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THE 3D INCOMPRESSIBLE BOUSSINESQ EQUATIONS WITH FRACTIONAL PARTIAL DISSIPATION*

WANRONG YANG†, QUANSEN JIU‡, AND JIAHONG WU§

Abstract. The system of the 3D Boussinesq equations is one of the most important models for geophysical fluids. The fundamental problem of whether or not reasonably smooth solutions to the 3D Boussinesq equations with the standard Laplacian dissipation can blow up in a finite time is an outstanding open problem. The Boussinesq equations with partial or fractional dissipation not only naturally generalize the classical Boussinesq equations, but also are physically relevant and mathematically important. This paper focuses on a system of the 3D Boussinesq equations with fractional partial dissipation and proves that any H^1-initial data always leads to a unique and global-in-time solution. The result of this paper is part of our efforts devoted to the global well-posedness problem on the Boussinesq equations with minimal dissipation.

Keywords. 3D Boussinesq equations; fractional partial dissipation; global regularity.

AMS subject classifications. 35Q35; 76D03.

1. Introduction

The Boussinesq equations model geophysical fluids such as atmospheric fronts and oceanic currents as well as fluids in our daily life such as the Rayleigh–Benard convection (see, e.g., [2–4,10,11]). The standard 3D incompressible Boussinesq equations are given by

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p + \nu\Delta u + \theta e_3, \\ \nabla\cdot u = 0, \\ \partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta, \end{cases} \tag{1.1}$$

where u denotes the velocity field, p the pressure, ν the viscosity, θ the temperature, e_3 the unit vertical vector and κ the thermal diffusivity. Given sufficiently smooth initial data

$$u(x,0) = u_0(x), \qquad \theta(x,0) = \theta_0(x),$$

the issue of whether (1.1) has a unique global-in-time solution is an outstanding open problem.

The global regularity problem on the 3D Boussinesq equations is supercritical in the sense that, if we replace the Laplacian operator in the velocity equation of (1.1) by a fractional Laplacian $-(-\Delta)^\alpha$ with $\alpha \geq \frac{5}{4}$, then the hyperdissipative Boussinesq equations

$$\begin{cases} \partial_t u + u\cdot\nabla u = -\nabla p - \nu(-\Delta)^\alpha u + \theta e_3, \\ \nabla\cdot u = 0, \\ \partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta \end{cases} \tag{1.2}$$

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†School of Mathematics and Information Sciences, Beifang University of Nationalities, Yinchuan, Ningxia 750021, P.R. China (yangwanrong1618@163.com).

‡School of Mathematics, Capital Normal University, Beijing 100037, P.R. China (jiuqs@mail.cnu.edu.cn).

§Department of Mathematics, Oklahoma State University, 401 Mathematical Sciences, Stillwater, OK 74078, USA (jiahong.wu@okstate.edu).

always possess a unique global solution. In fact, the global existence and regularity result actually holds for (1.2) with $\kappa=0$ (see, e.g., [5, 12, 19, 21]). We note that the critical exponent $\alpha=\frac{5}{4}$ makes the kinetic energy invariant under the natural scaling. Here the fractional Laplacian operator $(-\Delta)^\alpha$ is defined via the Fourier transform (see, e,g., [13])

$$\widehat{(-\Delta)^\alpha} f(\xi)=|\xi|^{2\alpha}\widehat{f}(\xi).$$

This paper attempts to reduce the dissipation in (1.2). We consider the following system of Boussinesq equations

$$\begin{cases} \partial_t u_1+(u\cdot\nabla)u_1=-\partial_1 p-\nu(\Lambda_1^{\frac{5}{2}}+\Lambda_2^{\frac{5}{2}})u_1, \\ \partial_t u_2+(u\cdot\nabla)u_2=-\partial_2 p-\nu(\Lambda_2^{\frac{5}{2}}+\Lambda_3^{\frac{5}{2}})u_2, \\ \partial_t u_3+(u\cdot\nabla)u_3=-\partial_3 p-\nu(\Lambda_1^{\frac{5}{2}}+\Lambda_3^{\frac{5}{2}})u_3+\theta, \\ \nabla\cdot u=0, \\ \partial_t\theta+u\cdot\nabla\theta=-\kappa\Lambda^{2\gamma}\theta. \end{cases} \tag{1.3}$$

Here $\Lambda=(-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator and Λ_k^γ with $\gamma>0$ and $k=1,2,3$ are directional fractional operators defined via the Fourier transform

$$\widehat{\Lambda_k^\gamma f}(\xi)=|\xi_k|^\gamma\widehat{f}(\xi), \qquad k=1,2,3.$$

We prove the following theorem.

THEOREM 1.1. *Assume $u_0\in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$, and $\theta_0\in H^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Assume $\gamma\geq\frac{17}{20}$. Then (1.3) has a unique global solution (u,θ) satisfying, for any $T>0$,*

$$\begin{aligned} &(u,\theta)\in L^\infty(0,T;H^1(\mathbb{R}^3)), \\ &(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\in L^2(\mathbb{R}^3\times(0,T)), && (\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\nabla u_1\in L^2(\mathbb{R}^3\times(0,T)), \\ &(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_2\in L^2(\mathbb{R}^3\times(0,T)), && (\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_2\in L^2(\mathbb{R}^3\times(0,T)), \\ &(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_3\in L^2(\mathbb{R}^3\times(0,T)), && (\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_3\in L^2(\mathbb{R}^3\times(0,T)), \\ &\Lambda^\gamma\theta\in L^2(\mathbb{R}^3\times(0,T)), && \Lambda^\gamma\nabla\theta\in L^2(\mathbb{R}^3\times(0,T)). \end{aligned}$$

Compared with (1.2), each of the equations of u_1, u_2 and u_3 in (1.3) only has two directional hyperdissipation. When there is no thermal diffusion, or $\kappa=0$, it does not appear possible to prove the global existence and regularity. The fractional dissipation in θ helps bound the nonlinearity in the temperature equation. It is clear that the global existence and regularity still holds when $\Lambda_k^{\frac{5}{2}}$ $(k=1,2,3)$ in (1.3) is replaced by $\Lambda_k^{2\sigma}$ with any $\sigma>\frac{5}{4}$.

This work was partially motivated by our recent result on the 3D Navier–Stokes equations with fractional partial dissipation [20]. We recently introduced the Navier–Stokes equations with directional hyperdissipation and proved the global regularity of the Navier–Stokes with directional hyperdissipation, namely (1.3) with $\theta=0$. [20] improves the classical result for the hyperdissipative Navier–Stokes equations with $(-\Delta)^\alpha u$ (see, e.g., [6, 9, 15]). In contrast to (1.2), Theorem 1.1 requires only $\gamma\geq\frac{17}{20}$, not $\gamma\geq 1$. It is worth mentioning two important papers, one by Tao [14] and one by Barbato, Morandin and Romito [1], on the Navier–Stokes equations with logarithmically supercritical hyperdissipation. The magneto-hydrodynamic equations with hyperdissipation have also been investigated (see, e.g., [16,18]).

The proof of Theorem 1.1 naturally divides into two main parts: the existence part and the uniqueness part. The global existence and regularity part boils down to showing the global H^1 *a priori* bound. The global L^2-bound follows directly from a standard energy estimate. However, the global H^1-bound is not a trivial consequence of energy estimate. One key idea is how to effectively make use of the reduced dissipation to bound the nonlinearity. Since the dissipation is only available in some directions, it is necessary to write the corresponding nonlinear terms, say

$$I \equiv \int \nabla((u\cdot\nabla)u)\cdot\nabla u\,dx$$

explicitly into components, due to $\nabla\cdot u=0$,

$$I=\sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_1\,\partial_i u_1\,dx+\sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_2\,\partial_i u_2\,dx+\sum_{i,j=1}^{3}\int \partial_i u_j\,\partial_j u_3\,\partial_i u_3\,dx.$$

Several tool lemmas are employed to facilitate the estimates of the terms above (see Section 2). Integration by parts and the divergence-free condition $\nabla\cdot u=0$ are repeatedly applied to rebalance the derivatives. The requirement $\gamma>\frac{17}{20}$ appears to be necessary in order to control the terms generated by the nonlinearity $u\cdot\nabla\theta$. The proof for the uniqueness makes use of the difference of two solutions in L^2 and we actually establish a stronger version than stated in Theorem 1.1.

The global regularity result presented here constitutes an important first step in our program on the global well-posedness of the Boussinesq equations with partial or fractional dissipation. Our aim here is the global regularity for the Boussinesq equations with minimal regularization. It is our hope that this program will help develop new techniques and sharpen classical tools. Our next step in this program is to show the global regularity for

$$\begin{cases}\partial_t u_1+(u\cdot\nabla)u_1=-\partial_1 p-\nu(\Lambda_1^{\frac52}+\Lambda_2^{\frac52})u_1,\\ \partial_t u_2+(u\cdot\nabla)u_2=-\partial_2 p-\nu(\Lambda_2^{\frac52}+\Lambda_3^{\frac52})u_2,\\ \partial_t u_3+(u\cdot\nabla)u_3=-\partial_3 p-\nu\Lambda^{\frac52}u_3+\theta,\\ \nabla\cdot u=0,\\ \partial_t\theta+u\cdot\nabla\theta=0,\end{cases}\tag{1.4}$$

which does not involve thermal diffusion. The approach for (1.4) is to consecutively prove global bounds in more and more regular spaces: H^1 and $H^{\frac54}$ and then H^s for general $s>\frac52$. The details appear to be very complex and we have not had a definite answer to the global regularity problem on (1.4) yet.

