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WELL-POSEDNESS FOR THE REGULARIZED INTERMEDIATE LONG-WAVE EQUATION*

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Abstract. In this work we prove local and global well-posedness of the Cauchy problem of the regularized intermediate long-wave (rILW) equation in periodic and nonperiodic Sobolev spaces.

Keywords. internal waves; dispersive models; regularized intermediate long-wave equation; well-posedness for PDEs; pseudodifferential operators.

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1. Introduction

The purpose of this work is to study the local and global well-posedness problem associated to the regularized intermediate long-wave (rILW) equation

$$\eta_t+\eta_x-\frac{3}{2}\alpha\eta\eta_x-\sqrt{\beta}\,\frac{\rho_2}{\rho_1}\mathcal{T}(\eta_{xt})=0.$$

The rILW equation is a nonlinear model for the wave evolution at the interface between two fluids of densities $\rho_1<\rho_2$. For stable stratification the upper layer has the lowest density. The displacement of such an interface is re-scaled to $\eta(x,t)$, where x and t are proportional to the horizontal space and time variable, respectively. Both fluids are considered inviscid, immiscible, incompressible and irrotational. The height of the unperturbed lower layer is comparable to the characteristic wavelength of the wave at the interface and it is greater than the unperturbed upper layer. The ratio between the unperturbed lower layer's height and the characteristic wavelength of the perturbed interface is $h>0$. This configuration corresponds to a shallow water regime for the upper layer and an intermediate regime for the lower layer. Subscripts x and t stand for partial derivatives, α and β are positive constants, called nonlinear parameter and dispersive parameter, respectively.

The operator $\mathcal{T}$, sometimes referred to as *Hilbert transform on the strip of height* h, is defined in the frequency domain by

$$\widehat{\mathcal{T}f}(k)=i\coth(hk)\widehat{f}(k),\qquad k\in\mathbb{R}\text{ (or }\mathbb{Z}),\ k\neq 0,$$

where $\widehat{\ }$ indicates the nonperiodic or periodic Fourier transform with respect to x, depending on the nonperiodic or periodic domain. Following [8], the definition adopted

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here for the Fourier transform of nonperiodic functions $f:\mathbb{R}\to\mathbb{C}$ is

$$\widehat{f}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx, \qquad \forall\ k\in\mathbb{R},$$

and of 2π-periodic functions is

$$\widehat{f}(k)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx, \qquad \forall\ k\in\mathbb{Z}.$$

In this physical setting, the rILW equation is asymptotically equivalent to the well known intermediate long-wave (ILW) equation

$$\eta_t+\eta_x-\frac{3}{2}\alpha\eta\eta_x+\sqrt{\beta}\,\frac{\rho_2}{\rho_1}\mathcal{T}[\eta_{xx}]=0, \tag{1.1}$$

first studied by Joseph [9] in 1977 and Kubota, Ko and Dobbs [11] in 1978. Specifically, Choi and Camassa [7], obtained equation (1.1) as a unidirectional model extracted from a bidirectional system that asymptotically approximates the Euler equations up to order $\sqrt{\beta}$, when the parameter α is considered of the same order than $\sqrt{\beta}$, that is, $\alpha=O(\sqrt{\beta})$. The rILW equation can be obtained in the same way, because the aforementioned bidirectional system leads to $\eta_t=-\eta_x+\mathcal{O}(\sqrt{\beta})$, therefore η_{xt} and $-\eta_{xx}$ can be exchanged within the order of approximation $\mathcal{O}(\sqrt{\beta})$. In fact, as shown in [3], solutions are very similar and the regularized version has the advantage of possessing bounded phase velocity, which makes it better suited for applications and numerical simulations than the ILW equation.

Considering the problem of well-posedness, some results for the ILW equation deserve attention. For Sobolev spaces H^s defined in Section 2, Abdelouhab, Bona, Felland and Saut [2], 1989, stated a well-posedness theorem for the ILW equation if $s>\frac{3}{2}$, based on the proof of well-posedness for the Benjamin–Ono (BO) equation. A detailed proof of well-posedness in weighted Sobolev spaces with $s>\frac{3}{2}$ can be found in [5] and [1]. In [6], Burq and Planchon suggested that their well-posedness result for the BO equation in H^s with $s>\frac{1}{4}$ can be adapted for the ILW equation taking into consideration the specificities of the operator $\mathcal{T}$.

