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RECOVERY OF ATTENUATION COEFFICIENTS FROM PHASELESS MEASUREMENTS FOR THE HELMHOLTZ EQUATION*

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Abstract. We consider the Helmholtz equation with a complex attenuation coefficient on a bounded, strictly convex domain in $\mathbb{R}^d$. We prove a Hölder conditional stability estimate for identifying attenuation coefficients from phaseless boundary value measurements, when the initial excitation state is in the form of a Gaussian bump. We use the Gaussian beam Ansatz and stability results for the X-ray transform on strictly convex domains to establish these estimates.

Keywords. phase less measurements; Helmholtz equation; Gaussian beams.

AMS subject classifications. 35R30.

1. Introduction

Interest in phase less measurements has seen an increasing rise in popularity in the theory of signal processing. When measuring a wave from its source, many times it is only possible to gain information about the modulus of the signal. Algorithms found in [6] and [7] center on decomposing a wave into a sequence of Fourier modes. In this article, we are interested in phaseless measurements of solutions to the Helmholtz equation. The goal of this article is slightly different. We will show having an idea of which partial differential equation the wave comes from is enough to give complete reconstruction of the coefficients in the partial differential equation.

As a practical application, in multi-wave tomography usually some type of wave is sent to the portion of the body being imaged. In electromagnetic or optical radiation tomography the wave interaction with the tissues of the patient is measured [3]. Naturally one cannot measure inside the patient, so some initial boundary value problem must be considered. One such mathematical model for the emitted ultrasound waves is the following Helmholtz equation with a high-frequency source term. The equation is

$$Lu = \Delta u + (i\lambda n^2(x) + \lambda^2)u = h(x,\lambda) \quad x \in \mathbb{R}^d. \tag{1.1}$$

The scalar λ is large, and $n^2(x)$ is the attenuation coefficient. For this article, we restrict ourself to a smooth, strictly convex bounded domain Ω. The attenuation coefficient is multiplying the lower order terms which is a difference to [12], as we want the geometric ray paths to be straight lines so we may apply the results in [15]. The source $h(x,\lambda)$, which emits the waves, we also assume to depend on λ and be compactly supported. Given a smooth, strictly convex bounded domain Ω, we assume that $n^2(x) \equiv 1$ on the boundary $\partial\Omega$. The measurements we consider are then of the form

$$\{|u(x)|, x \in \partial\Omega\}, \tag{1.2}$$

with the collection of initial excitation states $h(x,\lambda)$ varying over all $x \in \partial\Omega$ in the form of complex Gaussian bumps.

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In this paper, we derive a stability result for the attenuation coefficients of the Helmholtz equation for fully phaseless measurements. Stable reconstructions have been made from Robin conditions for lower order terms than considered here [4]. In the related case, for the acoustic wave equation for Dirichlet boundary conditions in [9,14] and Robin conditions in [5,20], the potential can also be recovered. However stability estimates from phaseless measurements have not been previously given.

In [10,11], uniqueness results in dimension 3 for lower order terms than the ones considered are derived from phaseless measurements. These papers are predicated on analyticity arguments, which require data in a small neighborhood of the source. The question of phaseless stability from internal measurements for the Schrödinger equation was also examined in [2]. These measurements are in contrast to the boundary data we require. We are not able to prove uniqueness results unless $\lambda \to \infty$. However, morally this is equivalent to setting the attenuation term equal to zero. The inverse problem of recovering the source for wave equations from Dirichlet boundary conditions is also examined in [8,16,17,19], and [18]. One might be tempted to think the problems are equivalent via the Fourier transform, however this is not the case on a compactly supported domain, as the Fourier transform of $C_c^\infty(\mathbb{R}^d)$ functions does not have compact support. This makes the previous works distinct.

In the first two sections we briefly review the construction of the Gaussian beam Ansatz found in [12] and the last section on observability estimates we give a new reconstruction algorithm from phases data for the attenuation coefficient.

2. Construction of solutions

We make more precise the explicit nature of the solutions. We will use the Ansatz,

$$U(x) = \sum_{j=0}^{l} \exp(i\lambda\psi(x)) a_j(x) \lambda^{-j} \tag{2.1}$$

to build asymptotic solutions to equation (1.1) in the high frequency limit, and then give estimates on the difference between these approximate solutions and the true solutions. We will follow [12] for the Gaussian beam Ansatz in constructing the phase a and amplitude ψ. We claim

LEMMA 2.1. *There is a zeroth order Gaussian beam which takes the form*

$$U^0(x) = (a_0(s) + \mathcal{O}(|x - x(s)|)) \exp(i\lambda\psi(x))\phi(x) + \mathcal{O}(\lambda^{-1}). \tag{2.2}$$

where $x(s)$ is a curve in space-time and ϕ is a cutoff defined below, which solves the Helmholtz equation in the free space with no source term.