The rest of this paper proves Theorem 1.1.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. Naturally the proof is divided into two parts: the existence part and the uniqueness part. The existence part boils down to a global *a priori* bound on (u,θ) in H^1 while the uniqueness part evaluates the difference of two solutions in the regularity class stated in Theorem 1.1. The rest of this section is divided into three subsections. The first subsection lists some of the tools used subsequently, the second shows the global *a priori* H^1 bound and the third proves the uniqueness.

Throughout the rest of this paper, we use $\|f\|_{L^2}$ to denote $\|f\|_{L^2(\mathbb{R}^3)}$ and $\|f\|_{L^2_{x_i}}$ to denote the one-dimensional L^2-norm (in terms of x_i), and $\|f\|_{L^2_{x_i x_j}}$ to denote the two-dimensional L^2-norm (in terms of x_i and x_j). In addition, we also use the notion

$$\|f\|_{L^r_{x_i} L^q_{x_j} L^p_{x_k}} \equiv \big\| \big\| \|f\|_{L^p_{x_k}} \big\|_{L^q_{x_j}} \big\|_{L^r_{x_i}}.$$

2.1. Preparations. This subsection states four tool lemmas to be used later. The first one is the following trace lemma. A proof can be found in [17].

Lemma 2.1. *Let $s>0$. Let $f=f(x_1,x')$ with $x'=(x_2,x_3,\cdots,x_d)$ be a d-dimensional function and $f\in H^{s+\frac{1}{2}}(\mathbb{R}^d)$. Then there exists a constant $C=C(d,s)$ such that*

$$\|\Lambda^s_{x'} f(x_1,x')\|_{L^2_{x'} L^\infty_{x_1}} \le C\|f\|_{H^{s+\frac{1}{2}}(\mathbb{R}^d)}.$$

The second one is a Sobolev embedding inequality involving one-dimensional functions. A simple proof can be found in [20].

Lemma 2.2. *Let $2\le p\le\infty$. Let $s>\frac{1}{2}-\frac{1}{p}$. Then, there exists a constant $C=C(p,s)$ such that, for any 1D functions $f\in H^s(\mathbb{R})$,*

$$\|f\|_{L^p(\mathbb{R})} \le C\|f\|_{L^2(\mathbb{R})}^{1-\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)} \|\Lambda^s f\|_{L^2(\mathbb{R})}^{\frac{1}{s}\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

The third one contains two well-known calculus inequalities (see, e.g., [7, p.334]). In this lemma $J=(I-\Delta)^{\frac{1}{2}}$ denotes the inhomogeneous differentiation operator.

Lemma 2.3. *Let $s>0$. Let $p,p_1,p_3\in(1,\infty)$ and $p_2,p_4\in[1,\infty]$ satisfying*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Then, for two constants C_1 and C_2,

$$\begin{aligned}
&\|J^s(fg)\|_{L^p} \le C_1\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|J^s g\|_{L^{p_3}}\|f\|_{L^{p_4}}\right),\\
&\|J^s(fg)-fJ^s g\|_{L^p} \le C_2\left(\|J^s f\|_{L^{p_1}}\|g\|_{L^{p_2}}+\|J^{s-1} g\|_{L^{p_3}}\|\nabla f\|_{L^{p_4}}\right).
\end{aligned}$$

The next tool lemma states one version of the Minkowski inequality, which is the foundation for exchanging two Lebesgue norms (see, e.g., [8]).

Lemma 2.4. *Let $f=f(x,y)$ with $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^n$ be a measurable function on $\mathbb{R}^m\times\mathbb{R}^n$. Let $1\le q\le p\le\infty$. Then*

$$\big\|\|f\|_{L^q_y(\mathbb{R}^n)}\big\|_{L^p_x(\mathbb{R}^m)} \le \big\|\|f\|_{L^p_x(\mathbb{R}^m)}\big\|_{L^q_y(\mathbb{R}^n)}.$$

2.2. Global H^1-bound for (u,θ). This subsection establishes a global H^1-bound on solutions of (1.3). More precisely, we prove the following proposition.

Proposition 2.1. *Assume $u_0\in H^1(\mathbb{R}^3)$ with $\nabla\cdot u_0=0$ and $\theta_0\in H^1(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Let (u,θ) be the corresponding solution of (1.3). Then, (u,θ) obeys Lemma 2.5 and the following global $\dot H^1$ bound, for any $t>0$,*

$$
\begin{aligned}
&(\|\nabla u(t)\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+\nu\int_0^t\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla u_1\|_{L^2}^2\,d\tau+\nu\int_0^t\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\nabla u_2\|_{L^2}^2\,d\tau\\
&\quad+\nu\int_0^t\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\nabla u_3\|_{L^2}^2\,d\tau+\kappa\int_0^t\|\Lambda^\gamma\nabla\theta\|_{L^2}^2\,d\tau\\
&\le C(\|u_0\|_{H^1},\|\theta_0\|_{H^1\cap L^\infty},t),
\end{aligned}
$$

where, for the sake of brevity, we have written

$$
\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla u_1\|_{L^2}\equiv\|\Lambda_1^{\frac54}\nabla u_1\|_{L^2}+\|\Lambda_2^{\frac54}\nabla u_1\|_{L^2}.
$$

A necessary step in the proof of Proposition 2.1 is the following global L^2-bound, which follows from a direct L^2 energy estimate involving (1.3).

Lemma 2.5. *Assume $u_0\in H^1$ with $\nabla\cdot u_0=0$ and $\theta_0\in L^2(\mathbb{R}^3)\cap L^\infty(\mathbb{R}^3)$. Let (u,θ) be the corresponding solution of (1.3). Then, for any $t\ge 0$,*

$$
\begin{aligned}
&\|u(t)\|_{L^2}^2+2\nu\int_0^t\Big(\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1(\tau)\|_{L^2}^2+\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2(\tau)\|_{L^2}^2\\
&\qquad\qquad+\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})u_3(\tau)\|_{L^2}^2\Big)\,d\tau\le(\|u_0\|_{L^2}+t\|\theta_0\|_{L^2})^2,\\
&\|\theta(t)\|_{L^2}^2+2\kappa\int_0^t\|\Lambda^\gamma\theta(\tau)\|_{L^2}^2\,d\tau\le\|\theta_0\|_{L^2}^2,\\
&\|\theta(t)\|_{L^q}\le\|\theta_0\|_{L^q},\qquad 2\le q\le\infty.
\end{aligned}
$$

We are now ready to prove Proposition 2.1.

Proof. **(Proof of Proposition 2.1.)** Taking the L^2-inner product of Δu with the first three equations of (1.3) and integrating by parts, we have

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\nabla u\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla u_1\|_{L^2}^2+\nu\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\nabla u_2\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\nabla u_3\|_{L^2}^2\\
&=-\sum_{i,j=1}^3\int\partial_iu_j\,\partial_ju_1\,\partial_iu_1\,dx-\sum_{i,j=1}^3\int\partial_iu_j\,\partial_ju_2\,\partial_iu_2\,dx-\sum_{i,j=1}^3\int\partial_iu_j\,\partial_ju_3\,\partial_iu_3\,dx\\
&\qquad+\int\nabla u_3\cdot\nabla\theta dx\\
&:=I_1+I_2+I_3+I_4.
\end{aligned}\tag{2.1}
$$

We now bound the terms above. Some of the estimates on the terms involving only u are similar to those in [20]. We first estimate I_1. To do so, we write out the nine terms explicitly,

$$
\begin{aligned}
I_1=-\int\Big(&(\partial_1u_1)^3+\partial_1u_1\,\partial_1u_2\,\partial_2u_1+\partial_1u_1\,\partial_1u_3\,\partial_3u_1\\
&+(\partial_2u_1)^2\,\partial_1u_1+(\partial_2u_1)^2\,\partial_2u_2+\partial_2u_1\,\partial_2u_3\,\partial_3u_1\\
&+(\partial_3u_1)^2\,\partial_1u_1+\partial_3u_1\,\partial_3u_2\,\partial_2u_1+(\partial_3u_1)^2\,\partial_3u_3\Big)\,dx.
\end{aligned}\tag{2.2}
$$

When we estimate the terms above, we keep in mind that we have the space and time L^2 integrability of the terms

$$\Lambda_1^{\frac{5}{4}}u_1,\quad \Lambda_2^{\frac{5}{4}}u_1,\quad \Lambda_2^{\frac{5}{4}}u_2,\quad \Lambda_3^{\frac{5}{4}}u_2,\quad \Lambda_1^{\frac{5}{4}}u_3,\quad \Lambda_3^{\frac{5}{4}}u_3$$

and the left hand side of (2.1) allows us to control the space and time L^2-norm of the terms

$$(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})(\partial_1,\partial_2,\partial_3)u_1,\quad (\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})(\partial_1,\partial_2,\partial_3)u_2,\quad (\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})(\partial_1,\partial_2,\partial_3)u_3.$$

The terms in (2.2) will be labelled as I_{11}, I_{12}, $\cdots$ according to the order they appear in (2.2).

