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ASYMPTOTIC EXPANSION WITH BOUNDARY LAYER ANALYSIS FOR STRONGLY ANISOTROPIC ELLIPTIC EQUATIONS*

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Abstract. In this article, we derive the asymptotic expansion, in theory, up to an arbitrary order for the solution of a two-dimensional elliptic equation with strongly anisotropic diffusion coefficients along different directions, subject to the Neumann boundary condition and the Dirichlet boundary condition on specific parts of the domain boundary, respectively. The ill-posedness arising from the Neumann boundary condition in the strongly anisotropic diffusion limit is handled by the decomposition of the solution into a mean part and a fluctuation part. The boundary layer analysis due to the Dirichlet boundary condition is conducted for each order in the expansion for the fluctuation part. Our results suggest that the leading order is the combination of the mean part and the composite approximation of the fluctuation part for the general Dirichlet boundary condition. We also apply this method to derive the results for the heterogeneous diffusion problems.

Keywords. strongly anisotropic elliptic equation; matched asymptotic expansion; boundary layer analysis.

AMS subject classifications. 35B25; 35C20; 35J25.

1. Introduction

The strongly anisotropic elliptic problem that we consider in this article is the following equation imposed in the domain $D=(0,\ 1)\times(0,\ 1)$ with mixed Dirichlet–Neumann boundary conditions:

$$\begin{cases} -\varepsilon^{-2}\partial_x^2 u_\varepsilon(x,y) - \partial_y^2 u_\varepsilon(x,y) = f(x,y), & \text{in } D, \\ \partial_x u_\varepsilon(0,y) = \partial_x u_\varepsilon(1,y) = 0, & \text{for } 0 \leqslant y \leqslant 1, \\ u_\varepsilon(x,0) = \phi_0(x), \quad u_\varepsilon(x,1) = \phi_1(x), & \text{for } 0 \leqslant x \leqslant 1. \end{cases} \tag{1.1}$$

The special feature for this equation is that ε is a small positive number, that is, the diffusion coefficient along the x-direction is very large. Here the Neumann boundary conditions are imposed on the left and right boundaries and the Dirichlet boundary conditions are imposed on the top and bottom boundaries of the rectangular domain.

The Equation (1.1) belongs to a large class of diffusion models with the strongly anisotropic diffusion coefficients from many applications, e.g., flows in porous media, semiconductor modeling, heat conduction in fusion plasmas, and so on. For instance, in magnetized plasmas, the particles are confined by the magnetic field along the field lines. In such cases, the distance between two successive collisions is extremely large compared to the mean free path in the perpendicular direction, which yields strongly anisotropic diffusion tensors. Note that we use ε^2 here rather than ε in the diffusion coefficients for a reason we will justify later. Furthermore, in this simplified model (1.1),

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the line field parallels to the x-axis. In more realistic models, the line field may not be so simple and could be a closed loop.

Our main interest is to examine the asymptotic behaviors of the solution u_ε such as the asymptotic expansion $u_\varepsilon \sim u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$. In this paper we first work on the Equation (1.1) and then generalize the results to the case where the diffusion coefficients are heterogeneous (see the Equation (5.1) below). We first show why the traditional formal asymptotic expansions, by directly plugging the ansatz into (1.1), fails to provide meaningful results. If this approach is applied, it is easy to see that all u_i's share the same homogeneous Neumann boundary conditions at $x=0$ and $x=1$. Furthermore, $\partial_x^2 u_0 = \partial_x^2 u_1 = 0$, $-\partial_x^2 u_2 - \partial_y^2 u_0 = f$, $-\partial_x^2 u_{k+2} - \partial_y^2 u_k = 0$ $\forall k \geqslant 1$. It follows that both u_0 and u_1 are constants along the x-direction, thus only depend on y. But these two functions $u_0(y)$ and $u_1(y)$ are undetermined, and it follows that $\{u_k : k \geqslant 2\}$ cannot to be determined either. There are no extra conditions to help resolve this issue.

The above ill-posedness also imposes severe numerical challenges. The traditional numerical methods for the elliptic equations, such as the standard five-point scheme, suffer from the large condition numbers for tiny ε. There have been a lot of efforts focusing on the efficient numerical methods for the strongly anisotropic elliptic problem. In particular, one class of asymptotic preserving (AP) method developed by P. Degond et al [5–8], is to decompose the solution u_ε into two parts, a *mean* part along the strongly diffusive direction and a *fluctuation* part. This mean-fluctuation decomposition reformulates the original equation into a coupled system of the equations for these two parts. Another idea proposed in [19] is to replace one of the Neumann boundary condition by the integration of the original equation along the field line.

We shall see below that such a mean-fluctuation decomposition in [7] is also essential to our asymptotic analysis. Besides the ill-posedness, due to the Neumann boundary condition, we point out another barrier in our analysis: the existence of the boundary layers around $y=0$ and $y=1$. This fact can be partially observed from the u_0 term in the above naive expansion: The derived conclusion that u_0 is a function of only y is inconsistent with the Dirichlet boundary conditions in (1.1) unless both functions ϕ_0 and ϕ_1 are constant. In general, a function of only the y variable (such as the so-called mean part) is not capable of describing the limiting solution in the whole domain because there exist boundary layers near each nonconstant Dirichlet boundary. Inside these boundary layers, the function of only the y variable has to be corrected to match the nonconstant Dirichlet boundary conditions. It is noteworthy that for existing numerical examples presented in previous works such as [8, 19], as far as the authors know, the Dirichlet boundary conditions are always homogeneous, and so there are no boundary layers. Actually, those numerical examples are intentionally constructed by choosing a true solution without boundary layers first and then defining the force term f accordingly. However, as we argued above, the emergence of the boundary layer is generic. This phenomena is a big challenge in our asymptotic analysis.

Elliptic equations are intrinsically connected to the probabilistic diffusion processes described by Itō stochastic differential equations (SDE) [14]. Traditionally, there are many classical research works applying the asymptotic methods to the elliptic equations associated with some small parameters to obtain the leading order quantities related to the corresponding diffusion processes. One well-known example is the exit problem of a particle in a potential well perturbed by small isotropic noise (e.g. [13,15,16,18]). These works used the singular perturbation techniques (e.g. the WKB method) to analyze the boundary layers at the homogenous Dirichlet boundaries induced by the small noise intensity. However, for our strongly anisotropic elliptic problem (1.1), the underlying stochastic dynamics is not the SDE with small isotropic noise, but the following slow-fast two-time-scale system: let (X_t,Y_t) be the position of a particle in D satisfying the following SDE

$$\begin{cases} \mathrm{d}X_t = \varepsilon^{-1}\mathrm{d}W_t, \\ \mathrm{d}Y_t = \mathrm{d}B_t, \end{cases} \tag{1.2}$$

subject to the reflection boundary condition on the left and right boundaries ($x=0$ and $x=1$) and the absorbing boundary condition on the top and bottom boundaries ($y=0$ and $y=1$). Here W_t and B_t are two independent (standard) Brownian motions. By the Feymann-Kac formula [14], the solution to (1.1) is represented by

$$u_\varepsilon(x,y) = \mathbb{E}\left[\phi_{Y_\tau}(X_\tau) + \frac{1}{2}\int_0^\tau f(X_t,Y_t)\,\mathrm{d}t \;\middle|\; (X_0,Y_0)=(x,y)\right], \tag{1.3}$$

where $\tau = \inf\{t>0 : Y_t = 0 \text{ or } 1\}$ is the absorption time of the Y process to the Dirichlet boundaries.

The solution to (1.2) is straightforward: $X_t = X_0 + \varepsilon^{-1}W_t \overset{\mathrm{d}}{=} X_0 + W_{t/\varepsilon^2}$ and $Y_t = Y_0 + B_t$ ("$\overset{\mathrm{d}}{=}$" means the equality in the sense of distributions). So X_t is a fast Brownian motion and Y_t is a slow one. The particle randomly moves drastically fast with the speed at the order $\mathcal{O}(\varepsilon^{-1})$ along the x-direction while with a normal speed at the order $\mathcal{O}(1)$ in the y-direction. By the averaging principle [11], the leading order dynamics as the limit of $\varepsilon\downarrow 0$ is the expectation of the slow dynamics for Y_t with respect to the invariant measure of the fast variable X_t, which happens to be a uniform distribution here. Thus the expectation of the integral part in (1.3) should have a limit independent of the x variable, which is exactly the so called *mean* part defined in [7]. In essence, the mean-fluctuation decomposition is the result of averaging principle and the ergodicity of the fast process. The boundary layer is generated by the first term ϕ_0,ϕ_1 in (1.3), whose expectation depends on the distribution of the absorbing point (X_τ,Y_τ). If the starting position (x,y) is away from the absorbing boundary, then the absorbing time τ is sufficient large compared to the $\mathcal{O}(\varepsilon)$ relaxation time to the equilibrium in the x-direction, so that the averaging principle holds, and the limit of (1.3) is a function of the variable y only. However, the averaging principle breaks down if the initial position (x,y) is very close to the absorbing boundary so that τ would be too short to allow the fast dynamics to relax to the equilibrium. It is easy to see that this occurs if the distance to the boundary is $\mathcal{O}(\varepsilon)$, thus the thickness of the boundary layers around $y=0$ and $y=1$ is $\mathcal{O}(\varepsilon)$.

Our main motivation is to give a more detailed understanding of the above probabilistic picture by the tool of asymptotic analysis. The goal is to derive a series of approximate functions to the solution u_ε up to an arbitrary order as $\varepsilon\downarrow 0$. Due to the different nature of the underlying stochastic dynamics, our asymptotic analysis is completely different from the traditional singular perturbation approach. For the general Dirichlet boundary conditions in (1.1), we shall show that each asymptotic term $u_0,u_1,u_2,\cdots$ exhibits the boundary layer effect. In particular, the leading order $u_0(x,y)$ is not simply the mean part $\bar{u}(y) = \int_0^1 u_\varepsilon(x,y)\,\mathrm{d}x$. To attack the ill-posedness arising from the Neumann boundary conditions, we utilize the framework of mean-fluctuation decomposition; to deal with the boundary layers originating from the nonconstant Dirichlet boundary conditions, we adopt the Van Dyke's method of matched asymptotic expansions [9] in which the outer expansion and the inner expansion are both conducted. The series in the outer expansion are described by the y-parametrized one-dimensional Neumann boundary value problem in the x variable, while the series in the inner expansion are in the form of the two-dimensional elliptic equations which are solved with the aid of the Fourier series.

The above method can also be applied to more general diffusion models where the diffusion coefficients are heterogeneous in space, with a more technical analysis of the boundary layers. Firstly, the equations for the mean and fluctuation parts now form a truly coupled system so that we will also need to perform the boundary layer analysis for the mean part. Secondly, after the change of variables in the inner regions within the boundary layers, the diffusion coefficients still explicitly depend on ε, and consequently we need the Taylor expansions of the diffusion coefficients. Lastly, the terms in the inner expansion can be expanded in the generalized eigenfunctions of a generalized eigenvalue problem arising from the application of separation of variables, but these generalized eigenfunctions cannot be expressed analytically (recall that we resort to the Fourier series to obtain analytical expressions for the simple model (1.1)).