The rILW equation, as its name indicates, involves a better behaved operator that allows a more straightforward approach in proving well-posedness, as done by Albert and Bona in [3] for a general class of regularized equations that includes global well-posedness for the rILW equation if $s>\frac{3}{2}$. The results presented here, though restricted to the rILW, improve their work to $s>\frac{1}{2}$. Regarding the local existence, the strong solution is obtained directly in H^s, without resorting to intermediate function spaces and for $s>\frac{1}{2}$. From this, the global result is also proved for $s>\frac{1}{2}$ with the help of a Brezis–Gallouet type inequality. Our proof adapted the one for the regularized Benjamin-Ono (rBO) equation presented in Kalisch and Bona [10] if $s>\frac{3}{2}$ and improved by Angulo, Scialom and Banquet [4] to $s>\frac{1}{2}$. Both cases, periodic and nonperiodic, are covered here.

In Section 2 the local well-posedness for the rILW equation is proved and Section 3 is devoted to the proof of global well-posedness. Some auxiliary results are compiled in the Appendix.

2. Local well-posedness for the rILW equation

In this Section we prove local well-posedness for the Cauchy problem associated to the rILW. To begin with, let us write the rILW equation as

$$\eta_t=G\eta.$$

Decompose the evolution operator G by means of the linear operator M as

$$G\eta = M\left(\eta - \frac{3}{4}\alpha\eta^2\right).$$

In the frequency domain this corresponds to

$$\widehat{\eta}_t(k) = m(k)\left(\widehat{\eta} - \frac{3}{4}\alpha\widehat{\eta^2}\right)(k),$$

with the linear operator M described as

$$m(k) = \frac{-ik}{A(k)}, \qquad A(k) = 1 + \sqrt{\beta}\frac{\rho_2}{\rho_1}k\coth(hk),$$

taking the limit value when $k=0$.

The next lemmas establish important properties of the operator G which are valid in the Sobolev space

$$H^s_{\mathbb{R}} = \left\{ f \in \mathcal{S}'(\mathbb{R});\ \widehat{f} \text{ is measurable and } \|f\|_s^2 = \int_{-\infty}^{\infty} (1+k^2)^s \left|\widehat{f}(k)\right|^2 dk < \infty \right\}$$

or

$$H^s_{\text{per}} = \left\{ f \in \mathcal{P}';\ \|f\|_s^2 = 2\pi \sum_{k=-\infty}^{\infty} (1+k^2)^s \left|\widehat{f}(k)\right|^2 < \infty \right\},$$

depending on the domain. As in [8], $\mathcal{S}'$ and $\mathcal{P}'$ denote the set of all temperate distributions and of all periodic distributions, respectively. For simplicity, both Sobolev spaces will be indicated by H^s, the results proved hereafter are valid for both nonperiodic and periodic cases and only when necessary different expressions are written.

We also consider

$$H^\infty = \bigcap_{s\geq 0} H^s.$$

LEMMA 2.1. *The operator M is bounded, that is, $M \in \mathcal{B}(H^s, H^s)$ and satisfies*

$$\|Mf\|_s \leq \frac{\rho_1}{\sqrt{\beta}\rho_2}\|f\|_s, \quad \forall\ f \in H^s, \quad s \in \mathbb{R}.$$

Proof. From Lemma 4.1 (see Appendix 4),

$$h|k| \leq hk\coth(hk),$$

thus

$$1 + \sqrt{\beta}\frac{\rho_2}{\rho_1}|k| \leq A(k).$$

Therefore,

$$|m(k)| = \frac{|k|}{A(k)} \leq \frac{|k|}{1+\sqrt{\beta}\frac{\rho_2}{\rho_1}|k|} = \frac{\rho_1}{\sqrt{\beta}\rho_2}\frac{|k|}{\left(\frac{\rho_1}{\sqrt{\beta}\rho_2} + |k|\right)} < \frac{\rho_1}{\sqrt{\beta}\rho_2},$$

which leads to

$$\|Mf\|_s^2 = \int_{-\infty}^{\infty} (1+k^2)^s \left| m(k)\widehat{f}(k)\right|^2 dk \le \left(\frac{\rho_1}{\sqrt{\beta}\rho_2}\right)^2 \|f\|_s^2.$$

