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GLOBAL CLASSICAL SOLUTIONS TO THE TWO-FLUID INCOMPRESSIBLE NAVIER–STOKES–MAXWELL SYSTEM WITH OHM'S LAW*

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Abstract. We study the global in time classical solutions to the two-fluid incompressible Navier–Stokes–Maxwell system with (solenoidal) Ohm's law with small initial data. This system is a coupling of the incompressible Navier–Stokes equations with the Maxwell equations through the Lorenz force and Ohm's law for the current. In this proof, we employ the decay properties of both the electric field E and the wave equation with linear damping of the divergence free magnetic field B.

Keywords. Navier–Stokes–Maxwell system; Ohm's law; global classical solution.

AMS subject classifications. 35A01; 35A09; 35B45; 35Q35; 35Q61; 76D05; 76K05.

1. Introduction

In this paper, we study the following two-fluid incompressible Navier–Stokes–Maxwell equations with Ohm's law:

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = \frac{1}{2}(nE + j\times B), \\ \partial_t E - \nabla\times B = -j, \\ \partial_t B + \nabla\times E = 0, \\ \mathrm{div} v = \mathrm{div} B = 0, \ \mathrm{div} E = n, \\ j = nv + \sigma(-\frac{1}{2}\nabla n + E + v\times B), \end{cases} \tag{1.1}$$

with the initial data

$$v|_{t=0} = v^{in}, \quad E|_{t=0} = E^{in}, \quad B|_{t=0} = B^{in},$$

which satisfy the comparability conditions $\mathrm{div} v^{in} = \mathrm{div} B^{in} = 0$. Here $v, E, B : \mathbb{R}^+\times\mathbb{R}^3 \to \mathbb{R}^3$ are the velocity of the fluid, the electric and magnetic fields, respectively, while $n : \mathbb{R}^+\times\mathbb{R}^3 \to \mathbb{R}$ is the electric charge. The vector field j is the electric current given by Ohm's law (the fifth equation of the system, where σ is the electric resistivity). μ is the viscosity of the fluid and the scalar function p stands for the pressure. The force under a quasi-neutrality assumption of the net charge carried by the fluid. Note that the pressure p can be recovered from v and $j\times B$ via an explicit Caldéron-Zygmund operator. The second equation in system (1.1) is the Ampère–Maxwell equation for an electric field E. The third equation is simply Faraday's law. For the more physical backgrounds of this system, see [3] and [4].

The system (1.1) can be derived from more microscopic models, say, two species Vlasov–Maxwell–Boltzmann (briefly VMB) equations under some suitable physical scalings. For example, [2] and [8]. In particular, in the recent remarkable breakthrough of Arsénio and Saint-Raymond [2], they justify the convergence from the global in time renormalized solutions of the VMB equations to weak (or dissipative) solutions of the

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several models of the incompressible viscous electro-magneto-hydrodynamics, including the system (1.1). We emphasize that system (1.1) can only derived from two species VMB. That is why it is called two-fluid Navier–Stokes–Maxwell equations.

We also study the following system which is called the two-fluid incompressible Navier–Stokes–Maxwell equations with solenoidal Ohm's law:

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = \frac{1}{2} j\times B, \\ \partial_t E - \nabla\times B = -j, \\ \partial_t B + \nabla\times E = 0, \\ j = \sigma(-\frac{1}{2}\nabla\bar{p} + E + v\times B), \\ \mathrm{div} v = \mathrm{div} B = \mathrm{div} E = \mathrm{div} j = 0, \end{cases} \tag{1.2}$$

where the scalar function $\bar{p}$ in system (1.2) is the Lagrangian multiplier for the divergence-free property of the electric current j. The initial data of the system (1.2) is given by

$$v|_{t=0} = v^{in},\ E|_{t=0} = E^{in},\ B|_{t=0} = B^{in},$$

which satisfy the comparability conditions $\mathrm{div} v^{in} = \mathrm{div} E^{in} = \mathrm{div} B^{in} = 0$. As for the system (1.1), the system (1.2) can be also formally derived from the two species VMB equations, see also [2]. As explained carefully in [2], in particular Chapter 2 and 3, the system (1.1) is derived from VMB for strong interactions, while system (1.2) is derived for the case weak interactions. Furthermore, the system (1.2) can be viewed as an asymptotic regime of the system (1.1).

On the other hand, there is another system which is also called incompressible Navier–Stokes–Maxwell system with Ohm's law in the literatures:

$$\begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla p = j\times B, \\ \partial_t E - \nabla\times B = -j, \\ \partial_t B + \nabla\times E = 0, \\ \mathrm{div} v = \mathrm{div} B = 0, \\ \sigma(E + v\times B) = j. \end{cases} \tag{1.3}$$

The system (1.3) is very similar to the system (1.1) except there is the electric charge $n(t,x)$ in (1.1). We emphasize that system (1.3) is not just letting $n\equiv 0$ in system (1.1), which will gives $\mathrm{div} E = 0$. The system (1.3) is not a special case of either system (1.1) or (1.2). However, we would like to state that these systems are quite similar from the point view of mathematical analysis, which is the main concern of the current paper.

The well-posedness of all the above systems (1.1), (1.2) and (1.3) is a highly non-trivial problem, both in the context of weak solutions and more regular frameworks. For weak solutions, the existence of global in time Leray type weak solutions are completely open, even in 2-dimension. Previous analytical studies are mainly on the system (1.3), although we believe that all the results on (1.3) can be also obtained for the system (1.1) and (1.2) by employing the similar methods. A first breakthrough comes from Masmoudi [9], who in 2-dimensional case proved the existence and uniqueness of global strong solutions to system (1.3) for the initial data $(v^{in}, E^{in}, B^{in}) \in L^2(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$ with $s > 0$. His proof relied on the energy inequality combined with a logarithmic inequality that enabled him to upper estimate the L^∞-norm of the velocity field by the energy norm and higher Sobolev norms. It is notable that in [9], the divergence-free condition of the magnetic field B or the decay property of the linear part coming from Maxwell's equations is *not* used.

Another line of research, again for the system (1.3), was carried out by Ibrahim and Keraani [6], who considered the data $(v^{in}, E^{in}, B^{in}) \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \times (\dot{H}^{1/2}(\mathbb{R}^3))^2$ for 3-dimension, and $(v_0, E_0, B_0) \in \dot{B}^0_{2,1}(\mathbb{R}^2) \times (L^2_{log}(\mathbb{R}^2))^2$ for 2-dimensional case. Later on, German, Ibrahim and Masmoudi [5] refines the previous results by running a fixed-point argument to obtain mild solutions of system (1.3), but taking the initial velocity field in the natural Navier–Stokes space $H^{d/2-1}$. Their results extended the earlier results in many respects. The regularity of the initial velocity and electromagnetic fields is lowered, and they unify the proof in both spacial dimensions 2 and 3. Furthermore, they employed an L^2L^∞-estimate on the velocity field, which significantly simplifies the fixed-point arguments used in [6]. For some other asymptotic problems related, say, the derivation of the MHD from the Navier–Stokes–Maxwell system in the context of weak solutions, see Arsenio–Ibrahim–Masmoudi [1].

As stated above, in Arsénio and Saint-Raymond's recent book [2], they justify the weak convergence from renormalized solutions of VMB equations to weak solutions of the several models of the incompressible viscous electro-magneto-hydrodyanmics, including the systems (1.1) and (1.2). On the other hand, it is natural to consider these asymptotic limits in the context of classical solutions. A first step in this project is to prove the global in time classical solutions of the systems (1.1) and (1.2) with small initial data. This is the goals of this paper. We can not expect the large initial data result at the current stage since analytically these systems are more like 3-dimensional incompressible Euler equations.

Because of the similarities between the systems (1.1)-(1.2) with (1.3), we employ many ideas from the previous works on the system (1.3), in particular [5]. The key point is to find enough dissipation or decay properties. We first observe the structure of the system (1.1). The second Ampère–Maxwell equation

$$\partial_t E - \nabla \times B + \sigma E - \tfrac{\sigma}{2}\nabla n = -nv - \sigma v \times B$$

has a decay term σE, where the term $-\sigma v \times B$ can be regarded as a source. Out of consideration for the cancellation

$$\big\langle \nabla^k(\nabla \times B), \nabla^k E\big\rangle - \big\langle \nabla^k(\nabla \times E), \nabla^k B\big\rangle = 0$$

for all $k \geq 0$, the third Faraday's law equation shall be considered together with the second Ampère–Maxwell equation while we derive the energy estimates, namely,

$$\begin{cases} \partial_t E - \nabla \times B + \sigma E - \tfrac{\sigma}{2}\nabla n = -nv - \sigma v \times B, \\ \partial_t B + \nabla \times E = 0. \end{cases}$$

We also notice that the third Faraday's law equation $\partial_t B + \nabla \times E = 0$ with $\mathrm{div} B = 0$ does not have explicit dissipative term. Fortunately, the divergence-free property of B enable us to gain a wave equation of B with a damping term $\sigma \partial_t B$

$$\partial_{tt} B - \Delta B + \sigma \partial_t B = \nabla \times (nv) + \sigma \nabla \times (v \times B),$$

in which we take ∂_t in the Faraday's law equation and make use of the Ampère–Maxwell equation and the equality $\nabla \times (\nabla \times B) = -\Delta B$ if $\mathrm{div} B = 0$. Furthermore, the divergence-free property of B implies $\mathrm{div}(\nabla \times B) = 0$. Then, by taking div on the second Ampère–Maxwell equation and the relation $\mathrm{div} E = n$, we gain

$$\partial_t n + \mathrm{div} j = 0,$$

which will be employed in proving the local solution. Then the Ohm's law (the fifth equation of system (1.1)) reduces to a parabolic equation of n with a damping σn

$$\partial_t n + v\cdot\nabla n - \tfrac{\sigma}{2}\Delta n + \sigma n = -\sigma \mathrm{div}(v\times B).$$

We emphasize that we do not utilize all the decay and dissipative properties displayed in the above arguments while we construct the local solution of system (1.1) with large initial data. We only require the basic energy law

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|_{L^2}^2 + \tfrac{1}{2}\|E\|_{L^2}^2 + \tfrac{1}{2}\|B\|_{L^2}^2 + \tfrac{1}{4}\|n\|_{L^2}^2\right) + \mu\|\nabla v\|_{L^2}^2 + \tfrac{\sigma}{2}\|-\tfrac{1}{2}\nabla n + E + v\times B\|_{L^2}^2 = 0,$$

on which based, we can derive a responded higher order energy estimate. Utilizing the above dissipative and decay properties enables us to gain a global in time higher order energy estimate. Thus, if the initial data is small, we can globally extend the local solution which has been constructed. We note that it is interesting that the global well-posedness with small initial data will fail if any one condition (divergence-free property of B or decay properties of E and n) is not taken into consideration.