The other extensions to the situations more relevant for applications may include the nonlinear strongly anisotropic diffusion problems, or the case where the field lines are not simply along the x-axis direction, but in the form of general curves. The theoretic asymptotic analysis is not well suitable for these complicated models. Several efficient numerical methods have been developed for these problems [1] [2], however, the boundary conditions imposed there are all pure Neumann boundary conditions. There are also some other asymptotic works on more general problems for elliptic equations. For instance, [3] focuses on the interface problem where the diffusion coefficients in different subdomains have huge differences, [4] conducts an asymptotic analysis of elliptic problems with perturbed domains or interfaces.

The rest of the paper is organized as follows. Section 2 presents our main result of the asymptotic expansion. Section 3 gives a rigorous proof of our formal expansion. Section 4 presents a specific example to illustrate the effect of boundary layer and the convergence order in ε. Section 5 illustrates how to generalize these results to the more representative models with the heterogeneous diffusion coefficients and the appendix collects some technical details for this section. The last section contains our concluding discussion.

2. Asymptotic results

2.1. Decomposing the solution into the mean value and the fluctuation.

For the solution u_ε to (1.1), we introduce the mean part $\bar{u}$ along the line field, i.e., the x-coordinate

$$\bar{u}(y) := \int_0^1 u_\varepsilon(x,y)\,\mathrm{d}x,$$

and denote the residual as the fluctuation part $\tilde{u}_\varepsilon$,

$$\tilde{u}_\varepsilon := u_\varepsilon - \bar{u}.$$

We shall show, in the following paragraph, that $\bar{u}$ is independent of ε here; so it is denoted by $\bar{u}$ instead of $\bar{u}_\varepsilon$.

By integrating both sides of the Equation (1.1) with respect to x over $[0,\ 1]$, we obtain

$$\begin{cases} -\bar{u}''(y) = \bar{f}(y), & \text{in } (0,\ 1), \\ \bar{u}(0) = \bar{\phi}_0, \quad \bar{u}(1) = \bar{\phi}_1, \end{cases} \tag{2.1}$$

where

$$\bar{f}(y)=\int_0^1 f(x,y)\,\mathrm{d}x,\quad \bar{\phi}_0=\int_0^1 \phi_0(x)\,\mathrm{d}x,\quad \bar{\phi}_1=\int_0^1 \phi_1(x)\,\mathrm{d}x.$$

Clearly, (2.1) is a well-posed linear two-point boundary value problem, and $\bar{u}$ can be solved uniquely. Formally,

$$\bar{u}(y)=y\left(\int_0^1\int_0^z \bar{f}(t)\,\mathrm{d}t\mathrm{d}z-\bar{\phi}_0+\bar{\phi}_1\right)-\int_0^y\int_0^z \bar{f}(t)\,\mathrm{d}t\mathrm{d}z+\bar{\phi}_0. \tag{2.2}$$

Subtracting (2.1) from (1.1) yields the elliptic problem for the fluctuating part:

$$\begin{cases} -\varepsilon^{-2}\partial_x^2\tilde{u}_\varepsilon-\partial_y^2\tilde{u}_\varepsilon=\tilde{f}, & \text{in } D,\\ \partial_x\tilde{u}_\varepsilon(0,y)=\partial_x\tilde{u}_\varepsilon(1,y)=0, & \text{for } 0\leqslant y\leqslant 1,\\ \tilde{u}_\varepsilon(x,0)=\tilde{\phi}_0(x),\quad \tilde{u}_\varepsilon(x,1)=\tilde{\phi}_1(x), & \text{for } 0\leqslant x\leqslant 1, \end{cases} \tag{2.3}$$

where

$$\tilde{f}(x,y)=f(x,y)-\bar{f}(y),\quad \tilde{\phi}_0(x)=\phi_0(x)-\bar{\phi}_0,\quad \tilde{\phi}_1(x)=\phi_1(x)-\bar{\phi}_1.$$

Note that by construction, we have

$$\begin{aligned} \int_0^1 \tilde{u}_\varepsilon(x,y)\,\mathrm{d}x=0,&\quad \text{for } 0\leqslant y\leqslant 1,\\ \int_0^1 \tilde{f}(x,y)\,\mathrm{d}x=0,&\quad \text{for } 0\leqslant y\leqslant 1,\\ \int_0^1 \tilde{\phi}_0(x)\,\mathrm{d}x=\int_0^1 \tilde{\phi}_1(x)\,\mathrm{d}x=0.& \end{aligned} \tag{2.4}$$

2.2. Asymptotic expansions of the fluctuation $\tilde{u}_\varepsilon$. Our main task is to seek an asymptotic expansion of the fluctuation $\tilde{u}_\varepsilon$. Formally, as $\varepsilon\downarrow 0$ in (2.3) and (2.4), the formal limit $\tilde{u}_0=\lim_{\varepsilon\downarrow 0}\tilde{u}_\varepsilon$ would satisfy

$$\begin{cases} \partial_x^2\tilde{u}_0=0, & \text{in } D,\\ \partial_x\tilde{u}_0(0,y)=\partial_x\tilde{u}_0(1,y)=0, & \text{for } 0\leqslant y\leqslant 1,\\ \int_0^1 \tilde{u}_0(x,y)\,\mathrm{d}x=0, & \text{for } 0\leqslant y\leqslant 1,\\ \tilde{u}_0(x,0)=\tilde{\phi}_0(x),\quad \tilde{u}_0(x,1)=\tilde{\phi}_1(x), & \text{for } 0\leqslant x\leqslant 1. \end{cases} \tag{2.5}$$

Clearly, this is an ill-posed problem unless $\tilde{\phi}_0(x)\equiv 0$ and $\tilde{\phi}_1(x)\equiv 0$, since the first three equations in (2.5) yield $\tilde{u}_0\equiv 0$. This inconsistency implies that we have a singular perturbation problem and anticipate the emergence of two boundary layer regions near the Dirichlet boundaries $y=0$ and $y=1$ respectively. We apply the Van Dyke's method of matched asymptotic expansions [9] to tackle this problem, i.e., first separately solve the problem in the inner regions within the boundary layers and in the outer region away from the boundary layers, then match them at the edges of the boundary layers.

2.2.1. Outer expansion. Assume the following outer expansion away from the Dirichlet boundaries $y=0$ and $y=1$:

$$\tilde{u}_\varepsilon^{\mathbf{ot}}(x,y)=\sum_{n=0}^{\infty}\varepsilon^n\tilde{u}_n^{\mathbf{ot}}(x,y).$$

Substituting this into the equation in (2.3) and equating coefficients, we obtain

$$-\partial_x^2\tilde{u}_0^{\mathbf{ot}}\equiv 0, \tag{2.6}$$

$$-\partial_x^2\tilde{u}_1^{\mathbf{ot}}\equiv 0, \tag{2.7}$$

$$-\partial_x^2\tilde{u}_2^{\mathbf{ot}}=\partial_y^2\tilde{u}_0^{\mathbf{ot}}+\tilde{f}, \tag{2.8}$$

$$-\partial_x^2\tilde{u}_n^{\mathbf{ot}}=\partial_y^2\tilde{u}_{n-2}^{\mathbf{ot}},\quad \text{for } n\geqslant 3. \tag{2.9}$$

These equations are a set of parametric one dimensional differential equations in the x variable and y is in the role of parameters. The outer expansion solutions must also satisfy the Neumann boundary condition as in (2.3) and the integral condition (2.4), which gives for any n

$$\begin{cases}\partial_x\tilde{u}_n^{\mathbf{ot}}(0,y)=\partial_x\tilde{u}_n^{\mathbf{ot}}(1,y)=0, & \text{for } 0\leqslant y\leqslant 1,\\ \displaystyle\int_0^1\tilde{u}_n^{\mathbf{ot}}(x,y)\,\mathrm{d}x=0, & \text{for } 0\leqslant y\leqslant 1.\end{cases} \tag{2.10}$$

Then each $\tilde{u}_n^{\mathbf{ot}}$ is the unique solution to these Neumann problems due to the second condition in (2.10). We can solve $\tilde{u}_n^{\mathbf{ot}}$ recursively from (2.6)∼(2.9) together with (2.10). In particular, we have

$$\tilde{u}_0^{\mathbf{ot}}(x,y)\equiv 0, \tag{2.11}$$

$$\tilde{u}_n^{\mathbf{ot}}(x,y)\equiv 0,\quad \text{for odd } n,$$

$$\tilde{u}_2^{\mathbf{ot}}(x,y)=-\tilde{F}_2(x,y)+\tilde{F}_3(1,y). \tag{2.12}$$

Here $\tilde{F}_n$ is defined recursively as

$$\tilde{F}_n(x,y)=\begin{cases}\displaystyle\int_0^x\tilde{F}_{n-1}(z,y)\,\mathrm{d}z, & \text{for } n\geqslant 1,\\ \tilde{f}(x,y), & \text{for } n=0.\end{cases} \tag{2.13}$$

2.2.2. Inner expansion near $y=0$. Next we explore the inner solution near $y=0$ in terms of the stretched variable $\xi=y/\varepsilon$ by assuming

$$\tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi)=\sum_{n=0}^{\infty}\varepsilon^n\tilde{u}_n^{\mathbf{in},0}(x,\xi).$$

In terms of ξ, the equation in (2.3) becomes

$$-\varepsilon^{-2}\partial_x^2\tilde{u}_\varepsilon(x,\xi)-\varepsilon^{-2}\partial_\xi^2\tilde{u}_\varepsilon(x,\xi)=\tilde{f}(x,\varepsilon\xi). \tag{2.14}$$

Thus the inner expansion $\tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi)$ near $y=0$ asymptotically satisfies the Equation (2.14) with the boundary conditions

$$\begin{cases}\partial_x\tilde{u}_\varepsilon^{\mathbf{in},0}(0,\xi)=\partial_x\tilde{u}_\varepsilon^{\mathbf{in},0}(1,\xi)=0, & \text{for } \xi\geqslant 0,\\ \tilde{u}_\varepsilon^{\mathbf{in},0}(x,0)=\tilde{\phi}_0(x), & \text{for } 0\leqslant x\leqslant 1,\end{cases}$$

and the integral condition

$$\int_0^1 \tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi)\,\mathrm{d}x = 0, \quad \text{for } \xi \geqslant 0.$$