In the periodic case, the integral must be substituted by a sum with a factor 2π. In both cases, this means that $M \in \mathcal{B}(H^s, H^s)$ and satisfies

$$\|Mf\|_s \le \frac{\rho_1}{\sqrt{\beta}\rho_2}\|f\|_s, \quad \forall\, f \in H^s.$$

□

LEMMA 2.2. *Let $s > \frac{1}{2}$. If $\eta \in H^s$ then $G\eta \in H^s$.*

Proof. Since $s > \frac{1}{2}$, H^s is a Banach algebra, $\eta - \frac{3}{4}\alpha\eta^2 \in H^s$ and Lemma 2.1 leads to

$$G\eta = M\left(\eta - \frac{3}{4}\alpha\eta^2\right) \in H^s.$$

□

Now we can prove the local well-posedness of the Cauchy problem for the rILW equation

$$\begin{cases} \eta \in C([-T,T], H^s) \\ \eta_t + \eta_x - \dfrac{3}{2}\alpha\eta\eta_x - \sqrt{\beta}\dfrac{\rho_2}{\rho_1}\mathcal{T}(\eta_{xt}) = 0 \text{ in } H^s \\ \eta(0) = \phi \in H^s, \end{cases} \tag{2.1}$$

when $s > \frac{1}{2}$.

Based on [8], we adopt the following definition of local well-posedness:

DEFINITION 2.1. *Let X, Y be two Banach spaces and $F: Y \to X$ a continuous function. The Cauchy problem*

$$\begin{cases} \eta_t(t) = F(\eta(t)) \text{ in } X \\ \eta(0) = \phi \in Y, \end{cases} \tag{2.2}$$

is locally well-posed in Y if

(a) *there exist $T > 0$ and a function $\eta \in C([-T,T];Y)$ such that $\eta(0) = \phi$ and the differential equation is satisfied in the following sense*

$$\lim_{h\to 0}\left\| \frac{\eta(t+h)-\eta(t)}{h} - F(\eta(t))\right\|_X = 0, \tag{2.3}$$

where the derivatives at $t = -T$ and $t = T$ are computed from the right and the left, respectively,

(b) *the problem (2.2) has at most one solution in $C([-T,T];Y)$,*

(c) *the map $\phi \longmapsto \eta$ is continuous, that is, given $(\phi_n)_{n\in\mathbb{N}} \subset Y$, such that $\phi_n \xrightarrow{Y} \phi^*$ and $\eta^* \in C([-T^*,T^*];Y)$ the solution for the initial condition ϕ^*. Then, for n sufficiently large, solutions η_n corresponding to ϕ_n can be defined on the interval $[-T^*,T^*]$ and the following limit holds,*

$$\lim_{n\to\infty} \sup_{[-T^*,T^*]} \|\eta_n(t) - \eta^*(t)\|_Y = 0.$$

Remark 2.1. Note that the definition adopted here includes the persistence property, that is, $\eta(t)\in Y$, $\forall\, t\in[-T,T]$.

Theorem 2.1. *There exist $T>0$ and a function $\eta\in C([-T,T];H^s)$ that satisfy the differential equation in the sense of the limit (2.3) and $\eta(0)=\phi$.*

Proof. We start by proving the existence of the solution for the integral problem,

$$\eta(t)=\phi-\int_0^t M\left(\frac{3}{4}\alpha\eta^2(\tau)-\eta(\tau)\right)d\tau. \tag{2.4}$$

In order to apply Banach's fixed point theorem, let us consider

$$\Lambda=\Lambda(T,R,\phi)=\{v\in C([-T,T];H^s);\ d(v,\Phi)\le R\},$$

which is a complete metric space, as proved in Lemma 4.3 (Appendix 4), for any $T>0$ and $R>0$. Here $\Phi:[-T,T]\to H^s$ is the constant path $\Phi(t)=\phi$ and for $u,v\in\Lambda$,

$$d(u,v)=\sup_{t\in[-T,T]}\{\|u(t)-v(t)\|_s\}.$$

Afterwards, consider the operator

$$\begin{aligned}J:\Lambda&\longrightarrow\Lambda\\ v&\longmapsto Jv:[-T,T]\longrightarrow H^s,\end{aligned}$$

defined by

$$Jv(t)=\phi-\int_0^t M\left(\frac{3}{4}\alpha v^2(\tau)-v(\tau)\right)d\tau,$$

that satisfies the following conditions:

