcal{G}$ with respect to their third argument, respectively.

LEMMA 2.2. *([12]) Assume that*

(1) $\lambda_1(u^*(x_1(t),t),v^*(x_1(t),t)) \le f(x_1(t),t,u^*(x_1(t),t),v^*(x_1(t),t))$,

(2) $\lambda_2(u^*(x_2(t),t),v^*(x_2(t),t)) \ge g(x_2(t),t,u^*(x_2(t),t),v^*(x_2(t),t))$,

(3) $f(0,0,\hat{u}^*,\hat{v}^*) < \lambda_2(\hat{u}^*,\hat{v}^*)$,

(4) $f(0,0,\hat{u}^*,\hat{v}^*) < g(0,0,\hat{u}^*,\hat{v}^*)$,

if

$$\| A \|_{min} < 1,$$

the free boundary problem (1.1) *with boundary conditions* (2.11) $\sim$ (2.14) *admits a unique solution on a shaped domain* $R(\epsilon)$, *where* $\epsilon > 0$ *is sufficiently small.*

3. The generalized Riemann problem and delta shock wave

In this section, we turn our efforts to study the instability of the delta shock wave. Thus, this section is devoted to the description of the generalized Riemann problem for the chromatography equations (1.1) with initial data (1.6). We pay more attention to the differences between the above generalized Riemann solutions and the corresponding Riemann solutions of problem (1.1),(2.1).

We now seek the solution to the generalized Riemann problem (1.1),(1.6). Using the method of characteristics, the classical solutions $(u^-(x,t),v^-(x,t))$ and $(u^+(x,t),v^+(x,t))$ can be defined in strip domains D_- and D_+ for local time, respectively, see Figure 3.1. Here the local smooth solutions$(u^-(x,t),v^-(x,t))$ and $(u^+(x,t),v^+(x,t))$ are given by solving the initial value problem (1.1) with initial data $(u_0^-(x),v_0^-(x))$ and $(u_0^+(x),v_0^+(x))$ on both sides of $x=0$, respectively.

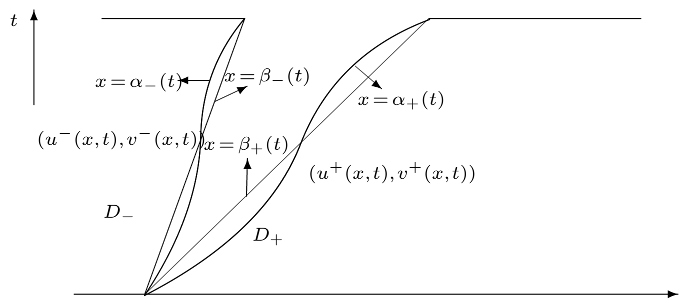


Fig. 3.1.

In the case $-\hat{u}^-+\hat{v}^-<0$, the right boundary of domain D_- is a $I-$characteristic $x=\alpha_-(t)$, namely

$$\begin{cases} \dfrac{v^-(\alpha_-(t),t)}{u^-(\alpha_-(t),t)}=\dfrac{\hat{v}^-}{\hat{u}^-}, \\ \dfrac{d\alpha_-(t)}{dt}=1+\dfrac{1}{1-u^-(\alpha_-(t),t)+v^-(\alpha_-(t),t)}. \end{cases} \tag{3.1}$$

However, in the case of $-\hat{u}^-+\hat{v}^->0$, the right boundary of domain D_- is a straight $II-$characteristic $x=\beta_-(t)$, namely,

$$\begin{cases} -u^-(\beta_-(t),t)+v^-(\beta_-(t),t)=-\hat{u}^-+\hat{v}^-, \\ \dfrac{d\beta_-(t)}{dt}=1+\dfrac{1}{(1-u^-(\beta_-(t),t)+v^-(\beta_-(t),t))^2}. \end{cases} \tag{3.2}$$

Similarly, in the case of $-\hat{u}^++\hat{v}^+<0$, the left boundary of domain D_+ is a straight $II-$characteristic $x=\beta_+(t)$, namely,

$$\begin{cases} -u^+(\beta_+(t),t)+v^+(\beta_+(t),t)=-\hat{u}^++\hat{v}^+, \\ \dfrac{d\beta_+(t)}{dt}=1+\dfrac{1}{(1-u^+(\beta_+(t),t)+v^+(\beta_+(t),t))^2}. \end{cases} \tag{3.3}$$

However, in the case of $-\hat{u}^++\hat{v}^+>0$, the left boundary of domain D_+ is a

$I-$characteristic $x=\alpha_+(t)$, namely,

$$\begin{cases} \dfrac{v^+(\alpha_+(t),t)}{u^+(\alpha_+(t),t)}=\dfrac{\hat{v}^+}{\hat{u}^+}, \\ \dfrac{d\alpha_+(t)}{dt}=1+\dfrac{1}{1-u^+(\alpha_+(t),t)+v^+(\alpha_+(t),t)}. \end{cases} \tag{3.4}$$

It is clear that the construction of the generalized Riemann solution for problem (1.1),(1.6) between the right boundary of domain D_- and the left boundary of domain D_+ is determined completely by the corresponding Riemann problem near the origin. According to the solutions to the corresponding Riemann problem (1.1),(2.1), we consider six different cases.

Case 1: $0<-\hat{u}^-+\hat{v}^-<-\hat{u}^++\hat{v}^+$**.**

As shown in Figure 3.2(a), the solution of the Riemann problem (1.1),(2.1) consists of a backward shock wave $\overleftarrow{S}$ from $(\hat{u}^-,\hat{v}^-)$ to $(\hat{u}^*,\hat{v}^*)$, a contact discontinuity J from $(\hat{u}^*,\hat{v}^*)$ to $(\hat{u}^+,\hat{v}^+)$. Here the backward shock wave is $O\hat{A}_1:\ x=(1+\dfrac{1}{(1-\hat{u}^-+\hat{v}^-)(1-\hat{u}^++\hat{v}^+)})t$, the contact discontinuity is $O\hat{A}_2:\ x=(1+\dfrac{1}{1-\hat{u}^++\hat{v}^+})t$ and the intermediate state is $(\hat{u}^*,\hat{v}^*)=(\dfrac{-\hat{u}^++\hat{v}^+}{-\hat{u}^-+\hat{v}^-}\hat{u}^-,\dfrac{-\hat{u}^++\hat{v}^+}{-\hat{u}^-+\hat{v}^-}\hat{v}^-)$.

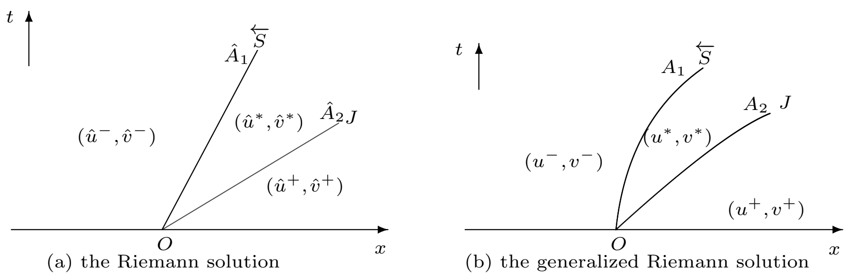


Fig. 3.2. *Case 1.*

In view of the above Riemann solution, we will prove that the generalized Riemann problem (1.1) and (1.6) admits a unique solution locally in time as shown in Figure 3.2(b). Here the backward shock wave

$$OA_1: x=x_s(t) \quad (x_s(0)=0)$$

and the contact discontinuity

$$OA_2: x=x_c(t) \quad (x_c(0)=0)$$

are free boundaries. On the left boundary curve $x=x_s(t)$, we have

$$\frac{dx_s(t)}{dt}=1+\frac{1}{(1-u^-+v^-)(1-u^*+v^*)}, \tag{3.5}$$

$$u^-v^*=u^*v^-. \tag{3.6}$$

On the right boundary curve $x=x_c(t)$, we have

$$\frac{dx_c(t)}{dt}=1+\frac{1}{1-u^*+v^*}, \tag{3.7}$$

$$-u^* + v^* = -u^+ + v^+. \tag{3.8}$$

The generalized Riemann solution to problem (1.1),(1.6) is $(u^-(x,t), v^-(x,t))$ on the domain $\{(x,t) \,|\, x < x_s(t), 0 \le t < \epsilon\}$, $\epsilon > 0$ small. The generalized Riemann solution is $(u^+(x,t), v^+(x,t))$ on the domain $\{(x,t) \,|\, x > x_c(t), 0 \le t < \epsilon\}$. However, the generalized Riemann solution to problem (1.1),(1.6) is unknown on the domain $\{(x,t) \,|\, x_s(t) < x < x_c(t), 0 \le t < \epsilon\}$. We denoted it by $(u^*(x,t), v^*(x,t))$. Moreover, the value

$$(u^*(0,0), v^*(0,0)) = (\hat{u}^*, \hat{v}^*) = \Big(\frac{-\hat{u}^+ + \hat{v}^+}{-\hat{u}^- + \hat{v}^-}\hat{u}^-, \frac{-\hat{u}^+ + \hat{v}^+}{-\hat{u}^- + \hat{v}^-}\hat{v}^-\Big), \tag{3.9}$$

which is determined uniquely from boundary conditions (3.6) and (3.8).

By using the fact that $(u^-(x,t), v^-(x,t))$ and $(u^+(x,t), v^+(x,t))$ are known smooth functions, the generalized Riemann problem (1.1),(1.6) is equivalent to the free boundary problem (1.1) with boundary conditions (3.5)$\sim$(3.8) on the fan-shaped domain $\{(x,t) \,|\, x_s(t) < x < x_c(t), 0 \le t < \epsilon\}$, $\epsilon > 0$ small. We turn our attention now to the above free boundary problem.

We shall rewrite the boundary conditions (3.6) and (3.8) by introducing the variables

$$\begin{pmatrix} \mathcal{U}(x,t) \\ \mathcal{V}(x,t) \end{pmatrix} = \begin{pmatrix} l_2(\hat{u}^*, \hat{v}^*) \\ l_1(\hat{u}^*, \hat{v}^*) \end{pmatrix} \begin{pmatrix} u^*(x,t) \\ v^*(x,t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ \hat{v}^* & -\hat{u}^* \end{pmatrix} \begin{pmatrix} u^*(x,t) \\ v^*(x,t) \end{pmatrix}. \tag{3.10}$$

On $x = x_s(t)$, the boundary condition (3.6) then reduces to

$$\mathcal{V} = \frac{(u^-\hat{v}^* - v^-\hat{u}^*)\mathcal{U}}{v^- - u^-}. \tag{3.11}$$

On $x = x_c(t)$, the boundary condition (3.8) can be written as

$$\mathcal{U} = -u^+ + v^+. \tag{3.12}$$

From Equations (3.11) and (3.12), we get the characterizing matrix A of the above free boundary problem [12]:

$$A = \begin{pmatrix} 0 & 0 \\ \dfrac{u^-\hat{v}^* - v^-\hat{u}^*}{v^- - u^-} & 0 \end{pmatrix}. \tag{3.13}$$

From Lemma 2.2, if the minimal characterizing number $\| A \|_{min} < 1$, the free boundary problem under consideration admits a unique piecewise smooth solution on the fan-shaped domain $\{(x,t) \,|\, x_s(t) < x < x_c(t), 0 \le t < \epsilon\}$. By Lemma 2.1, it is not difficult to check

$$\| A \|_{min} = 0 < 1 \tag{3.14}$$

Thus there exists a unique C^1 solution to the free boundary problem locally in time.

Based on the above discussion, we know that the generalized solution has the desired structure which is a backward shock wave followed by a contact discontinuity. That is, the solution to the Riemann problem is stable near the origin after perturbations (1.6) of the Riemann initial data.

Case 2: $-\hat{u}^- + \hat{v}^- \le 0 \le -\hat{u}^+ + \hat{v}^+$.