We first deal with I_{12}, the second term in (2.2). We will return to I_{11} later. Integration by parts yields

$$\begin{aligned} I_{12} &= -\int \partial_1 u_1\,\partial_1 u_2\,\partial_2 u_1\,dx = \int u_2\,\partial_1\partial_2 u_1\,\partial_1 u_1\,dx + \int u_2\,\partial_1\partial_1 u_1\,\partial_2 u_1\,dx \\ &:= I_{121} + I_{122}. \end{aligned}$$

By Hölder's inequality and Lemma 2.4,

$$|I_{121}| \le \|\partial_1\partial_2 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\,\|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\,\|u_2\|_{L^2_{x_3}L^4_{x_1x_2}}.$$

By Gagliardo–Nirenberg's inequality,

$$\|\partial_1\partial_2 u_1\|_{L^2_{x_1x_3}L^4_{x_2}} \le C\,\|\Lambda_2^{\frac{1}{4}}\partial_1\partial_2 u_1\|_{L^2}$$

and

$$\|u_2\|_{L^2_{x_3}L^4_{x_1x_2}} \le C\,\|u_2\|_{L^2}^{\frac{1}{2}}\,\|\nabla_h u_2\|_{L^2}^{\frac{1}{2}},$$

where $\nabla_h = (\partial_1,\partial_2)$. Applying Gagliardo–Nirenberg's inequality and Lemma 2.2 with $p=\infty$ and $s=1$, we have

$$\begin{aligned} \|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} &\le C\,\|\Lambda_1^{\frac{1}{4}}\partial_1 u_1\|_{L^2_{x_1x_2}L^\infty_{x_3}} \\ &\le C\,\|\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^{\frac{1}{2}}\,\|\Lambda_3\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^{\frac{1}{2}}. \end{aligned}$$

Combining the estimates above and applying Young's inequality yield

$$\begin{aligned} |I_{121}| &\le C\,\|\Lambda_2^{\frac{1}{4}}\partial_1\partial_2 u_1\|_{L^2}\,\|u_2\|_{L^2}^{\frac{1}{2}}\,\|\nabla_h u_2\|_{L^2}^{\frac{1}{2}}\,\|\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^{\frac{1}{2}}\,\|\Lambda_3\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^{\frac{1}{2}} \\ &\le \frac{\nu}{128}\,\|\Lambda_2^{\frac{5}{4}}\partial_1 u_1\|_{L^2}^2 + \frac{\nu}{128}\,\|\Lambda_1^{\frac{5}{4}}\partial_3 u_1\|_{L^2}^2 + C\,\|u_2\|_{L^2}^2\,\|\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^2\,\|\nabla_h u_2\|_{L^2}^2. \end{aligned}$$

Similarly,

$$|I_{122}| \le C\,\|\Lambda_2^{\frac{1}{4}}\partial_1\partial_1 u_1\|_{L^2}\,\|u_2\|_{L^2}^{\frac{1}{2}}\,\|\nabla_h u_2\|_{L^2}^{\frac{1}{2}}\,\|\Lambda_1^{\frac{1}{4}}\partial_2 u_1\|_{L^2}^{\frac{1}{2}}\,\|\Lambda_3\Lambda_1^{\frac{1}{4}}\partial_2 u_1\|_{L^2}^{\frac{1}{2}}.$$

Due to the elementary inequalities

$$|\xi_2|^{\frac{1}{4}}\,|\xi_1| \le \frac{4}{5}|\xi_1|^{\frac{5}{4}} + \frac{1}{5}|\xi_2|^{\frac{5}{4}},\qquad |\xi_1|^{\frac{1}{4}}\,|\xi_2| \le \frac{1}{5}|\xi_1|^{\frac{5}{4}} + \frac{4}{5}|\xi_2|^{\frac{5}{4}}$$

and Plancherel's theorem, we have

$$\|\Lambda_2^{\frac{1}{4}}\partial_1\partial_1 u_1\|_{L^2} \le C(\|\Lambda_1^{\frac{5}{4}}\partial_1 u_1\|_{L^2} + \|\Lambda_2^{\frac{5}{4}}\partial_1 u_1\|_{L^2}),$$
$$\|\Lambda_1^{\frac{1}{4}}\partial_2 u_1\|_{L^2} \le C(\|\Lambda_1^{\frac{5}{4}} u_1\|_{L^2} + \|\Lambda_2^{\frac{5}{4}} u_1\|_{L^2}).$$

Therefore, by Young's inequality,

$$\begin{aligned}|I_{122}| \le& \frac{\nu}{128}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_1 u_1\|_{L^2}^2 + \frac{\nu}{128}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_3 u_1\|_{L^2}^2\\ &+C\|u_2\|_{L^2}^2\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\|_{L^2}^2\|\nabla_h u_2\|_{L^2}^2.\end{aligned}$$

We return to estimate I_{11}, the first term in (2.1). I_{11} can be handled similarly as I_{12}. Integrating by parts and then bounding it as I_{121}, we have

$$\begin{aligned}I_{11} =& 2\int u_1\,\partial_1\partial_1 u_1\,\partial_1 u_1\,dx\\ \le& C\|\partial_1\partial_1 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\|\partial_1 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\|u_1\|_{L^2_{x_3}L^4_{x_1x_2}}\\ \le& \frac{\nu}{128}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_1 u_1\|_{L^2}^2 + \frac{\nu}{128}\|\Lambda_1^{\frac{5}{4}}\partial_3 u_1\|_{L^2}^2 + C\|u_1\|_{L^2}^2\|\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2.\end{aligned}$$

We now estimate I_{13}. Integrating by parts, we have

$$\begin{aligned}I_{13} =& -\int \partial_1 u_1\,\partial_1 u_3\,\partial_3 u_1\,dx = \int u_1\,\partial_3\partial_1 u_1\,\partial_1 u_3\,dx + \int u_1\,\partial_1 u_1\,\partial_1\partial_3 u_3\,dx\\ :=& I_{131} + I_{132}.\end{aligned}$$

In fact, as in the estimates of I_{121},

$$\begin{aligned}I_{131} \le& C\|\partial_3\partial_1 u_1\|_{L^2_{x_1x_3}L^4_{x_2}}\|\partial_1 u_3\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\|u_1\|_{L^2_{x_3}L^4_{x_1x_2}}\\ \le& C\|\Lambda_2^{\frac{1}{4}}\partial_3\partial_1 u_1\|_{L^2}\|\Lambda_1^{\frac{1}{4}}\partial_1 u_3\|_{L^2}^{\frac{1}{2}}\|\Lambda_3\Lambda_1^{\frac{1}{4}}\partial_1 u_3\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2}^{\frac{1}{2}}\|\nabla_h u_1\|_{L^2}^{\frac{1}{2}}\\ \le& C\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_3 u_1\|_{L^2}\|\Lambda_1^{\frac{5}{4}}u_3\|_{L^2}^{\frac{1}{2}}\|\Lambda_1^{\frac{5}{4}}\partial_3 u_3\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2}^{\frac{1}{2}}\|\nabla_h u_1\|_{L^2}^{\frac{1}{2}}\\ \le& \frac{\nu}{128}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_3 u_1\|_{L^2}^2 + \frac{\nu}{128}\|\Lambda_1^{\frac{5}{4}}\partial_3 u_3\|_{L^2}^2\\ &+C\|u_1\|_{L^2}^2\|\Lambda_1^{\frac{5}{4}}u_3\|_{L^2}^2\|\nabla_h u_1\|_{L^2}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}I_{132} \le& C\|\partial_1\partial_3 u_3\|_{L^2_{x_1x_2}L^4_{x_3}}\|\partial_1 u_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}}\|u_1\|_{L^2_{x_2}L^4_{x_1x_3}}\\ \le& C\|\Lambda_3^{\frac{5}{4}}\partial_1 u_3\|_{L^2}\|\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^{\frac{1}{2}}\|\Lambda_2\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^{\frac{1}{2}}\|u_1\|_{L^2}^{\frac{1}{2}}\|(\partial_1,\partial_3)u_1\|_{L^2}^{\frac{1}{2}}\\ \le& \frac{\nu}{128}\|\Lambda_3^{\frac{5}{4}}\partial_1 u_3\|_{L^2}^2 + \frac{\nu}{128}\|\Lambda_1^{\frac{5}{4}}\partial_2 u_1\|_{L^2}^2\\ &+C\|u_1\|_{L^2}^2\|\Lambda_1^{\frac{5}{4}}u_1\|_{L^2}^2\|(\partial_1,\partial_3)u_1\|_{L^2}^2.\end{aligned}$$

This settles the estimate of I_{13}. We turn to I_{14} and I_{15}. Integrating by parts and invoking the divergence-free condition $\nabla\cdot u = 0$, we have

$$I_{14} + I_{15} = -\int (\partial_2 u_1)^2\,\partial_1 u_1\,dx - \int (\partial_2 u_1)^2\,\partial_2 u_2\,dx$$

$$
\begin{aligned}
&= \int (\partial_2 u_1)^2\, \partial_3 u_3\, dx \\
&= -2 \int u_3\, \partial_3\partial_2 u_1\, \partial_2 u_1\, dx \\
&\le 2\|\partial_3\partial_2 u_1\|_{L^2_{x_1x_3} L^4_{x_2}} \|\partial_2 u_1\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \|u_3\|_{L^2_{x_3} L^4_{x_1x_2}} \\
&\le C\|\Lambda_2^{\frac{5}{4}} \partial_3 u_1\|_{L^2} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_1\|_{L^2}^{\frac{1}{2}} \|\Lambda_3(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_1\|_{L^2}^{\frac{1}{2}} \|u_3\|_{L^2}^{\frac{1}{2}} \|\nabla_h u_3\|_{L^2}^{\frac{1}{2}} \\
&\le \frac{\nu}{128} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_1\|_{L^2}^2 + C\|u_3\|_{L^2}^2 \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) u_1\|_{L^2}^2 \|\nabla_h u_3\|_{L^2}^2.
\end{aligned}
$$