The N^{th} order Gaussian beam Ansatz can be constructed accordingly. However, for the purposes of this paper the zeroth order terms are the most important, so we re-iterate part of the construction for completeness.

REMARK 2.1. One might be tempted to use a real-phase Ansatz. However, it is not possible to pick up the attenuation coefficient without the use of a weighted Sobolev space, since the Fadeev solutions pick up a factor of $|x - x_0|^{-1}$ which is singular. The Gaussian beam Ansatz was chosen because of the good control over the error estimates in higher Sobolev norms, which allows for the use of certain embedding theorems in the section on observability inequalities. Also, because of the presence of the tail, one does not have to construct solutions which are localized along a central ray, which is necessary for the estimates in (4.14) and (4.16) to work. See, for instance, [13] for a nice treatment of the real-phase Ansatz for the wave equation, with localized characteristic functions.

Proof. Every geometrics optics solution concentrates on an open set around the ray path $\{(s,x(s)):0\le s\le T\}$. The flow for the ray path of H is defined by the set of ordinary differential equations (ODEs)

$$\dot{x}=2p, \quad \dot{p}=0. \tag{2.3}$$

We then have that

$$LU=\sum_{j=0}^{l}\exp(i\lambda\psi(x))c_j(x)\lambda^{-j}, \tag{2.4}$$

The coefficients c_j for $j=0,1,\dots,l$ are defined recursively as follows:

$$\begin{aligned}
&c_{-2}=(1-|\nabla_x\psi|^2)a_0(x)=E(x)a_0,\\
&c_{-1}=in^2(x)a_0+\nabla_x\cdot(a_0\nabla_x\psi)+\nabla_x a_0\cdot\nabla_x\psi+Ea_1(x),\\
&c_j=in^2(x)a_{j+1}+\nabla_x\cdot(a_{j+1}\nabla_x\psi)+\nabla_x a_{j+1}\cdot\nabla_x\psi+E(x)a_{j+1}+\Delta_x a_j.
\end{aligned}$$

If we are Taylor expanding the coefficients $a_j(x)$ around the central ray $x(s)$ then we arrive at the following set of ordinary differential equations:

$$\dot{S}=2, \qquad \dot{a}_0=-n^2((x(s))a_0. \tag{2.5}$$

Here we assume that we have chosen c_{-2} and c_{-1} to vanish on the ray to third and first order respectively. For the stability estimates we prove the ODE which defines a_0 explicitly is the most important for identifying the attenuation coefficient $n^2(x)$. This leads to the following set of differential equations:

$$\dot{S}=2, \quad \dot{M}=-2M^2, \quad \dot{a}_0=-\mathrm{tr}(M(s))a_0-n^2(x(s))a_0. \tag{2.6}$$

Solving the system one sees that

$$a_0(s)=\exp\left(\int_0^s -n^2(x(t))-\mathrm{tr}M(t)\,dt\right). \tag{2.7}$$

The phase ψ needs to verify the conditions

$$\psi(x(s))=S(s), \quad \nabla\psi(x(s))=p(s), \quad D^2\psi(x(s))=M(s) \tag{2.8}$$

which can be done if we set

$$\begin{aligned}
\psi(x)=S(s)+(x-x(s))\cdot p(s)+\frac{1}{2}(x-x(s))\cdot M(s)(x-x(s))\\
a(x(s))=a_0(s).
\end{aligned}$$

We use initial data $S(0)=0$ and $M(0)$ such that

$$M(0)=M(0)^T, \qquad M(0)\dot{x}(0)=\dot{p}(0), \quad \Im M(0)\,\text{positive definite on}\,\dot{x}(0)^{\perp}. \tag{2.9}$$

cf. Section 2 of [12] for this choice.