The solution to the Riemann problem (1.1),(2.1) consists of a delta shock wave δS from $(\hat{u}^-,\hat{v}^-)$ to $(\hat{u}^+,\hat{v}^+)$. Here the speed of the delta shock wave is $1+\dfrac{1}{(1-\hat{u}^-+\hat{v}^-)(1-\hat{u}^++\hat{v}^+)}$, see Figure 3.3(a).

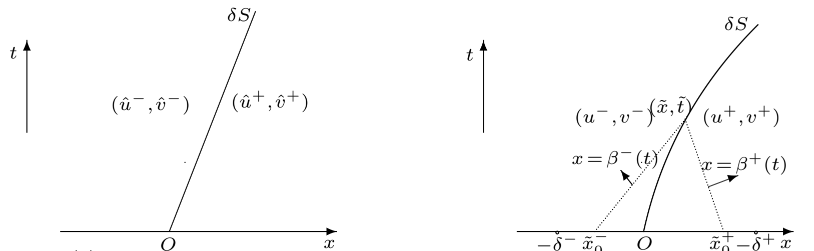


Fig. 3.3. *Subcase 2.1, Subcase 2.2 with $-\dot{u}_0^-(0)+\dot{v}_0^-(0)>0$ and Subcase 2.3 with $-\dot{u}_0^+(0)+\dot{v}_0^+(0)>0$.*

It is natural then to ask whether there is a delta shock wave in the solution of the generalized Riemann problem (1.1),(1.6). If so, we must choose u and v to satisfy

$$\begin{aligned} u &= u^+ + [u]H(-x+x_\delta(t)) + \omega(x_\delta(t),t)\delta(l), \\ v &= v^+ + [v]H(-x+x_\delta(t)) + \omega(x_\delta(t),t)\delta(l). \end{aligned} \tag{3.15}$$

Hereafter, we use the usual notation $[u]=u^- - u^+$ with u^- and u^+ the values of the function u on the left-hand and right-hand sides of the discontinuity $x=x_\delta(t)$ with $x_\delta(0)=0$. $H(x)$ is the Heaviside function that is 0 when $x<0$ and 1 when $x>0$. $\omega(x_\delta(t),t)$ and σ_δ are the weight and the tangential derivative of curve $l \triangleq \{(x_\delta(t),t): 0 \leqslant t < T\}$, which is given by

$$\begin{cases} \dfrac{d\sqrt{1+\sigma_\delta^2}\,\omega(x_\delta(t),t)}{dt} = -\sigma_\delta[u] + \left[\left(1+\dfrac{1}{1-u+v}\right)u\right], \\ \dfrac{d\sqrt{1+\sigma_\delta^2}\,\omega(x_\delta(t),t)}{dt} = -\sigma_\delta[v] + \left[\left(1+\dfrac{1}{1-u+v}\right)v\right], \\ \omega(0,0)=0. \end{cases} \tag{3.16}$$

By direct calculation, we find the propagating speed of the delta shock wave

$$\sigma_\delta = \frac{dx_\delta(t)}{dt} = 1 + \frac{1}{(1-u^-(x_\delta(t),t)+v^-(x_\delta(t),t))(1-u^+(x_\delta(t),t)+v^+(x_\delta(t),t))}. \tag{3.17}$$

In order to show the existence of the delta shock wave, we begin by a definition.

Definition 3.1. *If (u,v) satisfies Equation (3.15), then it is an admissible delta shock wave solution to the initial value problem (1.1),(1.6) in the sense of distributions on $[0,T)$, if (u,v) satisfies Definition 2.1 and the entropy condition*

$$\lambda_2(u^+,v^+) \le \lambda_1(u^+,v^+) \le \frac{dx_\delta(t)}{dt} \le \lambda_1(u^-,v^-) \le \lambda_2(u^-,v^-) \tag{3.18}$$

on the discontinuity $x=x_\delta(t)$.

The inequality (3.18) shows that all the four characteristic lines on both sides of the discontinuity $x=x_\delta(t)$ are not outgoing. According to Definition 3.1, we first check whether the delta shock wave solution (u,v) defined in Definition 3.1 satisfies problem (1.1),(1.6) in the sense of distributions on $[0,T)$. It is useful to state explicitly the following proposition.

PROPOSITION 3.1. *The delta shock wave solution (u,v) defined in Definition 3.1 satisfies problem (1.1),(1.6) in the sense of distributions on a domain $D(T)=\{(x,t)\,|\,-\infty<x<\infty, 0\le t<T\}$, where $T>0$ is a finite time.*

Proof. We define

$$U(x,t)=u^{+}+[u]H(-x+x_\delta(t)),\qquad V(x,t)=v^{+}+[v]H(-x+x_\delta(t)). \tag{3.19}$$

Then (3.15) implies

$$u=U(x,t)+\omega(x_\delta(t),t)\delta(l),\qquad v=V(x,t)+\omega(x_\delta(t),t)\delta(l). \tag{3.20}$$

Using (3.19) on the left-hand side of Equation (2.8), for any test function $\phi\in C_0^\infty((-\infty,+\infty)\times[0,T))$, it follows that

$$\begin{aligned}
&\int_0^T\int_{-\infty}^{+\infty}\Big(U\phi_t+\Big(U+\frac{U}{1-U+V}\Big)\phi_x\Big)dxdt\\
&\quad+\int_0^T\omega(x_\delta(t),t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt+\int_0^{+\infty}u_0(x)\phi(x,0)dx\\
&=\int_0^T\int_{-\infty}^{x_\delta(t)}(u^-\phi_t+\Big(u^-+\frac{u^-}{1-u^-+v^-}\Big)\phi_x)dxdtdxdt\\
&\quad+\int_0^T\int_{x_\delta(t)}^{+\infty}(u^+\phi_t+\Big(u^++\frac{u^+}{1-u^++v^+}\Big)\phi_x)dxdt\\
&\quad+\int_0^T\omega(x_\delta(t),t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt+\int_0^{+\infty}u_0(x)\phi(x,0)dx.
\end{aligned}$$

Since (u^-,v^-) is a C^1 solution to problem (1.1) with initial data $(u_0^-(x),v_0^-(x))$ in the domain D_-, using the divergence theorem, we have

$$\begin{aligned}
&\int_0^T\int_{-\infty}^{x_\delta(t)}(u^-\phi_t+\Big(u^-+\frac{u^-}{1-u^-+v^-}\Big)\phi_x)dxdt\\
&=\int_0^T\int_{-\infty}^{x_\delta(t)}((u^-\phi)_t+(\Big(u^-+\frac{u^-}{1-u^-+v^-}\Big)\phi)_x)dxdt\\
&=-\int_{-\infty}^0u_0^-(x)\phi(x,0)dx+\int_0^T\phi(x_\delta(t),t)(-u^-(x_\delta(t),t)\sigma_\delta(x_\delta(t),t)+u^-(x_\delta(t),t)\\
&\qquad+\frac{u^-(x_\delta(t),t)}{1-u^-(x_\delta(t),t)+v^-(x_\delta(t),t)})dt.
\end{aligned}$$

Similarly, since (u^+,v^+) is a C^1 solution to problem (1.1) with initial data $(u_0^+(x),v_0^+(x))$ in the domain D_+, we have

$$\begin{aligned}
&\int_0^T\int_{x_\delta(t)}^{+\infty}(u^+\phi_t+\Big(u^++\frac{u^+}{1-u^++v^+}\Big)\phi_x)dxdt\\
&=\int_0^T\int_{x_\delta(t)}^{+\infty}(u^+\phi)_t+(\Big(u^++\frac{u^+}{1-u^++v^+}\Big)\phi)_xdxdt
\end{aligned}$$

$$
\begin{aligned}
=&-\int_0^{+\infty} u_0^+(x)\phi(x,0)dx-\int_0^T \phi(x_\delta(t),t)(-u^+(x_\delta(t),t)\sigma_\delta(x_\delta(t),t)+u^+(x_\delta(t),t)\\
&+\frac{u^+(x_\delta(t),t)}{1-u^+(x_\delta(t),t)+v^+(x_\delta(t),t)})dt.
\end{aligned}
$$

Finally, it can be proved that

$$
\begin{aligned}
&\int_0^T\int_{-\infty}^{+\infty}\left(U\phi_t+\left(U+\frac{U}{1-U+V}\right)\phi_x\right)dxdt\\
&\quad+\int_0^T \omega(x_\delta(t),t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt+\int_{-\infty}^{+\infty}u_0(x)\phi(x,0)dx\\
=&-\int_{-\infty}^{+\infty}u_0(x)\phi(x,0)dx+\int_0^T\phi(x_\delta(t),t)([u+\frac{u}{1-u+v}]-[u]\sigma_\delta)dt\\
&\quad+\int_0^T \omega(x_\delta(t),t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt+\int_{-\infty}^{+\infty}u_0(x)\phi(x,0)dx\\
=&\int_0^T\phi(x_\delta(t),t)([u+\frac{u}{1-u+v}]-[u]\sigma_\delta)dt+\int_0^T \omega(x_\delta(t),t)\frac{\partial\phi(x,t)}{\partial l}\sqrt{1+\sigma_\delta^2}dt\\
=&\int_0^T\phi(x_\delta(t),t)([u+\frac{u}{1-u+v}]-[u]\sigma_\delta)dt+\int_0^T \omega(x_\delta(t),t)\sqrt{1+\sigma_\delta^2}d\phi(x_\delta(t),t)\\
=&\int_0^T\phi(x_\delta(t),t)([u+\frac{u}{1-u+v}]-[u]\sigma_\delta)dt+\omega(x_\delta(t),t)\sqrt{1+\sigma_\delta^2}\phi(x_\delta(t),t)\Big|_{t=0}^{t=T}\\
&\quad-\int_0^T\phi d(\sqrt{1+\sigma_\delta^2}\omega(x_\delta(t),t))=0.
\end{aligned}
$$

That is, the equality (2.8) holds. Substituting line (3.20) into the left-hand side of Equation (2.9), we can prove similarly that Equation (2.9) holds. ∎

Recalling Definition 3.1, the delta shock wave solution (3.15) should satisfy the entropy condition (3.18) on the discontinuity $x=x_\delta(t)$. We next investigate the validity of the entropy condition along the delta shock curve. There are three subcases: $-\hat{u}^-+\hat{v}^-<0<-\hat{u}^++\hat{v}^+$, $-\hat{u}^-+\hat{v}^-=0<-\hat{u}^++\hat{v}^+$ and $-\hat{u}^-+\hat{v}^-<0=-\hat{u}^++\hat{v}^+$.

Subcase 2.1: $-\hat{u}^-+\hat{v}^-<0<-\hat{u}^++\hat{v}^+$.

By virtue of $-\hat{u}^-+\hat{v}^-<0<-\hat{u}^++\hat{v}^+$, it can be proved that there exist constants $\delta^->0$ and $\delta^+>0$ so small that the C^1 functions $u_0^-(x), v_0^-(x), u_0^+(x)$ and $v_0^+(x)$ satisfy the inequality

$$
-u_0^-(x_0^-)+v_0^-(x_0^-)<0<-u_0^+(x_0^+)+v_0^+(x_0^+) \tag{3.21}
$$

for any $x_0^-\in(-\delta^-,0]$ and $x_0^+\in[0,-\delta^+)$.