- Condition 1: There exists $T_1>0$ such that for any $T<T_1$ the operator $J\colon\Lambda\to\Lambda$ is well defined, that is, $J(\Lambda)\subset\Lambda$.
- Condition 2: There exists $T_2>0$ such that for any $T<T_2$ the operator $J\colon\Lambda\to\Lambda$ is a contraction, that is, for $u,\ v\in\Lambda$,

$$\sup_{t\in[-T,T]}\|Ju(t)-Jv(t)\|_s\le q_T\sup_{t\in[-T,T]}\|u(t)-v(t)\|_s,$$

where $q_T\in[0,1)$.

In fact, from Lemma 2.1

$$\begin{aligned}\|Jv(t)-\phi\|_s&=\left\|\int_0^t M\left(\frac{3}{4}\alpha v^2(\tau)-v(\tau)\right)d\tau\right\|_s\\ &\le\frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t\left\|\frac{3}{4}\alpha v^2(\tau)-v(\tau)\right\|_s d\tau\right|\\ &\le\frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t\|v(\tau)\|_s\left(\frac{3}{4}\alpha C_s\|v(\tau)\|_s+1\right)d\tau\right|,\end{aligned}$$

where the constant $C_s>0$ comes from the inequality

$$\|fg\|_s\le C_s\|f\|_s\|g\|_s,\ \ \forall\ f,g\in H^s,$$

that turns H^s into a Banach algebra for $s > \frac{1}{2}$.

Furthermore, since $v \in \Lambda$,

$$\begin{aligned}\|Jv(t) - \phi\|_s &\leq \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t (R + \|\phi\|_s)\left(\frac{3}{4}\alpha C_s (R + \|\phi\|_s) + 1\right) d\tau\right| \\ &\leq \frac{\rho_1}{\sqrt{\beta}\rho_2} T (R + \|\phi\|_s)\left(\frac{3}{4}\alpha C_s (R + \|\phi\|_s) + 1\right).\end{aligned}$$

Setting

$$T_1 = \sqrt{\beta}\frac{\rho_2}{\rho_1}\left(\frac{R}{R + \|\phi\|_s}\right)\left(\frac{1}{\frac{3}{4}\alpha C_s (R + \|\phi\|_s) + 1}\right),$$

it follows that $\forall\, T < T_1$,

$$d(Jv, \Phi) = \sup_{t \in [-T,T]} \{\|Jv(t) - \phi\|_s\} \leq R,$$

and $Jv(t) \in H^s$, $\forall\, t \in [-T,T]$. Finally, $\forall\, t, \tilde{t} \in [-T,T]$,

$$\begin{aligned}\|Jv(t) - Jv(\tilde{t})\|_s &= \left\|\int_{\tilde{t}}^t M\left(\frac{3}{4}\alpha v^2(\tau) - v(\tau)\right) d\tau\right\|_s \\ &\leq \frac{\rho_1}{\sqrt{\beta}\rho_2}|t - \tilde{t}|(R + \|\phi\|_s)\left(\frac{3}{4}\alpha C_s (R + \|\phi\|_s) + 1\right),\end{aligned}$$

converges to zero when $|t - \tilde{t}| \to 0$. Therefore $Jv \in C([-T,T]; H^s)$ and

$$Jv \in \Lambda, \quad \forall\, v \in \Lambda,$$

as Condition 1 requires.

Likewise for Condition 2, if u, $v \in \Lambda$ then

$$\begin{aligned}\|Ju(t) - Jv(t)\|_s &\leq \left|\int_0^t \left\|M\left(\frac{3}{4}\alpha\left(u^2(\tau) - v^2(\tau)\right) - (u(\tau) - v(\tau))\right)\right\|_s d\tau\right| \\ &\leq \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t \left\|\frac{3}{4}\alpha\left(u^2(\tau) - v^2(\tau)\right) - (u(\tau) - v(\tau))\right\|_s d\tau\right| \\ &\leq \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t \|u(\tau) - v(\tau)\|_s \left(\frac{3}{4}\alpha C_s \|u(\tau) + v(\tau)\|_s + 1\right) d\tau\right|,\end{aligned}$$