Now we precisely state our main theorems as follows:

THEOREM 1.1 (Local well-posedness). *If the initial data fulfill $v^{in}, E^{in}, \mathrm{div}E^{in}, B^{in} \in H^s(\mathbb{R}^3)$ for $s\geq 2$, then there exist two positive constant T and C, depending on s, μ, σ and the all initial data, such that system (1.1) admits a unique solution (v,E,B) on the interval $[0,T]$ satisfying*

$$\begin{aligned} &v\in L^\infty(0,T;H^s(\mathbb{R}^3))\cap L^2(0,T;H^{s+1}(\mathbb{R}^3)),\\ &E,n,B\in L^\infty(0,T;H^s(\mathbb{R}^3)).\end{aligned}$$

Moreover, the following energy bound

$$\begin{aligned}&\sup_{t\in[0,T]}\left(\|v\|_{H^s}^2 + \|E\|_{H^s}^2 + \|B\|_{H^s}^2 + \|n\|_{H^2}^2\right)\\ &\quad + \int_0^T \left(\mu\|\nabla v\|_{H^s}^2 + \sigma\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n + \nabla^k E + (\nabla^k v)\times B\|_{L^2}^2\right)\mathrm{d}t \leq C\end{aligned}$$

holds.

THEOREM 1.2 (Global well-posedness). *Assume that the initial conditions $v^{in}\in H^s(\mathbb{R}^3)$, and $E^{in}, B^{in}\in H^{s+1}(\mathbb{R}^3)$ for $s\geq 2$. If there is a small positive constant ϵ_0, depending only upon s, the viscosity μ and the electric resistivity σ, such that the initial eneregy*

$$\begin{aligned}\mathcal{E}^{in} :=& \|v^{in}\|_{H^s(\mathbb{R}^3)}^2 + \|E^{in}\|_{H^s(\mathbb{R}^3)}^2 + \|\mathrm{div}E^{in}\|_{H^s(\mathbb{R}^3)}^2\\ &+ \|\nabla\times E^{in}\|_{H^s(\mathbb{R}^3)}^2 + \|B^{in}\|_{H^{s+1}(\mathbb{R}^3)}^2 \leq \epsilon_0,\end{aligned}$$

then system (1.1) admits a unique global in time solution (v,E,B) satisfying

$$\begin{aligned}&v\in L^\infty(\mathbb{R}^+;H^s(\mathbb{R}^3))\cap L^2(\mathbb{R}^+;\dot{H}^{s+1}(\mathbb{R}^3)),\\ &E\in L^\infty(\mathbb{R}^+;H^s(\mathbb{R}^3)),\ n(=\mathrm{div}E)\in L^\infty(\mathbb{R}^+;H^s(\mathbb{R}^3))\cap L^2(\mathbb{R}^+;\dot{H}^{s+1}(\mathbb{R}^3)),\\ &B\in L^\infty(\mathbb{R}^+;H^{s+1}(\mathbb{R}^3)),\ \partial_t B(=-\nabla\times E)\in L^\infty(\mathbb{R}^+;H^s(\mathbb{R}^3)).\end{aligned}$$

Furthermore, the following energy inequality

$$\sup_{t\geq 0}\left(\|v\|_{H^s(\mathbb{R}^3)}^2 + \|E\|_{H^s(\mathbb{R}^3)}^2 + \|n\|_{H^s(\mathbb{R}^3)}^2 + \|B\|_{H^{s+1}(\mathbb{R}^3)}^2 + \|\partial_t B\|_{H^s(\mathbb{R}^3)}^2\right)$$

$$+\int_0^\infty \big(\mu\|\nabla v\|^2_{H^s(\mathbb{R}^3)}+\sigma\|\nabla n\|^2_{H^s(\mathbb{R}^3)}\big)\mathrm{d}t$$
$$+\sigma\int_0^\infty\sum_{k=0}^s\|-\tfrac12\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|^2_{L^2(\mathbb{R}^3)}\mathrm{d}t\le C\mathcal{E}^{in}$$

holds for some constant $C=C(\mu,\sigma)>0$.

The organization of this paper is as follows: in the next section, we derive a priori estimate of the two-fluid incompressible Navier–Stokes–Maxwell system (1.1) with Ohm's law. Based on the last section, we prove the local well-posedness with large initial data in Section 3. In Section 4, by utilizing the decay property of the electric field E and the divergence-free property of the magnetic field B, we justify the global classical solution to system (1.1) with small initial data. In Section Appendix A, we give the well-posedness of the the two-fluid incompressible Navier–Stokes–Maxwell system (1.2) with solenoidal Ohm's law, which is highly similar to the system (1.1).

2. A priori estimates

In this section, we will derive the a priori estimate of the system (1.1), so that we can construct the local in time solution. We again emphasize that the divergence-free property of B and the decay properties of E and n do not need to be concerned in deriving the a priori estimates for the local well-posedness of system (1.1). The following energy functionals shall be first defined:

$$\mathcal{E}_L(t):=\|v\|^2_{H^s}+\tfrac12\|E\|^2_{H^s}+\tfrac12\|B\|^2_{H^s}+\tfrac14\|n\|^2_{H^s},$$
$$\mathcal{D}_L(t):=\mu\|\nabla v\|^2_{H^s}+\tfrac{\sigma}{2}\sum_{k=0}^s\|-\tfrac12\nabla^{k+1}n+\nabla^kE+(\nabla^kv)\times B\|^2_{L^2}.$$

Then, we articulate the following proposition:

PROPOSITION 2.1 (A priori estimate). *Assume that* (v,E,B) *is a sufficiently smooth solution to system* (1.1) *on the interval* $[0,T]$. *Then there is a positive constant* $C=C(s,\mu,\sigma)>0$, *such that*

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_L(t)+\mathcal{D}_L(t)\le C\mathcal{E}_L(t)[\mathcal{E}_L^{\frac12}(t)+\mathcal{E}_L(t)+\mathcal{E}_L^2(t)] \tag{2.1}$$

holds for all $t\in[0,T]$.

Proof. Notice that the cancelation $\langle\nabla^k(\nabla\times B),\nabla^kE\rangle-\langle\nabla^k(\nabla\times E),\nabla^kB\rangle=0$ holds for all integers $k\ge0$. Indeed, direct calculation enables us

$$\begin{aligned}&\langle\nabla^k(\nabla\times B),\nabla^kE\rangle-\langle\nabla^k(\nabla\times E),\nabla^kB\rangle\\
=&\langle\nabla^k(\varepsilon_{ijl}\partial_jB_l),\nabla^kE_i\rangle-\langle\nabla^k(\varepsilon_{ijl}\partial_jE_l),\nabla^kB_i\rangle\\
=&\langle\varepsilon_{ijl}\partial_j\nabla^kB_l,\nabla^kE_i\rangle+\langle\varepsilon_{ijl}\partial_j\nabla^kE_i,\nabla^kB_l\rangle\\
=&\langle\varepsilon_{ijl}\partial_j(\nabla^kB_l\nabla^kE_i),1\rangle-\langle\varepsilon_{ijl}\nabla^kB_l,\partial_j\nabla^kE_i\rangle+\langle\varepsilon_{ijl}\partial_j\nabla^kE_i,\nabla^kB_l\rangle\\
=&\langle\mathrm{div}(\nabla^kB\times\nabla^kE),1\rangle=0.\end{aligned} \tag{2.2}$$

Then, by the cancellation (2.2) and the evolution of n, hence $\partial_tn+\mathrm{div}j=0$, which is implied by the Ampère–Maxwell equation under the condition $\mathrm{div}B=0$, one can easily derive L^2-estimate of system (1.1)

$$\tfrac12\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\|v\|^2_{L^2}+\tfrac12\|E\|^2_{L^2}+\tfrac12\|B\|^2_{L^2}+\tfrac14\|n\|^2_{L^2}\big)+\mu\|\nabla v\|^2_{L^2}+\tfrac{\sigma}{2}\|-\tfrac12\nabla n+E+v\times B\|^2_{L^2}=0. \tag{2.3}$$

Now we derive the higher order energy estimates. For $k\geq 1$, we act the derivative operator ∇^k on the first fluid equations of system (1.1) and take L^2-inner product by dot with $\nabla^k v$, and then by integration by parts on $\mathbb{R}^3$ we gain

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k v\|_{L^2}^2+\mu\|\nabla^{k+1}v\|_{L^2}^2+\tfrac{\sigma}{2}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2\\
=&-\big\langle\nabla^k(v\cdot\nabla v),\nabla^k v\big\rangle+\tfrac{1}{2}\sum_{\substack{a+b=k\\a\geq 1}}\big\langle\nabla^a n\nabla^b E,\nabla^k v\big\rangle+\tfrac{1}{2}\sum_{\substack{a+b=k\\b\geq 1}}\big\langle\nabla^a j\times\nabla^b B,\nabla^k V\big\rangle\\
&-\tfrac{1}{2}\sum_{\substack{a+b=k\\a\geq 1}}\big\langle\nabla^a n\nabla^b v,-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\big\rangle+\tfrac{1}{2}\big\langle\nabla^k E,\nabla^k j\big\rangle\\
&-\tfrac{\sigma}{2}\sum_{\substack{a+b=k\\b\geq 1}}\big\langle\nabla^a v\times\nabla^b B,-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\big\rangle-\tfrac{1}{4}\big\langle\nabla^k j,\nabla^{k+1}n\big\rangle.
\end{aligned}
$$