Let $x=\beta^-(t)$ (resp. $x=\beta^+(t)$) be the downwards left (resp. right) $II-$characteristic from any point $(\tilde{x},\tilde{t})$ on the delta shock wave curve $x=x_\delta(t)$, see Figure 3.3(b). When the time $\tilde{t}$ is small enough, the $II-$characteristic $x=\beta^-(t)$ (resp. $x=\beta^+(t)$) starting at $(\tilde{x},\tilde{t})$ will intersect at a point $(\tilde{x}_0^-,0)$ (resp. $(\tilde{x}_0^+,0)$) on the initial axis $t=0$ with $0>\tilde{x}_0^->-\delta^-$ (resp. $0<\tilde{x}_0^+<\delta^+$). Along the $II-$characteristic $x=\beta^-(t)$ (resp. $x=\beta^+(t)$), the Riemann invariant $\zeta(u,v)=-u+v$ must be a constant. Then we get

$$
\zeta(u^-(\beta^-(t),t),v^-(\beta^-(t),t))=-u^-(\beta^-(t),t)+v^-(\beta^-(t),t)=-u^-(\tilde{x},\tilde{t})+v^-(\tilde{x},\tilde{t})
$$

$$= \zeta(u_0^-(\tilde{x}_0^-), v_0^-(\tilde{x}_0^-)) = -u_0^-(\tilde{x}_0^-) + v_0^-(\tilde{x}_0^-), \tag{3.22}$$

and

$$\begin{aligned}\zeta(u^+(\beta^+(t),t), v^+(\beta^+(t),t)) &= -u^+(\beta^+(t),t) + v^+(\beta^+(t),t) = -u^+(\tilde{x},\tilde{t}) + v^+(\tilde{x},\tilde{t}) \\ &= \zeta(u_0^+(\tilde{x}_0^+), v_0^+(\tilde{x}_0^+)) = -u_0^+(\tilde{x}_0^+) + v_0^+(\tilde{x}_0^+).\end{aligned} \tag{3.23}$$

Using Equations (3.22) and (3.23) in Rankine–Hugoniot condition (3.17), we have the propagation speed of the delta shock wave $x = x_\delta(t)$ at the point $(\tilde{x},\tilde{t})$

$$\begin{aligned}\sigma_\delta(\tilde{x},\tilde{t}) &= \frac{dx_\delta(t)}{dt}\Big|_{t=\tilde{t}} = 1 + \frac{1}{(1-u^-(\tilde{x},\tilde{t})+v^-(\tilde{x},\tilde{t}))(1-u^+(\tilde{x},\tilde{t})+v^+(\tilde{x},\tilde{t}))} \\ &= 1 + \frac{1}{(1-u_0^-(\tilde{x}_0^-)+v_0^-(\tilde{x}_0^-))(1-u_0^+(\tilde{x}_0^+)+v_0^+(\tilde{x}_0^+))}.\end{aligned} \tag{3.24}$$

This, together with inequality (3.21), gives

$$\begin{aligned}\lambda_2(u^+(\tilde{x},\tilde{t}), v^+(\tilde{x},\tilde{t})) &= 1 + \frac{1}{(1-u_0^+(\tilde{x}_0^+)+v_0^+(\tilde{x}_0^+))^2} < \lambda_1(u^+(\tilde{x},\tilde{t}), v^+(\tilde{x},\tilde{t})) \\ &= 1 + \frac{1}{1-u_0^+(\tilde{x}_0^+)+v_0^+(\tilde{x}_0^+)} < \sigma_\delta(\tilde{x},\tilde{t}) \\ &= 1 + \frac{1}{(1-u_0^-(\tilde{x}_0^-)+v_0^-(\tilde{x}_0^-))(1-u_0^+(\tilde{x}_0^+)+v_0^+(\tilde{x}_0^+))} \\ &< \lambda_1(u^-(\tilde{x},\tilde{t}), v^-(\tilde{x},\tilde{t})) = 1 + \frac{1}{1-u_0^-(\tilde{x}_0^-)+v_0^-(\tilde{x}_0^-)} \\ &= 1 + \frac{1}{1-u^-(\tilde{x},\tilde{t})+v^-(\tilde{x},\tilde{t})} \\ &< \lambda_2(u^-(\tilde{x},\tilde{t}), v^-(\tilde{x},\tilde{t})) = 1 + \frac{1}{(1-u_0^-(\tilde{x}_0^-)+v_0^-(\tilde{x}_0^-))^2} \\ &= 1 + \frac{1}{(1-u^-(\tilde{x},\tilde{t})+v^-(\tilde{x},\tilde{t}))^2}.\end{aligned} \tag{3.25}$$

From inequality (3.25), it is obvious that the entropy condition holds at any point $(\tilde{x},\tilde{t})$ on the curve $x = x_\delta(t)$ locally in time. We can now prove the following proposition.

PROPOSITION 3.2. *In the case $-\hat{u}^- + \hat{v}^- < 0 < -\hat{u}^+ + \hat{v}^+$, the generalized Riemann solution to problem (1.1),(1.6) is a delta shock wave locally in time, which is given by Equation (3.15). The delta shock wave curve $x = x_\delta(t)$ has the following property:*

(1) If $(1-\hat{u}^+ + \hat{v}^+)^2(-\dot{u}_0^-(0) + \dot{v}_0^-(0)) < (1-\hat{u}^- + \hat{v}^-)^2(-\dot{u}_0^+(0) + \dot{v}_0^+(0))$, the curve $x = x_\delta(t)$ is convex;

(2) If $(1-\hat{u}^+ + \hat{v}^+)^2(-\dot{u}_0^-(0) + \dot{v}_0^-(0)) > (1-\hat{u}^- + \hat{v}^-)^2(-\dot{u}_0^+(0) + \dot{v}_0^+(0))$, the curve $x = x_\delta(t)$ is concave.

The generalized Riemann problem has a solution in the form similar to the corresponding Riemann solution; see Figure 3.3.

Proof. Proposition 3.1 implies that the delta shock wave solution (3.15) satisfies problem (1.1),(1.6) in the sense of distributions. From inequality (3.25), we have that the entropy condition holds along the delta shock wave curve $x = x_\delta(t)$ locally in time. Hence, by Definition 3.1, we know that the generalized Riemann solution to problem (1.1),(1.6) is a delta shock wave in a neighborhood of the origin (0,0). Namely, the generalized Riemann problem has a solution in the form similar to the corresponding Riemann solution locally in time.

To determine the behavior of the delta shock wave curve $x=x_\delta(t)$ near the origin, we need to calculate the value of $\ddot{x}_\delta(0)$ as follows. From Equation (3.17), it follows that

$$(1-u^-+v^-)(1-u^++v^+)\frac{dx_\delta(\tilde{t})}{d\tilde{t}}=(1-u^-+v^-)(1-u^++v^+)+1. \tag{3.26}$$

Differentiating Equation (3.26) with respect to $\tilde{t}$ and let $\tilde{t}=0$, then one obtain

$$\begin{aligned}(1-\hat{u}^-+\hat{v}^-)(1-\hat{u}^++\hat{v}^+)\ddot{x}_\delta(0)=&(1-\dot{x}_\delta(0))(1-\hat{u}^++\hat{v}^+)\frac{d(-u^-+v^-)}{d\tilde{t}}\Big|_{\tilde{t}=0}\\&+(1-\dot{x}_\delta(0))(1-\hat{u}^-+\hat{v}^-)\frac{d(-u^++v^+)}{d\tilde{t}}\Big|_{\tilde{t}=0}.\end{aligned} \tag{3.27}$$

From Equation (3.27), we turn next to the values of $\frac{d(-u^-+v^-)}{d\tilde{t}}\Big|_{\tilde{t}=0}$ and $\frac{d(-u^++v^+)}{d\tilde{t}}\Big|_{\tilde{t}=0}$, respectively. Using Equation (1.1), we write

$$\begin{aligned}\frac{d(-u+v)}{dt}=&\frac{\partial(-u+v)}{\partial t}+\frac{\partial(-u+v)}{\partial x}\frac{dx}{dt}=\frac{\partial}{\partial x}\left\{\left(1+\frac{1}{1-u+v}\right)(u-v)\right\}\\&+\frac{\partial(-u+v)}{\partial x}\frac{dx}{dt}=(\frac{dx}{dt}-1-\frac{1}{(1-u+v)^2})\frac{\partial(-u+v)}{\partial x}.\end{aligned} \tag{3.28}$$

If we let $(u(x,t),v(x,t))=(u^-(x,t),v^-(x,t))$, along the delta shock wave curve $x=x_\delta(t)$, together with $\dot{x}_\delta(0)=1+\frac{1}{(1-\hat{u}^-+\hat{v}^-)(1-\hat{u}^++\hat{v}^+)}$, the equality (3.28) implies

$$\begin{aligned}\frac{d(-u^-+v^-)}{d\tilde{t}}\Big|_{\tilde{t}=0}=&(\dot{x}_\delta(0)-1-\frac{1}{(1-\hat{u}^-+\hat{v}^-)^2})(-\dot{u}_0^-(0)+\dot{v}_0^-(0))\\=&\frac{-\hat{u}^-+\hat{v}^-+\hat{u}^+-\hat{v}^+}{(1-\hat{u}^-+\hat{v}^-)^2(1-\hat{u}^++\hat{v}^+)}(-\dot{u}_0^-(0)+\dot{v}_0^-(0)).\end{aligned} \tag{3.29}$$

Similarly, letting $(u(x,t),v(x,t))=(u^-(x,t),v^-(x,t))$, we get

$$\begin{aligned}\frac{d(-u^++v^+)}{d\tilde{t}}\Big|_{\tilde{t}=0}=&(\dot{x}_\delta(0)-1-\frac{1}{(1-\hat{u}^++\hat{v}^+)^2})(-\dot{u}_0^+(0)+\dot{v}_0^+(0))\\=&\frac{-\hat{u}^++\hat{v}^++\hat{u}^--\hat{v}^-}{(1-\hat{u}^++\hat{v}^+)^2(1-\hat{u}^-+\hat{v}^-)}(-\dot{u}_0^+(0)+\dot{v}_0^+(0)).\end{aligned} \tag{3.30}$$

Using equalities (3.29) and (3.30) in Equation (3.27) yields

$$\begin{aligned}(1-\hat{u}^-+\hat{v}^-)(1-\hat{u}^++\hat{v}^+)\ddot{x}_\delta(0)=&(1-\dot{x}_\delta(0))\frac{-\hat{u}^-+\hat{v}^-+\hat{u}^+-\hat{v}^+}{(1-\hat{u}^-+\hat{v}^-)^2}(-\dot{u}_0^-(0)+\dot{v}_0^-(0))\\&+(1-\dot{x}_\delta(0))\frac{-\hat{u}^++\hat{v}^++\hat{u}^--\hat{v}^-}{(1-\hat{u}^++\hat{v}^+)^2}(-\dot{u}_0^+(0)+\dot{v}_0^+(0)).\end{aligned} \tag{3.31}$$

With inequalities $1-\hat{u}^-+\hat{v}^->0$, $1-\hat{u}^++\hat{v}^+>0$ and $1-\dot{x}_\delta(0)<0$ in mind, Equation (3.31) shows that the second derivative of the delta shock wave curve at the origin

$\ddot{x}(0)>0$ when $(1-\hat{u}^{+}+\hat{v}^{+})^2(-\dot{u}_0^-(0)+\dot{v}_0^-(0))<(1-\hat{u}^-+\hat{v}^-)^2(-\dot{u}_0^+(0)+\dot{v}_0^+(0))$, otherwise $\ddot{x}(0)<0$ when $(1-\hat{u}^{+}+\hat{v}^{+})^2(-\dot{u}_0^-(0)+\dot{v}_0^-(0))>(1-\hat{u}^-+\hat{v}^-)^2(-\dot{u}_0^+(0)+\dot{v}_0^+(0))$. ◻

Subcase 2.2: $-\hat{u}^-+\hat{v}^-=0<-\hat{u}^++\hat{v}^+$.