The estimates of I_{16} is more delicate. The integrand of I_{16},

$$
I_{16} = -\int \partial_2 u_3\, \partial_3 u_1\, \partial_2 u_1\, dx
$$

involves $\partial_3 u_1$ and $\partial_2 u_3$, but the first component equation of (1.3) involves no dissipation in the third direction and the third component equation involves no dissipation in the second direction. By integration by parts,

$$
\begin{aligned}
I_{16} &= \int u_1\, \partial_3\partial_2 u_3\, \partial_2 u_1\, dx + \int u_1\, \partial_2 u_3\, \partial_3\partial_2 u_1\, dx \\
&:= I_{161} + I_{162}.
\end{aligned}
$$

The first term I_{161} can be estimated similarly as I_{12}. In fact,

$$
\begin{aligned}
I_{161} &\le \|\partial_3\partial_2 u_3\|_{L^2_{x_1x_2} L^4_{x_3}} \|\partial_2 u_1\|_{L^2_{x_3} L^\infty_{x_1} L^4_{x_2}} \|u_1\|_{L^2_{x_1} L^4_{x_2x_3}} \\
&\le \frac{\nu}{128} \|\Lambda_3^{\frac{5}{4}} \partial_2 u_3\|_{L^2}^2 + \frac{\nu}{128} \|\Lambda_2^{\frac{5}{4}} \partial_1 u_1\|_{L^2}^2 \\
&\quad + C\|u_1\|_{L^2}^2 \|\Lambda_2^{\frac{5}{4}} u_1\|_{L^2}^2 \|(\partial_2, \partial_3) u_1\|_{L^2}^2,
\end{aligned}
$$

The estimates of I_{162} is different.

$$
\begin{aligned}
I_{162} &\le \|\partial_3\partial_2 u_1\|_{L^2_{x_2x_3} L^4_{x_1}} \|\partial_2 u_3\|_{L^2_{x_2} L^\infty_{x_3} L^4_{x_1}} \|u_1\|_{L^2_{x_1x_3} L^\infty_{x_2}} \\
&\le C\|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_1\|_{L^2} \|\Lambda_1^{\frac{1}{4}} \partial_2 u_3\|_{L^2}^{\frac{1}{2}} \|\Lambda_3 \Lambda_1^{\frac{1}{4}} \partial_2 u_3\|_{L^2}^{\frac{1}{2}} \|u_1\|_{L^2}^{\frac{3}{5}} \|\Lambda_2^{\frac{5}{4}} u_1\|_{L^2}^{\frac{2}{5}} \\
&\le C\|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_1\|_{L^2} \|\partial_2 u_3\|_{L^2}^{\frac{2}{5}} \|\Lambda_1^{\frac{5}{4}} \partial_2 u_3\|_{L^2}^{\frac{1}{10}} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_2 u_3\|_{L^2}^{\frac{1}{2}} \\
&\quad \times \|u_1\|_{L^2}^{\frac{3}{5}} \|\Lambda_2^{\frac{5}{4}} u_1\|_{L^2}^{\frac{2}{5}} \\
&\le \frac{\nu}{128} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_2^{\frac{5}{4}}) \partial_3 u_1\|_{L^2}^2 + \frac{\nu}{128} \|(\Lambda_1^{\frac{5}{4}}, \Lambda_3^{\frac{5}{4}}) \partial_2 u_3\|_{L^2}^2 \\
&\quad + C\|u_1\|_{L^2}^3 \|\Lambda_2^{\frac{5}{4}} u_1\|_{L^2}^2 \|\partial_2 u_3\|_{L^2}^2.
\end{aligned}
$$

We now turn to the last three terms in (2.2). Due to $\nabla \cdot u = 0$, the last three terms in (2.2) can regrouped into two terms,

$$
-\int \partial_3 u_1\, \partial_3 u_2\, \partial_2 u_1 + \int \partial_2 u_2 (\partial_3 u_1)^2\, dx := I_{17} + I_{18}.
$$

I_{17} can be estimated as I_{12}. By integration by parts,

$$
I_{17} = \int u_1\, \partial_3\partial_3 u_2\, \partial_2 u_1\, dx + \int u_1\, \partial_3\partial_2 u_1\, \partial_3 u_2\, dx := I_{171} + I_{172}.
$$

I_{171} can be estimated similarly as I_{12} and we have

$$\begin{aligned}|I_{171}| &\le \|\partial_3\partial_3 u_2\|_{L^2_{x_1x_3}L^4_{x_2}} \|\partial_2 u_1\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}} \|u_1\|_{L^2_{x_3}L^4_{x_1x_2}} \\ &\le C\|\Lambda_2^{\frac14}\partial_3\partial_3 u_2\|_{L^2} \|\Lambda_1^{\frac14}\partial_2 u_1\|_{L^2}^{\frac12} \|\Lambda_1^{\frac14}\partial_2\partial_3 u_1\|_{L^2}^{\frac12} \|u_1\|_{L^2}^{\frac12} \|(\partial_1,\partial_2)u_1\|_{L^2}^{\frac12} \\ &\le \frac{\nu}{128}\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\partial_3 u_2\|_{L^2}^2 + \frac{\nu}{128}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^2 \\ &\quad + C\|u_1\|_{L^2}^2 \|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1\|_{L^2}^2 \|(\partial_1,\partial_2)u_1\|_{L^2}^2.\end{aligned}$$

Similarly,

$$\begin{aligned}|I_{172}| \le &\frac{\nu}{128}\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\partial_3 u_2\|_{L^2}^2 + \frac{\nu}{128}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^2 \\ &+ C\|u_1\|_{L^2}^2 \|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2 \|(\partial_1,\partial_2)u_1\|_{L^2}^2.\end{aligned}$$

We deal with I_{18},

$$I_{18} = \int \partial_2 u_2 (\partial_3 u_1)^2\, dx.$$

Due to the appearance of $(\partial_3 u_1)^2$ and the lack of dissipation in the third direction in the equation of u_1, the handling of this term is more delicate. By integration by parts,

$$I_{18} = -2\int u_2\, \partial_3 u_1\, \partial_2\partial_3 u_1\, dx.$$

By Hölder's inequality and by Minkowski's inequality,

$$|I_{18}| \le 2\|\partial_2\partial_3 u_1\|_{L^2_{x_1x_3}L^4_{x_2}} \|u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}} \|\partial_3 u_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}}. \tag{2.3}$$

By Lemma 2.2 and an interpolation inequality,

$$\begin{aligned}\|\partial_3 u_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}} &\le C\|\Lambda_2^{\frac14}\partial_3 u_1\|_{L^2_{x_2x_3}L^\infty_{x_1}} \\ &\le C\|\Lambda_2^{\frac14}\partial_3 u_1\|_{L^2}^{\frac12} \|\Lambda_2^{\frac14}\Lambda_1\partial_3 u_1\|_{L^2}^{\frac12} \\ &\le C\|\partial_3 u_1\|_{L^2}^{\frac25} \|\Lambda_2^{\frac54}\partial_3 u_1\|_{L^2}^{\frac{1}{10}} \|\Lambda_2^{\frac14}\Lambda_1\partial_3 u_1\|_{L^2}^{\frac12} \\ &\le C\|\partial_3 u_1\|_{L^2}^{\frac25} \|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^{\frac35},\end{aligned} \tag{2.4}$$

where we have invoked the interpolation inequality

$$\|\Lambda_2^{\frac14}\partial_3 u_1\|_{L^2_{x_2}} \le C\|\partial_3 u_1\|_{L^2_{x_2}}^{\frac45} \|\Lambda_2^{\frac54}\partial_3 u_1\|_{L^2_{x_2}}^{\frac15}.$$

In addition,

$$\|u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}} \le C\|u_2\|_{L^2}^{\frac35} \|\Lambda_3^{\frac54} u_2\|_{L^2}^{\frac25}. \tag{2.5}$$

Inserting (2.4) and (2.5) in (2.3) yields

$$\begin{aligned}|I_{18}| &\le C\|\Lambda_2^{\frac54}\partial_3 u_1\|_{L^2} \|u_2\|_{L^2}^{\frac35} \|\Lambda_3^{\frac54} u_2\|_{L^2}^{\frac25} \|\partial_3 u_1\|_{L^2}^{\frac25} \|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^{\frac35} \\ &\le \frac{\nu}{128}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\partial_3 u_1\|_{L^2}^2 + C\|u_2\|_{L^2}^3 \|\Lambda_3^{\frac54} u_2\|_{L^2}^2 \|\partial_3 u_1\|_{L^2}^2.\end{aligned}$$

We remark that I_{16} and I_{18} could have been treated in a similar fashion. We intentionally estimated them differently to make available different approaches that serve the same purpose. We have finished estimating all terms in I_1 in (2.1). I_2 and I_3 in (2.1) can be similarly estimated as the terms in I_1 and we omit the details. Finally we deal with I_4,

$$I_4=\int \nabla u_3\cdot\nabla\theta dx\le\|\nabla u_3\|_{L^2}\|\nabla\theta\|_{L^2}\le\|\nabla u_3\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2,$$