Here we make use of the equality $\big\langle n\nabla^k v,(\nabla^k v)\times B\big\rangle=0$ for all $k\geq 0$ and the calculation

$$
\begin{aligned}
&\big\langle\nabla^k(nE+j\times B),\nabla^k v\big\rangle=\big\langle\nabla^k(nE_l)+\varepsilon_{lmn}\nabla^k(j_m B_n),\nabla^k v_l\big\rangle\\
=&\big\langle n\nabla^k E_l+\varepsilon_{lmn}\nabla^k j_m B_n,\nabla^k v_l\big\rangle\\
&+\sum_{\substack{a+b=k\\b\geq 1}}\big\langle\varepsilon_{lmn}\nabla^a j_m\nabla^b B_n,\nabla^k v_l\big\rangle+\sum_{\substack{a+b=k\\a\geq 1}}\big\langle\nabla^a n\nabla^b E,\nabla^k v\big\rangle\\
=&\big\langle\nabla^k E,n\nabla^k v\big\rangle-\big\langle\nabla^k j,(\nabla^k v)\times B\big\rangle\\
&+\sum_{\substack{a+b=k\\b\geq 1}}\big\langle(\nabla^a j)\times(\nabla^b B),\nabla^k v\big\rangle+\sum_{\substack{a+b=k\\a\geq 1}}\big\langle\nabla^a n\nabla^b E,\nabla^k v\big\rangle\\
=&\big\langle\nabla^k E,n\nabla^k v\big\rangle-\big\langle\nabla^k j-n\nabla^k v,\nabla^k E+(\nabla^k v)\times B\big\rangle\\
&+\sum_{\substack{a+b=k\\b\geq 1}}\big\langle(\nabla^a j)\times(\nabla^b B),\nabla^k v\big\rangle+\sum_{\substack{a+b=k\\a\geq 1}}\big\langle\nabla^a n\nabla^b E,\nabla^k v\big\rangle\\
=&\big\langle\nabla^k E,n\nabla^k v\big\rangle-\sigma\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2-\tfrac{1}{2}\big\langle\nabla^k j,\nabla^{k+1}n\big\rangle\\
&+\sum_{\substack{a+b=k\\b\geq 1}}\big\langle(\nabla^a j)\times(\nabla^b B),\nabla^k v\big\rangle+\sum_{\substack{a+b=k\\a\geq 1}}\big\langle\nabla^a n\nabla^b E,\nabla^k v\big\rangle\\
&-\sum_{\substack{a+b=k\\a\geq 1}}\big\langle\nabla^a n\nabla^b v-\sigma(\nabla^b v)\times(\nabla^a B),-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\big\rangle.
\end{aligned}
$$

For the second Ampère–Maxwell-equation and the third Faraday's law equation, we take k-order derivatives and take L^2-inner product by dot with $\frac{1}{2}\nabla^k E$ and $\frac{1}{2}\nabla^k B$, and then by the cancellation (2.2) we gain

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\tfrac{1}{2}\|\nabla^k E\|_{L^2}^2+\tfrac{1}{2}\|\nabla^k B\|_{L^2}^2\big)=-\tfrac{1}{2}\big\langle\nabla^k j,\nabla^k E\big\rangle.$$

Analogously, for the evolution of n: $\partial_t n+\mathrm{div}j=0$, which is implied by the Ampère–Maxwell equation and $\mathrm{div}B=0$, we have

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\big(\tfrac{1}{4}\|\nabla^k n\|_{L^2}^2\big)=\tfrac{1}{4}\big\langle\nabla^k j,\nabla^{k+1}n\big\rangle.$$

As a consequence, we have for $1\leq k\leq s(s\geq 2)$

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^k v\|_{L^2}^2+\tfrac{1}{2}\|\nabla^k E\|_{L^2}^2+\tfrac{1}{2}\|\nabla^k B\|_{L^2}^2+\tfrac{1}{4}\|\nabla^k n\|_{L^2}^2\Big)$$

$$
\begin{aligned}
&+\mu\|\nabla^{k+1}v\|_{L^2}^2+\tfrac{\sigma}{2}\|-\tfrac12\nabla^{k+1}n+\nabla^kE+(\nabla^kv)\times B\|_{L^2}^2\\
=&-\big\langle\nabla^k(v\cdot\nabla v),\nabla^kv\big\rangle+\tfrac12\sum_{\substack{a+b=k\\a\ge1}}\big\langle\nabla^an\nabla^bE,\nabla^kv\big\rangle\\
&+\tfrac12\sum_{\substack{a+b=k\\b\ge1}}\big\langle(\nabla^aj)\times(\nabla^bB),\nabla^kv\big\rangle-\tfrac12\sum_{\substack{a+b=k\\a\ge1}}\big\langle\nabla^an\nabla^bv,-\tfrac12\nabla^{k+1}n+\nabla^kE+(\nabla^kv)\times B\big\rangle\\
&-\tfrac{\sigma}{2}\sum_{\substack{a+b=k\\b\ge1}}\big\langle(\nabla^av)\times(\nabla^bB),-\tfrac12\nabla^{k+1}n+\nabla^kE+(\nabla^kv)\times B\big\rangle\\
:=&I_1+I_2+I_3+I_4+I_5.
\end{aligned}\tag{2.4}
$$

Now we estimate the terms $I_i(1\le i\le5)$ term by term. For the term I_1, we have

$$
\begin{aligned}
I_1=&-\sum_{\substack{a+b=k\\a\ge1}}\big\langle\nabla^av\nabla^{b+1}v,\nabla^kv\big\rangle\lesssim\sum_{\substack{a+b=k\\a\ge1}}\|\nabla^av\|_{L^4}\|\nabla^{b+1}v\|_{L^4}^2\|\nabla^kv\|_{L^2}^2\\
\lesssim&\sum_{\substack{a+b=k\\a\ge1}}\|\nabla v\|_{H^a}\|\nabla v\|_{H^{b+1}}\|v\|_{H^k}\lesssim\|\nabla v\|_{H^s}\|\nabla v\|_{H^{s-1}}\|v\|_{H^s}.
\end{aligned}\tag{2.5}
$$

Here we make use of the divergence-free property of the velocity v, Hölder inequality and Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^4(\mathbb{R}^3)$. For the term I_2, it is easy to be derived from the Hölder inequality and Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^4(\mathbb{R}^3)$, $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ that

$$
\begin{aligned}
I_2\lesssim&\|\nabla^kn\|_{L^2}\|E\|_{L^\infty}\|\nabla^kv\|_{L^2}+\sum_{\substack{a+b=k\\a,b\ge1}}\|\nabla^an\|_{L^4}\|\nabla^bE\|_{L^4}\|\nabla^kv\|_{L^2}\\
\lesssim&\|n\|_{H^s}\|E\|_{H^s}\|\nabla v\|_{H^{s-1}}.
\end{aligned}\tag{2.6}
$$

For the term I_3, Hölder inequalty, Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ $(p=3,4)$, $H^2(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$ and the Sobolev inequality $\|f\|_{L^6(\mathbb{R}^3)}\lesssim\|\nabla f\|_{L^2(\mathbb{R}^3)}$ (for $f\in H^1(\mathbb{R}^3)$) enable us to estimate

$$
\begin{aligned}
I_3=&\tfrac12\big\langle j\times\nabla^kB,\nabla^kv\big\rangle+\tfrac12\sum_{\substack{a+b=k\\a,b\ge1}}\big\langle\nabla^aj\times\nabla^bB,\nabla^kv\big\rangle\\
\lesssim&\|\nabla^kB\|_{L^2}\|\nabla^kv\|_{L^3}\|j\|_{L^6}+\sum_{\substack{a+b=k\\a,b\ge1}}\|\nabla^kv\|_{L^3}\|\nabla^aj\|_{L^2}\|\nabla^bB\|_{L^6}\\
\lesssim&\|B\|_{H^s}\|\nabla v\|_{H^s}\Big(\|j\|_{L^6}+\sum_{1\le ak-1}\|\nabla^aj\|_{L^2}\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
\|j\|_{L^6}\lesssim&\|v\|_{L^6}\|n\|_{L^\infty}+\sigma(\|\nabla n\|_{L^6}+\|E\|_{L^6}+\|v\|_{L^6}\|B\|_{L^\infty})\\
\lesssim&\|\nabla v\|_{L^2}\|n\|_{H^2}+\sigma(\|\nabla^2n\|_{L^2}+\|\nabla E\|_{L^2}+\|\nabla v\|_{L^2}\|B\|_{H^s})\\
\lesssim&\|\nabla v\|_{H^{s-1}}\|n\|_{H^s}+\sigma(\|\nabla n\|_{H^{s-1}}+\|E\|_{H^s}+\|\nabla v\|_{H^{s-1}}\|B\|_{H^s}),
\end{aligned}
$$

and for $1\le a\le k-1$

$$
\begin{aligned}
\|\nabla^aj\|_{L^2}\lesssim&\|\nabla^a(nv)\|_{L^2}+\sigma(\|\nabla^{a+1}n\|_{L^2}+\|\nabla^aE\|_{L^2}+\|\nabla^a(v\times B)\|_{L^2})\\
\lesssim&\|\nabla v\|_{H^{s-1}}\|n\|_{H^s}+\|\nabla n\|_{H^{s-1}}\|v\|_{H^s}\\
&+\sigma(\|\nabla n\|_{H^{s-1}}+\|E\|_{H^s}+\|\nabla v\|_{H^{s-1}}\|B\|_{H^s}).
\end{aligned}
$$

Thus, we have

$$I_3 \lesssim (1+\sigma)\|B\|_{H^s}\|\nabla v\|_{H^s}\big[\|\nabla v\|_{H^{s-1}}(\|n\|_{H^s}+\|B\|_{H^s}) \\ +\|\nabla n\|_{H^{s-1}}(1+\|v\|_{H^s})+\|E\|_{H^s}\big]. \tag{2.7}$$