We classify the discussion into two subcases:$-\dot{u}_0^-(0)+\dot{v}_0^-(0)>0$ and $-\dot{u}_0^-(0)+\dot{v}_0^-(0)<0$. We first consider the subcase $-\dot{u}_0^-(0)+\dot{v}_0^-(0)>0$. Due to this subcase sondition and $-\hat{u}^-+\hat{v}^-=0<-\hat{u}^++\hat{v}^+$, it can be proved that there exist constants $\delta^->0$ and $\delta^+>0$ so small that the C^1 functions $u_0^-(x),v_0^-(x),u_0^+(x)$ and $v_0^+(x)$ satisfy the inequality (3.21) for any $x_0^-\in(-\delta^-,0)$ and $x_0^+\in(0,-\delta^+)$. Similar to the **Subcase 2.1**, we find that the delta shock wave δS solution (3.15) satisfies the entropy condition (3.18) along the delta shock wave curve $x=x_\delta(t)$ locally in time. Furthermore, Proposition 3.1 implies that the delta shock wave solution (3.15) satisfies problem (1.1),(1.6) in the sense of distributions. Based on the above discussion and the Definition 3.1, we see that the generalized Riemann solution of problem (1.1),(1.6) is a delta shock wave locally in time. We depict the generalized Riemann solution and the corresponding Riemann solution in Figure 3.3, which implies that the delta shock wave solution to the Riemann problem can retain its form near the origin after the perturbation of Riemann initial data in this subcase.

Next consider the subcase $-\dot{u}_0^-(0)+\dot{v}_0^-(0)<0$. In the same way as **Subcase 2.1**, we have (3.22)∼(3.24) correspondingly for this subcase. Since $-\hat{u}^-+\hat{v}^-=0$ and $-\dot{u}_0^-(0)+\dot{v}_0^-(0)<0$, we find that there exist constants $\delta^->0$ and $\delta^+>0$ small so that the C^1 functions $u_0^-(x),v_0^-(x),u_0^+(x)$ and $v_0^+(x)$ satisfy

$$0<-u_0^-(x_0^-)+v_0^-(x_0^-)<-u_0^+(x_0^+)+v_0^+(x_0^+) \tag{3.32}$$

for any $x_0^-\in(-\delta^-,0)$ and $x_0^+\in(0,-\delta^+)$. From the above inequality, we get

$$\begin{aligned}\lambda_1(u^+(\tilde{x},\tilde{t}),v^+(\tilde{x},\tilde{t}))&=1+\frac{1}{1-u_0^+(\tilde{x}_0^+)+v_0^+(\tilde{x}_0^+)}>\sigma_\delta(\tilde{x},\tilde{t})\\&=1+\frac{1}{(1-u_0^-(\tilde{x}_0^-)+v_0^-(\tilde{x}_0^-))(1-u_0^+(\tilde{x}_0^+)+v_0^+(\tilde{x}_0^+))}\end{aligned} \tag{3.33}$$

on the delta shock curve $x=x_\delta(t)$. Comparing the inequality (3.33) with (3.18), we see that in this subcase the entropy condition fails to hold on the curve $x=x_\delta(t)$. According to the Definition 3.1, we conclude that the generalized Riemann solution is no longer a delta shock wave for this subcase.

For the subcase $-\dot{u}_0^-(0)+\dot{v}_0^-(0)<0$, we will prove that the generalized Riemann problem (1.1),(1.6) is resolved by a backward shock wave and a contact discontinuity, as depicted in Figure 3.4(b). To see this, the backward shock wave $\overleftarrow{S}:x=x_s(t)$ and the contact discontinuity $J:x=x_c(t)$ satisfy the boundary conditions (3.5)∼(3.6) and the boundary conditions (3.7)∼(3.8), respectively. $(u^-(x,t),v^-(x,t))$ and $(u^+(x,t),v^+(x,t))$ are known smooth functions. The intermediate state $(u^*(x,t),v^*(x,t))$ is an unknown solution to system (1.1) with initial value (1.6). Due to $-\hat{u}^-+\hat{v}^-=0$, instead of Equation (3.9), in this subcase we have

$$\hat{u}^*=\lim_{(x,t)\to(0,0)}u^*(x,t)=+\infty,\qquad \hat{v}^*=\lim_{(x,t)\to(0,0)}v^*(x,t)=+\infty.$$

We now consider the free boundary problem (1.1) with boundary conditions (3.5)∼(3.8) on the fan-shaped domain $\{(x,t)\,|\,x_s(t)<x<x_c(t),0\le t<\epsilon\}$, $\epsilon>0$ small. As we have analyzed in **Case 1**, the characterizing matrix of this free boundary problem is (3.13), which is completely analogous to **Case 1**. Then we have the inequality (3.14) for this subcase. This means that the above free boundary problem admits a unique solution on the above fan-shaped domain. We obtain the following proposition.

Proposition 3.3. *In case of* $-\hat{u}^- + \hat{v}^- = 0 < -\hat{u}^+ + \hat{v}^+$ *and* $-\dot{u}_0^-(0) + \dot{v}_0^-(0) < 0$, *the generalized Riemann solution to problem* (1.1),(1.6) *is a backward shock wave followed by a contact discontinuity locally in time. The generalized solution is dramatically different from the corresponding Riemann solution of problem* (1.1),(2.1), *which is a delta shock wave, see Figure 3.4.*

It is possible to give another proof of the proposition. We let $x = \beta^*(t)$ be the upwards right $II-$characteristic from any point (x,t) on the shock wave curve $x = x_s(t)$ (as depicted in Figure 3.4(b)). The $II-$characteristic curve $x = \beta^*(t)$ intersects the contact discontinuity $x = x_c(t)$ at the point (x_0, t_0). It is known that the Riemann invariant $\zeta(u,v) = -u + v$ must be a constant along $II-$characteristic, so we have

$$\begin{aligned}\zeta(u^*,v^*) &= -u^*(x,t) + v^*(x,t) = -u^*(\beta^*(t),t) + v^*(\beta^*(t),t) = -u^*(x_0,t_0) + v^*(x_0,t_0)\\ &= -u^*(x_s(t),t) + v^*(x_s(t),t) = -u^*(x_c(t_0),t_0) + v^*(x_c(t_0),t_0).\end{aligned} \tag{3.34}$$

Using Equation (3.34), we find

$$\frac{d\beta^*(t)}{dt} = 1 + \frac{1}{(1 - u^*(\beta^*(t),t) + v^*(\beta^*(t),t))^2} = 1 + \frac{1}{(1 - u^*(x_0 t_0) + v^*(x_0,t_0))^2}. \tag{3.35}$$

This means that the propagating speed of $II-$characteristic is a constant. Namely, the characteristic $x = \beta^*(t)$ is a straight line. From the above equality, we have

$$\frac{x_s(t) - x_c(t_0)}{t - t_0} = 1 + \frac{1}{(1 - u^*(x_s(t),t) + v^*(x_s(t),t))^2} = 1 + \frac{1}{(1 - u^*(x_c(t_0),t_0) + v^*(x_c(t_0),t_0))^2}. \tag{3.36}$$

We may write the equality (3.36) in the form

$$\begin{aligned}&(x_s(t) - x_c(t_0))(1 - u^*(x_s(t),t) + v^*(x_s(t),t))^2\\ =&(t - t_0)(1 - u^*(x_s(t),t) + v^*(x_s(t),t))^2 + t - t_0.\end{aligned} \tag{3.37}$$

Differentiating the above equation with respect to t and let $t = 0$, then one obtains

$$\begin{aligned}&(\dot{x}_s(0) - \dot{x}_c(0)\frac{dt_0}{dt}\Big|_{t=0})(1 - \hat{u}^* + \hat{v}^*)^2\\ &\quad + 2(x_s(0) - x_c(0))(1 - \hat{u}^* + \hat{v}^*)\frac{d(-u^*(x_s(t),t) + v^*(x_s(t),t))}{dt}\Big|_{t=0}\\ =&(1 - \frac{dt_0}{dt}\Big|_{t=0})(1 - \hat{u}^* + \hat{v}^*)^2 + 1 - \frac{dt_0}{dt}\Big|_{t=0}.\end{aligned} \tag{3.38}$$

Substituting

$$x_s(0) = x_c(0) = 0 \tag{3.39}$$

and

$$\dot{x}_s(0) = \dot{x}_c(0) = 1 + \frac{1}{(1 - \hat{u}^- + \hat{v}^-)(1 - \hat{u}^+ + \hat{v}^+)} = 1 + \frac{1}{1 - \hat{u}^+ + \hat{v}^+} \tag{3.40}$$

into Equation (3.38), and using $-\hat{u}^+ + \hat{v}^+ > 0$, we have

$$\frac{dt_0}{dt}\Big|_{t=0} = 1. \tag{3.41}$$

Moreover, differentiating the last equality in Equation (3.34) with respect to t and letting $t=0$, we obtain

$$\frac{d(-u^*(x_s(t),t)+v^*(x_s(t),t))}{dt}\Big|_{t=0}=\frac{d(-u^*(x_c(t_0),t_0)+v^*(x_c(t_0),t_0))}{dt_0}\Big|_{t_0=0}\frac{dt_0}{dt}\Big|_{t=0}. \tag{3.42}$$

Using equality (3.41) in Equation (3.42), we get

$$\frac{d(-u^*(x_s(t),t)+v^*(x_s(t),t))}{dt}\Big|_{t=0}=\frac{d(-u^*(x_c(t_0),t_0)+v^*(x_c(t_0),t_0))}{dt_0}\Big|_{t_0=0}. \tag{3.43}$$

In what follows, we will compute two important values $\ddot{x}_s(0)$ and $\ddot{x}_c(0)$. The first is the second derivative of the shock wave curve at the origin, and the second is the second derivative of the contact discontinuity wave curve at the origin. Firstly, we compute the value $\ddot{x}_s(0)$. Along $x=x_s(t)$, from equality (3.5), it can be easily checked that

$$(1-u^-+v^-)(1-u^*+v^*)\frac{dx_s(t)}{dt}=(1-u^-+v^-)(1-u^*+v^*)+1. \tag{3.44}$$

Differentiating the above equality with respect to t and letting $t=0$, together with condition (3.8) and Equation (3.34), we have

$$\begin{aligned}
&(1-\hat{u}^++\hat{v}^+)\ddot{x}_s(0)\\
=&\frac{d(-u^-(x_s(t),t)+v^-(x_s(t),t))}{dt}\Big|_{t=0}(1-\hat{u}^++\hat{v}^+)+\frac{d(-u^*(x_s(t),t)+v^*(x_s(t),t))}{dt}\Big|_{t=0}\\
&-\frac{d(-u^-(x_s(t),t)+v^-(x_s(t),t))}{dt}\Big|_{t=0}(1-\hat{u}^++\hat{v}^+)\dot{x}_s(0)\\
&-\frac{d(-u^*(x_s(t),t)+v^*(x_s(t),t))}{dt}\Big|_{t=0}\dot{x}_s(0).
\end{aligned} \tag{3.45}$$

On the one hand, in view of equality (3.28), we obtain

$$\frac{d(-u^-(x_s(t),t)+v^-(x_s(t),t))}{dt}=\Big(\frac{dx_s(t)}{dt}-1-\frac{1}{(1-u^-+v^-)^2}\Big)\frac{\partial(-u^-+v^-)}{\partial x}. \tag{3.46}$$

Using Equations (3.40) and (3.46), together with $-\hat{u}^-+\hat{v}^-=0$, one can get that

$$\frac{d(-u^-(x_s(t),t)+v^-(x_s(t),t))}{dt}\Big|_{t=0}=\frac{\hat{u}^+-\hat{v}^+}{1-\hat{u}^++\hat{v}^+}(-\dot{u}_0^-(0)+\dot{v}_0^-(0)). \tag{3.47}$$

On the other hand, due to Equations (3.43) and (3.8), we have

$$\begin{aligned}
\frac{d(-u^*(x_s(t),t)+v^*(x_s(t),t))}{dt}\Big|_{t=0}&=\frac{d(-u^*(x_c(t_0),t_0)+v^*(x_c(t_0),t_0))}{dt_0}\Big|_{t_0=0}\\
&=\frac{d(-u^+(x_c(t_0),t_0)+v^+(x_c(t_0),t_0))}{dt_0}\Big|_{t_0=0}.
\end{aligned} \tag{3.48}$$