We can write the Taylor expansion of the fluctuation part of the external force:

$$\tilde{f}(x,\varepsilon\xi) = \sum_{n=0}^{\infty} \frac{\varepsilon^n \xi^n}{n!} \partial_y^n \tilde{f}(x,0),$$

then equate coefficients of the same powers of ε to obtain the following two-dimensional elliptic equations on the domain $(x,\xi) \in D' := (0,\ 1) \times (0,\ +\infty)$:

$$\begin{cases} -\partial_x^2 \tilde{u}_0^{\mathbf{in},0} - \partial_\xi^2 \tilde{u}_0^{\mathbf{in},0} = 0, & \text{in } D', \\ \partial_x \tilde{u}_0^{\mathbf{in},0}(0,\xi) = \partial_x \tilde{u}_0^{\mathbf{in},0}(1,\xi) = 0, & \text{for } \xi \geqslant 0, \\ \tilde{u}_0^{\mathbf{in},0}(x,0) = \tilde{\phi}_0(x), & \text{for } 0 \leqslant x \leqslant 1, \\ \displaystyle\int_0^1 \tilde{u}_0^{\mathbf{in},0}(x,\xi)\,\mathrm{d}x = 0, & \text{for } \xi \geqslant 0, \end{cases} \tag{2.15}$$

$$\begin{cases} -\partial_x^2 \tilde{u}_1^{\mathbf{in},0} - \partial_\xi^2 \tilde{u}_1^{\mathbf{in},0} = 0, & \text{in } D', \\ \partial_x \tilde{u}_1^{\mathbf{in},0}(0,\xi) = \partial_x \tilde{u}_1^{\mathbf{in},0}(1,\xi) = 0, & \text{for } \xi \geqslant 0, \\ \tilde{u}_1^{\mathbf{in},0}(x,0) = 0, & \text{for } 0 \leqslant x \leqslant 1, \\ \displaystyle\int_0^1 \tilde{u}_1^{\mathbf{in},0}(x,\xi)\,\mathrm{d}x = 0, & \text{for } \xi \geqslant 0, \end{cases} \tag{2.16}$$

and for $n \geqslant 2$,

$$\begin{cases} -\partial_x^2 \tilde{u}_n^{\mathbf{in},0} - \partial_\xi^2 \tilde{u}_n^{\mathbf{in},0} = \dfrac{\xi^{n-2}}{(n-2)!} \partial_y^{n-2} \tilde{f}(x,0), & \text{in } D', \\ \partial_x \tilde{u}_n^{\mathbf{in},0}(0,\xi) = \partial_x \tilde{u}_n^{\mathbf{in},0}(1,\xi) = 0, & \text{for } \xi \geqslant 0, \\ \tilde{u}_n^{\mathbf{in},0}(x,0) = 0, & \text{for } 0 \leqslant x \leqslant 1, \\ \displaystyle\int_0^1 \tilde{u}_n^{\mathbf{in},0}(x,\xi)\,\mathrm{d}x = 0, & \text{for } \xi \geqslant 0. \end{cases} \tag{2.17}$$

These problems (2.15), (2.16) and (2.17) do not have the uniqueness of the solutions even the integral conditions $\int_0^1 \tilde{u}_n^{\mathbf{in},0}(x,\xi)\,\mathrm{d}x = 0$ are imposed. The uniqueness comes from matching the outer solutions, as we will show below.

We next solve $\tilde{u}_n^{\mathbf{in},0}$ by the method of separation of variables because of the simple geometry of the domain. We first work on the lowest order at $n = 0$, and the following terms will be determined in a next subsection. We look at the solutions that can be expanded into the form

$$\tilde{u}_0^{\mathbf{in},0}(x,\xi) = \sum_k A_k(\xi) B_k(x).$$

The equation and boundary conditions in (2.15) show that

$$B_k(x) = \cos(k\pi x), \quad \text{for } k \geqslant 0,$$

which form a complete orthogonal basis for the Hilbert space $L^2([0,\ 1])$. Thus we can expand $\tilde{u}_0^{\mathbf{in},0}(\cdot,\xi)$ and $\tilde{\phi}_0$ respectively in terms of these Fourier cosine series:

$$\tilde{u}_0^{\mathbf{in},0}(x,\xi)=\sum_{k=1}^{+\infty}A_k(\xi)\cos(k\pi x),\qquad \tilde{\phi}_0(x)=\sum_{k=1}^{+\infty}\phi_{0,k}\cos(k\pi x),$$

where the coefficients for $k\geqslant 1$ are respectively

$$A_k(\xi)=2\int_0^1\tilde{u}_0^{\mathbf{in},0}(x,\xi)\cos(k\pi x)\,\mathrm{d}x,\qquad \tilde{\phi}_{0,k}=2\int_0^1\tilde{\phi}_0(x)\cos(k\pi x)\,\mathrm{d}x.$$

Note that the terms for $k=0$ in these Fourier cosine series disappear since

$$A_0(\xi)=\int_0^1\tilde{u}_0^{\mathbf{in},0}(x,\xi)\,\mathrm{d}x=0,\qquad \tilde{\phi}_{0,0}=\int_0^1\tilde{\phi}_0(x)\,\mathrm{d}x=0.$$

Substituting these Fourier cosine expansions into (2.17), we deduce that for each $k\geqslant 1$, $A_k(\xi)$ satisfies

$$\begin{cases}-A_k''(\xi)+k^2\pi^2A_k(\xi)=0, & \text{in } (0,\ +\infty),\\ A_k(0)=\tilde{\phi}_{0,k}.\end{cases}$$

Hence we have that for $k\geqslant 1$,

$$A_k(\xi)=(c_k+\tilde{\phi}_{0,k})\mathrm{e}^{-k\pi\xi}-c_k\mathrm{e}^{k\pi\xi},$$

with the constants c_k to be determined later by matching the outer and inner solutions.

REMARK 2.1. We comment a bit on the possibility of generalizing the above calculations to a general line field. One can work in the curvilinear coordinate of the field line and obtain the equations for the outer expansions straightforwardly. But since in general the line field may not match the Dirichlet boundary like in our model (1.1), then the boundary layer may not be a rectangular band with a uniform width ε, and consequently, the stretching variable ξ would not be simply equal to y/ε; the geometric property of the field line and the Dirichlet boundary should be incorporated to derive the equations for the inner expansion near the Dirichlet boundary.

2.2.3. Matching. To determine the constants c_k's in the first-term approximation of the boundary layer solution near $y=0$, we make use of the essential point that the inner solution $\tilde{u}_0^{\mathbf{in},0}(x,\xi)$ and the outer solution $\tilde{u}_0^{\mathbf{ot}}(x,y)$ should match on the boundary of the layer near $y=0$, that is,

$$\lim_{\xi\to+\infty}\tilde{u}_0^{\mathbf{in},0}(x,\xi)=\lim_{y\to 0+}\tilde{u}_0^{\mathbf{ot}}(x,y)=0.$$

This gives $c_k=0$ for $k\geqslant 1$, and so $A_k(\xi)=\tilde{\phi}_{0,k}\mathrm{e}^{-k\pi\xi}$ for $k\geqslant 1$. Consequently,

$$\tilde{u}_0^{\mathbf{in},0}(x,\xi)=\sum_{k=1}^{+\infty}\tilde{\phi}_{0,k}\mathrm{e}^{-k\pi\xi}\cos(k\pi x). \tag{2.18}$$

2.2.4. Inner expansion near $y=1$. For the other Dirichlet boundary at $y=1$, we proceed in the exactly same way to derive the inner solution. Assume the inner expansion near $y=1$ in terms of the stretched variable $\eta=(1-y)/\varepsilon$,

$$\tilde{u}_\varepsilon^{\mathbf{in},1}(x,\eta)=\sum_{n=0}^{\infty}\varepsilon^n\tilde{u}_n^{\mathbf{in},1}(x,\eta).$$

In terms of η, the equation in (2.3) becomes

$$-\varepsilon^{-2}\partial_x^2\tilde{u}_\varepsilon(x,\eta)-\varepsilon^{-2}\partial_\eta^2\tilde{u}_\varepsilon(x,\eta)=\tilde{f}(x,1-\varepsilon\eta).$$

Thus the inner expansion $\tilde{u}_\varepsilon^{\mathbf{in},1}(x,\eta)$ near $y=1$ must asymptotically satisfy this equation and the boundary and integral conditions

$$\begin{cases}\partial_x\tilde{u}_\varepsilon^{\mathbf{in},1}(0,\eta)=\partial_x\tilde{u}_\varepsilon^{\mathbf{in},1}(1,\eta)=0, & \text{for } \eta\geqslant 0,\\ \tilde{u}_\varepsilon^{\mathbf{in},1}(x,0)=\tilde{\phi}_1, & \text{for } 0\leqslant x\leqslant 1,\\ \displaystyle\int_0^1\tilde{u}_\varepsilon^{\mathbf{in},1}(x,\eta)\,\mathrm{d}x=0, & \text{for } \eta\geqslant 0.\end{cases}$$

Again, using Taylor expansion and then equating coefficients of like powers, we obtain

$$\begin{cases}-\partial_x^2\tilde{u}_0^{\mathbf{in},1}-\partial_\eta^2\tilde{u}_0^{\mathbf{in},1}=0, & \text{in } D',\\ \partial_x\tilde{u}_0^{\mathbf{in},1}(0,\eta)=\partial_x\tilde{u}_0^{\mathbf{in},1}(1,\eta)=0, & \text{for } \eta\geqslant 0,\\ \tilde{u}_0^{\mathbf{in},1}(x,0)=\tilde{\phi}_1, & \text{for } 0\leqslant x\leqslant 1,\\ \displaystyle\int_0^1\tilde{u}_0^{\mathbf{in},1}(x,\eta)\,\mathrm{d}x=0, & \text{for } \eta\geqslant 0,\end{cases}$$

$$\begin{cases}-\partial_x^2\tilde{u}_1^{\mathbf{in},1}-\partial_\eta^2\tilde{u}_1^{\mathbf{in},1}=0, & \text{in } D',\\ \partial_x\tilde{u}_1^{\mathbf{in},1}(0,\eta)=\partial_x\tilde{u}_1^{\mathbf{in},1}(1,\eta)=0, & \text{for } \eta\geqslant 0,\\ \tilde{u}_1^{\mathbf{in},1}(x,0)=0, & \text{for } 0\leqslant x\leqslant 1,\\ \displaystyle\int_0^1\tilde{u}_1^{\mathbf{in},1}(x,\eta)\,\mathrm{d}x=0, & \text{for } \eta\geqslant 0,\end{cases}$$

and for $n\geqslant 2$,

$$\begin{cases}-\partial_x^2\tilde{u}_n^{\mathbf{in},1}-\partial_\eta^2\tilde{u}_n^{\mathbf{in},1}=\dfrac{(-1)^n\eta^{n-2}}{(n-2)!}\partial_y^{n-2}\tilde{f}(x,1), & \text{in } D',\\ \partial_x\tilde{u}_n^{\mathbf{in},1}(0,\eta)=\partial_x\tilde{u}_n^{\mathbf{in},1}(1,\eta)=0, & \text{for } \eta\geqslant 0,\\ \tilde{u}_n^{\mathbf{in},1}(x,0)=0, & \text{for } 0\leqslant x\leqslant 1,\\ \displaystyle\int_0^1\tilde{u}_n^{\mathbf{in},1}(x,\eta)\,\mathrm{d}x=0, & \text{for } \eta\geqslant 0.\end{cases}$$