For the estimation of the term I_4, we utilize the Hölder inequality, Sobolev embedding theory to deduce

$$\begin{aligned} I_4 &\lesssim \Big(\|\nabla^a n\|_{L^2}\|v\|_{L^\infty}+\sum_{\substack{a+b=k\\a,b\geq 1}}\|\nabla^a n\|_{L^4}\|\nabla^b v\|_{L^4}\Big)\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2} \\ &\lesssim \Big(\|\nabla n\|_{H^{k-1}}\|v\|_{H^2}+\sum_{\substack{a+b=k\\a,b\geq 1}}\|\nabla^a n\|_{H^1}\|\nabla^b v\|_{H^1}\Big)\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2} \\ &\lesssim \|\nabla n\|_{H^{s-1}}\|v\|_{H^s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}. \end{aligned} \tag{2.8}$$

For the estimation of I_5, we deduce that by the Hölder inequality, Sobolev embedding theory

$$\begin{aligned} I_5 &\lesssim \sigma\Big(\|v\|_{L^\infty}\|\nabla^k B\|_{L^2}+\sum_{\substack{a+b=k\\a,b\geq 1}}\|\nabla^a v\|_{L^4}\|\nabla^b v\|_{L^4}\Big)\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2} \\ &\lesssim \Big(\|v\|_{H^2}\|B\|_{H^k}+\sum_{\substack{a+b=k\\a,b\geq 1}}\|\nabla^a v\|_{H^1}\|\nabla^b v\|_{H^1}\Big)\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2} \\ &\lesssim \sigma\|v\|_{H^s}\|B\|_{H^s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}. \end{aligned} \tag{2.9}$$

We figure out that all the norms with the form $\|\nabla f\|_{H^{s-1}}$ occurred in the above estimates will be controlled as $\|\nabla f\|_{H^{s-1}}\leq\|f\|_{H^s}$ when we prove the local solution and the inequality $\|\nabla f\|_{H^{s-1}}\leq\|\nabla f\|_{H^s}$ will be employed in proving the global classical solution with small initial data. Plugging the inequalities (2.5), (2.6), (2.7), (2.8) and (2.9) into equation (2.4), summing up for all integer $1\leq k\leq s$ and combining the equality (2.3) reduce to

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|_{H^s}^2+\tfrac{1}{2}\|E\|_{H^s}^2+\tfrac{1}{2}\|B\|_{H^s}^2+\tfrac{1}{4}\|n\|_{H^s}^2\Big) \\ &\quad+\mu\|\nabla v\|_{H^s}^2+\frac{\sigma}{2}\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2 \\ &\lesssim \|\nabla v\|_{H^s}\|v\|_{H^s}^2+\|v\|_{H^s}\|E\|_{H^s}\|n\|_{H^s} \\ &\quad(1+\sigma)\|\nabla v\|_{H^s}\|B\|_{H^s}(\|v\|_{H^s}\|n\|_{H^s}+\|v\|_{H^s}\|B\|_{H^s}+\|n\|_{H^s}+\|E\|_{H^s}) \\ &\quad+(\|n\|_{H^s}\|v\|_{H^s}+\sigma\|v\|_{H^s}\|B\|_{H^s})\Big[\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{2.10}$$

Recalling the definition of the energy functionals $\mathcal{E}_L(t)$ and $\mathcal{D}_L(t)$, we derive the inequality (2.1) from the energy inequality (2.10) and we finish the proof of Proposition 2.1. □

REMARK 2.1. We emphasize that if we justify global well-posedness with small initial data, the estimation of the terms I_5 should be different from that in the proof of

Proposition 2.1. More precisely,

$$
\begin{aligned}
I_5 \lesssim & \sigma\Big(\|v\|_{L^6}\|\nabla^k B\|_{L^3} + \sum_{\substack{a+b=k\\ a,b\geq 1}} \|\nabla^a v\|_{L^4}\|\nabla^b v\|_{L^4}\Big)\| -\tfrac{1}{2}\nabla^{k+1} n + \nabla^k E + (\nabla^k v)\times B\|_{L^2}\\
\lesssim & \Big(\|\nabla v\|_{L^2}\|\nabla B\|_{H^k} + \sum_{\substack{a+b=k\\ a,b\geq 1}} \|\nabla^a v\|_{H^1}\|\nabla^b v\|_{H^1}\Big)\| -\tfrac{1}{2}\nabla^{k+1} n + \nabla^k E + (\nabla^k v)\times B\|_{L^2}\\
\lesssim & \sigma\|\nabla v\|_{H^s}\|\nabla B\|_{H^s}\| -\tfrac{1}{2}\nabla^{k+1} n + \nabla^k E + (\nabla^k v)\times B\|_{L^2},
\end{aligned} \tag{2.11}
$$

where we make use of the Sobolev inequality $\|f\|_{L^6(\mathbb{R}^3)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^3)}$ if $f \in H^1(\mathbb{R}^3)$. Then by substituting the inequalities (2.5), (2.7) and (2.11) into equation (2.4), summing up for all integer $1 \leq k \leq s$ and combining with the L^2-estimate (2.3), we gain

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|_{H^s}^2 + \tfrac{1}{2}\|E\|_{H^s}^2 + \tfrac{1}{2}\|B\|_{H^s}^2 + \tfrac{1}{4}\|n\|_{H^s}^2\Big)\\
&+ \mu\|\nabla v\|_{H^s}^2 + \tfrac{\sigma}{2}\sum_{k=0}^{s}\| -\tfrac{1}{2}\nabla^{k+1} n + \nabla^k E + (\nabla^k v)\times B\|_{L^2}^2\\
\lesssim & (1+\sigma)\|\nabla v\|_{H^s}(\|\nabla v\|_{H^s} + \|\nabla n\|_{H^s})\\
&\times(\|v\|_{H^s} + \|B\|_{H^s} + \|n\|_{H^s})(1 + \|v\|_{H^s} + \|B\|_{H^s} + \|n\|_{H^s})\\
&+ (1+\sigma)\Big[\sum_{k=0}^{s}\| -\tfrac{1}{2}\nabla^{k+1} n + \nabla^k E + (\nabla^k v)\times B\|_{L^2}^2\Big]^{\frac{1}{2}}\\
&\times(\|\nabla v\|_{H^s}\|\nabla B\|_{H^s} + \|\nabla n\|_{H^s}\|v\|_{H^s}),
\end{aligned} \tag{2.12}
$$

which will be applied in verifying global well-posedness of system (1.1) with small initial data.

3. Local well-posedness with large initial data

In this section, based on a priori estimate derived in Section 2, we prove the local well-posedness of system (1.1) with large initial data, namely, prove Theorem 1.1. We employ mollifier method to achieve our goal.

Proof. (**Proof of Theorem 1.1.**) We first define a mollifier

$$
\mathcal{J}_\epsilon f := \mathcal{F}^{-1}\big(\mathbf{1}_{|\xi|\leq \frac{1}{\epsilon}}\mathcal{F}(f)\big),
$$

where the symbol $\mathcal{F}$ stands for the Fourier transform operator and $\mathcal{F}^{-1}$ is its inverse transform. We remark that the mollifier $\mathcal{J}_\epsilon$ has property $\mathcal{J}_\epsilon^2 = \mathcal{J}_\epsilon$. Then the approximate system of (1.1) can be constructed as follows:

$$
\begin{cases}
\partial_t v^\epsilon + \mathcal{J}_\epsilon(\mathcal{J}_\epsilon v^\epsilon \cdot \nabla \mathcal{J}_\epsilon v^\epsilon) - \mu\Delta \mathcal{J}_\epsilon v^\epsilon + \nabla p^\epsilon = \tfrac{1}{2}\mathcal{J}_\epsilon(\mathcal{J}_\epsilon n^\epsilon \mathcal{J}_\epsilon E^\epsilon + \mathcal{J}_\epsilon j^\epsilon \times \mathcal{J}_\epsilon B^\epsilon),\\
\partial_t E^\epsilon - \nabla\times \mathcal{J}_\epsilon B^\epsilon = -\mathcal{J}_\epsilon j^\epsilon,\\
\partial_t B^\epsilon + \nabla\times \mathcal{J}_\epsilon E^\epsilon = 0,\\
\mathrm{div} v^\epsilon = \mathrm{div} B^\epsilon = 0,\ \mathrm{div} E^\epsilon = n^\epsilon,\\
j^\epsilon = \mathcal{J}_\epsilon n^\epsilon \mathcal{J}_\epsilon v^\epsilon + \sigma(-\tfrac{1}{2}\nabla n^\epsilon + E^\epsilon + \mathcal{J}_\epsilon v^\epsilon \times \mathcal{J}_\epsilon B^\epsilon),\\
\partial_t n^\epsilon + \mathrm{div}\mathcal{J}_\epsilon j^\epsilon = 0,\\
(v^\epsilon, E^\epsilon, B^\epsilon)|_{t=0} = (\mathcal{J}_\epsilon v^{in}, \mathcal{J}_\epsilon E^{in}, \mathcal{J}_\epsilon B^{in}).
\end{cases} \tag{3.1}
$$

By ODE theory, we know that there is a maximal $T_\epsilon > 0$ such that the approximate system (3.1) has a unique solution $(v^\epsilon, E^\epsilon, B^\epsilon, n^\epsilon) \in C([0,T_\epsilon); H^s(\mathbb{R}^3))$. Since the mollifier