Along the contact discontinuity wave curve $x=x_c(t)$, it follows from Equations (3.28), (3.40), and (3.48)

$$\frac{d(-u^*(x_s(t),t)+v^*(x_s(t),t))}{dt}\Big|_{t=0}=\frac{d(-u^*(x_c(t_0),t_0)+v^*(x_c(t_0),t_0))}{dt_0}\Big|_{t_0=0}$$

$$= \frac{-\hat{u}^+ + \hat{v}^+}{(1-\hat{u}^+ + \hat{v}^+)^2}(-\dot{u}_0^+(0) + \dot{v}_0^+(0)). \tag{3.49}$$

Then, using Equations (3.40), (3.47), and (3.49) in Equation (3.45), we get

$$(1-\hat{u}^+ + \hat{v}^+)\ddot{x}_s(0) = \frac{\hat{v}^+ - \hat{u}^+}{1-\hat{u}^+ + \hat{v}^+}(-\dot{u}_0^-(0) + \dot{v}_0^-(0)) - \frac{\hat{v}^+ - \hat{u}^+}{(1-\hat{u}^+ + \hat{v}^+)^3}(-\dot{u}_0^+(0) + \dot{v}_0^+(0)). \tag{3.50}$$

Secondly, we now compute the value $\ddot{x}_c(0)$. Along $x = x_c(t)$, from equality (3.7), it is easy to see that

$$(1-u^* + v^*)\frac{dx_c(t_0)}{dt_0} = 2 - u^* + v^*. \tag{3.51}$$

Differentiating Equation (3.51) with respect to t_0 and letting $t_0 = 0$ yields

$$\begin{aligned} &\frac{d(-u^*(x_c(t_0),t_0) + v^*(x_c(t_0),t_0))}{dt_0}\Big|_{t_0=0}\dot{x}_c(0) + (1-\hat{u}^+ + \hat{v}^+)\ddot{x}_c(0) \\ &= \frac{d(-u^*(x_c(t_0),t_0) + v^*(x_c(t_0),t_0))}{dt_0}\Big|_{t_0=0}. \end{aligned} \tag{3.52}$$

Using Equations (3.40) and (3.49) in Equation (3.52), we get

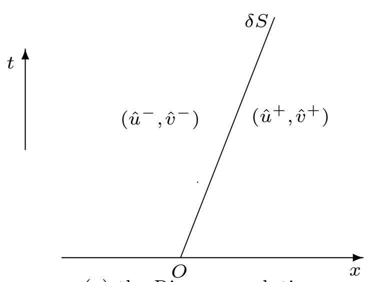
$$(1-\hat{u}^+ + \hat{v}^+)\ddot{x}_c(0) = -\frac{\hat{v}^+ - \hat{u}^+}{(1-\hat{u}^+ + \hat{v}^+)^3}(-\dot{u}_0^+(0) + \dot{v}_0^+(0)). \tag{3.53}$$

Thirdly, since we have the two values $\ddot{x}_s(0)$ and $\ddot{x}_c(0)$, combining Equations (3.50) and (3.53) gives

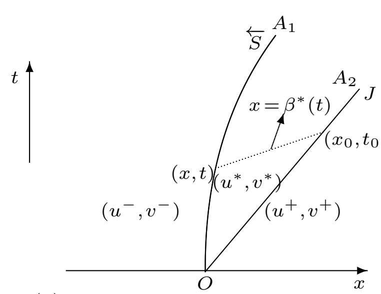
$$(1-u^+ + v^+)(\ddot{x}_s(0) - \ddot{x}_c(0)) = \frac{\hat{v}^+ - \hat{u}^+}{1-\hat{u}^+ + \hat{v}^+}(-\dot{u}_0^-(0) + \dot{v}_0^-(0)) < 0. \tag{3.54}$$

This implies that $\ddot{x}_s(0) < \ddot{x}_c(0)$.

Finally, by virtue of Equations (3.39), (3.40), and (3.54), it can be proven that the generalized Riemann solution to problem (1.1),(1.6) in this subcase clearly consists of a backward shock $\overleftarrow{S}$ from (u^-,v^-) to (u^*,v^*), followed by a contact discontinuity J from (u^*,v^*) to (u^+,v^+) near the origin, see Figure 3.4(b). Thus we have completed the construction and proof of the generalized Riemann solution for this subcase.

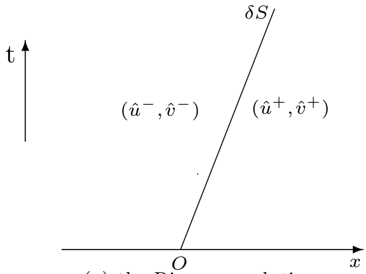


(a) the Riemann solution

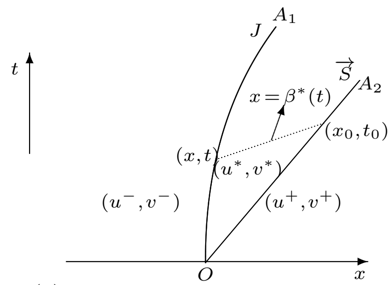


(b) the generalized Riemann solution

Fig. 3.4. *Subcase 2.2 with* $-\dot{u}_0^-(0) + \dot{v}_0^-(0) < 0$.



(a) the Riemann solution



(b) the generalized Riemann solution

Fig. 3.5. *Subcase 2.3 with* $-\dot{u}_0^+(0)+\dot{v}_0^+(0)<0$.

Subcase 2.3: $-\hat{u}^-+\hat{v}^-<0=-\hat{u}^++\hat{v}^+$.

We classify the discussion into two subcases:$-\dot{u}_0^+(0)+\dot{v}_0^+(0)>0$ and $-\dot{u}_0^+(0)+\dot{v}_0^+(0)<0$. We first consider the subcase $-\dot{u}_0^+(0)+\dot{v}_0^+(0)>0$. Due to this subcase condition and $-\hat{u}^-+\hat{v}^-<0=-\hat{u}^++\hat{v}^+$, it can be proved that there exist constants $\delta^->0$ and $\delta^+>0$ so small that the C^1 functions $u_0^-(x),v_0^-(x),u_0^+(x)$ and $v_0^+(x)$ satisfy the inequality (3.21) for any $x_0^-\in(-\delta^-,0)$ and $x_0^+\in(0,-\delta^+)$. In what follows, the discussion is the same as in **Case 1**. We prove that the generalized Riemann solution to problem (1.1),(1.6) is a delta shock wave locally in time, which has a structure similar to the Riemann solution to problem (1.1),(2.1) for this subcase, see Figure 3.3. We omit the discussion.

We turn our efforts to the second subcase $-\dot{u}_0^+(0)+\dot{v}_0^+(0)<0$. Since $-\hat{u}^++\hat{v}^+=0$ and $-\dot{u}_0^+(0)+\dot{v}_0^+(0)<0$, we find that there exist constants $\delta^->0$ and $\delta^+>0$ so small that the C^1 functions $u_0^-(x),v_0^-(x),u_0^+(x)$ and $v_0^+(x)$ satisfy

$$-u_0^-(x_0^-)+v_0^-(x_0^-)<-u_0^+(x_0^+)+v_0^+(x_0^+)<0 \tag{3.55}$$

for any $x_0^-\in(-\delta^-,0)$ and $x_0^+\in(0,-\delta^+)$. From the above inequality, we get

$$\begin{aligned}1+\frac{1}{(1-u_0^-(\tilde{x}_0^-)+v_0^-(\tilde{x}_0^-))(1-u_0^+(\tilde{x}_0^+)+v_0^+(\tilde{x}_0^+))}&=\sigma_\delta(\tilde{x},\tilde{t})>\lambda_1(u^-(\tilde{x},\tilde{t}),v^-(\tilde{x},\tilde{t}))\\&=1+\frac{1}{1-u_0^-(\tilde{x}_0^-)+v_0^-(\tilde{x}_0^-)}\end{aligned} \tag{3.56}$$

on the delta shock curve $x=x_\delta(t)$. Comparing inequality (3.56) with (3.18), we see that in this subcase the entropy condition fails to hold on the curve $x=x_\delta(t)$. According to Definition 3.1, we conclude that the generalized Riemann solution is no longer a delta shock wave for this subcase.

For the subcase $-\dot{u}_0^+(0)+\dot{v}_0^+(0)<0$, we will prove that the generalized Riemann problem (1.1),(1.6) is resolved by a contact discontinuity followed by a forward shock wave, as depicted in Figure 3.5(b). For the contact discontinuity $OA_1:x=x_c(t)$ ($x_c(0)=0$), on which we have

$$\frac{dx_c(t)}{dt}=1+\frac{1}{1-u^-+v^-}, \tag{3.57}$$

$$-u^-+v^-=-u^*+v^*. \tag{3.58}$$

For the forward shock wave $OA_2:x=x_s(t)(x_s(0)=0)$ on which we have

$$\frac{dx_s(t)}{dt}=1+\frac{1}{(1-u^*+v^*)(1-u^++v^+)}, \tag{3.59}$$

$$u^+v^* = u^*v^+. \tag{3.60}$$

The generalized Riemann solution to problem (1.1),(1.6) is $(u^-(x,t),v^-(x,t))$ on the domain $\{(x,t)\,|\,x<x_c(t),0\le t<\epsilon\}$ ($\epsilon>0$ so small). The generalized Riemann solution is $(u^+(x,t),v^+(x,t))$ on the domain $\{(x,t)\,|\,x>x_s(t),0\le t<\epsilon\}$. However, the generalized Riemann solution to problem (1.1),(1.6) is unknown on the domain $\{(x,t)\,|\,x_c(t)<x<x_s(t),0\le t<\epsilon\}$. We denote it by $(u^*(x,t),v^*(x,t))$. The value of $(\hat{u}^*,\hat{v}^*)$ is determined uniquely from boundary conditions (3.58) and (3.60). Due to $-\hat{u}^+ + \hat{v}^+ = 0$, we have

$$\hat{u}^* = \lim_{(x,t)\to(0,0)} u^*(x,t) = +\infty, \qquad \hat{v}^* = \lim_{(x,t)\to(0,0)} v^*(x,t) = +\infty.$$

In what follows, we have to solve the free boundary problem (1.1) with boundary conditions (3.57)$\sim$(3.60) on the fan-shaped domain $\{(x,t)\,|\,x_c(t)<x<x_s(t),0\le t<\epsilon\}$ ($\epsilon>0$ so small). We introduce the change of variables (2.15). Boundary condition (3.58) on $x=x_c(t)$ then reduces to

$$\mathcal{V} = -u^- + v^-. \tag{3.61}$$

Boundary condition (3.60) on $x=x_s(t)$ can be written as

$$\mathcal{U} = \frac{(u^+\hat{v}^* - v^+\hat{u}^*)\mathcal{V}}{v^+ - u^+}. \tag{3.62}$$

From Equations (3.61) and (3.62), we find the characterizing matrix A of above free boundary problem is [12]:

$$A = \begin{pmatrix} 0 & \dfrac{u^+\hat{v}^* - v^+\hat{u}^*}{v^+ - u^+} \\ 0 & 0 \end{pmatrix}.$$

By Lemma 2.1, it is easy to prove that

$$\|A\|_{min} = 0 < 1.$$

Using the local existence and uniqueness theorem proposed by Li Ta-tsien and Yu Wen-ci [12], we see that the free boundary problem under consideration admits a unique piecewise C^1 solution on the fan-shaped domain $\{(x,t)\,|\,x_c(t)<x<x_s(t),0\le t<\epsilon\}$, $\epsilon>0$ small. Then the generalized Riemann solution has the desired structure, as depicted in Figure 3.5(b). We obtain the following proposition.