As before, we solve the above equations by Fourier cosine series and use the matching procedure to determine the constants, then we obtain the lowest order

$$\tilde{u}_0^{\mathbf{in},1}(x,\eta)=\sum_{k=1}^{+\infty}\tilde{\phi}_{1,k}\mathrm{e}^{-k\pi\eta}\cos(k\pi x), \tag{2.19}$$

where

$$\tilde{\phi}_{1,k}=2\int_0^1\tilde{\phi}_1(x)\cos(k\pi x)\,\mathrm{d}x.$$

2.2.5. Composite expansion. Now we can get the leading order term of u_ε which is valid on the whole domain. Expressing all the three pieces of expansions in terms of x and y, and combining them by adding them together and then subtracting their common parts, eventually we obtain the following composite approximation by noting (2.2), (2.11), (2.18) and (2.19),

$$\begin{aligned}
u_\varepsilon(x,y) = &\ \bar{u}(y) + \tilde{u}_\varepsilon(x,y) \\
\sim &\ \bar{u}(y) + \big(\tilde{u}_0^{\mathbf{ot}}(x,y) + \tilde{u}_0^{\mathbf{in},0}(x,y/\varepsilon) + \tilde{u}_0^{\mathbf{in},1}(x,(1-y)/\varepsilon) \\
&- \lim_{\xi\to+\infty} \tilde{u}_0^{\mathbf{in},0}(x,\xi) - \lim_{\eta\to+\infty} \tilde{u}_0^{\mathbf{in},1}(x,\eta)\big) \\
= &\ y\left(\int_0^1\int_0^z \bar{f}(t)\,\mathrm{d}t\mathrm{d}z - \bar{\phi}_0 + \bar{\phi}_1\right) - \int_0^y\int_0^z \bar{f}(t)\,\mathrm{d}t\mathrm{d}z + \bar{\phi}_0 \\
&+ \sum_{k=1}^{+\infty}\left(\tilde{\phi}_{0,k}\mathrm{e}^{-k\pi y/\varepsilon} + \tilde{\phi}_{1,k}\mathrm{e}^{-k\pi(1-y)/\varepsilon}\right)\cos(k\pi x) \\
=:&\, u^{[0]}(x,y).
\end{aligned} \tag{2.20}$$

2.2.6. Higher order approximations. Higher order approximations can be obtained similarly by the Van Dyke's method of matched asymptotic expansions [9]. Let us compute the second order expansion to demonstrate the technique. To this end, we need to solve the next two orders in the inner expansion near the boundaries $y=0,1$, i.e., $\tilde{u}_n^{\mathbf{in},0}$ and $\tilde{u}_n^{\mathbf{in},1}$ for $n=1,2$.

Using the method of separation of variables again, by expanding in the Fourier cosine series, we obtain

$$\tilde{u}_1^{\mathbf{in},0}(x,\xi) = \sum_{k=1}^{+\infty} a_k(\mathrm{e}^{-k\pi\xi} - \mathrm{e}^{k\pi\xi})\cos(k\pi x),$$

$$\tilde{u}_2^{\mathbf{in},0}(x,\xi) = \sum_{k=1}^{+\infty}\left[\frac{\tilde{f}_k(0)}{k^2\pi^2} + b_k\mathrm{e}^{k\pi\xi} - \left(b_k + \frac{\tilde{f}_k(0)}{k^2\pi^2}\right)\mathrm{e}^{-k\pi\xi}\right]\cos(k\pi x),$$

where

$$\tilde{f}_k(y) = 2\int_0^1 \tilde{f}(x,y)\cos(k\pi x)\,\mathrm{d}x,$$

a_k and b_k are undetermined constants. Clearly, the $\mathrm{e}^{k\pi\xi}$ terms should disappear since they exponentially blow up, this implies that $a_k=0$ and $b_k=0$. Hence, we have the first three terms of the inner solution $\tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi)$:

$$\begin{aligned}
\tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi) &= \tilde{u}_0^{\mathbf{in},0}(x,\xi) + \varepsilon\tilde{u}_1^{\mathbf{in},0}(x,\xi) + \varepsilon^2\tilde{u}_2^{\mathbf{in},0}(x,\xi) + \mathcal{O}(\varepsilon^3) \\
&= \sum_{k=1}^{+\infty}\left[\tilde{\phi}_{0,k}\mathrm{e}^{-k\pi\xi} + \varepsilon^2\frac{\tilde{f}_k(0)}{k^2\pi^2}(1-\mathrm{e}^{-k\pi\xi})\right]\cos(k\pi x) + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

Note that the outer expansion is

$$\begin{aligned}
\tilde{u}_\varepsilon^{\mathbf{ot}}(x,y) &= \tilde{u}_0^{\mathbf{ot}}(x,y) + \varepsilon\tilde{u}_1^{\mathbf{ot}}(x,y) + \varepsilon^2\tilde{u}_2^{\mathbf{ot}}(x,y) + \mathcal{O}(\varepsilon^3) \\
&= \varepsilon^2\big(-\tilde{F}_2(x,y) + \tilde{F}_3(1,y)\big) + \mathcal{O}(\varepsilon^3).
\end{aligned}$$

To see that these two expansions do match up to the given order, we write the outer expansion $\tilde{u}_\varepsilon^{\mathbf{ot}}(x,y)$ in terms of the inner variables (x,ξ) and denoted by $\big(\tilde{u}_\varepsilon^{\mathbf{ot}}(x,y)\big)^{\mathbf{in},0}$; on the other hand, write $\tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi)$ in terms of (x,y) and denoted it by $\big(\tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi)\big)^{\mathbf{ot}}$. By dropping the asymptotically negligible $\mathrm{e}^{-k\pi\xi}$ terms as $\xi\to+\infty$, we have the $\mathcal{O}(\varepsilon^3)$ approximations:

$$\big(\tilde{u}_\varepsilon^{\mathbf{ot}}(x,y)\big)^{\mathbf{in},0}\approx\varepsilon^2\big(-\tilde{F}_2(x,0)+\tilde{F}_3(1,0)\big), \tag{2.21}$$

$$\big(\tilde{u}_\varepsilon^{\mathbf{in},0}(x,\xi)\big)^{\mathbf{ot}}\approx\sum_{k=1}^{+\infty}\varepsilon^2\frac{\tilde{f}_k(0)}{k^2\pi^2}\cos(k\pi x). \tag{2.22}$$

To show that matching (to this order) has been accomplished, we only need to check that the right hand sides of (2.22) and (2.21) are equal. In fact, we have for every y, by (2.10) and (2.12),

$$\int_0^1\big(-\tilde{F}_2(x,y)+\tilde{F}_3(1,y)\big)\,\mathrm{d}x=0;$$

from (2.13) and the integration by parts twice, for $k\geqslant 1$ we have that

$$\begin{aligned}2\int_0^1\big(-\tilde{F}_2(x,y)+\tilde{F}_3(1,y)\big)\cos(k\pi x)\,\mathrm{d}x&=\frac{2}{k\pi}\int_0^1\tilde{F}_1(x,y)\sin(k\pi x)\,\mathrm{d}x\\&=\frac{2}{k^2\pi^2}\int_0^1\tilde{f}(x,y)\cos(k\pi x)\,\mathrm{d}x=\frac{\tilde{f}_k(y)}{k^2\pi^2}.\end{aligned}$$

Note that in the second equality, we also used the simple fact of the integral condition

$$\tilde{F}_1(x,y)=\int_0^x\tilde{f}(z,y)\,\mathrm{d}z=0,\quad\text{for } x=0,\ 1.$$

Analogously, we can solve

$$\begin{aligned}&\tilde{u}_1^{\mathbf{in},1}(x,\eta)\equiv 0,\\&\tilde{u}_2^{\mathbf{in},1}(x,\eta)=\frac{\tilde{f}_k(1)}{k^2\pi^2}(1-\mathrm{e}^{-k\pi\eta}),\end{aligned}$$

and from

$$\begin{aligned}&\big(\tilde{u}_\varepsilon^{\mathbf{ot}}(x,\xi)\big)^{\mathbf{in},1}\approx\varepsilon^2\big(-\tilde{F}_2(x,1)+\tilde{F}_3(1,1)\big),\\&\big(\tilde{u}_\varepsilon^{\mathbf{in},1}(x,\xi)\big)^{\mathbf{ot}}\approx\sum_{k=1}^{+\infty}\varepsilon^2\frac{\tilde{f}_k(1)}{k^2\pi^2}\cos(k\pi x),\end{aligned}$$

we also see that matching (to this order) has been accomplished.

The last step is to combine the three expansions into a composite expansion

$$\begin{aligned}u_\varepsilon(x,y)\sim\ &\bar{u}(y)+\tilde{u}_0^{\mathbf{ot}}(x,y)+\varepsilon\tilde{u}_1^{\mathbf{ot}}(x,y)+\varepsilon^2\tilde{u}_2^{\mathbf{ot}}(x,y)\\&+\tilde{u}_0^{\mathbf{in},0}(x,y/\varepsilon)+\varepsilon\tilde{u}_1^{\mathbf{in},0}(x,y/\varepsilon)+\varepsilon^2\tilde{u}_2^{\mathbf{in},0}(x,y/\varepsilon)\\&+\tilde{u}_0^{\mathbf{in},1}(x,(1-y)/\varepsilon)+\varepsilon\tilde{u}_1^{\mathbf{in},1}(x,(1-y)/\varepsilon)+\varepsilon^2\tilde{u}_2^{\mathbf{in},1}(x,(1-y)/\varepsilon)\\&-\big(\tilde{u}_\varepsilon^{\mathbf{ot}}(x,\xi)\big)^{\mathbf{in},0}-\big(\tilde{u}_\varepsilon^{\mathbf{ot}}(x,\xi)\big)^{\mathbf{in},1}\end{aligned}$$

$$\begin{aligned}
&\approx u^{[0]}(x,y)+\varepsilon^2\sum_{k=1}^{+\infty}\left[\frac{\tilde{f}_k(y)}{k^2\pi^2}-\frac{\tilde{f}_k(0)}{k^2\pi^2}\mathrm{e}^{-k\pi y/\varepsilon}-\frac{\tilde{f}_k(1)}{k^2\pi^2}\mathrm{e}^{-k\pi(1-y)/\varepsilon}\right]\cos(k\pi x)\\
&=: u^{[2]}(x,y),
\end{aligned} \tag{2.23}$$

where $u^{[0]}(x,y)$ is the leading order given in (2.20).

Clearly, using the above method, we may proceed to derive the asymptotic expansions of u_ε to any order, and in general, the form of the asymptotic expansion up to $\mathcal{O}(\varepsilon^{2n})$, $n=0,1,\cdots$, is

$$\begin{aligned}
u_\varepsilon(x,y)\sim\ & u^{[0]}(x,y)+\sum_{m=1}^{n}\varepsilon^{2m}\sum_{k=1}^{+\infty}\left[\frac{\tilde{f}_k^{(2m-2)}(y)}{(k\pi)^{2m}}-\frac{\tilde{f}_k^{(2m-2)}(0)}{(k\pi)^{2m}}\mathrm{e}^{-k\pi y/\varepsilon}\right.\\
&\left.-\frac{\tilde{f}_k^{(2m-2)}(1)}{(k\pi)^{2m}}\mathrm{e}^{-k\pi(1-y)/\varepsilon}\right]\cos(k\pi x)\\
=:&u^{[2n]}(x,y).
\end{aligned} \tag{2.24}$$

The justification of the approximation orders of these asymptotic expansions will be given in the next section.