Putting all these estimates together, we have

$$\begin{aligned}&\frac{d}{dt}\|\nabla u\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla u_1\|_{L^2}^2+\nu\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\nabla u_2\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\nabla u_3\|_{L^2}^2\\ &\le C\big(\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1\|_{L^2}^2+\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2\|_{L^2}^2+\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})u_3\|_{L^2}^2\big)\\ &\qquad\times(\|u\|_{L^2}^2+\|u\|_{L^2}^3)\|\nabla u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2.\end{aligned}\tag{2.6}$$

Taking the L^2-inner product of $\Delta\theta$ with the fifth equation of (1.3) and integrating by parts, we have

$$\begin{aligned}&\frac12\frac{d}{dt}\|\nabla\theta\|_{L^2}^2+\kappa\|\Lambda^\gamma\nabla\theta\|_{L^2}^2\\ &=-\sum_{j=1}^3\int\partial_1u_j\,\partial_j\theta\,\partial_1\theta\,dx-\sum_{j=1}^3\int\partial_2u_j\,\partial_j\theta\,\partial_2\theta\,dx-\sum_{j=1}^3\int\partial_3u_j\,\partial_j\theta\,\partial_3\theta\,dx\\ &:=K_1+K_2+K_3.\end{aligned}\tag{2.7}$$

We first estimate K_1. To do so, we write out the three terms explicitly,

$$\begin{aligned}K_1&=-\int\partial_1u_1(\partial_1\theta)^2\,dx-\int\partial_1u_2\,\partial_1\theta\,\partial_2\theta\,dx-\int\partial_1u_3\,\partial_1\theta\,\partial_3\theta\,dx\\ &:=K_{11}+K_{12}+K_{13}.\end{aligned}\tag{2.8}$$

We first deal with K_{12}, the second term in (2.8). We will return to K_{11} and K_{13} later. Integration by parts yields

$$\begin{aligned}K_{12}&=-\int\partial_1u_2\,\partial_1\theta\,\partial_2\theta\,dx=\int\theta\,\partial_1u_2\,\partial_2\partial_1\theta\,dx+\int\theta\,\partial_2\partial_1u_2\,\partial_1\theta\,dx\\ &:=K_{121}+K_{122}.\end{aligned}$$

By Hölder's inequality and Lemma 2.3,

$$\begin{aligned}K_{121}&=\int\Lambda_2^{1-\gamma}(\theta\partial_1u_2)\Lambda_2^\gamma\partial_1\theta\,dx\\ &\le C\|\Lambda_2^\gamma\partial_1\theta\|_{L^2}(\|\theta\|_{L^\infty}\|\Lambda_2^{1-\gamma}\partial_1u_2\|_{L^2}\\ &\quad+\|\partial_1u_2\|_{L^2_{x_1}L^\infty_{x_3}L^\mu_{x_2}}\|\Lambda_2^{1-\gamma}\theta\|_{L^\infty_{x_1}L^2_{x_3}L^{\tilde\mu}_{x_2}}),\end{aligned}\tag{2.9}$$

where $\frac1\mu+\frac1{\tilde\mu}=\frac12$ with $\mu>2$ and $\tilde\mu>2$. (2.9) is obtained by repeatedly applying Lemma 2.3 to $\Lambda_2^{1-\gamma}(\theta\partial_1u_2)$ first as a function of x_2, then as a function of x_3 and of x_1. By Gagliardo–Nirenberg's inequality,

$$\|\partial_1u_2\|_{L^2_{x_1}L^\infty_{x_3}L^\mu_{x_2}}\le C\|\Lambda_2^\rho\partial_1u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}},$$

where $\frac{1}{\mu}=\frac{1}{2}-\rho$. Applying Lemma 2.2 with $p=\infty$ and $s=\frac{5}{4}-\rho$, we have

$$\|\Lambda_2^\rho\partial_1 u_2\|_{L^2_{x_1x_2}L^\infty_{x_3}}\le C\|\Lambda_2^\rho\partial_1 u_2\|_{L^2}^{1-\frac{1}{2(\frac{5}{4}-\rho)}}\|\Lambda_3^{\frac{5}{4}-\rho}\Lambda_2^\rho\partial_1 u_2\|_{L^2}^{\frac{1}{2(\frac{5}{4}-\rho)}}.$$

By Gagliardo–Nirenberg's inequality,

$$\|\Lambda_2^\rho\partial_1 u_2\|_{L^2}\le\|\partial_1 u_2\|_{L^2}^{1-\frac{4\rho}{5}}\|\Lambda_2^{\frac{5}{4}}\partial_1 u_2\|_{L^2}^{\frac{4\rho}{5}}.$$

Due to the elementary inequalities

$$|\xi_3|^{\frac{5}{4}-\rho}|\xi_2|^\rho\le\frac{5-4\rho}{5}|\xi_3|^{\frac{5}{4}}+\frac{4\rho}{5}|\xi_2|^{\frac{5}{4}}$$

and Plancherel's theorem, we have

$$\|\Lambda_3^{\frac{5}{4}-\rho}\Lambda_2^\rho\partial_1 u_2\|_{L^2}\le C(\|\Lambda_2^{\frac{5}{4}}\partial_1 u_2\|_{L^2}+\|\Lambda_3^{\frac{5}{4}}\partial_1 u_2\|_{L^2}).$$

Then

$$\begin{aligned}\|\partial_1 u_2\|_{L^2_{x_1}L^\infty_{x_3}L^\mu_{x_2}}&\le C\|\partial_1 u_2\|_{L^2}^{(1-\frac{4\rho}{5})(1-\frac{1}{2(\frac{5}{4}-\rho)})}\|\Lambda_2^{\frac{5}{4}}\partial_1 u_2\|_{L^2}^{\frac{4\rho}{5}(1-\frac{1}{2(\frac{5}{4}-\rho)})}\\&\quad\times\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\partial_1 u_2\|_{L^2}^{\frac{1}{2(\frac{5}{4}-\rho)}}\\&\le C\|\partial_1 u_2\|_{L^2}^{\frac{3-4\rho}{5}}\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\partial_1 u_2\|_{L^2}^{\frac{4\rho+2}{5}}.\end{aligned}$$

By an interpolation inequality,

$$\|\Lambda_2^{1-\gamma}\theta\|_{L^{\widetilde{\mu}}_{x_2}}\le C\|\theta\|_{L^\infty_{x_2}}^{1-\frac{2}{\widetilde{\mu}}}\|\Lambda_2^{\frac{(1-\gamma)\widetilde{\mu}}{2}}\theta\|_{L^2_{x_2}}^{\frac{2}{\widetilde{\mu}}}.$$

Therefore, by Hölder's inequality and Lemma 2.1,

$$\begin{aligned}\|\Lambda_2^{1-\gamma}\theta\|_{L^\infty_{x_1}L^2_{x_3}L^{\widetilde{\mu}}_{x_2}}&\le C\Big\|\|\theta\|_{L^\infty_{x_2}}^{1-\frac{2}{\widetilde{\mu}}}\|\Lambda_2^{\frac{(1-\gamma)\widetilde{\mu}}{2}}\theta\|_{L^2_{x_2}}^{\frac{2}{\widetilde{\mu}}}\Big\|_{L^\infty_{x_1}L^2_{x_3}}\\&\le C\|\theta\|_{L^\infty}^{1-\frac{2}{\widetilde{\mu}}}\|\Lambda_2^{\frac{(1-\gamma)\widetilde{\mu}}{2}}\theta\|_{L^\infty_{x_1}L^2_{x_2x_3}}^{\frac{2}{\widetilde{\mu}}}\\&\le C\|\theta_0\|_{L^\infty}^{1-\frac{2}{\widetilde{\mu}}}\|\theta\|_{H^{\frac{1}{2}+\frac{1-\gamma}{2}\widetilde{\mu}}}^{\frac{2}{\widetilde{\mu}}}.\end{aligned}$$

Combining the estimates above and applying Young's inequality yield

$$\begin{aligned}K_{121}&\le C\|\Lambda_2^\gamma\partial_1\theta\|_{L^2}\Big(\|\theta\|_{L^\infty}\|\Lambda_2^{1-\gamma}\partial_1 u_2\|_{L^2}+\|\partial_1 u_2\|_{L^2}^{\frac{3-4\rho}{5}}\\&\quad\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\partial_1 u_2\|_{L^2}^{\frac{4\rho+2}{5}}\|\theta_0\|_{L^\infty}^{1-\frac{2}{\widetilde{\mu}}}\|\theta\|_{H^{\frac{1}{2}+\frac{1-\gamma}{2}\widetilde{\mu}}}^{\frac{2}{\widetilde{\mu}}}\Big)\\&\le\frac{\kappa}{128}\|\Lambda_2^\gamma\partial_1\theta\|_{L^2}^2+\frac{\nu}{128}\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\partial_1 u_2\|_{L^2}^2\\&\quad+C\left(\|\Lambda^{\frac{1}{2}+\frac{(1-\gamma)\widetilde{\mu}}{2}}\theta\|_{L^2}^{\frac{2}{\widetilde{\mu}}\cdot\frac{10}{3-4\rho}}+1\right)\|\partial_1 u_2\|_{L^2}^2,\end{aligned}$$

where μ, $\widetilde{\mu}$ and ρ have been selected to obey the constraints

$$\begin{cases}\frac{1-\gamma}{2}\widetilde{\mu}+\frac{1}{2}\le\gamma,\\ \frac{2}{\widetilde{\mu}}\cdot\frac{10}{3-4\rho}\le 2,\\ \frac{1}{\mu}+\frac{1}{\widetilde{\mu}}=\frac{1}{2},\\ \frac{1}{\mu}=\frac{1}{2}-\rho.\end{cases}\tag{2.10}$$