Proposition 3.4. *In case of $-\hat{u}^- + \hat{v}^- < 0 = -\hat{u}^+ + \hat{v}^+$ and $-\dot{u}_0^+(0) + \dot{v}_0^+(0) < 0$, the generalized Riemann solution to problem (1.1),(1.6) is a contact discontinuity followed by a forward shock wave locally in time. The generalized solution is different from the corresponding Riemann solution of problem (1.1),(2.1), which is a delta shock wave, see Figure 3.5.*

It is possible to give another proof of the proposition. We let $x=\beta^*(t)$ be the upwards right $II-$characteristic from any point (x,t) on the contact discontinuity curve $x=x_c(t)$ (as depicted in Figure 3.5). The $II-$characteristic curve $x=\beta^*(t)$ intersects the shock wave curve $x=x_s(t)$ at the point (x_0,t_0). It is known that the Riemann invariant $\zeta(u,v)=-u+v$ must be a constant along $II-$characteristic, so we have

$$\zeta(u^*,v^*) = -u^*(x,t)+v^*(x,t) = -u^*(\beta^*(t),t)+v^*(\beta^*(t),t) = -u^*(x_0,t_0)+v^*(x_0,t_0)$$

$$=-u^*(x_c(t),t)+v^*(x_c(t),t)=-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0). \tag{3.63}$$

Using Equation (3.63), we have

$$\frac{d\beta^*(t)}{dt}=1+\frac{1}{(1-u^*(\beta^*(t),t)+v^*(\beta^*(t),t))^2}=1+\frac{1}{(1-u^*(x_0.t_0)+v^*(x_0,t_0))^2}. \tag{3.64}$$

This means that the propagating speed of $II-$characteristic is a constant. Namely, the characteristic $x=\beta^*(t)$ is a straight line. From the above equality, we have

$$\begin{aligned}\frac{x_c(t)-x_s(t_0)}{t-t_0}=&1+\frac{1}{(1-u^*(x_c(t),t)+v^*(x_c(t),t))^2}\\ =&1+\frac{1}{(1-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0))^2}.\end{aligned} \tag{3.65}$$

We may write equality (3.65) in the form

$$\begin{aligned}&(x_c(t)-x_s(t_0))(1-u^*(x_c(t),t)+v^*(x_c(t),t))^2\\ =&(t-t_0)(1-u^*(x_c(t),t)+v^*(x_c(t),t))^2+t-t_0.\end{aligned} \tag{3.66}$$

Differentiating the above equation with respect to t and let $t=0$, one obtains

$$\begin{aligned}&(\dot{x}_c(0)-\dot{x}_s(0)\frac{dt_0}{dt}\Big|_{t=0})(1-\hat{u}^-+\hat{v}^-)^2\\ &\qquad+2(x_c(0)-x_s(0))(1-\hat{u}^-+\hat{v}^-)\frac{d(-u^-(x_c(t),t)+v^-(x_c(t),t))}{dt}\Big|_{t=0}\\ =&(1-\frac{dt_0}{dt}\Big|_{t=0})(1-\hat{u}^-+\hat{v}^-)^2+1-\frac{dt_0}{dt}\Big|_{t=0}.\end{aligned} \tag{3.67}$$

Substituting

$$x_c(0)=x_s(0)=0 \tag{3.68}$$

and

$$\dot{x}_s(0)=\dot{x}_c(0)=1+\frac{1}{(1-\hat{u}^-+\hat{v}^-)(1-\hat{u}^++\hat{v}^+)}=1+\frac{1}{1-\hat{u}^-+\hat{v}^-} \tag{3.69}$$

into Equation (3.67), and using $-\hat{u}^-+\hat{v}^-<0$, we get

$$\frac{dt_0}{dt}\Big|_{t=0}=1. \tag{3.70}$$

Moreover, differentiating the last equality in (3.63) with respect to t and letting $t=0$, we obtain

$$\frac{d(-u^*(x_c(t),t)+v^*(x_c(t),t))}{dt}\Big|_{t=0}=\frac{d(-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0}\frac{dt_0}{dt}\Big|_{t=0}. \tag{3.71}$$

If we use equality (3.70) in Equation (3.71), we derive

$$\frac{d(-u^*(x_c(t),t)+v^*(x_c(t),t))}{dt}\Big|_{t=0}=\frac{d(-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0}. \tag{3.72}$$

In what follows, we will compute two important values $\ddot{x}_c(0)$ and $\ddot{x}_s(0)$. The first is the second derivative of the contact discontinuity wave curve at the origin, and the second is the second derivative of the shock wave curve at the origin. Firstly, we compute the value $\ddot{x}_c(0)$. Along $x=x_c(t)$, with equality (3.57), it holds that

$$(1-u^-+v^-)\frac{dx_c(t)}{dt}=2-u^-+v^-. \tag{3.73}$$

Differentiating the above equality with respect to t and letting $t=0$, yields

$$\begin{aligned}&\frac{d(-u^-(x_c(t),t)+v^-(x_c(t),t))}{dt}\Big|_{t=0}\dot{x}_c(0)+(1-\hat{u}^-+\hat{v}^-)\ddot{x}_c(0)\\ =&\frac{d(-u^-(x_c(t),t)+v^-(x_c(t),t))}{dt}\Big|_{t=0}.\end{aligned} \tag{3.74}$$

From Equation (3.28), we have

$$\begin{aligned}&\frac{d(-u^-(x_c(t),t)+v^-(x_c(t),t))}{dt}\\ =&\frac{\partial(-u^-+v^-)}{\partial t}+\frac{\partial(-u^-+v^-)}{\partial x}\frac{dx_c(t)}{dt}\\ =&\frac{\partial}{\partial x}\left\{\left(1+\frac{1}{1-u^-+v^-}\right)(u^--v^-)\right\}\\ &+\frac{\partial(-u^-+v^-)}{\partial x}\frac{dx_c(t)}{dt}=\Big(\frac{dx_c(t)}{dt}-1-\frac{1}{(1-u^-+v^-)^2}\Big)\frac{\partial(-u^-+v^-)}{\partial x}.\end{aligned} \tag{3.75}$$

Noting Equations (3.69) and (3.75), together with $-\hat{u}^++\hat{v}^+=0$, we have

$$\frac{d(-u^-(x_c(t),t)+v^-(x_c(t),t))}{dt}\Big|_{t=0}=\frac{-\hat{u}^-+\hat{v}^-}{(1-\hat{u}^-+\hat{v}^-)^2}(-\dot{u}_0^-(0)+\dot{v}_0^-(0)). \tag{3.76}$$

Inserting Equations (3.69) and (3.76) into Equation (3.74) yields

$$(1-\hat{u}^-+\hat{v}^-)\ddot{x}_c(0)=\frac{\hat{u}^--\hat{v}^-}{(1-\hat{u}^-+\hat{v}^-)^3}(-\dot{u}_0^-(0)+\dot{v}_0^-(0)). \tag{3.77}$$

Secondly, we compute the value $\ddot{x}_s(0)$. Along $x=x_s(t)$, with (3.59), it holds that

$$(1-u^*+v^*)(1-u^++v^+)\frac{dx_s(t_0)}{dt_0}=(1-u^*+v^*)(1-u^++v^+)+1. \tag{3.78}$$

By Equations (3.58) and (3.63), differentiating Equation (3.78) with respect to t_0 and letting $t_0=0$ yields

$$\begin{aligned}(1-\hat{u}^-+\hat{v}^-)\ddot{x}_s(0)=&\frac{d(-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0}\\ &+(1-\hat{u}^-+\hat{v}^-)\frac{d(-u^+(x_s(t_0),t_0)+v^+(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0}\\ &-\frac{d(-u^+(x_s(t_0),t_0)+v^+(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0}(1-\hat{u}^-+\hat{v}^-)\dot{x}_s(0)\\ &-\frac{d(-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0}\dot{x}_s(0).\end{aligned} \tag{3.79}$$

In view of Equations (3.58) and (3.72), we get

$$\begin{aligned}\frac{d(-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0} &= \frac{d(-u^*(x_c(t),t)+v^*(x_c(t),t))}{dt}\Big|_{t=0} \\ &= \frac{d(-u^-(x_c(t),t)+v^-(x_c(t),t))}{dt}\Big|_{t=0}. \end{aligned} \tag{3.80}$$

Along the delta shock wave curve $x=x_s(t)$, it follows from Equations (3.28), (3.69), and (3.80) that

$$\begin{aligned}\frac{d(-u^*(x_s(t_0),t_0)+v^*(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0} &= \frac{d(-u^*(x_c(t),t)+v^*(x_c(t),t))}{dt}\Big|_{t=0} \\ &= \frac{-\hat{u}^-+\hat{v}^-}{(1-\hat{u}^-+\hat{v}^-)^2}(-\dot{u}_0^-(0)+\dot{v}_0^-(0)). \end{aligned} \tag{3.81}$$

On the other hand, based on Equations (3.28) and (3.69), we obtain

$$\frac{d(-u^+(x_s(t_0),t_0)+v^+(x_s(t_0),t_0))}{dt_0}\Big|_{t_0=0} = \left(\frac{1}{1-\hat{u}^-+\hat{v}^-}-1\right)(-\dot{u}_0^+(0)+\dot{v}_0^+(0)). \tag{3.82}$$

Substituting Equations (3.69), (3.81), and (3.82) into Equation (3.79), we get

$$(1-\hat{u}^-+\hat{v}^-)\ddot{x}_s(0) = \frac{\hat{u}^--\hat{v}^-}{(1-\hat{u}^-+\hat{v}^-)^3}(-\dot{u}_0^-(0)+\dot{v}_0^-(0)) - \frac{\hat{u}^--\hat{v}^-}{1-\hat{u}^-+\hat{v}^-}(-\dot{u}_0^+(0)+\dot{v}_0^+(0)). \tag{3.83}$$

Thirdly, combing Equations (3.77) and (3.83), it follows that

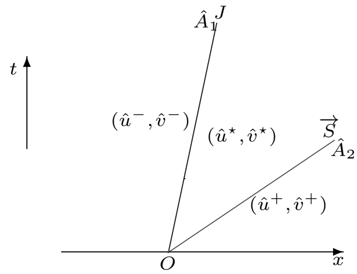
$$(1-u^-+v^-)(\ddot{x}_s(0)-\ddot{x}_c(0)) = \frac{\hat{v}^--\hat{u}^-}{1-\hat{u}^-+\hat{v}^-}(-\dot{u}_0^+(0)+\dot{v}_0^+(0)) > 0. \tag{3.84}$$

This implies that $\ddot{x}_s(0) > \ddot{x}_c(0)$.

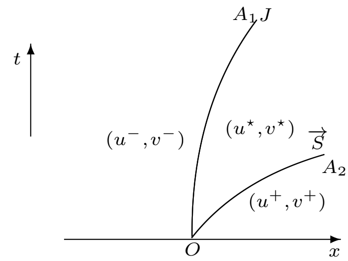
Finally, by virtue of Equations (3.68), (3.69), and (3.84), we obtain that the generalized Riemann solution to problem (1.1),(1.6) in this subcase clearly consists of a contact discontinuity J from (u^-,v^-) to (u^*,v^*), followed by a forward shock $\overrightarrow{S}$ from (u^*,v^*) to (u^+,v^+) near the origin. Thus we have completed the construction and proof of the generalized Riemann solution for this subcase as depicted in Figure 3.5.

Case 3: $-\hat{u}^-+\hat{v}^- < -\hat{u}^++\hat{v}^+ < 0$.

The Riemann solution to problem (1.1),(1.6) is to connect the state $(\hat{u}^-,\hat{v}^-)$ on the left to the intermediate state $(\hat{u}^\star,\hat{v}^\star) = \left(\frac{-\hat{u}^-+\hat{v}^-}{-\hat{u}^++\hat{v}^+}\hat{u}^+, \frac{-\hat{u}^-+\hat{v}^-}{-\hat{u}^++\hat{v}^+}\hat{v}^+\right)$ through a contact discontinuity J, and then connect the intermediate state $(\hat{u}^\star,\hat{v}^\star)$ to the state $(\hat{u}^+,\hat{v}^+)$ on the right through a forward shock wave $\overrightarrow{S}$. We depict the structure of the Riemann solution in Figure 3.6(a), where the contact discontinuity curve is $O\hat{A}_1: x = (1+\frac{1}{1-\hat{u}^-+\hat{v}^-})t$ and the shock wave curve is $O\hat{A}_2: x = (1+\frac{1}{(1-\hat{u}^-+\hat{v}^-)(1-\hat{u}^++\hat{v}^+)})t$.