We now define a cutoff function $\phi_\lambda \in C^\infty(\mathbb{R}^n)$ for $\lambda > 0$ by

$$\phi_\lambda(x) = \begin{cases} 0 & \text{if} \quad x \in \{x : |x - x(s)| > 2\lambda^{-\frac{1}{2d}} \quad 0 \le s \le T\} \\ 1 & \text{if} \quad x \in \{x : |x - x(s)| < \lambda^{-\frac{1}{2d}} \quad 0 \le s \le T\} = A_\lambda. \end{cases} \tag{2.10}$$

One can arrange so that there is a constant C such that for all $m \in \mathbb{N}$:

$$|\nabla_x^m \phi_\lambda| \le C\lambda^{-m}. \tag{2.11}$$

We drop the subscript λ for the rest of this paper. We write $f \sim g$ if there exists a constant $C > 0$ with $C^{-1}g(x) \le f(x) \le Cg(x)$ for all arguments x. The following result may also be found in [21], Corollary 5.

Corollary 2.1. *Let $\psi(x)$ correspond to a zeroth order beam. We then have*

$$\exp(-2\lambda \Im\psi(x)) \sim \exp(-\lambda C|x - x(s)|^2),$$

where C is independent of λ. Notice that we are taking more care to the construction of the cutoffs than in [12] as they are rather cavalier about this aspect of their construction. As a consequence if we let B denote the set

$$B = \{x : |x - x(s)| > \lambda^{-(\frac{1}{2} - \sigma)},\ 0 \le t \le T\}, \qquad \sigma > 0,$$

then since $2\Im\psi(x) \sim |x - x(s)|^2$, $\exp(-2\lambda\Im\psi(x))$ is exponentially decreasing in λ for all $x \in B$.

Proof. We need only observe that $M(s)$ is a bounded and positive definite matrix. From the form of the constructed phase functions the desired result follows. □

The construction of the localized cutoff finishes the construction of $U_\lambda(x)$. □

3. Introduction of the source terms

We now include the source functions. In [12], Section 2.1, the following is asserted:

Theorem 3.1. *There exists a Gaussian beam solution, which solves equation* (1.1) *and is found by solving the Dirichlet problem with the true solution restricted to a hyperplane containing the source point as initial data.*

This is the same principle as in [21] for the wave equation. The proof is the same argument as in Section 2.1 of [12] and is repeated for completeness. We let ρ be a function such that $|\nabla\rho| = 1$ on the set $\{x : \rho(x) = 0\}$ and let the hypersurface $\{x : \rho(x) = 0\}$ be denoted as Σ. We let x_0 be a point in Σ and we let $(x(s), p(s)))$ be the solution path—in other words the nul-bicharacteristics with $(x(0), p(0)) = (x_0, \nabla\rho(x_0))$. The hypersurface Σ is given by $s = \sigma(y)$ with $\sigma(0) = 0$ and $\nabla\sigma(0) = 0$, where $x = (s, y)$ and $y = (y_1, ..., y_{d-1})$ is transversal. We let the optics Ansatz $U(x)$ have initial data $(x(0), p(0))$ and be defined in this tubular neighbourhood. We let U^+ be the restriction of U to the surface $\{x : \rho(x) \ge 0\}$. Because we need to have a source term which is a multiple of $\delta(\rho)$ we also need a second 'outgoing' solution U^- which is defined on $\{x : \rho(x) \le 0\}$ which is then equal to U^+ on the hypersurface Σ. For example, we can write the ingoing and outgoing optics solutions as

$$U^+ = A^+(x, \lambda)\exp(i\lambda\psi^+(x)) \quad U^- = A^-(x, \lambda)\exp(i\lambda\psi^-(x)) \tag{3.1}$$

were we have set $\psi^+ = \psi^-$ and $A^+ = A^-$ on Σ, then the requirement that their Taylor series coincide on the boundary is equivalent to setting $\psi^+(\sigma(y)) = \psi^-(\sigma(y))$ and differentiating with respect to y and evaluating at $y=0$. We extend U^+ to be 0 on $\{x:\rho(x)<0\}$ and U^- to be 0 on $\{x:\rho(x)>0\}$. We define our geometric optics Ansatz solution U to be $U=U^+ + U^-$. We set $A=A^+=A^-$ on Σ. In order to add the source terms we notice that

$$\begin{aligned} LU &= i\lambda\left(\left(\frac{\partial\psi^+}{\partial\nu}-\frac{\partial\psi^-}{\partial\nu}\right)A(x,\lambda)+\frac{\partial A^+}{\partial\nu}(x,\lambda)-\frac{\partial A^-}{\partial\nu}(x,\lambda)\right)\exp(i\lambda\psi^+)\delta(\rho)+f_{gb} \\ &= g_0\delta(\rho)+f_{gb} \end{aligned} \tag{3.2}$$