for all $t \in [-T,T]$. Furthermore, $\|u(\tau)\|_s + \|v(\tau)\|_s \leq 2(R + \|\phi\|_s)$ leads to

$$\|Ju(t) - Jv(t)\|_s \leq d(u,v)\frac{\rho_1}{\sqrt{\beta}\rho_2}\left(\frac{3}{2}\alpha C_s (R + \|\phi\|_s) + 1\right) T,$$

and taking

$$T_2 = \sqrt{\beta}\frac{\rho_2}{\rho_1}\frac{1}{\left(\frac{3}{2}\alpha C_s (R + \|\phi\|_s) + 1\right)},$$

we obtain

$$\sup_{t \in [-T,T]} \|Ju(t) - Jv(t)\|_s \leq q_T \sup_{t \in [-T,T]} \|u(t) - v(t)\|_s, \quad \forall\, T < T_2,$$

where $q_T = T/T_2$. It follows from Banach's fixed point theorem that for $0 < T < \min\{T_1, T_2\}$, there exists a unique function $\eta \in \Lambda$, such that $J\eta = \eta$, that is, there exists $\eta \in C([-T,T]; H^s)$, a solution for the integral problem (2.4) and consequently, a strong solution of the original problem (see [14], p. 125).

To complete the proof of (a), η must satisfy condition (2.3). In fact,

$$\lim_{h\to 0}\left\|\frac{\eta(t+h)-\eta(t)}{h} - G(\eta(t))\right\|_s$$
$$= \lim_{h\to 0}\left\|\frac{1}{h}\int_t^{t+h} M\left(\frac{3}{4}\alpha\eta^2(\tau) - \eta(\tau)\right)d\tau - M\left(\frac{3}{4}\alpha\eta^2(t)-\eta(t)\right)\right\|_s = 0,$$

since the integrand does not depend on t. □

Uniqueness in Λ does not assure uniqueness in the whole space $C([-T,T], H^s)$. That is why the following theorem, based on Gronwall's inequality, is established in order to prove (b).

THEOREM 2.2. *Problem (2.1) has at most one solution in H^s.*

Proof. Consider two solutions $\eta_i \in C([-T_i, T_i], H^s)$, $i = 1,2$, for the Cauchy problems (2.1) with data ϕ_i, $i = 1,2$, respectively. Then,

$$\eta_i(t) = \phi_i - \int_0^t M\left(\frac{3}{4}\alpha\eta_i^2(\tau) - \eta_i(\tau)\right)d\tau.$$

Define $T = \min\{T_1,\ T_2\}$.

Similar estimates to those used in Theorem 2.1 provide the following inequality for the norm of the difference between those solutions,

$$\|\eta_1(t)-\eta_2(t)\|_s$$
$$\leq \|\phi_1-\phi_2\|_s + \left|\int_0^t \left\|M\left(\frac{3}{4}\alpha\left(\eta_1^2(\tau)-\eta_2^2(\tau)\right) - (\eta_1(\tau)-\eta_2(\tau))\right)\right\|_s d\tau\right|$$
$$\leq \|\phi_1-\phi_2\|_s + \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t \|\eta_1(\tau)-\eta_2(\tau)\|_s\left(\frac{3}{4}\alpha C_s\|\eta_1(\tau)+\eta_2(\tau)\|_s + 1\right)d\tau\right|$$
$$\leq \|\phi_1-\phi_2\|_s + \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t \|\eta_1(\tau)-\eta_2(\tau)\|_s\left(\frac{3}{4}\alpha C_s\left(2\tilde{R} + \|\phi_1\|_s + \|\phi_2\|_s\right)+1\right)d\tau\right|$$
$$\leq \|\phi_1-\phi_2\|_s + \tilde{\beta}\left|\int_0^t \|\eta_1(\tau)-\eta_2(\tau)\|_s\, d\tau\right|,$$

where

$$\tilde{R} = \max\left\{\sup_{t\in[-T,T]}\|\eta_1(t)-\phi_1\|_s,\ \sup_{t\in[-T,T]}\|\eta_2(t)-\phi_2\|_s\right\}$$

and

$$\tilde{\beta} = \frac{\rho_1}{\sqrt{\beta}\rho_2}\left(\frac{3}{4}\alpha C_s\left(2\tilde{R}+\|\phi_1\|_s+\|\phi_2\|_s\right)+1\right).$$

Now we apply Gronwall's inequality to obtain

$$\|\eta_1(t)-\eta_2(t)\|_s \leq \|\phi_1-\phi_2\|_s\ e^{\tilde{\beta}|t|}, \qquad \forall\ t\in[-T,T].$$

In particular, if $\phi_1=\phi_2$ then $\eta_1=\eta_2$. Therefore, the solution is unique in $C([-T,T],H^s)$. ∎

The next step is to prove item (c) from Definition 2.1.