(a) the Riemann solution



(b) the generalized Riemann solution

Fig. 3.6. *Case 3.*

Motivated by the corresponding Riemann solution, we will prove that the generalized Riemann solution to problem (1.1),(1.6) admits a unique local solution, which is a contact discontinuity and a forward shock wave together. As shown in Figure 3.6(b), on the contact discontinuity curve $x=x_c(t)$, we have

$$\frac{dx_c(t)}{dt}=1+\frac{1}{1-u^\star+v^\star}, \tag{3.85}$$

$$-u^-+v^-=-u^\star+v^\star. \tag{3.86}$$

On the shock wave curve $x=x_s(t)$, we have

$$\frac{dx_s(t)}{dt}=1+\frac{1}{(1-u^\star+v^\star)(1-u^++v^+)}, \tag{3.87}$$

$$u^+v^\star=u^\star v^+. \tag{3.88}$$

Both of them are free boundaries. Here the unknown generalized Riemann solution of problem (1.1),(1.6) on the domain $\{(x,t)\,|\,x_c(t)<x<x_s(t), 0\le t<\epsilon\}$ is denoted by $(u^\star(x,t),v^\star(x,t))$. From the boundary conditions (3.86) and (3.88), we find the value $(u^\star(0,0),v^\star(0,0))=(\hat{u}^\star,\hat{v}^\star)=(\frac{-\hat{u}^-+\hat{v}^-}{-\hat{u}^++\hat{v}^+}\hat{u}^+,\frac{-\hat{u}^-+\hat{v}^-}{-\hat{u}^++\hat{v}^+}\hat{v}^+)$.

In this case, the generalized Riemann problem is equivalent to the free boundary problem (1.1) with the boundary conditions (3.85)~(3.88). We turn next to the above free boundary problem. By a standard method, we introduce the change of invariant

$$\begin{pmatrix}\mathcal{U}(x,t)\\ \mathcal{V}(x,t)\end{pmatrix}=\begin{pmatrix}l_1(\hat{u}^\star,\hat{v}^\star)\\ l_2(\hat{u}^\star,\hat{v}^\star)\end{pmatrix}\begin{pmatrix}u^\star(x,t)\\ v^\star(x,t)\end{pmatrix}=\begin{pmatrix}\hat{v}^\star & -\hat{u}^\star\\ -1 & 1\end{pmatrix}\begin{pmatrix}u^\star(x,t)\\ v^\star(x,t)\end{pmatrix}. \tag{3.89}$$

Using this change of invariant, on $x=x_c(t)$, the boundary condition (3.86) can be rewritten as

$$\mathcal{V}=-u^-+v^-. \tag{3.90}$$

On $x=x_s(t)$, the boundary condition (3.88) can be written as

$$\mathcal{U}=\frac{(u^+\hat{v}^\star-v^+\hat{u}^\star)\mathcal{V}}{v^+-u^+}. \tag{3.91}$$

Hence, the characterizing matrix A of this free boundary problem is of the form [12]:

$$A=\begin{pmatrix}0 & \dfrac{u^+\hat{v}^\star-v^+\hat{u}^\star}{v^+-u^+}\\ 0 & 0\end{pmatrix}. \tag{3.92}$$

By Lemma 2.1, it is easy to prove that $\|A\|_{min}=0<1$ for this subcase. Using Lemma 2.2, it follows that the above free boundary problem admits a unique piecewise C^1 solution locally in time. As depicted in Figure 3.6(b)), the generalized Riemann solution has the same structure as the corresponding Riemann solution in this case. This implies that the Riemann solution has a local structure stability near the origin after the perturbation of Riemann initial data.

Case 4: $-\hat{u}^+ + \hat{v}^+ < 0 < -\hat{u}^- + \hat{v}^-$.

As shown in Figure 3.7(a), the corresponding Riemann solution to problem (1.1),(2.1) consists of two rarefaction waves $\overleftarrow{R}_1$ and $\overrightarrow{R}_2$, which is separated by a vacuum intermediate state (0,0). Here

$$O\hat{A}_1 : x = (1 + \frac{1}{(1-\hat{u}^- + \hat{v}^-)^2})t \tag{3.93}$$

and

$$O\hat{A}_2 : x = (1 + \frac{1}{(1-\hat{u}^+ + \hat{v}^+)^2})t \tag{3.94}$$

are II-characteristics.

Motivated by the Riemann problem, we solve the generalized Riemann problem (1.1),(2.1) in the hope of obtaining a solution containing two centred waves near the origin. The generalized Riemann solution has the construction shown in Figure 3.7(b). On the boundary $OA_1 : x = \beta^-(t)$, we have

$$\frac{d\beta^-(t)}{dt} = (1 + \frac{1}{(1-u(\beta^-(t),t)+v(\beta^-(t),t))^2})t, \tag{3.95}$$

$$-u^*(\beta^-(t),t)+v^*(\beta^-(t),t) = -u^-(\beta^-(t),t)+v^-(\beta^-(t),t). \tag{3.96}$$

Equation (3.95) shows that $x=\beta^-(t)$ actually is a known left most II-characteristic curve. On the boundary condition $OA_2 : x = \beta^+(t)$, we have

$$\frac{d\beta^+(t)}{dt} = (1 + \frac{1}{(1-u(\beta^+(t),t)+v(\beta^+(t),t))^2})t, \tag{3.97}$$

$$-u^*(\beta^+(t),t)+v^*(\beta^+(t),t) = -u^+(\beta^+(t),t)+v^+(\beta^+(t),t). \tag{3.98}$$

Equation (3.97) implies that $x=\beta^+(t)$ is also a known right most II-characteristic curve. We denote the generalized Riemann solution on the triangle-like domain A_1OA_2 by (u^*,v^*), which is an unknown regular solution to system (1.1).

In order to solve the free boundary problem (1.1) with boundary conditions (3.95)$\sim$(3.98), we introduce the Riemann invariants (2.6) as new unknown functions. The system (1.1) can be expressed in the following diagonal form

$$\begin{cases} \varsigma_t + (1 + \dfrac{1}{1+\zeta})\varsigma_x = 0, \\ \zeta_t + (1 + \dfrac{1}{(1+\zeta)^2})\zeta_x = 0. \end{cases} \tag{3.99}$$

In terms of the Riemann invariants (ζ,ς), the preceding boundary condition (3.96) on OA_1 can be rewritten in the form of

$$\zeta(u^*(\beta^-(t),t),v^*(\beta^-(t),t)) = \zeta(u^-(\beta^-(t),t),v^-(\beta^-(t),t)), \tag{3.100}$$

and boundary condition (3.98) on OA_2 can be rewritten as

$$\zeta(u^*(\beta^+(t),t),v^*(\beta^+(t),t))=\zeta(u^+(\beta^+(t),t),v^+(\beta^+(t),t)). \tag{3.101}$$

Solving the characteristic problem (3.99) with data (3.100) and (3.101), we obtain a group of II-characteristics drawn from the origin O, which is a centred wave with centre O. In this case, the generalized Riemann problem (1.1),(1.6) admits a unique piecewise C^1 solution locally in time, and this solution has two centred waves. As depicted in Figure 3.7, the structure of the solution to the Riemann problem is stable near the origin after the perturbation of Riemann initial data.

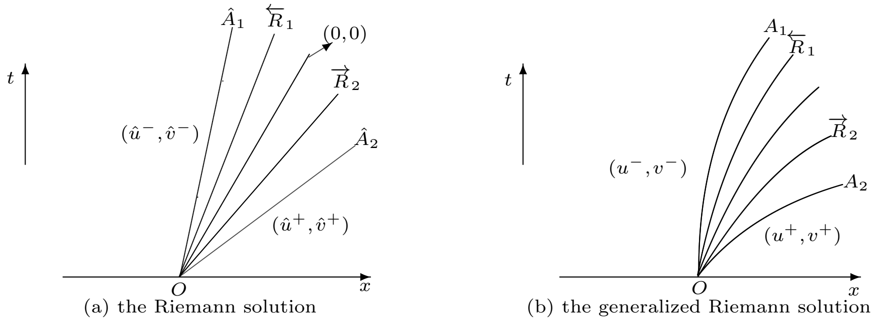


Fig. 3.7. *Case 4.*

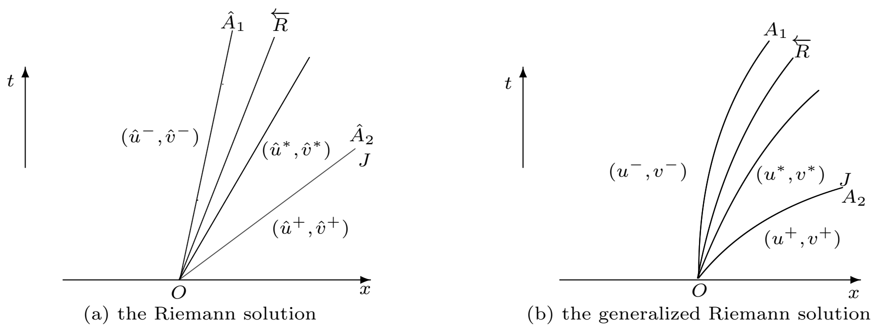


Fig. 3.8. *Case 5.*

Case 5: $0\le -\hat{u}^+ + \hat{v}^+ < -\hat{u}^- + \hat{v}^-$.

For this case, the corresponding Riemann solution to problem (1.1),(2.1) consists of a backward rarefaction wave $\overleftarrow{R}$ followed by a contact discontinuity J. As shown in Figure 3.8(a),

$$O\hat{A}_1 : x=(1+\frac{1}{(1-\hat{u}^-+\hat{v}^-)^2})t \tag{3.102}$$

is a II-characteristic;

$$O\hat{A}_2 : x=(1+\frac{1}{1-\hat{u}^+ + \hat{v}^+})t \tag{3.103}$$

is a contact discontinuity and the intermediate state
$(\hat{u}^*,\hat{v}^*)=(\frac{-\hat{u}^+ + \hat{v}^+}{-\hat{u}^- + \hat{v}^-}\hat{u}^-, \frac{-\hat{u}^+ + \hat{v}^+}{-\hat{u}^- + \hat{v}^-}\hat{v}^-)$.

It is natural to try to solve the generalized Riemann problem (1.1),(1.6) by a similar structure near the origin. We depict the solution in Figure 3.8(b). $OA_1 : x = \beta^-(t)$ is actually a known left most characteristic curve, on which we have

$$\frac{d\beta^-(t)}{dt} = 1 + \frac{1}{(1-u^- + v^-)^2}, \tag{3.104}$$

$$-u^- + v^- = -u^* + v^*. \tag{3.105}$$

$OA_2 : x = x_c(t)$ is a contact discontinuity, which satisfies the boundary conditions (3.7) and (3.8) as **Case 1** completely. $(u^-(x,t), v^-(x,t))$ and $(u^+(x,t), v^+(x,t))$ are known smooth functions, The intermediate state (u^*, v^*) is an unknown regular solution to the generalized Riemann problem (1.1),(1.6). Moreover, the value

$$(u^*(0,0), v^*(0,0)) = (\hat{u}^*, \hat{v}^*) = (\frac{-\hat{u}^+ + \hat{v}^+}{-\hat{u}^- + \hat{v}^-}\hat{u}^-, \frac{-\hat{u}^+ + \hat{v}^+}{-\hat{u}^- + \hat{v}^-}\hat{v}^-),$$

is determined uniquely from boundary conditions (3.8) and (3.105).