2.3. Discussion. For our approximations (2.24), it is observed that the boundary layer terms would disappear if

$$\tilde{\phi}_{0,k}=\tilde{\phi}_{1,k}=\tilde{f}_k^{(2m-2)}(0)=\tilde{f}_k^{(2m-2)}(1)=0,\quad \text{for } 1\leqslant m\leqslant n,\ k\geqslant 1,$$

i.e., if

$$\phi_0(x),\ \phi_1(x),\ \partial_y^{2m}f(x,0),\ \partial_y^{2m}f(x,1),\quad \text{for } 0\leqslant m\leqslant n-1,$$

all happen to be constant functions independent of x. Note that this condition is stronger than the homogeneous Dirichlet boundary conditions since it also involves the even order normal derivatives up to $2n-2$ of the external force on the Dirichlet boundaries. In this case, free of the boundary layers, we only need to compute the outer expansions by solving the Equations (2.8) and (2.9) up to $2n$ to get the approximation $u^{[2n]}$. Note that each equation in (2.8) and (2.9) is actually a system parametrized by the y variable of independent ordinary differential equations in the x variable rather than a two-dimensional partial differential equation. In computation, each equation can be solved by a numerical integrator in parallel at all grid points of the y variable which is now viewed as a parameter. This can reduce the computational cost to linear scaling. However, in general, the appearance of the boundary layers seems inevitable; then many existing algorithms may need further improvements to approximate the solution on the whole domain.

3. Theoretical justification

We prove the following estimates of the errors in this section.

THEOREM 3.1. *Define the remainders in the asymptotic expansions of u_ε:*

$$r_{2n}=u_\varepsilon-u^{[2n]},\quad \textit{for } n=0,1,\cdots,$$

where the approximations $u^{[2n]}$'s are given in (2.24). Then we have

$$\|r_{2n}\|_\infty=\mathcal{O}\big(\varepsilon^{2(n+1)}\big).$$

To justify these estimates, we first establish a modified version of the maximum principle for elliptic equations with mixed boundary value conditions.

LEMMA 3.1. *Let*

$$\mathcal{L}=\sum_{i,j}a_{ij}(\boldsymbol{x})\partial^2_{x_ix_j}+\sum_i b_i(\boldsymbol{x})\partial_{x_i}$$

be a uniformly elliptic operator on a connected, bounded, open domain Ω*. Suppose that* $w\in C^2(\Omega)\cap C(\bar\Omega)$ *satisfies* $\mathcal{L}w\leqslant 0$ *in* Ω *and* $\partial_{\boldsymbol{n}}w\leqslant 0$ *on* $\Gamma_N\subset\partial\Omega$*, where* $\boldsymbol{n}$ *is the outer unit normal to* Ω *on the boundary* $\partial\Omega$*. Also assume that* Ω *satisfies the interior ball condition at every* $\boldsymbol{x}\in\Gamma_N$*. Let* $\Gamma_D=\partial\Omega\setminus\Gamma_N$*. Then*

$$\max_{\bar\Omega} w=\sup_{\Gamma_D} w.$$

Proof. It suffices to show that w attains its maximum over $\bar\Omega$ on Γ_D. Let us assume w is not constant within Ω, as otherwise the proof is trivial. Then the strong maximum principle [10] states that w cannot attain its maximum over $\bar\Omega$ at any interior point. Furthermore, w cannot attain its maximum over $\bar\Omega$ at any boundary point $\boldsymbol{x}_0\in\Gamma_N$ either, since otherwise Hopf's lemma [10] would imply $\partial_{\boldsymbol{n}}w(\boldsymbol{x}_0)>0$, which contradicts the assumption $\partial_{\boldsymbol{n}}w\leqslant 0$ on Γ_N. Hence w has to attain its maximum over $\bar\Omega$ on Γ_D. ∎

The proof of Theorem 3.1 relies on the following lemma, which is a consequence of the above modified version of the maximum principle.

LEMMA 3.2. *For any* $\varepsilon>0$*, let* $\mathcal{L}=-\partial_x^2-\varepsilon^2\partial_y^2$*. Suppose that* $u\in C^2(D)\cap C(\bar D)$ *satisfies*

$$\begin{cases}\mathcal{L}u=g, & \text{in } D,\\ \partial_x u(0,y)=\partial_x u(1,y)=0, & \text{for } 0\leqslant y\leqslant 1,\\ u(x,0)=\phi_0(x),\quad u(x,1)=\phi_1(x), & \text{for } 0\leqslant x\leqslant 1.\end{cases}$$

Then

$$\|u\|_\infty\leqslant \Phi+\frac{G}{2},$$

where $G=\sup_{(x,y)\in D}|g(x,y)|$ *and* $\Phi=\max_{i=0,1}\sup_{0\leqslant x\leqslant 1}|\phi_i(x)|$*.*

Proof. Let $w(x,y)=u(x,y)-v(x)$, where

$$v(x)=\Phi+\frac{G}{2}(1-x)^2.$$

Then our reader can check that

$$\begin{aligned}&\mathcal{L}w=g-G\leqslant 0,\quad \text{in } D,\\ &\partial_x w(x,y)=\partial_x u(x,y)+G(1-x)=\begin{cases}G\geqslant 0, & \text{for } x=0,\\ 0, & \text{for } x=1,\end{cases}\\ &w(x,0)\leqslant \phi_0(x)-\Phi\leqslant 0,\quad w(x,1)\leqslant\phi_1(x)-\Phi\leqslant 0.\end{aligned}$$

From the modified version of the maximum principle (Lemma 3.1), we conclude that $w(x,y)\leqslant 0$ in D, thus

$$u(x,y)\leqslant \Phi+\frac{G}{2}(1-x)^2\leqslant \Phi+\frac{G}{2},\quad \text{in } D.$$

Applying the above argument to $-u$ yields

$$-u(x,y)\leqslant \Phi+\frac{G}{2},\quad \text{in } D.$$

Then we obtain the desired inequality. ∎

Now we give the proof of Theorem 3.1.

Proof. **(Proof of Theorem 3.1.)** Direct calculation shows that

$$\begin{aligned}
\mathcal{L}r_{2n} &= \mathcal{L}u_\varepsilon - \mathcal{L}u^{[2n]}\\
&= \varepsilon^2 f - \varepsilon^2 \bar{f} - \sum_{m=1}^{n} \varepsilon^{2m} \sum_{k=1}^{+\infty} \frac{\tilde{f}_k^{(2m-2)}(y)}{(k\pi)^{2m-2}} \cos(k\pi x) + \sum_{m=1}^{n} \varepsilon^{2m+2} \sum_{k=1}^{+\infty} \frac{\tilde{f}_k^{(2m)}(y)}{(k\pi)^{2m}} \cos(k\pi x)\\
&= \varepsilon^2 \tilde{f} - \varepsilon^2 \sum_{k=1}^{+\infty} \tilde{f}_k(y)\cos(k\pi x) + \varepsilon^{2n+2} \sum_{k=1}^{+\infty} \frac{\tilde{f}_k^{(2n)}(y)}{(k\pi)^{2n}} \cos(k\pi x)\\
&= \mathcal{O}\big(\varepsilon^{2(n+1)}\big),
\end{aligned}$$

therefore r_{2n} satisfies

$$\begin{cases}
\mathcal{L}r_{2n} = \mathcal{O}\big(\varepsilon^{2(n+1)}\big), & \text{in } D,\\
\partial_x r_{2n}(0,y) = \partial_x r_{2n}(1,y) = 0, & \text{for } 0\leqslant y\leqslant 1,\\
r_{2n}(x,0) = \mathcal{O}\big(\mathrm{e}^{-\pi/\varepsilon}\big),\quad r_{2n}(x,1) = \mathcal{O}\big(\mathrm{e}^{-\pi/\varepsilon}\big), & \text{for } 0\leqslant x\leqslant 1,
\end{cases}$$

where $\mathcal{L}$ is the same as in Lemma 3.2. Then the asserted error estimates follow from Lemma 3.2. ∎

4. Numerical demonstration

To demonstrate the consequence of the boundary layer and to illustrate the convergence orders in Theorem 3.1, we study a specific example in this section. The main purpose is to numerically demonstrate the correct order of convergence of the derived expansions in L_∞ norm sense. Thus all solutions here are computed in a brute-force way by the finite difference method with sufficiently fine mesh grids for a given ε. In this example, we choose the source term

$$f(x,y) = \sin(\pi(x^2+y^2))$$

and the Dirichlet boundary conditions

$$\phi_0(x) = \cos(\pi x),\qquad \phi_1(x) = 16x^2(x-1)^2.$$

Note that the Dirichlet boundary conditions should be consistent with the Neumann boundary conditions in (1.1), which dictates the compatibility conditions

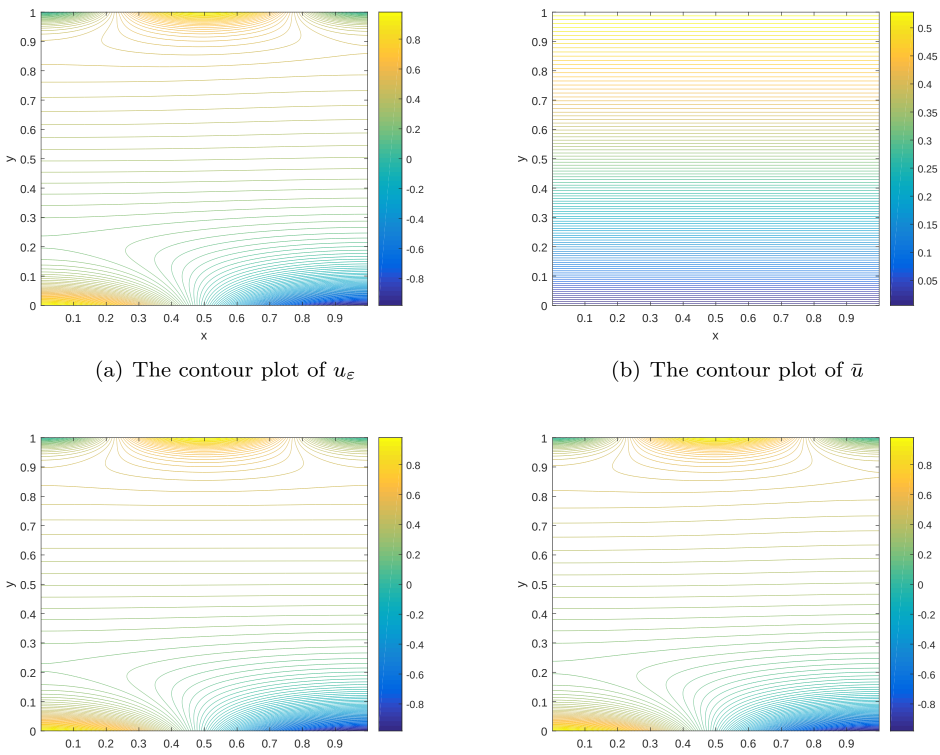
$$\phi_0'(0) = \phi_0'(1) = \phi_1'(0) = \phi_1'(1) = 0.$$

Clearly, these compatibility conditions are satisfied in this example.