When γ satisfies the condition of Theorem 1.1,

$$\gamma \geq \frac{17}{20},$$

we can choose μ, $\widetilde{\mu}$ and ρ satisfying (2.10), for example, if $\gamma = \frac{17}{20}$, then

$$\rho = \frac{3}{14}, \quad \mu = \frac{7}{2}, \quad \widetilde{\mu} = \frac{14}{3}$$

would fulfill the requirements. By Hölder's inequality and Young's inequality,

$$\begin{aligned}
|K_{122}| &\leq C\|\partial_2\partial_1 u_2\|_{L^2}\|\partial_1\theta\|_{L^2}\|\theta\|_{L^\infty} \\
&\leq C\|\partial_1 u_2\|_{L^2}^{\frac{1}{5}}\|\Lambda_2^{\frac{5}{4}}\partial_1 u_2\|_{L^2}^{\frac{4}{5}}\|\partial_1\theta\|_{L^2}\|\theta_0\|_{L^\infty} \\
&\leq \frac{\nu}{128}\|\Lambda_2^{\frac{5}{4}}\partial_1 u_2\|_{L^2}^2 + C\|\partial_1 u_2\|_{L^2}^{\frac{1}{3}}\|\partial_1\theta\|_{L^2}^{\frac{5}{3}}\|\theta_0\|_{L^\infty}^{\frac{5}{3}} \\
&\leq \frac{\nu}{128}\|\Lambda_2^{\frac{5}{4}}\partial_1 u_2\|_{L^2}^2 + C(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{L^2}^2).
\end{aligned}$$

Therefore, for $\gamma \geq \frac{17}{20}$,

$$\begin{aligned}
|K_{12}| \leq& \frac{\kappa}{128}\|\Lambda_2^{\gamma}\partial_1\theta\|_{L^2}^2 + \frac{\nu}{64}\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\partial_1 u_2\|_{L^2}^2 \\
&+ C(\|\Lambda^{\gamma}\theta\|_{L^2}^2 + 1)(\|\partial_1 u_2\|_{L^2}^2 + \|\partial_1\theta\|_{L^2}^2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
|K_{11}| \leq& \frac{\kappa}{128}\|\Lambda_1^{\gamma}\partial_1\theta\|_{L^2}^2 + \frac{\nu}{64}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\partial_1 u_1\|_{L^2}^2 \\
&+ C(\|\Lambda^{\gamma}\theta\|_{L^2}^2 + 1)(\|\partial_1 u_1\|_{L^2}^2 + \|\partial_1\theta\|_{L^2}^2)
\end{aligned}$$

and

$$\begin{aligned}
|K_{13}| \leq& \frac{\kappa}{128}\|\Lambda_3^{\gamma}\partial_1\theta\|_{L^2}^2 + \frac{\nu}{64}\|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\partial_1 u_3\|_{L^2}^2 \\
&+ C(\|\Lambda^{\gamma}\theta\|_{L^2}^2 + 1)(\|\partial_1 u_3\|_{L^2}^2 + \|\partial_1\theta\|_{L^2}^2).
\end{aligned}$$

We have finished estimating all terms in K_1 in (2.8). K_2 and K_3 in (2.7) can be similarly estimated as K_1 and we omit the details. Putting all these estimates together, we have

$$\begin{aligned}
&\frac{d}{dt}\|\nabla\theta\|_{L^2}^2 + \kappa\|\Lambda^{\gamma}\nabla\theta\|_{L^2}^2 \\
\leq& \frac{\nu}{64}\Big(\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\nabla u_1\|_{L^2}^2 + \|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_2\|_{L^2}^2 + \|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_3\|_{L^2}^2\Big) \\
&+ C(\|\Lambda^{\gamma}\theta\|_{L^2}^2 + 1)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned} \tag{2.11}$$

Combining (2.6) with (2.11), we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2) + \nu\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\nabla u_1\|_{L^2}^2 + \nu\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_2\|_{L^2}^2 \\
&\quad + \nu\|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\nabla u_3\|_{L^2}^2 + \kappa\|\Lambda^{\gamma}\nabla\theta\|_{L^2}^2 \\
\leq& C\Big((\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})u_1\|_{L^2}^2 + \|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_2\|_{L^2}^2 + \|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})u_3\|_{L^2}^2) \\
&\quad \times(\|u\|_{L^2}^2 + \|u\|_{L^2}^3) + \|\Lambda^{\gamma}\theta\|_{L^2}^2 + 1\Big)(\|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2).
\end{aligned}$$

Gronwall's inequality then yields the desired global bound. This completes the proof of Proposition 2.1. □

2.3. Uniqueness. This section proves the uniqueness part of Theorem 1.1. In fact, we prove a proposition that is slightly stronger than the desired uniqueness. The uniqueness in the following proposition does not require both solutions are in the regularity class induced by the existence part.

PROPOSITION 2.2. *Let $T>0$. Assume that $(u^{(1)},\theta^{(1)})$ and $(u^{(2)},\theta^{(2)})$ are two solutions of (1.3) satisfying,*

$$
\begin{aligned}
&(u^{(i)},\theta^{(i)})\in L^\infty(0,T;H^1(\mathbb{R}^3)), \quad \textit{for } i=1,2,\\
&(\Lambda_1^{\frac54},\Lambda_2^{\frac54})u_1^{(2)}\in L^2(\mathbb{R}^3\times(0,T)), \qquad (\Lambda_1^{\frac54},\Lambda_2^{\frac54})\nabla u_1^{(2)}\in L^2(\mathbb{R}^3\times(0,T)),\\
&(\Lambda_2^{\frac54},\Lambda_3^{\frac54})u_2^{(2)}\in L^2(\mathbb{R}^3\times(0,T)), \qquad (\Lambda_2^{\frac54},\Lambda_3^{\frac54})\nabla u_2^{(2)}\in L^2(\mathbb{R}^3\times(0,T)),\\
&(\Lambda_1^{\frac54},\Lambda_3^{\frac54})u_3^{(2)}\in L^2(\mathbb{R}^3\times(0,T)), \qquad (\Lambda_1^{\frac54},\Lambda_3^{\frac54})\nabla u_3^{(2)}\in L^2(\mathbb{R}^3\times(0,T)),\\
&\Lambda^\gamma\nabla\theta^{(2)}\in L^2(\mathbb{R}^3\times(0,T)).
\end{aligned}
$$

Then $(u^{(1)},\theta^{(1)})=(u^{(2)},\theta^{(2)})$ on $\mathbb{R}^3\times(0,T)$.

Proof. Let $p^{(1)}$ and $p^{(2)}$ be the pressures associated with $u^{(1)}$ and $u^{(2)}$, respectively. Then the differences $\widetilde{u}=u^{(1)}-u^{(2)}$, $\widetilde{p}=p^{(1)}-p^{(2)}$ and $\widetilde{\theta}=\theta^{(1)}-\theta^{(2)}$ satisfy

$$
\begin{cases}
\partial_t\widetilde{u}_1+(u^{(1)}\cdot\nabla)\widetilde{u}_1+(\widetilde{u}\cdot\nabla)u_1^{(2)}=-\partial_1\widetilde{p}-\nu(\Lambda_1^{\frac52}+\Lambda_2^{\frac52})\widetilde{u}_1,\\
\partial_t\widetilde{u}_2+(u^{(1)}\cdot\nabla)\widetilde{u}_2+(\widetilde{u}\cdot\nabla)u_2^{(2)}=-\partial_2\widetilde{p}-\nu(\Lambda_2^{\frac52}+\Lambda_3^{\frac52})\widetilde{u}_2,\\
\partial_t\widetilde{u}_3+(u^{(1)}\cdot\nabla)\widetilde{u}_3+(\widetilde{u}\cdot\nabla)u_3^{(2)}=-\partial_3\widetilde{p}-\nu(\Lambda_1^{\frac52}+\Lambda_3^{\frac52})\widetilde{u}_3+\widetilde{\theta},\\
\partial_t\widetilde{\theta}+(u^{(1)}\cdot\nabla)\widetilde{\theta}+(\widetilde{u}\cdot\nabla)\theta^{(2)}=-\kappa\Lambda^{2\gamma}\widetilde{\theta},\\
\nabla\cdot\widetilde{u}=0,\\
\widetilde{u}(x,0)=\widetilde{u}_0(x),\quad \widetilde{\theta}(x,0)=\widetilde{\theta}_0(x).
\end{cases}
\tag{2.12}
$$

Dotting (2.12) with $\widetilde{u}$ and invoking the divergence-free conditions $\nabla\cdot u^{(1)}=0$ and $\nabla\cdot\widetilde{u}=0$, we obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\widetilde{u}_1\|_{L^2}^2+\nu\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\widetilde{u}_2\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\widetilde{u}_3\|_{L^2}^2\\
&=-\int(\widetilde{u}\cdot\nabla)u^{(2)}\cdot\widetilde{u}\,dx+\int\widetilde{u}_3\widetilde{\theta}\,dx\\
&=-\int(\widetilde{u}\cdot\nabla)u_1^{(2)}\widetilde{u}_1\,dx-\int(\widetilde{u}\cdot\nabla)u_2^{(2)}\widetilde{u}_2\,dx-\int(\widetilde{u}\cdot\nabla)u_3^{(2)}\widetilde{u}_3\,dx+\int\widetilde{u}_3\widetilde{\theta}\,dx\\
&:=J_1+J_2+J_3+J_4.
\end{aligned}
$$