$\mathcal{J}_\epsilon$ satisfies $\mathcal{J}_\epsilon^2=\mathcal{J}_\epsilon$, we observe that $(\mathcal{J}_\epsilon v^\epsilon,\mathcal{J}_\epsilon E^\epsilon,\mathcal{J}_\epsilon B^\epsilon,\mathcal{J}_\epsilon n^\epsilon)$ is also a solution to the system (3.1). Then the uniqueness implies that $(\mathcal{J}_\epsilon v^\epsilon,\mathcal{J}_\epsilon E^\epsilon,\mathcal{J}_\epsilon B^\epsilon,\mathcal{J}_\epsilon n^\epsilon)=(v^\epsilon,E^\epsilon,B^\epsilon,n^\epsilon)$. As a consequence, the solution $(v^\epsilon,E^\epsilon,B^\epsilon,n^\epsilon)$ to the approximate system (3.1) also solves the system

$$\begin{cases}\partial_t v^\epsilon+\mathcal{J}_\epsilon(v^\epsilon\cdot\nabla v^\epsilon)-\mu\Delta v^\epsilon+\nabla p^\epsilon=\frac{1}{2}\mathcal{J}_\epsilon(n^\epsilon E^\epsilon+j^\epsilon\times B^\epsilon),\\ \partial_t E^\epsilon-\nabla\times B^\epsilon=-\mathcal{J}_\epsilon j^\epsilon,\\ \partial_t B^\epsilon+\nabla\times E^\epsilon=0,\\ \mathrm{div}v^\epsilon=\mathrm{div}B^\epsilon=0,\ \mathrm{div}E^\epsilon=n^\epsilon,\\ j^\epsilon=n^\epsilon v^\epsilon+\sigma(-\frac{1}{2}\nabla n^\epsilon+E^\epsilon+v^\epsilon\times B^\epsilon),\\ \partial_t n^\epsilon+\mathrm{div}\mathcal{J}_\epsilon j^\epsilon=0,\\ (v^\epsilon,E^\epsilon,B^\epsilon)|_{t=0}=(\mathcal{J}_\epsilon v^{in},\mathcal{J}_\epsilon E^{in},\mathcal{J}_\epsilon B^{in}).\end{cases}\tag{3.2}$$

As shown in the proceeding of the derivation of a priori estimate in Section 2, we can derive the energy estimate of the approximate system (3.2)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_{L,\epsilon}(t)+\mathcal{D}_{L,\epsilon}(t)\le C\mathcal{E}_{L,\epsilon}(t)[\mathcal{E}_{L,\epsilon}^{\frac{1}{2}}(t)+\mathcal{E}_{L,\epsilon}(t)+\mathcal{E}_{L,\epsilon}^2(t)]$$

for all $t\in[0,T_\epsilon)$, where the energy functionals $\mathcal{E}_{L,\epsilon}(t)$ and $\mathcal{D}_{L,\epsilon}(t)$ are

$$\mathcal{E}_{L,\epsilon}(t)=\|v^\epsilon\|_{H^s}^2+\tfrac{1}{2}\|E^\epsilon\|_{H^s}^2+\tfrac{1}{2}\|B^\epsilon\|_{H^s}^2+\tfrac{1}{4}\|n^\epsilon\|_{H^s}^2,$$
$$\mathcal{D}_{L,\epsilon}(t)=\mu\|\nabla v^\epsilon\|_{H^s}^2+\tfrac{\sigma}{2}\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E^\epsilon+(\nabla^k v^\epsilon)\times B^\epsilon\|_{L^2}^2.$$

Then continuity arguments imply that there is a $T=T(s,\mu,\sigma)>0$ such that the energy bound

$$\sup_{t\in[0,T]}\Big(\|v^\epsilon\|_{H^s}^2+\tfrac{1}{2}\|E^\epsilon\|_{H^s}^2+\tfrac{1}{2}\|B^\epsilon\|_{H^s}^2+\tfrac{1}{4}\|n^\epsilon\|_{H^s}^2\Big)$$
$$+\int_0^T\Big(\mu\|\nabla v^\epsilon\|_{H^s}^2+\tfrac{\sigma}{2}\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E^\epsilon+(\nabla^k v^\epsilon)\times B^\epsilon\|_{L^2}^2\Big)\mathrm{d}t\le C$$

uniformly holds, where the positive constant C depends only upon s, μ, σ and the initial data. By compactness arguments we finish the proof of Theorem 1.1. □

4. Global well-posedness with small initial data

In this section, we mainly prove the global classical solution to system (1.1) with small initial data. The *key point* is to use the following properties:

- the decay property of the electric field E coming from the linear part of the Ampère–Maxwell equation,
- the divergence-free property of the magnetic field B,
- the dissipative property of n coming from the Ampère–Maxwell equation by taking ‘div’ and the fifth Ohm's law equation.

These properties result to enough energy dissipation. More precisely, we consider the equations

$$\begin{cases}\partial_t E-\nabla\times B+\sigma E-\frac{\sigma}{2}\nabla n=-nv-\sigma v\times B,\\ \partial_t B+\nabla\times E=0,\\ E|_{t=0}=E^{in},\ B|_{t=0}=B^{in},\end{cases}\tag{4.1}$$

where the term $-nv-\sigma v\times B$ is regarded as a source term. The damping term σE will supply energy dissipation in the energy estimation. Additionally, the divergence-free condition of the magnetic B yields that

$$\nabla\times(\nabla\times B)=-\Delta B.$$

Then, by combining the second Ampère–Maxwell equation, the third Faraday's law equation reduces to

$$\begin{cases}\partial_{tt}B-\Delta B+\sigma\partial_t B=\nabla\times(nv)+\sigma\nabla\times(v\times B),\\ B|_{t=0}=B^{in},\\ \partial_t B|_{t=0}=-\nabla\times E^{in},\end{cases}\tag{4.2}$$

which is a wave equation with a damping $\sigma\partial_t B$. We remark that the last initial condition in system (4.2) is the compatibility condition from the Faraday's law (the third equation of (1.1)). Furthermore, the divergence-free property of B implies $\mathrm{div}(\nabla\times B)=0$. Then, by taking div on the second Ampère–Maxwell equation and the relation $\mathrm{div}E=n$, we gain

$$\partial_t n+\mathrm{div}j=0.$$

Then the Ohm's law (the fifth equation of system (1.1)) reduces to a parabolic equation of n with a damping σn

$$\begin{cases}\partial_t n+v\cdot\nabla n-\frac{\sigma}{2}\Delta n+\sigma n=-\sigma\mathrm{div}(v\times B),\\ n|_{t=0}=\mathrm{div}E^{in}.\end{cases}\tag{4.3}$$

So, we now find enough energy dissipation gaining from the decay terms σE, $\sigma\partial_t B$ and $-\frac{\sigma}{2}\Delta n+\sigma n$.

We define the following energy functionals:

$$\begin{aligned}\mathcal{E}_G(t):=&\|v\|_{H^s}^2+\tfrac{3}{2}\|E\|_{H^s}^2+(\tfrac{3}{2}+\delta\sigma-\sigma)\|B\|_{H^s}^2+\tfrac{5}{4}\|n\|_{H^s}^2\\&+(1-\delta)\|\partial_t B\|_{H^s}^2+\|\nabla B\|_{H^s}^2+\delta\|\partial_t B+B\|_{H^s}^2,\\ \mathcal{D}_G(t):=&\mu\|\nabla v\|_{H^s}^2+\tfrac{\sigma}{2}\|E\|_{H^s}^2+\tfrac{3}{8}\sigma\|\nabla n\|_{H^s}^2+\sigma\|n\|_{H^s}^2+(\sigma-\delta)\|\partial_t B\|_{H^s}^2\\&+\delta\|\nabla B\|_{H^s}^2+\tfrac{\sigma}{2}\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2,\end{aligned}$$

where $\delta=\frac{1}{2}\min\{1,\sigma\}\in(0,1)$. Then we can articulate the following proposition, which is utilized to prove the global well-posedness of system (1.1) with small initial data.

Proposition 4.1. *Assume that (v,E,B) is a sufficiently smooth solution to the Navier–Stokes–Maxwell system* (1.1). *Then the energy inequality*

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_G(t)+\mathcal{D}_G(t)\le C\mathcal{D}_G(t)[\mathcal{E}_G^{\frac{1}{2}}(t)+\mathcal{E}_G(t)]\tag{4.4}$$

holds for some constant $C>0$, which depends only upon s, the viscosity σ and the electric resistivity σ.

Proof. We have proven the inequality (2.12) in Remark 2.1 for verifying the global well-posedness. It remains to perform the H^s-energy estimates for the equations (4.1) and (4.2) for $s\ge 2$.

We first consider the equations (4.1). Multiplying E in the first equation of system (4.1) and B in the second equation of (4.1), respectively, and taking L^2-inner product yields that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|E\|_{L^2}^2+\|B\|_{L^2}^2\right)+\sigma\|E\|_{L^2}^2-\frac{\sigma}{2}\langle\nabla n,E\rangle=-\langle nv,E\rangle-\sigma\langle v\times B,E\rangle\\
&\lesssim\|v\|_{L^6}\|n\|_{L^3}\|E\|_{L^2}+\sigma\|v\|_{L^6}\|B\|_{L^3}\|E\|_{L^2}\\
&\lesssim\|\nabla v\|_{L^2}\|n\|_{H^1}\|E\|_{L^2}+\sigma\|\nabla v\|_{L^2}\|B\|_{H^1}\|E\|_{L^2}.
\end{aligned}\tag{4.5}$$

Here we make use of the cancellation (2.2), the Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$ and the inequality $\|v\|_{L^6(\mathbb{R}^3)}\lesssim\|\nabla v\|_{L^2(\mathbb{R}^3)}$ if $v\in H^1(\mathbb{R}^3)$.