where $\nu(x)=\nabla\rho(x)$ is the unit normal to Σ. We consider the singular part of Lu_{gb}, that is $g_0\delta(\rho)$, to be the same as source term and f_{gb} the error. We see that

$$f_{gb}=\exp(i\lambda\psi^+)\sum_{j=-2}^{l}c_j^+(x)\lambda^{-j}+\exp(i\lambda\psi^-)\sum_{j=-2}^{l}c_j^-(x)\lambda^{-j} \tag{3.3}$$

whenever c_j^+ are extended to be zero if $\rho(x)<0$ and c_j^- are extended to be zero when $\rho(x)>0$. We know by construction that $c_{-2}^{\pm}=\mathcal{O}(|x-x(s)|^3)$ and $c_{-1}^{\pm}=\mathcal{O}(|x-x(s)|)$ respectively.

Remark 3.1. While this procedure may seem a backwards and a bit ad hoc, it is useful for deriving good error estimates which are needed for the final proof. Otherwise a real-phase Anstaz or a Fadeev-type fundamental solution needs to be constructed, and this adds an additional weight of $|x-x_0|^{-1}$ with singularities into the construction.

In [12] they also prove for N^{th} order Gaussian beams:

Lemma 3.1. *We have the following estimate for the error of the Gaussian beam:*

$$||f_{gb}(x)||^2_{H^m(|x|<R)}\le\lambda^{-N+2+(1-d)/2+2m}. \tag{3.4}$$

Remark 3.2. Notice that this error now includes a function which is localized in a neighborhood of $\mathcal{O}(1/\sqrt{\lambda})$ around a curve $x(s)$ which is more precise than the estimate in [12], although the details follow the same way. The reason for the precise cut-off function ϕ_λ is twofold: we need a localized solution for property (4.16) to hold, and we also are considering a bounded domain.

The extension of the Gaussian beam Ansatz to $\mathbb{R}^d$ will give us the necessary means to make observations further away from the source and still identify source terms. We briefly review the results of [12]. They prove:

Lemma 3.2. *There exists an extension $\tilde{U}$ of the Gaussian beam solution to the whole space of the problem* (1.1) *such that*

$$\left|\left|u-\tilde{U}\right|\right|_{H^m(\mathbb{R}^d)}\le C\lambda^{-1}||f_{gb}||_{H^m(|x|<R)}. \tag{3.5}$$

4. Observability inequalities

We use the notation in [1] and again consider the problem

$$(-\Delta-\lambda^2-i\lambda n^2(x))u^{x_0,\omega_0}=h^{x_0,\omega_0}(x,\lambda), \tag{4.1}$$

where we let x_0 denote the position of the center of the plane wave source. Sources are indexed by x_0 and ω_0. We define the subset of the co-sphere bundle as

$$\partial\mathcal{S}\Omega^+=\{(x_0,\omega_0):x_0\in\partial\Omega,\langle\nu,\omega_0\rangle>0\}, \tag{4.2}$$

where ν denotes the outward unit normal to the boundary.

THEOREM 4.1. *Let $N \geq (1+d)/2+4$ and $\epsilon_0 > 0$, then there exists a constant C_1 which depends on diam(Ω) the $C(\Omega)$ norm of $n_i^2(x), i=1,2$ and a constant C_2 which depends on the diam(Ω) and the $C^{N+1}(\Omega)$ norm of $n_i^2(x)$, $i=1,2$ such that if $u_1^{x_0,\omega_0}$ and $u_2^{x_0,\omega_0}$ solve the radiation problem with attenuation coefficients n_1^2 and n_2^2 respectively then it follows that if*

$$\lambda^{-1} < \epsilon_0, \qquad \delta = \sup_{\partial\mathcal{S}\Omega^+} ||u_1^{x_0,\omega_0}| - |u_2^{x_0,\omega_0}|| < \epsilon_0, \tag{4.3}$$

then this implies

$$||n_2^2 - n_1^2||_{H^{-1/2}(\Omega)} \leq C_1 \left(\frac{C_2}{\lambda^{\beta'}} + \delta \right) \tag{4.4}$$

for some $\beta' \in (0,(2d)^{-1})$.

The uniqueness corollary follows immediately.