We choose a grid for which the grid points are half-integered in the x-direction and integered in the y-direction, that is,

$$x_i = \left(i - \frac{1}{2}\right)\Delta x, \qquad y_j = (j-1)\Delta y,$$

where $\Delta x = 1/N$, $\Delta y = 1/M$, and $i = 1,2,\cdots,N$, $j = 1,2,\cdots,M+1$. Then the solution u_ε to (1.1) is solved numerically by the standard five-point finite difference method with difference choices of M and N for different values of ε. For example, at $\varepsilon^2 = 0.05$, our mesh size is chosen as $M \times N = 512 \times 2048$. Note that $\Delta y = 1/2048 = 4.9 \times 10^{-4}$ and the theory shows the thickness of the boundary layer is roughly $\varepsilon = \sqrt{0.05} \approx 0.22$. Since there are around 450 grid points in y direction inside each boundary layer, the boundary layer is well resolved here. For other smaller values of ε, we numerically ensure N (as well as M) is sufficiently large by comparing the difference resulted from N and $2N$. The final values of N are listed in Table 4.1. The mean part $\bar{u}$ is analytically computed thanks to (2.2). The first order asymptotic approximation $u^{[0]}$ and the higher order asymptotic approximation $u^{[2]}$ are computed respectively by a truncation of the infinite summation in (2.20) and (2.23) at some sufficiently large point.



(a) The contour plot of u_ε (b) The contour plot of $\bar{u}$

(c) The contour plot of $u^{[0]}$ (d) The contour plot of $u^{[2]}$

FIG. 4.1. *The contour plots of the numerical solutions at $\varepsilon^2 = 0.05$. (a) the exact solution u_ε; (b) the mean part $\bar{u}$; (c) the asymptotic approximation $u^{[0]}$ and (d) the asymptotic approximation $u^{[2]}$.*

Figure 4.1 shows respectively the contour plots of the numerical results at $\varepsilon^2 = 0.05$

of the exact solution u_ε, the mean part $\bar{u}$, the first order asymptotic approximation $u^{[0]}$, and the higher order asymptotic approximation $u^{[2]}$. It is observed that there exist two boundary layers near the boundaries $y=0$ and $y=1$ respectively, and the thickness of each boundary layer is roughly $\varepsilon=\sqrt{0.05}\approx 0.22$. This is consistent with the result of the asymptotic analysis in last section. Clearly, as shown in the subfigure (b), the mean solution $\bar{u}$ fails to capture the leading order solution inside these two boundary layers. The first order approximation $u^{[0]}$ is very close to the true solution while the next order approximation $u^{[2]}$ is almost identical to the true solution, as indicated from the subfigures (c) and (d). We can notice that the improvement for $u^{[2]}$ is that in the region of the boundary layers, compared with subfigure (a) for u_ε, the counter lines in the subfigure (d) for $u^{[2]}$ are more accurate than those in the subfigure (c) for $u^{[0]}$.

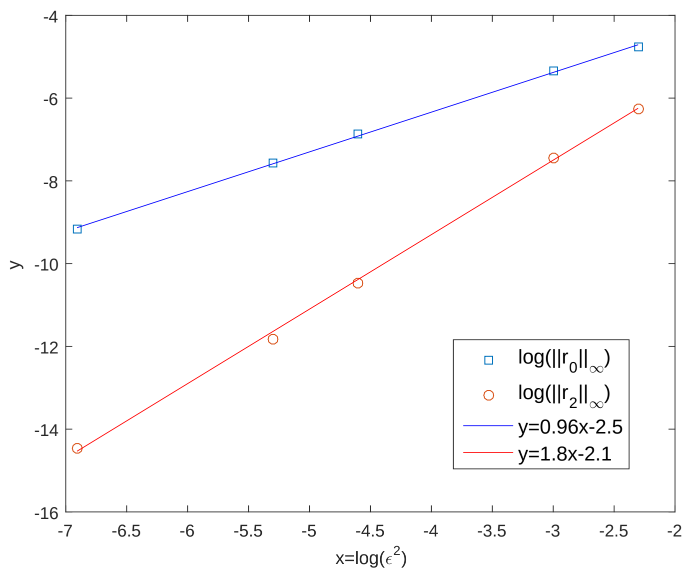


FIG. 4.2. *The plot on logarithmic scales of the discrete L_∞ norms of the errors r_0 (in the square-shaped markers) and r_2 (in the circle-shaped markers) versus ε^2. The best fitting straight lines through each set of data points are also shown. Their equations are indicated in the legend.*

ε^2	$\Vert r_0\Vert_\infty$	$\Vert r_2\Vert_\infty$	M	N
0.001	1.0533E−04	5.2834E−07	512	32768
0.005	5.2222E−04	7.3587E−06	512	16384
0.01	1.0335E−03	2.8475E−05	512	8192
0.05	4.7441E−03	5.7746E−04	512	2048
0.1	8.6241E−03	1.9240E−03	512	1024

TABLE 4.1. *The discrete L_∞ norms of the errors r_0 and r_2 for different values of ε. M and N respectively indicate the mesh grid size in x and y directions used in computing the true solution u_ε. Since the boundary layers are along the y direction whose width changes with ε, we need to change N to resolve the fine structures within the boundary layers. However, in the x direction, there are no boundary layers, so M does not change with ε.*

This example intuitively indicates that the mean part $\bar{u}$ is not sufficient to approximate the true solution u_ε in the L_∞ norm, i.e., $\|u_\varepsilon - \bar{u}\|_\infty$ does not converge. The fluctuation $\tilde{u}_\varepsilon$ has the non-negligible contribution to the true leading order $u^{[0]}$, in particularly near the Dirichlet boundaries.

To further demonstrate the convergence order, we compute the errors in L_∞ norm. Table 4.1 shows the discrete L_∞ norms of the errors r_0 and r_2 for different values of ε, and Figure 4.2 is the convergence plot of these errors versus ε^2 on logarithmic scales. The best fitting straight lines through each set of data points are also plotted in Figure 4.2. From the equations of these fitting lines, we observe that the orders of the approximation errors r_0 and r_2 are roughly $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon^4)$ respectively. These numerical results are consistent with the theoretical assertion in Theorem 3.1.

5. Generalization to the heterogeneous diffusions

The Equation (1.1) we studied so far has the constant diffusion coefficients. In this section, we show how the same approach can be generalized to the heterogeneous diffusions, i.e., the case of variable diffusion coefficients. Specifically, we consider the following elliptic equation

$$-\varepsilon^2\partial_x\big(a_1(x,y)\partial_x u_\varepsilon\big) - \partial_y\big(a_2(x,y)\partial_y u_\varepsilon\big) = f, \quad \text{in } D. \tag{5.1}$$

The same boundary conditions as in (1.1) are used here. The diffusion coefficients $a_i(x,y)$, $i=1,2$, satisfy the ellipticity condition

$$0 < M_1 < a_i(x,y) < M_2 < \infty, \ \forall\ (x,y) \in D,$$

with two positive constants M_1 and M_2. For ease of exposition, we further assume that $\phi_1 = 0$, so that the boundary layer only appears near the bottom side $y=0$. In addition, to illustrate the idea, we only derive the expansion $u_\varepsilon \sim u_0 + \varepsilon u_1$ up to the order $\mathcal{O}(\varepsilon)$.

5.1. Decomposing the solution into the mean and the fluctuation parts.

As before, we first decompose the solution u_ε into the mean part $\bar{u}_\varepsilon$ and the fluctuation part $\tilde{u}_\varepsilon$. Then the pair $\big(\bar{u}_\varepsilon(y), \tilde{u}_\varepsilon(x,y)\big)$ satisfies the following coupled ODE-PDE system [7]:

$$\begin{cases}
-\big(\bar{a}_2(y)\bar{u}_\varepsilon'(y)\big)' = \bar{f}(y) + \displaystyle\int_0^1 \partial_y\big(a_2(x,y)\partial_y\tilde{u}_\varepsilon(x,y)\big)\mathrm{d}x, & \text{in } (0,\ 1),\\
\bar{u}_\varepsilon(0) = \bar{\phi}_0, \quad \bar{u}_\varepsilon(1) = 0; & \\
-\partial_x\big(a_1(x,y)\partial_x\tilde{u}_\varepsilon(x,y)\big) - \varepsilon^2\partial_y\big(a_2(x,y)\partial_y\tilde{u}_\varepsilon(x,y)\big) & \\
\qquad = \varepsilon^2 f(x,y) + \varepsilon^2\partial_y\big(a_2(x,y)\bar{u}_\varepsilon'(y)\big), & \text{in } D,\\
\partial_x\tilde{u}_\varepsilon(0,y) = \partial_x\tilde{u}_\varepsilon(1,y) = 0, & \text{for } 0 \leqslant y \leqslant 1,\\
\tilde{u}_\varepsilon(x,0) = \tilde{\phi}_0(x), \quad \tilde{u}_\varepsilon(x,1) = 0, & \text{for } 0 \leqslant x \leqslant 1,\\
\displaystyle\int_0^1 \tilde{u}_\varepsilon(x,y)\mathrm{d}x = 0, & \text{for } 0 \leqslant y \leqslant 1.
\end{cases} \tag{5.2}$$

Here by the convention, $\bar{a}_i(y) := \int_0^1 a_i(x,y)\mathrm{d}x$ and $\tilde{a}_i := a_i - \bar{a}_i$ refer to the mean and fluctuation part of the function $a_i(x,y)$, $i=1,2$, respectively.

It is easy to see that if the fluctuation $\tilde{a}_2$ vanishes, the above system becomes decoupled, i.e., $\bar{u}_\varepsilon$ and $\tilde{u}_\varepsilon$ satisfy their own closed equations, respectively. For any non-vanishing $\tilde{a}_2$, we face with a coupled ODE-PDE system, however, we shall show later that the equation for the fluctuation part of each term in the asymptotic expansion can still be closed. The above coupled system (5.2) is also the form for the outer solution except the boundary conditions at $y=0$.

5.2. Asymptotic expansions of the mean and fluctuation terms.

5.2.1. Outer expansion. We work on the outer solution first. To ease the notation, we still use the little letters $u_\varepsilon, \bar{u}_\varepsilon, \tilde{u}_\varepsilon$, etc., to denote the outer solutions.