For $1\le k\le s$, we act the derivative operator ∇^k on the two equations of system (4.1), take L^2-inner product by dot product with $\nabla^k E$ and $\nabla^k B$, respectively, and the we deduce that by the cancellation (2.2) and Sobolev embedding theory

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^k E\|_{L^2}^2+\|\nabla^k B\|_{L^2}^2\Big)+\sigma\|\nabla^k E\|_{L^2}^2-\frac{\sigma}{2}\left\langle\nabla^{k+1}n,\nabla^k E\right\rangle\\
&=-\left\langle\nabla^k(nv),\nabla^k E\right\rangle-\sigma\left\langle\nabla^k(v\times B),\nabla^k E\right\rangle\\
&\lesssim\Big(\|\nabla^k n\|_{L^2}\|v\|_{L^\infty}+\|\nabla^k v\|_{L^2}\|n\|_{L^\infty}+\sum_{\substack{a+b=k\\a,b\ge1}}\|\nabla^a n\|_{L^4}\|\nabla^b v\|_{L^4}\Big)\|\nabla^k E\|_{L^2}\\
&\quad+\sigma\Big(\|v\|_{L^6}\|\nabla^k B\|_{L^3}+\|\nabla^k v\|_{L^2}\|B\|_{L^\infty}+\sum_{\substack{a+b=k\\a,b\ge1}}\|\nabla^a v\|_{L^4}\|\nabla^b B\|_{L^4}\Big)\|\nabla^k E\|_{L^2}\\
&\lesssim(\|\nabla n\|_{H^s}\|v\|_{H^s}+\|\nabla v\|_{H^s}\|n\|_{H^s})\|E\|_{H^s}+\sigma\|\nabla v\|_{H^s}(\|B\|_{H^s}+\|\nabla B\|_{H^s})\|E\|_{H^s}.
\end{aligned}\tag{4.6}$$

Here we make use of the cancellation (2.2), Hölder's inequality, Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^p(\mathbb{R}^3)$ for $p=3,4$ and the inequality $\|v\|_{L^6(\mathbb{R}^3)}\lesssim\|\nabla v\|_{L^2(\mathbb{R}^3)}$ for $v\in H^1(\mathbb{R}^3)$. We observe that

$$\frac{\sigma}{2}\left\langle\nabla^{k+1}n,\nabla^k E\right\rangle\le\frac{\sigma}{2}\|\nabla^k E\|_{L^2}^2+\frac{\sigma}{8}\|\nabla^{k+1}n\|_{L^2}^2$$

holds for all $k\ge0$. Then, summing up for all $1\le k\le s$ in inequality (4.6) and adding them to the inequality (4.5) reduce to

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|E\|_{H^s}^2+\|B\|_{H^s}^2\Big)+\frac{\sigma}{2}\|E\|_{H^s}^2-\frac{\sigma}{8}\|\nabla n\|_{H^s}^2\\
&\lesssim(\|\nabla n\|_{H^s}\|v\|_{H^s}+\|\nabla v\|_{H^s}\|n\|_{H^s})\|E\|_{H^s}+\sigma\|\nabla v\|_{H^s}(\|B\|_{H^s}+\|\nabla B\|_{H^s})\|E\|_{H^s}.
\end{aligned}\tag{4.7}$$

We now consider the wave equation (4.2). Taking L^2-inner product by dot with $\partial_t B$ and integrating by parts over $\mathbb{R}^3$, one knows

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t B\|_{L^2}^2+\|\nabla B\|_{L^2}^2\Big)+\sigma\|\partial_t B\|_{L^2}^2\\
&=\langle\nabla\times(nv),\partial_t B\rangle+\sigma\langle\nabla\times(v\times B),\partial_t B\rangle\\
&\lesssim(\|\nabla n\|_{L^3}\|v\|_{L^6}+\|n\|_{L^\infty}\|\nabla v\|_{L^2})\|\partial_t B\|_{L^2}\\
&\quad+\sigma(\|\nabla v\|_{L^2}\|B\|_{L^\infty}+\|v\|_{L^6}\|\nabla B\|_{L^3})\|\partial_t B\|_{L^2}\\
&\lesssim(\|\nabla n\|_{H^1}\|\nabla v\|_{L^2}+\|n\|_{H^2}\|\nabla v\|_{L^2})\|\partial_t B\|_{L^2}+\sigma\|\nabla v\|_{L^2}\|B\|_{H^2}\partial_t B\|_{L^2}\\
&\lesssim(1+\sigma)\|\nabla v\|_{H^s}\|\partial_t B\|_{H^s}(\|n\|_{H^s}+\|B\|_{H^s}).
\end{aligned}\tag{4.8}$$

For $1\le k\le s$, we take k-order derivative on the wave equation (4.2), and then take L^2-inner product by dot with $\nabla^k\partial_t B$, and then we estimate that by Hölder's inequality and Sobolev embedding theory

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^k\partial_t B\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2\Big)+\sigma\|\nabla^k\partial_t B\|_{L^2}^2\Big)+\sigma\|\nabla^k\partial_t B\|_{L^2}^2\\
=&\big\langle\nabla^k[\nabla\times(nv)],\nabla^k\partial_t B\big\rangle+\sigma\big\langle\nabla^k[\nabla\times(v\times B)],\nabla^k\partial_t B\big\rangle\\
\lesssim&\Big(\|n\|_{L^\infty}\|\nabla^{k+1}v\|_{L^2}+\|v\|_{L^\infty}\|\nabla^{k+1}n\|_{L^2}+\sum_{\substack{a+b=k+1\\a,b\geq 1}}\|\nabla^a n\|_{L^4}\|\nabla^b v\|_{L^4}\Big)\|\nabla^k\partial_t B\|_{L^2}\\
&+\sigma\Big(\|v\|_{L^\infty}\|\nabla^{k+1}B\|_{L^2}+\|B\|_{L^\infty}\|\nabla^{k+1}v\|_{L^2}+\sum_{\substack{a+b=k+1\\a,b\geq 1}}\|\nabla^a v\|_{L^4}\|\nabla^b B\|_{L^4}\Big)\|\nabla^k\partial_t B\|_{L^2}\\
\lesssim&(\|\nabla v\|_{H^s}\|n\|_{H^s}+\|\nabla n\|_{H^s}\|v\|_{H^s})\|\partial_t B\|_{H^s}\\
&+\sigma(\|\nabla B\|_{H^s}\|v\|_{H^s}+\|\nabla v\|_{H^s}\|B\|_{H^s})\|\partial_t B\|_{H^s}.
\end{aligned}\tag{4.9}$$

Thus we sum up for $1\leq k\leq s$ in inequality (4.9) and add them to inequality (4.8), and then we gain

$$\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t B\|_{H^s}^2+\|\nabla B\|_{H^s}^2\Big)+\sigma\|\partial_t B\|_{H^s}^2\\
\lesssim&(1+\sigma)\|\partial_t B\|_{H^s}(\|\nabla v\|_{H^s}+\|\nabla n\|_{H^s}+\|\nabla B\|_{H^s})(\|v\|_{H^s}+\|B\|_{H^s}+\|n\|_{H^s}).
\end{aligned}\tag{4.10}$$

By now, we do not find enough dissipation to control the energy term

$$(1+\sigma)\|\nabla B\|_{H^s}\|\partial_t B\|_{H^s}(\|v\|_{H^s}+\|B\|_{H^s}+\|n\|_{H^s})$$

in the right-hand side of inequality (4.10), while justifying the global well-posedness with small initial data. Thanks to the term ΔB in the wave equation (4.2), which will display energy dissipation if we dot with B (or higher order derivatives of B) in (4.2). Thus we possess enough dissipation to prove the global well-posedness. We take L^2-inner product in equation (4.2) with B, integrate by parts over $\mathbb{R}^3$ and then we gain

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\langle\partial_t B,B\rangle-\|\partial_t B\|_{L^2}^2+\|\nabla B\|_{L^2}^2+\tfrac{\sigma}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|B\|_{L^2}^2\\
=&\langle\nabla\times(nv),B\rangle+\sigma\langle\nabla\times(v\times B),B\rangle\\
\lesssim&\langle|\nabla n||v|+|n||\nabla v|,|B|\rangle+\sigma\langle|\nabla v||B|,|v||\nabla B|,|B|\rangle\\
\lesssim&(\|\nabla n\|_{L^2}\|v\|_{L^6}+\|n\|_{L^6}\|\nabla v\|_{L^2})\|B\|_{L^3}\\
&+\sigma(\|\nabla v\|_{L^2}\|B\|_{L^6}+\|v\|_{L^6}\|\nabla B\|_{L^2})\|B\|_{L^3}\\
\lesssim&\|\nabla n\|_{L^2}\|\nabla v\|_{L^2}\|B\|_{H^1}+\sigma\|\nabla v\|_{L^2}\|\nabla B\|_{L^2}\|B\|_{H^1}\\
\lesssim&(1+\sigma)(\|\nabla B\|_{H^s}+\|\nabla n\|_{H^s})\|\nabla v\|_{H^s}\|B\|_{H^s}.
\end{aligned}\tag{4.11}$$

Here we make use of the inequality $\|f\|_{L^6(\mathbb{R}^3)}\lesssim\|\nabla f\|_{L^2(\mathbb{R}^3)}$ for $f\in H^1(\mathbb{R}^3)$ and Sobolev embedding $H^1(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)$.

For $1\leq k\leq s$, by taking k-order derivatives on equation (4.2) and taking L^2-inner product with $\nabla^k B$, integrating by parts over $\mathbb{R}^3$, we derive from the Hölder inequality and Sobolev embedding theory that

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\big\langle\nabla^k\partial_t B,\nabla^k B\big\rangle-\|\nabla^k\partial_t B\|_{L^2}^2+\|\nabla^{k+1}B\|_{L^2}^2+\tfrac{\sigma}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k B\|_{L^2}^2\\
=&\big\langle\nabla^k[\nabla\times(nv)],\nabla^k B\big\rangle+\sigma\big\langle\nabla^k[\nabla\times(v\times B)],\nabla^k B\big\rangle\\
\lesssim&\Big(\|n\|_{L^\infty}\|\nabla^{k+1}v\|_{L^2}+\|\nabla^{k+1}n\|_{L^2}\|v\|_{L^\infty}+\sum_{\substack{a+b=k+1\\a,b\geq 1}}\|\nabla^a n\|_{L^4}\|\nabla^b v\|_{L^4}\Big)\|\nabla^k B\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&+\sigma\Big(\|v\|_{L^\infty}\|\nabla^{k+1}B\|_{L^2}+\|\nabla^{k+1}v\|_{L^2}\|B\|_{L^\infty}+\sum_{\substack{a+b=k+1\\a,b\geq 1}}\|\nabla^a v\|_{L^4}\|\nabla^b B\|_{L^4}\Big)\|\nabla^k B\|_{L^2}\\
&\lesssim(\|\nabla v\|_{H^s}\|n\|_{H^s}+\|\nabla n\|_{H^s}\|v\|_{H^s})\|\nabla B\|_{H^s}\\
&\quad+\sigma(\|\nabla B\|_{H^s}\|v\|_{H^s}+\|B\|_{H^s}\|\nabla v\|_{H^s})\|\nabla B\|_{H^s}\\
&\lesssim(1+\sigma)\|\nabla B\|_{H^s}(\|\nabla v\|_{H^s}+\|\nabla n\|_{H^s}+\|\nabla B\|_{H^s})(\|v\|_{H^s}+\|n\|_{H^s}+\|B\|_{H^s}), \qquad (4.12)
\end{aligned}
$$

where we utilize the inequality $\|f\|_{L^6(\mathbb{R}^3)}\lesssim\|\nabla f\|_{L^2(\mathbb{R}^3)}$ for $f\in H^1(\mathbb{R}^3)$. We sum up for all $1\leq k\leq s$ in inequality (4.12) and combine with the inequality (4.11), and then we gain

$$
\begin{aligned}
&\sum_{k=0}^{s}\tfrac{\mathrm{d}}{\mathrm{d}t}\big\langle\nabla^k\partial_t B,\nabla^k B\big\rangle-\|\partial_t B\|_{H^s}^2+\|\nabla B\|_{H^s}^2+\tfrac{\sigma}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|B\|_{H^s}^2\\
&\lesssim(1+\sigma)\|\nabla B\|_{H^s}(\|\nabla v\|_{H^s}+\|\nabla n\|_{H^s}+\|\nabla B\|_{H^s})(\|v\|_{H^s}+\|n\|_{H^s}+\|B\|_{H^s}).
\end{aligned}
$$