We estimate J_1 and write its terms explicitly,

$$
\begin{aligned}
J_1&=-\int\widetilde{u}_1\partial_1u_1^{(2)}\widetilde{u}_1\,dx-\int\widetilde{u}_2\partial_2u_1^{(2)}\widetilde{u}_1\,dx-\int\widetilde{u}_3\partial_3u_1^{(2)}\widetilde{u}_1\,dx\\
&:=J_{11}+J_{12}+J_{13}.
\end{aligned}
$$

By Hölder's inequality and Lemma 2.2,

$$
\begin{aligned}
|J_{11}|&\le\|\widetilde{u}_1\|_{L^2}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}}\|\partial_1u_1^{(2)}\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\\
&\le\|\widetilde{u}_1\|_{L^2}\|\Lambda_1^{\frac14}\widetilde{u}_1\|_{L^2_{x_1x_3}L^\infty_{x_2}}\|\Lambda_1^{\frac14}\partial_1u_1^{(2)}\|_{L^2_{x_1x_2}L^\infty_{x_3}}\\
&\le C\|\widetilde{u}_1\|_{L^2}\|\Lambda_1^{\frac14}\widetilde{u}_1\|_{L^2}^{\frac12}\|\Lambda_1^{\frac14}\Lambda_2\widetilde{u}_1\|_{L^2}^{\frac12}\|\Lambda_1^{\frac14}\partial_1u_1^{(2)}\|_{L^2}^{\frac12}\|\Lambda_1^{\frac14}\Lambda_3\partial_1u_1^{(2)}\|_{L^2}^{\frac12}.
\end{aligned}
$$

Inserting the interpolation inequality above,

$$\|\Lambda_1^{\frac14}\widetilde{u}_1\|_{L^2_{x_1}}\le C\|\widetilde{u}_1\|_{L^2_{x_1}}^{\frac45}\|\Lambda_1^{\frac54}\widetilde{u}_1\|_{L^2_{x_1}}^{\frac15},$$

we obtain

$$\begin{aligned}|J_{11}|&\le C\|\widetilde{u}_1\|_{L^2}^{\frac75}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\widetilde{u}_1\|_{L^2}^{\frac35}\|\Lambda_1^{\frac54}u_1^{(2)}\|_{L^2}^{\frac12}\|\Lambda_1^{\frac54}\Lambda_3u_1^{(2)}\|_{L^2}^{\frac12}\\&\le\frac{\nu}{128}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\widetilde{u}_1\|_{L^2}^2+C\|\widetilde{u}_1\|_{L^2}^2\|\Lambda_1^{\frac54}u_1^{(2)}\|_{L^2}^{\frac57}\|\Lambda_1^{\frac54}\Lambda_3u_1^{(2)}\|_{L^2}^{\frac57}.\end{aligned}$$

By Hölder's inequality and Lemma 2.2,

$$\begin{aligned}|J_{12}|&\le\|\partial_2u_1^{(2)}\|_{L^2}\|\widetilde{u}_2\|_{L^2_{x_1}L^\infty_{x_3}L^4_{x_2}}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_1}L^4_{x_2}}\\&\le C\|\partial_2u_1^{(2)}\|_{L^2}\|\Lambda_2^{\frac14}\widetilde{u}_2\|_{L^2_{x_1x_2}L^\infty_{x_3}}\|\Lambda_2^{\frac14}\widetilde{u}_1\|_{L^2_{x_2x_3}L^\infty_{x_1}}\\&\le C\|\partial_2u_1^{(2)}\|_{L^2}\|\Lambda_2^{\frac14}\widetilde{u}_2\|_{L^2}^{\frac12}\|\Lambda_2^{\frac14}\partial_3\widetilde{u}_2\|_{L^2}^{\frac12}\|\Lambda_2^{\frac14}\widetilde{u}_1\|_{L^2}^{\frac12}\|\Lambda_2^{\frac14}\partial_1\widetilde{u}_1\|_{L^2}^{\frac12}.\end{aligned}$$

Invoking the interpolation inequalities

$$\|\Lambda_2^{\frac14}\widetilde{u}_2\|_{L^2_{x_2}}\le C\|\widetilde{u}_2\|_{L^2_{x_2}}^{\frac45}\|\Lambda_2^{\frac54}\widetilde{u}_2\|_{L^2_{x_2}}^{\frac15},\qquad\|\Lambda_2^{\frac14}\widetilde{u}_1\|_{L^2_{x_2}}\le C\|\widetilde{u}_1\|_{L^2_{x_2}}^{\frac45}\|\Lambda_2^{\frac54}\widetilde{u}_1\|_{L^2_{x_2}}^{\frac15}$$

and thus

$$\|\Lambda_2^{\frac14}\widetilde{u}_2\|_{L^2}\le C\|\widetilde{u}_2\|_{L^2}^{\frac45}\|\Lambda_2^{\frac54}\widetilde{u}_2\|_{L^2}^{\frac15},\qquad\|\Lambda_2^{\frac14}\widetilde{u}_1\|_{L^2}\le C\|\widetilde{u}_1\|_{L^2}^{\frac45}\|\Lambda_2^{\frac54}\widetilde{u}_1\|_{L^2}^{\frac15},$$

we have

$$\begin{aligned}|J_{12}|&\le C\|\partial_2u_1^{(2)}\|_{L^2}\|\widetilde{u}_1\|_{L^2}^{\frac25}\|\widetilde{u}_2\|_{L^2}^{\frac25}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\widetilde{u}_1\|_{L^2}^{\frac35}\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\widetilde{u}_2\|_{L^2}^{\frac35}\\&\le\frac{\nu}{128}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\widetilde{u}_1\|_{L^2}^2+\frac{\nu}{128}\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\widetilde{u}_2\|_{L^2}^2\\&\quad+C\|\partial_2u_1^{(2)}\|_{L^2}^{\frac52}\|\widetilde{u}_1\|_{L^2}\|\widetilde{u}_2\|_{L^2}.\end{aligned}$$

The estimate for J_{13} is similar to that for J_{12}. In fact,

$$\begin{aligned}|J_{13}|&\le\|\partial_3u_1^{(2)}\|_{L^2}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}}\|\widetilde{u}_3\|_{L^2_{x_2}L^\infty_{x_3}L^4_{x_1}}\\&\le\frac{\nu}{128}\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\widetilde{u}_1\|_{L^2}^2+\frac{\nu}{128}\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\widetilde{u}_3\|_{L^2}^2\\&\quad+C\|\partial_3u_1^{(2)}\|_{L^2}^{\frac52}\|\widetilde{u}_1\|_{L^2}\|\widetilde{u}_3\|_{L^2}.\end{aligned}$$

Due to the symmetry, the estimates of J_2 and J_3 are similar to those for J_1 and we omit further details. Finally we deal with J_4,

$$J_4=\int\widetilde{u}_3\widetilde{\theta}\,dx\le\|\widetilde{u}_3\|_{L^2}\|\widetilde{\theta}\|_{L^2}\le\|\widetilde{u}_3\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2.$$

Collecting the bounds for J_1,J_2,J_3 and J_4, we obtain

$$\begin{aligned}&\frac{d}{dt}\|\widetilde{u}\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_2^{\frac54})\widetilde{u}_1\|_{L^2}^2+\nu\|(\Lambda_2^{\frac54},\Lambda_3^{\frac54})\widetilde{u}_2\|_{L^2}^2+\nu\|(\Lambda_1^{\frac54},\Lambda_3^{\frac54})\widetilde{u}_3\|_{L^2}^2\\&\le C\|\nabla u^{(2)}\|_{L^2}^{\frac52}\|\widetilde{u}\|_{L^2}^2+\|\widetilde{u}_3\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2\end{aligned}$$

$$+C\|\widetilde{u}_1\|_{L^2}^2\|\Lambda_1^{\frac{5}{4}}u_1^{(2)}\|_{L^2}^{\frac{5}{7}}\|\Lambda_1^{\frac{5}{4}}\partial_3 u_1^{(2)}\|_{L^2}^{\frac{5}{7}}$$
$$+C\|\widetilde{u}_2\|_{L^2}^2\|\Lambda_2^{\frac{5}{4}}u_2^{(2)}\|_{L^2}^{\frac{5}{7}}\|\Lambda_1^{\frac{5}{4}}\partial_1 u_2^{(2)}\|_{L^2}^{\frac{5}{7}}$$
$$+C\|\widetilde{u}_3\|_{L^2}^2\|\Lambda_3^{\frac{5}{4}}u_3^{(2)}\|_{L^2}^{\frac{5}{7}}\|\Lambda_3^{\frac{5}{4}}\partial_2 u_3^{(2)}\|_{L^2}^{\frac{5}{7}}. \tag{2.13}$$

Dotting (2.12) with $\widetilde{\theta}$ and invoking the divergence-free conditions $\nabla\cdot u^{(1)}=0$ and $\nabla\cdot\widetilde{u}=0$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}\|_{L^2}^2+\kappa\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^2 &= -\int(\widetilde{u}\cdot\nabla)\theta^{(2)}\cdot\widetilde{\theta}\,dx\\ &= -\int\widetilde{u}_1\partial_1\theta^{(2)}\widetilde{\theta}\,dx-\int\widetilde{u}_2\partial_2\theta^{(2)}\widetilde{\theta}\,dx-\int\widetilde{u}_3\partial_3\theta^{(2)}\widetilde{\theta}\,dx\\ &:= L_1+L_2+L_3.\end{aligned}$$