Theorem 2.3. *The map $\phi\longmapsto\eta$ is continuous, that is, given $(\phi_n)_{n\in\mathbb{N}}\subset H^s$, such that $\phi_n\xrightarrow{H^s}\phi^*$ and $\eta^*\in C([-T^*,T^*];H^s)$ the solution for the initial condition ϕ^*. Then, for n sufficiently large, solutions η_n corresponding to ϕ_n can be defined on the interval $[-T^*,T^*]$ and the following limit holds,*

$$\lim_{n\to\infty}\sup_{[-T^*,T^*]}\|\eta_n(t)-\eta^*(t)\|_s=0.$$

Proof. Let $\bar{T}_n=\min\{T_n,T^*\}$, where $T_n>0$ gives the maximal symmetric interval $(-T_n,T_n)$ in which solution η_n is defined, then for $t\in(-\bar{T}_n,\bar{T}_n)$ we have

$$\begin{aligned}
&\|\eta_n(t)-\eta^*(t)\|_s\\
&\le\|\phi_n-\phi^*\|_s+\frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t\|\eta_n(\tau)-\eta^*(\tau)\|_s\left(\frac{3}{4}\alpha C_s\|\eta_n(\tau)+\eta^*(\tau)\|_s+1\right)d\tau\right|\\
&\le\|\phi_n-\phi^*\|_s+\frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t\|\eta_n(\tau)-\eta^*(\tau)\|_s\left(\frac{3}{4}\alpha C_s\|\eta_n(\tau)-\eta^*(\tau)\|_s+\frac{3}{2}\alpha C_sR^*+1\right)d\tau\right|\\
&\le\|\phi_n-\phi^*\|_s+\left|\int_0^t\left(C_0\|\eta_n(\tau)-\eta^*(\tau)\|_s+C_1\|\eta_n(\tau)-\eta^*(\tau)\|_s^2\right)d\tau\right|,
\end{aligned}$$

where $R^*=\sup_{t\in[-T^*,T^*]}\|\eta^*(t)\|_s$, $C_0=\frac{\rho_1}{\sqrt{\beta}\rho_2}\left(\frac{3}{2}\alpha C_sR^*+1\right)$ and $C_1=\frac{\rho_1}{\sqrt{\beta}\rho_2}\frac{3}{4}\alpha C_s$.

Let us define

$$\Psi(t)=\|\phi_n-\phi^*\|_s+\left|\int_0^t\left(C_0\|\eta_n(\tau)-\eta^*(\tau)\|_s+C_1\|\eta_n(\tau)-\eta^*(\tau)\|_s^2\right)d\tau\right|,$$

it follows that

$$|\Psi'(t)|\le C_0\Psi(t)+C_1\Psi^2(t)\quad\text{and}\quad\Psi(0)=\|\phi_n-\phi^*\|_s.$$

Integration from 0 to t leads to

$$\log\left(\frac{C_0\Psi(t)}{C_0+C_1\Psi(t)}\right)-\log\left(\frac{C_0\Psi(0)}{C_0+C_1\Psi(0)}\right)\le C_0|t|.$$

Taking exponential and performing a straightforward calculation one gets that for $t\in(-\bar{T}_n,\bar{T}_n)\cap(-T_n^*,T_n^*)$

$$\|\eta_n(t)-\eta^*(t)\|_s\le\Psi(t)\le\frac{C_0\|\phi_n-\phi^*\|_s\,e^{C_0|t|}}{C_0+C_1\|\phi_n-\phi^*\|_s\,\left(1-e^{C_0|t|}\right)},\tag{2.5}$$

where $T_n^*=C_0^{-1}\log\left(\frac{C_0+C_1\|\phi_n-\phi^*\|_s}{C_1\|\phi_n-\phi^*\|_s}\right)$.