Noticing that

$$
\tfrac{\mathrm{d}}{\mathrm{d}t}\big\langle\nabla^k\partial_t B,\nabla^k B\big\rangle=\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla^k\partial_t B+\nabla^k B\|_{L^2}^2-\|\nabla^k\partial_t B\|_{L^2}^2-\|\nabla^k B\|_{L^2}^2\Big),
$$

we, consequently, have

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|\partial_t B+B\|_{H^s}^2-\|\partial_t B\|_{H^s}^2+(\sigma-1)\|B\|_{H^s}^2\Big)-\|\partial_t B\|_{H^s}^2+\|\nabla B\|_{H^s}^2\\
&\lesssim(1+\sigma)\|\nabla B\|_{H^s}(\|\nabla v\|_{H^s}+\|\nabla n\|_{H^s}+\|\nabla B\|_{H^s})(\|v\|_{H^s}+\|n\|_{H^s}+\|B\|_{H^s}). \qquad (4.13)
\end{aligned}
$$

For the evolution (4.3) of n, we take L^2-inner product by dot with n and integrate by parts over $\mathbb{R}^3$, and then we gain by Hölder's inequality and Sobolev embedding theory that

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|n\|_{L^2}^2+\tfrac{\sigma}{2}\|\nabla n\|_{L^2}^2+\sigma\|n\|_{L^2}^2=\sigma\langle v\times B,\nabla n\rangle\\
&\lesssim\sigma\|v\|_{L^6}\|B\|_{L^3}\|\nabla n\|_{L^2}\lesssim\sigma\|\nabla v\|_{L^2}\|\nabla n\|_{L^2}\|B\|_{H^1}\lesssim\sigma\|\nabla v\|_{H^s}\|\nabla n\|_{H^s}\|B\|_{H^s}. \qquad (4.14)
\end{aligned}
$$

For $1\leq k\leq s$, we act the derivative operator ∇^k on equation (4.3) and take L^2-inner product by dot with $\nabla^k n$, and then we have

$$
\begin{aligned}
&\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k n\|_{L^2}^2+\tfrac{\sigma}{2}\|\nabla^{k+1}n\|_{L^2}^2+\sigma\|\nabla^k n\|_{L^2}^2=\sigma\big\langle\nabla^k(v\times B),\nabla^{k+1}n\big\rangle\\
&\lesssim\sigma\Big(\|v\|_{L^6}\|\nabla^k B\|_{L^3}+\sum_{\substack{a+b=k\\a\geq 1}}\|\nabla^a v\|_{L^6}\|\nabla^b B\|_{L^3}\Big)\|\nabla^{k+1}n\|_{L^2}\\
&\lesssim\sigma\|\nabla v\|_{H^s}\|\nabla n\|_{H^s}(\|B\|_{H^s}+\|\nabla B\|_{H^s}). \qquad (4.15)
\end{aligned}
$$

Summing up for all integer $1\leq k\leq s$ in the above inequality (4.15) and combining with the inequality (4.14) enable us to gain

$$
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|n\|_{H^s}^2+\tfrac{\sigma}{2}\|\nabla n\|_{H^s}^2+\sigma\|n\|_{H^s}^2\lesssim\sigma\|\nabla v\|_{H^s}\|\nabla n\|_{H^s}(\|B\|_{H^s}+\|\nabla B\|_{H^s}). \qquad (4.16)
$$

Now we choose a number $\delta=\frac{1}{2}\min\{\sigma,1\}\in(0,1)$. It is implied that by multiplying δ in inequality (4.13) and adding it to the inequalities (2.12), (4.7), (4.10) and (4.16)

$$
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\Big(\|v\|_{H^s}^2+\tfrac{3}{2}\|E\|_{H^s}^2+(\tfrac{3}{2}+\delta\sigma-\delta)\|B\|_{H^s}^2+\tfrac{5}{4}\|n\|_{H^s}^2
$$

$$
\begin{aligned}
&+(1-\delta)\|\partial_t B\|_{H^s}^2+\|\nabla B\|_{H^s}^2+\delta\|\partial_t B+B\|_{H^s}^2\Big)\\
&+\mu\|\nabla v\|_{H^s}^2+\tfrac{\sigma}{2}\|E\|_{H^s}^2+\tfrac{3}{8}\sigma\|\nabla n\|_{H^s}^2+\sigma\|n\|_{H^s}^2+(\sigma-\delta)\|\partial_t B\|_{H^s}^2\\
&+\delta\|\nabla B\|_{H^s}^2+\tfrac{\sigma}{2}\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2\\
\lesssim&(1+\sigma)(\|\partial_t B\|_{H^s}+\|\nabla B\|_{H^s})(\|v\|_{H^s}+\|n\|_{H^s}+\|B\|_{H^s})\\
&\quad\times(\|\nabla v\|_{H^s}+\|\nabla n\|_{H^s}+\|\nabla B\|_{H^s})\\
&+(1+\sigma)(\|\nabla v\|_{H^s}+\|E\|_{H^s})(\|\nabla n\|_{H^s}+\|E\|_{H^s})\\
&\quad\times(\|n\|_{H^s}+\|v\|_{H^s}+\|B\|_{H^s}+\|\nabla B\|_{H^s})\\
&+(1+\sigma)\|\nabla v\|_{H^s}(\|\nabla v\|_{H^s}+\|\nabla n\|_{H^s}+\|E\|_{H^s})\\
&\quad\times(\|v\|_{H^s}+\|B\|_{H^s}+\|n\|_{H^s})(1+\|v\|_{H^s}+\|B\|_{H^s}+\|n\|_{H^s})\\
&+(1+\sigma)\Big[\sum_{k=0}^{2}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2\Big]^{\frac{1}{2}}\\
&\quad\times(\|\nabla v\|_{H^s}\|\nabla B\|_{H^s}+\|\nabla n\|_{H^s}\|v\|_{H^s}).
\end{aligned}
$$

We remark that the chosen constant $\delta\in(0,1)$ is such that $1-\delta>0$ and $\sigma-\delta>0$. Recalling the definition of the energy functionals $\mathcal{E}_G(t)$ and $\mathcal{D}_G(t)$, we derive the inequality (4.4) and thus finish the proof of Proposition 4.1. ∎

Proof. (**Proof of Theorem 1.2.**) Based on the energy estimate (4.4), we globally extend the solution constructed in Theorem 1.1 under small initial data condition.

One observe that

$$
\begin{aligned}
\mathcal{E}_G(0)=&\|v^{in}\|_{H^s}^2+\tfrac{3}{2}\|E^{in}\|_{H^s}^2+(\tfrac{3}{2}+\delta\sigma-\delta)\|B^{in}\|_{H^s}^2+\tfrac{5}{4}\|\mathrm{div}E^{in}\|_{H^s}^2\\
&+(1-\delta)\|\nabla\times E^{in}\|_{H^s}^2+\|\nabla B^{in}\|_{H^s}^2+\delta\|\nabla\times E^{in}+B^{in}\|_{H^s}^2\\
\le&2(2+\delta\sigma+\delta)(\|v^{in}\|_{H^s}^2+\|E^{in}\|_{H^s}^2+\|\mathrm{div}E^{in}\|_{H^s}^2+\|\nabla\times E^{in}\|_{H^s}^2+\|B^{in}\|_{H^{s+1}}^2)\\
=&2(2+\delta\sigma+\delta)\mathcal{E}^{in}.
\end{aligned}
$$

Denote $\mathcal{P}(\mathcal{E}_G(t)):=C[\mathcal{E}_G^{\frac{1}{2}}(t)+\mathcal{E}_G(t)]$, hence the inequality (4.4) can be rewritten as

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_G(t)+\mathcal{D}_G(t)\le\mathcal{D}_G(t)\mathcal{P}(\mathcal{E}_G(t)),$$

where $\mathcal{P}(0)=0$ and $\mathcal{P}(\cdot)$ is strictly increasing. Then there is an $\epsilon_0=\epsilon_0(\mu,\sigma,s)>0$ such that, if $\mathcal{E}^{in}\le\epsilon_0$, then

$$\mathcal{P}(\mathcal{E}_G(0))\le\mathcal{P}(2(2+\delta\sigma+\delta)\mathcal{E}^{in})\le\tfrac{1}{4}.$$

Now we define

$$T=\sup\Big\{\tau\ge 0;\ \sup_{t\in[0,\tau]}\mathcal{P}(\mathcal{E}_G(t))\le\tfrac{1}{2}\Big\}\ge 0.$$

By the continuity of $\mathcal{E}_G(t)$ we have $T>0$.