For any parameters a, b, p and q satisfying

$$a>0,\quad b>0,\quad a+b=\frac{1}{2},\quad \frac{1}{p}=\frac{1}{2}-a,\quad \frac{1}{q}=\frac{1}{2}-b,$$

we have, by Hölder's inequality and Lemma 2.4,

$$|L_1|\le\|\partial_1\theta^{(2)}\|_{L^2_{x_1x_2}L^p_{x_3}}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}}\|\widetilde{\theta}\|_{L^2_{x_2}L^q_{x_3}L^4_{x_1}}. \tag{2.14}$$

By Sobolev's inequality and an interpolation inequality,

$$\begin{aligned}\|\partial_1\theta^{(2)}\|_{L^2_{x_1x_2}L^p_{x_3}} &\le C\|\Lambda_3^a\partial_1\theta^{(2)}\|_{L^2}\\ &\le C\|\partial_1\theta^{(2)}\|_{L^2}^{1-\frac{a}{\gamma}}\|\Lambda_3^\gamma\partial_1\theta^{(2)}\|_{L^2}^{\frac{a}{\gamma}}.\end{aligned}$$

By Sobolev's inequality, Lemma 2.2 and an interpolation inequality,

$$\begin{aligned}\|\widetilde{u}_1\|_{L^2_{x_3}L^\infty_{x_2}L^4_{x_1}} &\le C\|\Lambda_1^{\frac{1}{4}}\widetilde{u}_1\|_{L^2_{x_1x_3}L^\infty_{x_2}}\\ &\le C\|\Lambda_1^{\frac{1}{4}}\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\|\Lambda_1^{\frac{1}{4}}\partial_2\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\\ &\le C\|\widetilde{u}_1\|_{L^2}^{\frac{2}{5}}\|\Lambda_1^{\frac{5}{4}}\widetilde{u}_1\|_{L^2}^{\frac{1}{10}}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^{\frac{1}{2}}\\ &\le C\|\widetilde{u}_1\|_{L^2}^{\frac{2}{5}}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^{\frac{3}{5}}.\end{aligned}$$

By Sobolev's inequality and an interpolation inequality,

$$\begin{aligned}\|\widetilde{\theta}\|_{L^2_{x_2}L^q_{x_3}L^4_{x_1}} \le C\|\Lambda_1^{\frac{1}{4}}\widetilde{\theta}\|_{L^2_{x_1x_2}L^q_{x_3}} &\le C\|\Lambda_1^{\frac{1}{4}}\Lambda_3^b\widetilde{\theta}\|_{L^2}\\ &\le C\|\widetilde{\theta}\|_{L^2}^{1-\frac{\frac{1}{4}+b}{\gamma}}\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^{\frac{\frac{1}{4}+b}{\gamma}}.\end{aligned}$$

Inserting the bounds above in (2.14) and applying Young's inequality yield

$$\begin{aligned}|L_1| &\le C\|\partial_1\theta^{(2)}\|_{L^2}^{1-\frac{a}{\gamma}}\|\Lambda_3^\gamma\partial_1\theta^{(2)}\|_{L^2}^{\frac{a}{\gamma}}\|\widetilde{u}_1\|_{L^2}^{\frac{2}{5}}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^{\frac{3}{5}}\|\widetilde{\theta}\|_{L^2}^{1-\frac{\frac{1}{4}+b}{\gamma}}\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^{\frac{\frac{1}{4}+b}{\gamma}}\\ &\le\frac{\kappa}{128}\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^2+\frac{\nu}{128}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^2\\ &\quad+C\|\partial_1\theta^{(2)}\|_{L^2}^{(1-\frac{a}{\gamma})\sigma}\|\Lambda^\gamma\partial_1\theta^{(2)}\|_{L^2}^{\frac{a}{\gamma}\sigma}\|\widetilde{u}_1\|_{L^2}^{\frac{2}{5}\sigma}\|\widetilde{\theta}\|_{L^2}^{(1-\frac{\frac{1}{4}+b}{\gamma})\sigma},\end{aligned} \tag{2.15}$$

σ satisfies

$$\frac{3}{10}+\frac{\frac{1}{4}+b}{2\gamma}+\frac{1}{\sigma}=1 \quad \text{or} \quad \sigma=\frac{10\gamma}{7\gamma-5b-\frac{5}{4}}.$$

For $\gamma\geq\frac{17}{20}$, we have $\frac{a}{\gamma}\sigma\leq 2$ and thus

$$\|\Lambda^\gamma\partial_1\theta^{(2)}\|_{L^2}^{\frac{a}{\gamma}\sigma}\leq C(1+\|\Lambda^\gamma\partial_1\theta^{(2)}\|_{L^2}^2).$$

It is easy to check that $\frac{2}{5}\sigma+(1-\frac{\frac{1}{4}+b}{\gamma})\sigma=2$. Therefore,

$$\|\widetilde{u}_1\|_{L^2}^{\frac{2}{5}\sigma}\|\widetilde{\theta}\|_{L^2}^{(1-\frac{\frac{1}{4}+b}{\gamma})\sigma}\leq C(\|\widetilde{u}_1\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2).$$

It then follows from (2.15) that

$$\begin{aligned}|L_1|\leq{}&\frac{\kappa}{128}\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^2+\frac{\nu}{128}\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^2\\&+C\|\partial_1\theta^{(2)}\|_{L^2}^{(1-\frac{a}{\gamma})\sigma}(1+\|\Lambda^\gamma\partial_1\theta^{(2)}\|_{L^2}^2)(\|\widetilde{u}_1\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2).\end{aligned}$$

Similarly,

$$\begin{aligned}|L_2|\leq{}&\frac{\kappa}{128}\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^2+\frac{\nu}{128}\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^2\\&+C\|\partial_2\theta^{(2)}\|_{L^2}^{(1-\frac{a}{\gamma})\sigma}(1+\|\Lambda^\gamma\partial_2\theta^{(2)}\|_{L^2}^2)(\|\widetilde{u}_2\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)\end{aligned}$$

and

$$\begin{aligned}|L_3|\leq{}&\frac{\kappa}{128}\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^2+\frac{\nu}{128}\|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\widetilde{u}_3\|_{L^2}^2\\&+C\|\partial_3\theta^{(2)}\|_{L^2}^{(1-\frac{a}{\gamma})\sigma}(1+\|\Lambda^\gamma\partial_3\theta^{(2)}\|_{L^2}^2)(\|\widetilde{u}_3\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2).\end{aligned}$$

Collecting the bounds for L_1,L_2 and L_3, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\widetilde{\theta}\|_{L^2}^2+\kappa\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^2\\&\leq\frac{\nu}{128}\Big(\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^2+\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^2+\|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\widetilde{u}_3\|_{L^2}^2\Big)\\&\quad+C\|\nabla\theta^{(2)}\|_{L^2}^{(1-\frac{a}{\gamma})\sigma}(1+\|\Lambda^\gamma\nabla\theta^{(2)}\|_{L^2}^2)(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2).\end{aligned}\tag{2.16}$$

Combining (2.13) with (2.16), we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)+\kappa\|\Lambda^\gamma\widetilde{\theta}\|_{L^2}^2\\&\qquad+\nu\|(\Lambda_1^{\frac{5}{4}},\Lambda_2^{\frac{5}{4}})\widetilde{u}_1\|_{L^2}^2+\nu\|(\Lambda_2^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\widetilde{u}_2\|_{L^2}^2+\nu\|(\Lambda_1^{\frac{5}{4}},\Lambda_3^{\frac{5}{4}})\widetilde{u}_3\|_{L^2}^2\\&\leq C(\|\nabla\theta^{(2)}\|_{L^2}^{(1-\frac{a}{\gamma})\sigma}(1+\|\Lambda^\gamma\nabla\theta^{(2)}\|_{L^2}^2)+\|\nabla u^{(2)}\|_{L^2}^{\frac{5}{2}}+1)(\|\widetilde{u}\|_{L^2}^2+\|\widetilde{\theta}\|_{L^2}^2)\\&\qquad+C\|\widetilde{u}_1\|_{L^2}^2\|\Lambda_1^{\frac{5}{4}}u_1^{(2)}\|_{L^2}^{\frac{5}{7}}\|\Lambda_1^{\frac{5}{4}}\partial_3u_1^{(2)}\|_{L^2}^{\frac{5}{7}}\\&\qquad+C\|\widetilde{u}_2\|_{L^2}^2\|\Lambda_2^{\frac{5}{4}}u_2^{(2)}\|_{L^2}^{\frac{5}{7}}\|\Lambda_2^{\frac{5}{4}}\partial_1u_2^{(2)}\|_{L^2}^{\frac{5}{7}}\\&\qquad+C\|\widetilde{u}_3\|_{L^2}^2\|\Lambda_3^{\frac{5}{4}}u_3^{(2)}\|_{L^2}^{\frac{5}{7}}\|\Lambda_3^{\frac{5}{4}}\partial_2u_3^{(2)}\|_{L^2}^{\frac{5}{7}}.\end{aligned}$$

Gronwall's inequality then implies that $(u^{(1)},\theta^{(1)})=(u^{(2)},\theta^{(2)})$ if $(u_0^{(1)},\theta_0^{(1)})=(u_0^{(2)},\theta_0^{(2)})$. This completes the proof of Proposition 2.2. ∎

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