What remains is to solve the free boundary problem (1.1) with boundary conditions (3.7)∼(3.8) and (3.104)∼(3.105) on the fan-shaped domain $\{(x,t) \,|\, \beta^-(t) < x < x_s(t), 0 \leq t < \epsilon\}$, $\epsilon > 0$ small. We turn next to this problem. By a standard method, we introduce the change of invariant (3.10). The boundary condition (3.105) on $x = \beta^-(t)$ can be reduced to the following form

$$\mathcal{V} = u^*\hat{v}^* - v^*\hat{u}^* = u^-\hat{v}^* - v^-\hat{u}^*, \tag{3.106}$$

the right side of which is actually a known function of t. The boundary condition (3.8) on $x = x_c(t)$ still can be rewritten as (3.12). Hence, for this case, the characterizing matrix A of the above free boundary problem has a simpler form

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We get that the minimal characterizing number

$$\| A \|_{min} = 0 < 1.$$

By Lemma 2.2, the free boundary problem under consideration admits a unique piecewise smooth solution locally in time. This shows that the generalized Riemann solution to problem (1.1),(1.6) near the origin is a centred wave and a contact discontinuity together as depicted in Figure 3.8(b).

Comparing the generalized Riemann solution with corresponding Riemann solution in this case, we see that the Riemann solution has a local structure stability after the perturbation of Riemann initial data.

Case 6: $-\hat{u}^+ + \hat{v}^+ < -\hat{u}^- + \hat{v}^- \leq 0$.

As depicted in Figure 3.9(a), the corresponding Riemann solution to problem (1.1),(2.1) consists of a contact discontinuity J from $(\hat{u}^-, \hat{v}^-)$ to $(\hat{u}^\star, \hat{v}^\star)$, and a forward rarefaction wave $\overrightarrow{R}$ from $(\hat{u}^\star, \hat{v}^\star)$ to $(\hat{u}^+, \hat{v}^+)$ for this case. Here the intermediate state $(\hat{u}^\star, \hat{v}^\star) = (\frac{-\hat{u}^- + \hat{v}^-}{-\hat{u}^+ + \hat{v}^+}\hat{u}^+, \frac{-\hat{u}^- + \hat{v}^-}{-\hat{u}^+ + \hat{v}^+}\hat{v}^+)$. The contact discontinuity J is defined by

$$O\hat{A}_1 : x = (1 + \frac{1}{1 - \hat{u}^- + \hat{v}^-})t,$$

and the II-characteristic is given by

$$O\hat{A}_2 : x = (1 + \frac{1}{(1-\hat{u}^+ + \hat{v}^+)^2})t.$$

Noting the above Riemann solution, we still hope that the generalized solution of problem (1.1),(1.6) consists of a contact discontinuity J from (u^-, v^-) to $(u^\star, v^\star)$, and a centred wave $\overrightarrow{R}$ from $(u^\star, v^\star)$ to (u^+, v^+). Here the intermediate state $(u^\star, v^\star)$ is an unknown regular solution to system (1.1),(1.6); see Figure 3.9(b). On the boundary $OA_1 : x = x_c(t)(x_c(0) = 0)$, we have

$$\frac{dx_c(t)}{dt} = 1 + \frac{1}{1 - u^- + v^-}, \tag{3.107}$$

$$-u^- + v^- = -u^\star + v^\star, \tag{3.108}$$

which is a contact discontinuity. On the boundary $OA_2 : x = \beta^+(t)$, we have

$$\frac{d\beta^+(t)}{dt} = 1 + \frac{1}{(1 - u^+ + v^+)^2}, \tag{3.109}$$

$$-u^\star + v^\star = -u^+ + v^+, \tag{3.110}$$

which is actually a known right most characteristic curve.

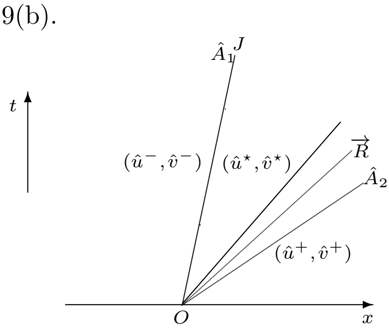
In this case, the generalized Riemann problem is equivalent to the free boundary problem (1.1) with boundary conditions (3.107)∼(3.110). Then it remains to determine whether the above free boundary problem has a unique solution on on the triangle-like domain A_1OA_2 locally in time. By a similar method as in **Case 3**, we introduce the change of variables (3.89). With this, on $x = x_c(t)$, the boundary condition (3.108) then reduces to (3.90), and the boundary condition (3.110) on can be reduced to the following form

$$\mathcal{U} = u^\star \hat{v}^\star - v^\star \hat{u}^\star = u^+ \hat{v}^\star - v^+ \hat{u}^\star. \tag{3.111}$$

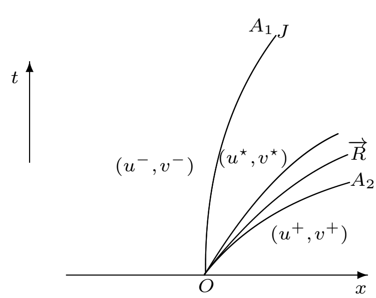
By virtue of Equations (3.90) and (3.111), it is easy to check that the boundary condition (3.90) and (3.111) are actually known functions of t. Hence, for this case, the characterizing matrix is

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which implies the minimal characterizing number $\| A \|_{min} = 0 < 1$. With this fact, the local solvability of the generalized solution to problem (1.1),(1.6) is established, see Figure 3.9(b).



(a) the Riemann solution



(b) the generalized Riemann solution

Fig. 3.9. *Case 6.*

The generalized Riemann solution shows that the Riemann solution in this case is stable near the origin after the perturbation of Riemann initial data.

Thus, the construction of the solution for the generalized Riemann problem (1.1),(1.6) is completed. In fact, we complete our proof of the following theorem.

THEOREM 3.1. *For any wave strength* $\|(\hat{u}^+,\hat{v}^+)-(\hat{u}^-,\hat{v}^-)\|$, *system* (1.1) *with initial data* (1.6) *always admits a unique local solution on a domain*

$$R(\epsilon)=\{(x,t)\,|\,-\infty<x<\infty, 0\leq t<\epsilon\},$$

where $\epsilon>0$ *is small. In a neighborhood of the origin* $(0,0)$ *in the* $x-t$ *plane, this solution has a similar structure to that of the corresponding Riemann problem* (1.1),(2.1) *in all of the cases except* **Case 2**: $-\hat{u}^-+\hat{v}^-\leq 0\leq -\hat{u}^++\hat{v}^+$. *In* **Case 2**, *we have three different subcases:*

(1) If $-\hat{u}^-+\hat{v}^-<0<-\hat{u}^++\hat{v}^+$, *or* $-\hat{u}^-+\hat{v}^-=0<-\hat{u}^++\hat{v}^+$ *with* $-\dot{u}_0^-(0)+\dot{v}_0^-(0)>0$, *or* $-\hat{u}^-+\hat{v}^-<0=-\hat{u}^++\hat{v}^+$ *with* $-\dot{u}_0^+(0)+\dot{v}_0^+(0)>0$, *then the solution of the generalized Riemann problem* (1.1),(1.6) *is a delta shock wave.*

(2) If $-\hat{u}^-+\hat{v}^-=0<-\hat{u}^++\hat{v}^+$ *with* $-\dot{u}_0^-(0)+\dot{v}_0^-(0)<0$, *then the solution of the generalized Riemann problem* (1.1),(1.6) *is a contact discontinuity followed by a shock wave.*

(3) If $-\hat{u}^-+\hat{v}^-<0=-\hat{u}^++\hat{v}^+$ *with* $-\dot{u}_0^+(0)+\dot{v}_0^+(0)<0$, *then the solution of the generalized Riemann problem* (1.1),(1.6) *is a shock wave followed by a contact discontinuity.*

Theorem 3.1 shows that a delta shock wave in the corresponding Riemann solution may turn into a combination of a shock wave and a contact discontinuity after the perturbations of Riemann initial data. The local structure stability of the Riemann solution fails, which is a new development on the generalized Riemann problem. Furthermore, the result allows us to better investigate the internal mechanism and instability of a delta shock wave.

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REFERENCES

[1] L. Ambrosio, G. Crippa, A. Figalli, and L. Spinolo, *Some new well-posedness results for continuity and transport equations, and applications to the chromatography system*, SIAM J. Math. Anal., 41:1890–1920, 2009.

[2] F. Bouchut, *On zero pressure gas dynamics, Advances in Kinetic Theory and Computing*, Selected Papers, Series on Advances in Mathematics for Applied Sciences, World Scientific, 22:171–190, 1994.

[3] F. Bouchut and F. James, *Duality solutions for pressureless gases, monotone scalar conservation laws, and uniqueness*, Comm. Part. Diff. Eqs., 24:2173–2189, 1999.

[4] Y. Brenier and E. Grenier, *Sticky particles and scalar conservation laws*, SIAM J. Num. Anal., 35(6):2317–2328, 1998.

[5] A. Bressan and W. Shen, *Uniqueness of discontinuous ODE and conservation laws*, Nonlinear anal., 34:637–652, 1998.

[6] H. Cheng and H. Yang, *Delta shock waves in chromatogrphy equations*, J. Math. Anal. Appl., 380:475–485, 2011.

[7] W. E, Y.G. Rykov, and Y.G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for systems of conservation laws arising in adhesion particle dynamics*, Comm. Math. Phys., 177:349–380, 1996.

[8] E. Grenier, *Existence globale pour le système des gaz sans pression*, C.R. Acad. Sci. Paris, Série I, 321:171–174, 1995.

[9] L.H. Guo, L.J Pan, and G. Yin, *The perturbed Riemann problem and delta contact discontinuity in chromatography equations*, Nonlinear Analysis-Theory Methods & Applications, 106:110–123, 2014.

[10] X.L. Han and L.J. Pan, *Formation and transition of delta shock to the chromatography equations*, 2018, submitted.

[11] T.T. Li, D. Serre, and H. Zhang, *The generalized Riemann problem for the motion of elastic strings*, SIAM J. Math. Anal., 23(5):1189–1203, 1992.

[12] T.T. Li and W.C. Yu, *Boundary value problems for quasilinear hyperbolic systems*, Duck. Univ. Math. Ser. V, Durham, NC, 1985.

[13] M. Mazzotti, *Nonclassical composition fronts in nonlinear chromatography: delta-shock*, Ind. Eng. Chem. Res., 48:7733–7752, 2009.

[14] M. Mazzotti, A. Tarafder, J. Cornel, F. Gritti, and G. Guiochon, *Experimental evidence of a delta-shock in nonlinear chromatography*, J. Chromatogr. A, 1217:2002–2012, 2010.

[15] C. Shen, *The Riemann problem for the Chaplygin gas equations with a source term*, Z. Angew. Math. Mech., 96(6):681–695, 2016.

[16] C. Shen, *Wave interactions and stability of the Riemann solutions for the chromatography equations*, J. Math. Anal. Appl., 365:609–618, 2010.

[17] W. Sheng and T. Zhang, *The Riemann Problem for the Transportation Equations in Gas Dynamics*, Mem. Amer. Math. Soc., 137:654, 1999.

[18] M. Sun, *Delta shock waves for the chromatography equations as self-similar viscosity limits*, Q. Appl. Math., 69:425–443, 2011.

[19] H. Yang and Y. Zhang, *New developments of delta shock waves and its applications in systems of conservation laws*, J. Diff. Eqs., 252:5951–5993, 2012.

[20] Y.B. Zeldovich and A.D. Myshkis, *Elements of Mathematical Physics: Medium Consisting of Noninteracting Particles (in Russian)*, Nauka, Moscow, 252, MR0351195(50:3684), 1973.