We claim that $T=+\infty$. Indeed, if $T<+\infty$, then for all $t\in[0,T]$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_G(t)+\mathcal{D}_G(t)\le 0,$$

which immediately means

$$\sup_{t\in[0,T]}\mathcal{E}_G(t)+\int_0^T\mathcal{D}_G(t)\mathrm{d}t\le\mathcal{E}_G(0).$$

Then the above bound reduces to

$$\sup_{t\in[0,T]} \mathcal{P}(\mathcal{E}_G(t)) \leq \mathcal{P}(\mathcal{E}_G(0)) \leq \tfrac{1}{4}.$$

By the continuity of $\mathcal{E}_G(t)$, there is a $t^*>0$ such that for all $t\in[0,T+t^*]$

$$\mathcal{P}(\mathcal{E}_G(t)) \leq \tfrac{1}{2},$$

which contradict to the definition of T. Thus the claim holds.

Consequently, by the definition of the energy functionals $\mathcal{E}_G(t)$ and $\mathcal{D}_G(t)$, we have the following bound

$$\begin{aligned}\sup_{t\geq 0}&\Big(\|v\|^2_{H^s(\mathbb{R}^3)}+\|E\|^2_{H^s(\mathbb{R}^3)}+\|\mathrm{div}E\|^2_{H^s(\mathbb{R}^3)}+\|B\|^2_{H^{s+1}(\mathbb{R}^3)}+\|\partial_t B\|^2_{H^s(\mathbb{R}^3)}\Big)\\ &+\int_0^\infty \Big(\mu\|\nabla v\|^2_{H^s(\mathbb{R}^3)}+\sigma\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}n+\nabla^k E+(\nabla^k v)\times B\|^2_{L^2(\mathbb{R}^3)}\Big)\mathrm{d}t\leq C(\sigma)\mathcal{E}^{in},\end{aligned}$$

which is uniformly bounded in time. Therefore, we can extend the solution constructed in Theorem 1.1 to the time interval $[0,+\infty)$, and we finish the proof of Theorem 1.2. □

Appendix A. The two-fluid incompressible Navier–Stokes–Maxwell equations with solenoidal Ohm's law. In this section, we concern the two-fluid incompressible Navier–Stokes–Maxwell equations with solenoidal Ohm's law (1.2) with the initial data

$$v|_{t=0}=v^{in},\ E|_{t=0}=E^{in},\ B|_{t=0}=B^{in},$$

which satisfy the comparability conditions $\mathrm{div}v^{in}=\mathrm{div}E^{in}=\mathrm{div}B^{in}=0$.

The solenoidal system (1.2) is very similar with the system (1.1). So, based on the arguments of the well-posedness for the system (1.1), we can prove the well-posedness for the system (1.2). We first articulate the local and global well-posedness theorems of (1.2):

THEOREM A.1 (Local well-posedness). *If the initial data fulfill $v^{in},E^{in},B^{in}\in H^s(\mathbb{R}^3)$ for $s\geq 2$, then there exist two positive constant T and C, depending on s, μ, σ and the all initial data, such that system* (1.2) *admits a unique solution (v,E,B) on the interval $[0,T]$ satisfying*

$$\begin{aligned}&v\in L^\infty(0,T;H^s(\mathbb{R}^3))\cap L^2(0,T;H^{s+1}(\mathbb{R}^3)),\\ &E,B\in L^\infty(0,T;H^s(\mathbb{R}^3)).\end{aligned}$$

Moreover, the following energy bound

$$\begin{aligned}\sup_{t\in[0,T]}&\Big(\|v\|^2_{H^s}+\|E\|^2_{H^s}+\|B\|^2_{H^s}\Big)\\ &+\int_0^T\Big(\mu\|\nabla v\|^2_{H^s}+\sigma\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}\bar{p}+\nabla^k E+(\nabla^k v)\times B\|^2_{L^2}\Big)\mathrm{d}t\leq C\end{aligned}$$

holds.

THEOREM A.2 (Global well-posedness). *Assume that the initial conditions $v^{in}\in H^s(\mathbb{R}^3)$, and $E^{in},B^{in}\in H^{s+1}(\mathbb{R}^3)$ for $s\geq 2$. If there is a small positive constant ϵ_0,* *depending only upon s, the viscosity μ and the electric resistivity σ, such that the initial eneregy*

$$\mathcal{E}^{in} := \|v^{in}\|^2_{H^s(\mathbb{R}^3)} + \|E^{in}\|^2_{H^s(\mathbb{R}^3)} + \|\nabla\times E^{in}\|^2_{H^s(\mathbb{R}^3)} + \|B^{in}\|^2_{H^{s+1}(\mathbb{R}^3)} \leq \epsilon_0,$$

then system (1.2) *admits a unique global in time solution* (v,E,B) *satisfying*

$$\begin{aligned}
&v \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s+1}(\mathbb{R}^3)),\\
&E \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^3)),\\
&B \in L^\infty(\mathbb{R}^+; H^{s+1}(\mathbb{R}^3)),\ \partial_t B (= -\nabla\times E) \in L^\infty(\mathbb{R}^+; H^s(\mathbb{R}^3)).
\end{aligned}$$

Furthermore, the following energy inequality

$$\begin{aligned}
&\sup_{t\geq 0}\Big(\|v\|^2_{H^s(\mathbb{R}^3)} + \|E\|^2_{H^s(\mathbb{R}^3)} + \|B\|^2_{H^{s+1}(\mathbb{R}^3)} + \|\partial_t B\|^2_{H^s(\mathbb{R}^3)} \Big)\\
&\quad + \int_0^\infty \Big(\mu\|\nabla v\|^2_{H^s(\mathbb{R}^3)} + \sigma \sum_{k=0}^s \| -\tfrac{1}{2}\nabla^{k+1}\bar{p} + \nabla^k E + (\nabla^k v)\times B\|^2_{L^2(\mathbb{R}^3)} \Big) \mathrm{d}t \leq C\mathcal{E}^{in}
\end{aligned}$$

holds for some constant $C = C(\mu,\sigma) > 0$.

The basic energy law of the system (1.2) is

$$\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2_{L^2} + \tfrac{1}{2}\|E\|^2_{L^2} + \tfrac{1}{2}\|B\|^2_{L^2}) + \mu\|\nabla v\|^2_{L^2} + \tfrac{\sigma}{2}\| -\tfrac{1}{2}\nabla\bar{p} + E + v\times B\|^2_{L^2} = 0,$$

on which based, we can derive the higher order energy estimate of system (1.2). More precisely, if we define the energy functionals

$$\begin{aligned}
&\widehat{\mathcal{E}}_L(t) := \|v\|^2_{H^s} + \tfrac{1}{2}\|E\|^2_{H^s} + \tfrac{1}{2}\|B\|^2_{H^s},\\
&\widehat{\mathcal{D}}_L(t) := \mu\|\nabla v\|^2_{H^s} + \tfrac{\sigma}{2}\sum_{k=0}^s \| -\tfrac{1}{2}\nabla^{k+1}\bar{p} + \nabla^k E + (\nabla^k v)\times B\|^2_{L^2},
\end{aligned}$$

for $s \geq 2$, then we gain the energy inequality

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{E}}_L(t) + \widehat{\mathcal{D}}_L(t) \leq C\widehat{\mathcal{E}}_L(t)[\widehat{\mathcal{E}}_L^{\frac{1}{2}}(t) + \widehat{\mathcal{E}}_L(t) + \widehat{\mathcal{E}}_L^2(t)]$$

for some constant $C = C(s,\sigma,\mu) > 0$, which immediately implies the local well-posedness of system (1.2), hence Theorem A.1.

For the global well-posedness of the solenoidal system (1.2), we must consider the decay property of the electric field E and the divergence-free property of the magnetic field B. Specifically, we consider the equations

$$\begin{cases}
\partial_t E - \nabla\times B + \sigma E = \tfrac{\sigma}{2}\nabla\bar{p} - \sigma v\times B,\\
\partial_t B + \nabla\times E = 0,\\
\mathrm{div} E = 0,\\
(E,B)|_{t=0} = (E^{in}, B^{in}),
\end{cases}$$

and

$$\begin{cases}
\partial_{tt} B - \Delta B + \sigma\partial_t B = \sigma\nabla\times(v\times B),\\
(B,\partial_t B)|_{t=0} = (B^{in}, -\nabla\times E^{in}),
\end{cases}$$

which is implied by the condition $\mathrm{div} B=0$. Thus, by the similar arguments in Section 4, we can yield that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{E}}_G(t)+\widehat{\mathcal{D}}_G(t)\leq C(\mu,\sigma,s)\widehat{\mathcal{D}}_G(t)[\widehat{\mathcal{E}}_G^{\frac{1}{2}}(t)+\widehat{\mathcal{E}}_G(t)], \tag{A.1}$$

where the energy functionals $\widehat{\mathcal{E}}_G(t)$ and $\widehat{\mathcal{D}}_G(t)$ are defined as

$$\begin{aligned}\widehat{\mathcal{E}}_G(t):=&\|v\|_{H^s}^2+\tfrac{3}{2}\|E\|_{H^s}^3+(\tfrac{3}{2}+\delta\sigma-\delta)\|B\|_{H^s}^2\\&+(1-\delta)\|\partial_t B\|_{H^s}^2+\|\nabla B\|_{H^s}^2+\delta\|\partial_t B+B\|_{H^s}^2,\\ \widehat{\mathcal{D}}_G(t):=&\mu\|\nabla v\|_{H^s}^2+\sigma\|E\|_{H^s}^2+(\sigma-\delta)\|\partial_t B\|_{H^s}^2+\delta\|\nabla B\|_{H^s}^2\\&+\tfrac{\sigma}{2}\sum_{k=0}^{s}\|-\tfrac{1}{2}\nabla^{k+1}\bar{p}+\nabla^k E+(\nabla^k v)\times B\|_{L^2}^2\end{aligned}$$

for $\delta=\frac{1}{2}\min\{1,\sigma\}\in(0,1)$. Then the inequality can imply the conclusion of the global well-posedness, i.e., Theorem A.2. We emphasize that all the arguments of the system (1.2) are highly similar to that of the system (1.1), and we omit the details here.

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