Since $\|\phi_n-\phi^*\|_s$ goes to zero as n goes to infinity, by choosing n sufficiently large we get that $T_n^*>T^*$ so inequality (2.5) is valid for $t\in(-\bar{T}_n,\bar{T}_n)$. Moreover, if $T_n\le T^*$

$$||\eta_n(t)||_s\le||\eta_n(t)-\eta^*(t)||_s+||\eta^*(t)||_s,\quad\forall t\in(-T_n,T_n)$$

it immediately follows that $\|\eta_n(t)\|_s$ is bounded in $(-T_n,T_n)$ which contradicts the maximality of T_n. Therefore, $T_n > T^*$ for sufficiently large n and estimate (2.5) remains true for $t \in [-T^*,T^*]$, and it yields

$$\lim_{n\to\infty} \sup_{[-T^*,T^*]} \|\eta_n(t) - \eta^*(t)\|_s = 0.$$

□

In this way, local well-posedness is achieved:

THEOREM 2.4. *Let $s > \frac{1}{2}$ and $\phi \in H^s$, then there exists $T = T(s,\|\phi\|_s) > 0$ such that the nonlinear Cauchy problem*

$$\begin{cases} \eta \in C([-T,T],H^s) \\ \eta_t + \eta_x - \dfrac{3}{2}\alpha\eta\eta_x - \sqrt{\beta}\dfrac{\rho_2}{\rho_1}\mathcal{T}(\eta_{xt}) = 0 \text{ in } H^s \\ \eta(0) = \phi \in H^s, \end{cases}$$

is locally well-posed in the sense of Definition 2.1.

3. Global well-posedness

A Cauchy problem is globally well-posed if the Definition 2.1 is satisfied for any $T > 0$. To ensure global existence, it is sufficient to combine the Extension Principle with some global a priori estimates for the local solutions in the H^s-norm, as described in [8]. In order to achieve this, let us begin proving the following lemma:

LEMMA 3.1. *If η satisfies equation (2.1) in the sense of distributions and $\eta \in C([-T,T],H^s)$ for some $s > \frac{1}{2}$, then for all $t \in [-T,T]$ the following norm is preserved:*

$$|||\eta(t)|||_{\frac{1}{2}} = |||\phi|||_{\frac{1}{2}},$$

where $|||\cdot|||_{\frac{1}{2}}$ stands for the norm

$$|||f|||_{\frac{1}{2}} = \left(\int_{-\infty}^{\infty} A(k)\left|\hat{f}(k)\right|^2 dk\right)^{\frac{1}{2}},$$

in the nonperiodic case and for 2π-periodic functions

$$|||f|||_{\frac{1}{2}} = \left(2\pi \sum_{k=-\infty}^{\infty} A(k)\left|\hat{f}(k)\right|^2\right)^{\frac{1}{2}}.$$

Proof. The proof follows the one presented in [10] but using the proper norm which is equivalent to the usual $\|\cdot\|_{\frac{1}{2}}$ norm in $H^{\frac{1}{2}}$, see Lemma 4.2.

Since $\eta - \frac{3}{4}\alpha\eta^2 \in C([-T,T],H^s)$, by Lemma 2.2, $\eta_t = G(\eta - \frac{3}{4}\alpha\eta^2) \in H^s$. Therefore $\eta \in C^1([-T,T],H^s)$. Consider a sequence $(\eta_n)_{n=1}^{\infty} \subset C^1([-T,T],H^{\infty})$ converging to a solution $\eta \in C^1([-T,T],H^s)$. Define

$$F : C^1([-T,T],H^s) \to C([-T,T],H^{s-1}),$$

such that $F(v) = v_t + v_x - \frac{3}{2}\alpha v v_x - \sqrt{\beta}\frac{\rho_2}{\rho_1}\mathcal{T}(v_{xt})$. Since F is continuous and $F(\eta) = 0$,

$$\lim_{n\to\infty} F(\eta_n) = 0,$$

in $C([-T,T],H^{s-1})$. Using the duality bracket $\langle f,g\rangle$ for $f\in H^{\frac{1}{2}}$ and $g\in H^{-\frac{1}{2}}$, we obtain

$$|\langle \eta_n, F(\eta_n)\rangle| \leq \|\eta_n\|_{\frac{1}{2}}\|F(\eta_n)\|_{-\frac{1}{2}} \leq \|\eta_n\|_s\|F(\eta_n)\|_{s-1}\to 0,$$

as $n\to\infty$ in $C([-T,T],\mathbb{R})$.