From the ansatz in the form $u_\varepsilon = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$ for the mean $\bar{u}_\varepsilon$ and the fluctuation $\tilde{u}_\varepsilon$, respectively, the straightforward calculation shows that

$$\tilde{u}_0 \equiv 0, \quad \text{and} \quad \tilde{u}_n \equiv 0, \quad \text{for odd } n, \tag{5.3}$$

which are the same as in the simplified case ($a_1 = a_2 \equiv 1$) in previous sections. The mean part $\bar{u}_0$ of the leading order of the outer solution satisfies

$$\begin{cases} -\big(\bar{a}_2(y)\bar{u}_0'(y)\big)' = \bar{f}(y), & \text{in } (0,\ 1), \\ \bar{u}_0(1) = 0, \end{cases}$$

thus it has a closed form

$$\bar{u}_0(y) = c_0 - \int_0^y \frac{\int_0^{y'} \bar{f}(z)\mathrm{d}z + \alpha_0(c_0)}{\bar{a}_2(y')} \mathrm{d}y' \tag{5.4}$$

with a constant $c_0 = \bar{u}_0(0)$ undetermined. Here $\alpha_0(c_0)$ is the implicit function of c_0 defined by $\bar{u}_0(1) = 0$ in (5.4). The constant c_0 has to be determined later by matching with the inner solution inside the boundary layer near $y = 0$.

5.2.2. Inner expansion. Now we examine the inner solution, which will be written in the capitalised letters, $U_\varepsilon(x,\xi)$, where $\xi = y/\varepsilon$. The calculation is straightforward but tedious, and we defer the details to Appendix A. The results we obtained are

$$\tilde{U}_0(x,\xi) = \sum_{n=1}^{+\infty} \beta_n B_n(x) \mathrm{e}^{-\sqrt{\lambda_n}\xi}, \tag{5.5}$$

$$\bar{U}_0(\xi) = \sum_{n=1}^{+\infty} \beta_n \left(1 - \mathrm{e}^{-\sqrt{\lambda_n}\xi}\right) \int_0^1 \frac{a_2(x,0)}{\bar{a}_2(0)} B_n(x)\mathrm{d}x + C_0\xi + \bar{\phi}_0, \tag{5.6}$$

where $\lambda_n > 0$ and $B_n(x)$, $n \geqslant 1$, are all the generalized eigenvalues and the corresponding eigenfunctions of two self-adjoint positive operators T_1 and T_2, defined by (A.5) and (A.6) in Appendix A, on the Hilbert space $\mathring{L}^2([0,\ 1]) := \{B \in L^2([0,\ 1]) : \int_0^1 B(x)\mathrm{d}x = 0\}$, the coefficients β_n can be determined by the boundary condition $\tilde{U}_0(x,0) = \tilde{\phi}_0(x)$, and C_0 is a constant to be determined in the matching procedure later. Note that $\tilde{U}_0(x,\xi)$ decays exponentially fast as $\xi \to +\infty$.

5.2.3. Matching. Making use of the matching condition

$$\bar{U}_0(\xi \to \infty) = \bar{u}(y \to 0) = c_0,$$

we derive the values of two constants from (5.4) and (5.6):

$$C_0 = 0, \qquad c_0 = \sum_{n=1}^{+\infty} \beta_n \int_0^1 \frac{a_2(x,0)}{\bar{a}_2(0)} B_n(x)\mathrm{d}x + \bar{\phi}_0. \tag{5.7}$$

This finishes the derivations of all four leading order terms: $\bar{u}_0(y), \tilde{u}_0(x,y)$, $\bar{U}_0(\xi)$ and $\tilde{U}_0(x,\xi)$.

5.3. First-order terms. The detailed derivation of the $\mathcal{O}(\varepsilon)$ terms can be found in Appendix B. We summarize the main results here:

$$\tilde{u}_1 \equiv 0, \qquad \bar{u}_1(y) = c_1\left[1 - \left(\int_0^1 \frac{\mathrm{d}y'}{\bar{a}_2(y')}\right)^{-1}\left(\int_0^y \frac{\mathrm{d}y'}{\bar{a}_2(y')}\right)\right], \tag{5.8}$$

$$\tilde{U}_1(x,\xi) = \sum_{n=1}^{+\infty} H_n(\xi)B_n(x), \tag{5.9}$$

$$\bar{U}_1(\xi) = \sum_{n=1}^{+\infty}\omega_n H_n(\xi) + \sum_{n=1}^{+\infty} l_n(\xi)\mathrm{e}^{-\sqrt{\lambda_n}\xi} + C_1\xi + D_1, \tag{5.10}$$

where

$$H_n(\xi) = \sum_{m=1}^{+\infty} p_{nm}(\xi)\mathrm{e}^{-\sqrt{\lambda_m}\xi},$$

$$C_1 = -\frac{\alpha_0(c_0)}{\bar{a}_2(0)}, \qquad c_1 = D_1 = -\sum_{n=1}^{+\infty} l_n(0), \tag{5.11}$$

and $\{p_{nm}(\xi)\}$ are certain known polynomials with order $\leqslant 2$. Hereinafter an object is referred to as a known quantity if it can be expressed explicitly in terms of a_1, a_2, f, ϕ_0, $\{\lambda_n : n \geqslant 1\}$ and $\{B_n : n \geqslant 1\}$. We have that $\{\omega_n\}$ are known coefficients, and $\{l_n(\xi)\}$ are known linear functions.

5.4. Composite expansion. In summary, we have obtained the expansions of the inner and outer solutions to the Equation (5.1) respectively up to the first order terms. Then the composite approximation with truncation error $\mathcal{O}(\varepsilon^2)$ can be written down as in the previous sections:

$$\begin{aligned} u_\varepsilon(x,y) \sim\; & \bar{u}_0(x,y) + \tilde{u}_0(x,y) + \bar{U}_0(x,y/\varepsilon) + \tilde{U}_0(x,y/\varepsilon) \\ & + \varepsilon\left(\bar{u}_1(x,y) + \tilde{u}_1(x,y) + \bar{U}_1(x,y/\varepsilon) + \tilde{U}_1(x,y/\varepsilon)\right) \\ & - (c_0 + C_1 y + \varepsilon D_1). \end{aligned}$$

Refer to (5.3)-(5.10) for the outer and inner expansions in the first two lines and (B.5)(5.7)(5.11) for the common part in the last line.

REMARK 5.1. The more general case than (5.1) is the non-diagonal diffusion tensor. In this difficult case, the line fields will not parallel to the coordinate axes, which will critically affect the formation of the boundary layers. Such complexity will hinder the feasibility of analytical studies.

6. Conclusion and discussion

We have presented a formal expansion of the solution to the strongly anisotropic diffusion Equation (1.1) in the simple rectangular domain. Our expansions take into account the boundary layers near the Dirichlet boundaries. We proved the rigorous convergence of the formal expansion by the maximum principle. We also illustrated that such developments can be generalized to more representative problems, e.g., the diffusion coefficients are functions of the space variables.

Throughout the paper, we have stressed much the challenges due to the emergence of the boundary layers. One might ask a natural question: whether the existing asymptotic preserving (AP) methods such as [8] are still applicable for the case here with the existence of the boundary layers. Our answer is yes if some numerical techniques like local mesh adaptivity can be implemented in the AP framework. It is easy to see that if a uniform mesh is used, then the number of grid points surely increases with a decreasing ε in order to resolve the boundary layers, even when the AP scheme is applied. Recall that in our illustrative example in Section 4, we used a very naive mesh refining strategy: a uniform but finer mesh in y direction so that $\Delta y \ll \Delta x$ and Δy decreases with ε. Certainly, more advanced adaptive techniques can be implemented, such as the moving mesh [17, 20] or adaptive mesh refinement based on either the heuristic criteria of the norm of gradients or the posterior error estimates [12, 21] as in many adaptive finite element methods. For the models of strong anisotropy with more general form of diffusion tensor and the line fields, it is conceivable that both the mean part and the fluctuation part require the adaptive strategy.

In summary, we have studied a diffusion model with strong anisotropy, which is analytically tractable, to illustrate how the different boundary conditions generate various types of challenges, especially, the emergence of the boundary layers which is quite common in singular perturbations. Our derivation of asymptotic expansion is formal but the convergence result is rigorous. Our numerical demonstration calls for a further adaptive improvement within the existing AP framework.

Appendix A. Derivation for the leading order term of the inner solution. Here, we derive the leading order term of the inner solution. The equations within the boundary layer are the following coupled system on the domain $(x,\xi)\in D'$:

$$\begin{cases} -\left(\bar{a}_2(\varepsilon\xi)\bar{U}_\varepsilon'(\xi)\right)' = \varepsilon^2 \bar{f}(\varepsilon\xi) + \int_0^1 \partial_\xi \left(a_2(x,\varepsilon\xi)\partial_\xi \tilde{U}_\varepsilon(x,\xi)\right) \mathrm{d}x, \\ -\partial_x \left(a_1(x,\varepsilon\xi)\partial_x \tilde{U}_\varepsilon(x,\xi)\right) - \partial_\xi \left(a_2(x,\varepsilon\xi)\partial_\xi \tilde{U}_\varepsilon(x,\xi)\right) \\ \qquad\qquad = \varepsilon^2 f(x,\varepsilon\xi) + \partial_\xi \left(a_2(x,\varepsilon\xi)\bar{U}_\varepsilon'(\xi)\right). \end{cases}$$

The boundary conditions at $\xi=0$ are $\bar{U}_\varepsilon(0)=\bar{\phi}_0$ and $\tilde{U}_\varepsilon(x,0)=\tilde{\phi}_0(x)$. $\tilde{U}_\varepsilon$ also satisfies the homogeneous Neumann conditions at $x=0$ and $x=1$.

After expanding $a_i(x,\varepsilon\xi)=a_i(x,0)+\varepsilon\partial_y a_i(x,0)\xi+O(\varepsilon^2)$, $i=1,2$, and applying the ansatz $U_\varepsilon = U_0+\varepsilon U_1+\varepsilon^2 U_2+\cdots$, we obtain that the mean part $\bar{U}_0$ of the leading order term of the inner solution satisfies

$$\begin{cases} -\bar{a}_2(0)\bar{U}_0''(\xi) = \int_0^1 a_2(x,0)\partial_\xi^2 \tilde{U}_0(x,\xi)\,\mathrm{d}x, \quad \text{in } (0,\ +\infty), \\ \bar{U}_0(0)=\bar{\phi}_0, \end{cases} \tag{A.1}$$

while the fluctuation part $\tilde{U}_0(x,\xi)$ satisfies

$$\begin{aligned} &-\partial_x \left(a_1(x,0)\partial_x \tilde{U}_0(x,\xi)\right) - a_2(x,0)\partial_\xi^2 \tilde{U}_0(x,\xi) \\ =\ & a_2(x,0)\bar{U}_0''(\xi) = -\frac{a_2(x,0)}{\bar{a}_2(0)} \int_0^1 a_2(x,0)\partial_\xi^2 \tilde{U}_0(x,\xi)\,\mathrm{d}x, \end{aligned} \tag{A.2}$$

where in the last equality we have used (A.1). So (A.2) is closed for $\tilde{U}_0$. Consequently, (A.2) with the original boundary condition $\tilde{U}_0(x,0)=\tilde{\phi}_0(x)$ and the matching boundary condition

$$\lim_{\xi\to+\infty} \tilde{U}_0(x,\xi) = \lim_{y\to 0+} \tilde{u}_0(x,y) = 0$$

plus the homogeneous Neumann conditions at $x=0$ and $x=1$ will uniquely determine the solution $\tilde{U}_0$. We solve (A.2) by the method of separation of variables. Take $\tilde{U}_0(x,\xi)=A(\xi)B(x)$ and plug it into (A.2) to get

$$\frac{-\big(a_1(x,0)B'(x)\big)'}{a_2(x,0)B(x)-\dfrac{a_2(x,0)}{\bar{a}_2(0)}\displaystyle\int_0^1 a_2(x,0)B(x)\,\mathrm{d}x}=\frac{A''(\xi)}{A(\xi)},$$

which equals a separation constant λ. Therefore

$$-\big(a_1(x,0)B'(x)\big)'=\lambda\left(a_2(x,0)B(x)-\frac{a_2(x,0)}{\bar{a}_2(0)}\int_0^1 a_2(x,0)B(x)\mathrm{d}x\right), \tag{A.3}$$

$$A''(\xi)=\lambda A(\xi). \tag{A.4}$$

Note that $\int_0^1 B(x)\mathrm{d}x=0$ and the (natural) boundary condition for $B(x)$ is $B'(0)=B'(1)=0$. In other words, λ and $B(x)$ satisfying (A.3) are respectively known as the generalized eigenvalue and eigenfunction of the linear operators

$$T_1(B):=-\Big(a_1(x,0)B'(x)\Big)', \tag{A.5}$$

$$T_2(B):=a_2(x,0)B(x)-\frac{a_2(x,0)}{\bar{a}_2(0)}\int_0^1 a_2(x,0)B(x)\mathrm{d}x. \tag{A.6}$$