COROLLARY 4.1. *Let δ be defined as in Theorem 4.1 above. Let $\delta = 0$ and $\lambda \to \infty$, then this implies $n_1^2 = n_2^2$.*

We proceed with the proof of the main theorem. We consider our globally defined complex optics solutions $\tilde{U}_1$ and $\tilde{U}_2$, which were constructed previously. We know these solutions exist by using Σ as the boundary of $\partial\Omega$ using Theorem 3.1, and the Lemmas 3.2 and 3.1. Dropping the superscripts x_0,ω_0, where it is understood, from our approximation the main term of interest obeys the following bound

$$\big||U_1^0| - |U_2^0|\big| \leq ||u_1| - |u_2|| + C\lambda^{-\beta'} \tag{4.5}$$

for $x \in \partial\Omega$. Here C is a generic constant which depends on the $C^N(\Omega)$ norm of n_i^2, $i=1,2$. This estimate is a result of constructing an approximate solution with N sufficiently large. Indeed, one sees that

$$u_i = \tilde{U}_i + f_{gb,i} = U_i^0 + \tilde{f}_{gb,i} \quad i=1,2 \tag{4.6}$$

where $f_{gb,i}$ is bounded in $C(\Omega)$ norm by $C\lambda^{-\beta'}$ with $\beta' \in (0,1)$ as a consequence of Lemma 3.1, with $m = d/2+1$, and N satisfying the hypothesis of Theorem 4.1. Here have used Sobolev embedding as $H^s(\Omega) \subseteq L^\infty(\Omega)$ when $s > d/2$, and the fact all the functions have compact support. By the triangle inequality, it remains to bound $\tilde{f}_{gb,i} - f_{gb,i} = \tilde{U}_i - U_i^0$ uniformly. From the error estimate for the Gaussian beam extension in Lemma 3.2 we conclude

$$\left|\left|\tilde{U}_1 - u_1\right|\right|_{H^1(\mathbb{R}^n)} \leq \frac{C}{\lambda}. \tag{4.7}$$

The constant C is a generic constant which depends on the $C^{N+1}(\Omega)$ norm of n_1^2. We use the fact that $\tilde{U}_1$ and u_1 are bounded in $C^{N+1}(\Omega)$ norm to obtain

$$\left|\left|\tilde{U}_1 - u_1\right|\right|_{C^0(\mathbb{R}^n)} \leq \frac{C}{\lambda^{\beta'}} \tag{4.8}$$

for some $\beta' \in (0,1)$. This result again follows from Sobolev embedding and the fact all the functions have compact support. We then use the estimate on the first order terms, equation (2.2), and we see that

$$||U_1 - a_0 \exp(i\lambda\psi)||_{C^0(\mathbb{R}^n)} \leq \frac{C}{\lambda^{1/2d}}. \tag{4.9}$$

Note that these estimates are the same for the subscript 2 solutions. Combining the estimates (4.9) and (4.8) gives the estimate (4.5).

Now we need to combine the estimates to recover the X-ray transform. We start with the following Lemma:

LEMMA 4.1. *Let A,B be positive functions in $C^0(\mathbb{R})$ and $\epsilon \in (0,1)$ such that*

$$||\exp(-A(x)) - \exp(-B(x))||_{C^0(\mathbb{R})} < \epsilon. \tag{4.10}$$

Then there is a constant C which depends on the $C^0(\mathbb{R})$ norm of A and B such that

$$||A(x) - B(x)||_{C^0(\mathbb{R})} < C\epsilon. \tag{4.11}$$

Proof. By the mean value theorem, for each fixed x there exists an r_* between $B(x)$ and $A(x)$ such that

$$|(\exp(-A) - \exp(-B))| = \left|\left(-\int_B^A \exp(-r)\,dr\right)\right| = |((B-A)\exp(-r^*))|. \tag{4.12}$$

Taking the supremum over x and then applying condition (4.10), yields the desired result. ∎