It is easy to verify that T_1 and T_2 are both (densely defined) self-adjoint operators on the Hilbert space $\mathring{L}^2([0,\ 1]):=\{B\in L^2([0,\ 1]):\int_0^1 B(x)\mathrm{d}x=0\}$. In addition, an application of integration by parts and the Cauchy-Schwarz inequality can show that T_1 and T_2 are both positive operators. It follows that the generalized eigenvalue λ is always positive. Therefore the bounded solution for (A.4) has the form $A(\xi)=c\mathrm{e}^{-\sqrt{\lambda}\xi}$, where c is an arbitrary constant. Assume that λ_n and $B_n(x)$, $n\geqslant 1$, are all the generalized eigenvalues and the corresponding eigenfunctions of T_1 and T_2. Then all λ_n are strictly positive and $\{B_n(x)\}$ is a complete set of the T_2-orthogonal basis in $\mathring{L}^2([0,\ 1])$. Thus the solution $\tilde{U}_0(x,\xi)$ for (A.2) can be represented by the series

$$\tilde{U}_0(x,\xi)=\sum_{n=1}^{+\infty}\beta_n B_n(x)\mathrm{e}^{-\sqrt{\lambda_n}\xi},$$

where the coefficients β_n are given by the boundary condition $\tilde{U}_0(x,0)=\tilde{\phi}_0(x)$. The Equation (A.1) for $\bar{U}_0(\xi)$ thus reduces to the form

$$-\bar{U}_0''(\xi)=\sum_{n=1}^{+\infty}\beta_n\lambda_n\mathrm{e}^{-\sqrt{\lambda_n}\xi}\int_0^1\frac{a_2(x,0)}{\bar{a}_2(0)}B_n(x)\,\mathrm{d}x,$$

with the boundary condition $\bar{U}_0(0)=\bar{\phi}_0$. Then the solution is

$$\bar{U}_0(\xi)=\sum_{n=1}^{+\infty}\beta_n\left(1-\mathrm{e}^{-\sqrt{\lambda_n}\xi}\right)\int_0^1\frac{a_2(x,0)}{\bar{a}_2(0)}B_n(x)\,\mathrm{d}x+C_0\xi+\bar{\phi}_0,$$

where C_0 is an undetermined constant.

Appendix B. Derivation for the first order term.

B.1. Outer solution. Note that $\tilde{u}_1=0$ by (5.3). Then the equation for $\bar{u}_1$ reads $-(\bar{a}_2(y)\bar{u}_1'(y))'=0$ with $\bar{u}_1(1)=0$. The solution has the form

$$\bar{u}_1(y)=c_1-\alpha_1(c_1)\int_0^y \frac{\mathrm{d}y'}{\bar{a}_2(y')}, \tag{B.1}$$

where the constant $c_1=\bar{u}_1(0)$ is to be determined later by the matching condition, and $\alpha_1(c_1)$ is the implicit function defined by $\bar{u}_1(1)=0$.

B.2. Inner solution. The first order term $(\bar{U}_1(\xi),\tilde{U}_1(x,\xi))$ of the inner expansion satisfies

$$\begin{cases} -\bar{a}_2(0)\bar{U}_1''(\xi)=\bar{a}_2'(0)\left(\xi\bar{U}_0'(\xi)\right)' \\ \qquad +\displaystyle\int_0^1 \partial_\xi\left(a_2(x,0)\partial_\xi\tilde{U}_1(x,\xi)+\partial_y a_2(x,0)\xi\partial_\xi\tilde{U}_0(x,\xi)\right)\mathrm{d}x; \\ -\partial_x\left(a_1(x,0)\partial_x\tilde{U}_1(x,\xi)\right)-a_2(x,0)\partial_\xi^2\tilde{U}_1(x,\xi) \\ =\partial_x\left(\partial_y a_1(x,0)\xi\partial_x\tilde{U}_0(x,\xi)\right)+\partial_y a_2(x,0)\partial_\xi\left(\xi\partial_\xi\tilde{U}_0(x,\xi)\right) \\ \qquad +\, a_2(x,0)\bar{U}_1''(\xi)+\partial_y a_2(x,0)\left(\xi\bar{U}_0'(\xi)\right)'. \end{cases} \tag{B.2}$$

For the second equation in (B.2), by the same trick as before, we can use the first equation to eliminate $a_2(x,0)\bar{U}_1''(\xi)$ so that it becomes a closed equation for $\tilde{U}_1(x,\xi)$. Then by using the T_2-orthogonal basis $\{B_n(x):n\geqslant 1\}$ of $\mathring{L}^2([0,\ 1])$, we expand

$$\tilde{U}_1(x,\xi)=\sum_{n=1}^{+\infty}H_n(\xi)B_n(x). \tag{B.3}$$

To derive H_n, we plug (B.3) together with (5.5)(5.6) into the closed equation of $\tilde{U}_1$ and thus obtain

$$H_n''(\xi)-\lambda_n H_n(\xi)=\sum_{m=1}^{+\infty}h_{nm}(\xi)\mathrm{e}^{-\sqrt{\lambda_m}\xi},$$

where $\{h_{nm}(\xi)\}$ are some known linear functions. The general solution is of the form

$$H_n(\xi)=\gamma_n^-\mathrm{e}^{-\sqrt{\lambda_n}\xi}+\gamma_n^+\mathrm{e}^{\sqrt{\lambda_n}\xi}+\sum_{m=1}^{+\infty}p_{nm}(\xi)\mathrm{e}^{-\sqrt{\lambda_m}\xi},$$

where $\{p_{nm}(\xi)\}$ are known polynomials with order $\leqslant 2$. Note that γ_n^+ must vanish since otherwise H_n would blow up exponentially as $\xi\to+\infty$. γ_n^- can be determined by the boundary condition $H_n(0)=0$ since $\tilde{U}_1(x,0)=0$. The term $\gamma_n^-\mathrm{e}^{-\sqrt{\lambda_n}\xi}$ can be absorbed into the series $\sum_{m=1}^{+\infty}p_{nm}(\xi)\mathrm{e}^{-\sqrt{\lambda_m}\xi}$, and thus we eventually have

$$H_n(\xi)=\sum_{m=1}^{+\infty}p_{nm}(\xi)\mathrm{e}^{-\sqrt{\lambda_m}\xi},$$

with $p_{nn}(\xi)$ redefined. A useful fact used later is that $H_n(\xi)$ decays exponentially fast as $\xi\to+\infty$.

Once $\tilde{U}_1(x,\xi)$ is obtained, we solve the first equation in (B.2) with the boundary condition $\bar{U}_1(0)=0$ to get

$$\bar{U}_1(\xi)=\sum_{n=1}^{+\infty}\omega_n H_n(\xi)+\sum_{n=1}^{+\infty}l_n(\xi)\mathrm{e}^{-\sqrt{\lambda_n}\xi}+C_1\xi+D_1, \tag{B.4}$$

where $\{\omega_n\}$ are known coefficients, $\{l_n(\xi)\}$ are known linear functions, and

$$D_1=-\sum_{n=1}^{+\infty}l_n(0),$$

but C_1 is an undetermined constant.

B.3. Matching. To proceed the matching condition to determine the two constants c_1 in (B.1) and C_1 in (B.4), we need to consider the outer solution expansion (in terms of the inner variables (x,ξ))

$$\begin{aligned} u_\varepsilon(x,y) &\sim \bar{u}_0(x,\varepsilon\xi)+\tilde{u}_0(x,\varepsilon\xi)+\varepsilon\bar{u}_1(x,\varepsilon\xi)+\varepsilon\tilde{u}_1(x,\varepsilon\xi)+\cdots \\ &=\bar{u}_0(x,\varepsilon\xi)+\varepsilon\bar{u}_1(x,\varepsilon\xi)+\cdots, \end{aligned}$$

where $\bar{u}_0$ and $\tilde{u}_1$ are given by (5.4) and (B.1) respectively, and the inner solution expansion (in terms of the original variables (x,y))

$$U_\varepsilon(x,\xi)\sim\bar{U}_0(x,y/\varepsilon)+\tilde{U}_0(x,y/\varepsilon)+\varepsilon\bar{U}_1(x,y/\varepsilon)+\varepsilon\tilde{U}_1(x,y/\varepsilon)+\cdots,$$

refer to (5.6) (5.5) (B.3) (B.4). After expanding the above two expressions in terms of ε, and noting the exponential decay of $\tilde{U}_0$ and H_n, we have the following approximations on the order $\mathcal{O}(\varepsilon^2)$:

$$\begin{aligned} \left(u_\varepsilon\right)^{\mathbf{in}}(x,\xi) &\approx c_0-\varepsilon\xi\frac{\alpha_0(c_0)}{\bar{a}_2(0)}+\varepsilon c_1, \\ \left(U_\varepsilon\right)^{\mathbf{ot}}(x,y) &\approx c_0+yC_1+\varepsilon D_1, \end{aligned} \tag{B.5}$$

where the sup-indices $\mathbf{in},\mathbf{ot}$ refer to the types of variables written in the expansions. Since $y=\varepsilon\xi$, matching (to this order $\mathcal{O}(\varepsilon^2)$) will be accomplished by selecting

$$C_1=-\frac{\alpha_0(c_0)}{\bar{a}_2(0)}, \qquad c_1=D_1=-\sum_{n=1}^{+\infty}l_n(0).$$

Now we have derived the values of the two undetermined constants in $\bar{u}_1$ and $\bar{U}_1$. It is noted that in determining these constants by matching, only the mean parts have contributions since the fluctuation parts either vanish or decay exponentially.

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