We would like to use the zeroth order coefficients to reconstruct the attenuation coefficient. We know using equation (2.2) that

$$\left| a(x)\exp(i\lambda\psi) - a_0(0)\exp\left(-\int_0^s (n_1^2(x(t))\,dt - \int_0^s \mathrm{tr}M(t)\,dt\right)\exp(\lambda\underline{\psi(x)})\phi(x)\right|$$
$$= \mathcal{O}(\lambda^{-1/2d}), \tag{4.13}$$

by choice of the cutoff function in definition (2.10). Examining the bound (4.5), we approximate the left-hand side using estimate (4.13) by

$$|a_0(0)\left(\exp\left(-\int_0^s (n_1^2(x(t))\,dt\right) - \exp\left(-\int_0^s (n_2^2(x(t))\,dt\right)\right)$$
$$\times \exp\left(-\int_0^s \mathrm{tr}M(t)\,dt\right)\exp(\lambda\underline{\psi(x)})\phi(x)| \tag{4.14}$$

where

$$\underline{\psi(x)} = (-(x-x(s))\Im M(s)\cdot(x-x(s))). \tag{4.15}$$

Because $\Im M(s)$ is a positive definite matrix, by Corollary (2.1) we have

$$\sup_{x\in\Omega}\left|\exp(\lambda\underline{\psi(x)})\phi(x)\right| = 1 \tag{4.16}$$

since we know $x(s)$ reaches the boundary and is contained in A_λ. Because the remaining coefficients in expression (4.14) are independent of x, we obtain that the supremum of (4.14) over $\partial\Omega$ is equal to

$$C(s)\left|a_0(0)\left(\exp\left(-\int_0^s(n_1^2(x(t))\,dt\right)-\exp\left(-\int_0^s(n_2^2(x(t))\,dt\right)\right)\right|, \tag{4.17}$$

where

$$C(s)=\exp\left(-\int_0^s \mathrm{tr}M(t)\,dt\right). \tag{4.18}$$

We apply Lemma 4.1, which gives

$$||I(n_1^2-n_2^2)||_{C^0(\partial\mathcal{S}\Omega^+)}\leq C_1\left(\delta+\frac{C_2}{\lambda^{\beta'}}\right) \tag{4.19}$$

where C_1 denotes a generic constant depending on the $C^0(\Omega)$ norm of n_1^2, n_2^2, while C_2 depends on the $C^{N+1}(\Omega)$ norm. Now we set $n_1^2-n_2^2=\tilde{n}^2$, which has compact support. Because we have assumed that Ω is strictly convex, it follows from estimate (4.19) that

$$\sup_{x\in\Omega,\theta\in\mathbb{S}^{d-1}}\left|\int_{-\infty}^{\infty}\tilde{n}^2(x+s\theta)\,ds\right|\leq C_1\left(\delta+\frac{C_2}{\lambda^{\beta'}}\right). \tag{4.20}$$

For $x\in\mathbb{R}^d$ and $\theta\in\mathbb{S}^{n-1}$, we set

$$Xf(x,\theta)=\int_{-\infty}^{\infty}\tilde{n}^2(x+s\theta)\,ds. \tag{4.21}$$

Because $\tilde{n}^2$ is compactly supported, we conclude

$$\left|\int_{\mathbb{S}^{d-1}}\int_{x\in\theta^\perp}|X\tilde{n}^2(x,\theta)|^2\,dx\,d\theta\right|^{1/2}\leq C_1\left(\delta+\frac{C_2}{\lambda^{\beta'}}\right). \tag{4.22}$$

As per [15], we define the following Sobolev norm for all $\alpha_0\in\mathbb{R}$:

$$||g||_{H^{\alpha_0}(T)}=\int_{\mathbb{S}^{d-1}}\int_{\omega_0^\perp}(1+|\eta|^2)^{\frac{\alpha_0}{2}}|\hat{g}(\theta,\eta)|^2\,d\eta\,d\theta, \tag{4.23}$$

where $\hat{g}$ denotes the Fourier transform of g.

The monograph [15] gives a stability estimate for functions in terms of the transform Xf.

THEOREM 4.2 ([15], Theorem 2.18). *For every α there are positive constants $c(\alpha,d,\Omega)$ and $C(\alpha,d,\Omega)$ such that for $f\in C_0^\infty(\Omega)$ we have*

$$c(\beta,d)\,||f||_{H_0^\alpha(\Omega)}\leq ||Xf||_{H^{\alpha+1/2}(T)}\leq C(\alpha,d)\,||f||_{H_0^\alpha(\Omega)}. \tag{4.24}$$

We can now give a reconstruction theorem, recalling that the Fourier transform is an isometry on L^2. To finish the proof of Theorem 4.1 we use estimate (4.22) along with the stability estimate above with $\alpha=-1/2$. We notice that $\tilde{n}^2\in C_0^\infty(\Omega)$ by choice of $n_1^2=n_2^2\equiv 1$ on $\partial\Omega$.

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