Since all η_n are smooth it is straightforward to compute

$$\begin{aligned} 2\Re\langle \eta_n, F(\eta_n)\rangle &= 2\Re\left\langle \eta_n, \partial_t\eta_n - \sqrt{\beta}\frac{\rho_2}{\rho_1}\mathcal{T}(\partial_{xt}\eta_n)\right\rangle \\ &= \int_{-\infty}^{\infty}\left(1+\sqrt{\beta}\frac{\rho_2}{\rho_1}k\coth(hk)\right)2\Re\left(\widehat{\eta_n}(k)\partial_t\overline{\widehat{\eta_n}(k)}\right)dk \\ &= \int_{-\infty}^{\infty} A(k)\partial_t\left|\widehat{\eta_n}(k)\right|^2 dk = \frac{d}{dt}|||\eta_n|||_{\frac{1}{2}}^2. \end{aligned}$$

Integrating from 0 to t, we have

$$2\Re\left|\int_0^t \langle \eta_n, F(\eta_n)\rangle d\tau\right| = |||\eta_n(t)|||_{\frac{1}{2}}^2 - |||\eta_n(0)|||_{\frac{1}{2}}^2.$$

for each $t\in[-T,T]$, Taking the limit as $n\to\infty$ gives,

$$|||\eta(t)|||_{\frac{1}{2}} - |||\phi|||_{\frac{1}{2}} = 0.$$

□

Theorem 3.1. *Let $s>\frac{1}{2}$ and $\phi\in H^s$, then the nonlinear Cauchy problem*

$$\begin{cases} \eta\in C(\mathbb{R},H^s) \\ \eta_t+\eta_x-\frac{3}{2}\alpha\eta\eta_x-\sqrt{\beta}\frac{\rho_2}{\rho_1}\mathcal{T}(\eta_{xt})=0 \text{ in } H^s \\ \eta(0)=\phi\in H^s, \end{cases}$$

is globally well-posed.

Proof. From the local solution we have

$$\|\eta(t)\|_s \leq \|\phi\|_s + \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t \left\|\eta-\frac{3}{4}\alpha\eta^2\right\|_s d\tau\right|.$$

Since $\|\eta^2\|_s \leq 2C_s\|\eta\|_s\|\eta\|_\infty$ ([12], page 50),

$$\|\eta(t)\|_s \leq \|\phi\|_s + \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t \|\eta\|_s + \frac{3}{2}\alpha C_s\|\eta\|_\infty\|\eta\|_s d\tau\right|$$

and using the Brezis–Gallouet type inequality $\|\eta\|_\infty \leq C\left(1+\sqrt{\log(1+\|\eta\|_s)}\right)\|\eta\|_{\frac{1}{2}}$, (see lemma 3.2 in [4]),

$$\|\eta(t)\|_s \leq \|\phi\|_s + \frac{\rho_1}{\sqrt{\beta}\rho_2}\left|\int_0^t \|\eta\|_s + \frac{3}{2}\alpha C_s C\left(1+\sqrt{\log(1+\|\eta\|_s)}\right)\|\eta\|_{\frac{1}{2}}\|\eta\|_s d\tau\right|.$$

By Lemma 3.1 and Lemma 4.2 there exists a constant $\mathcal{C}>0$ such that $\|\eta(t)\|_{\frac{1}{2}} \leq \mathcal{C}\|\phi\|_{\frac{1}{2}}$, therefore,

$$\|\eta(t)\|_s \leq \|\phi\|_s + C_0\left|\int_0^t \left(2+\sqrt{\log(1+\|\eta\|_s)}\right)\|\eta\|_s d\tau\right|,$$

where $C_0 = \frac{\rho_1}{\sqrt{\beta}\rho_2} \cdot \max\left\{1, \frac{3}{2}\alpha C_s C \mathcal{C} \|\phi\|_{\frac{1}{2}}\right\}$.

Following [4], for $t \in [-T,T]$, let us define

$$\Phi(t) = \|\phi\|_s + C_0 \left| \int_0^t \left(2 + \sqrt{\log(1 + \|\eta(\tau)\|_s)}\right) \|\eta(\tau)\|_s \, d\tau \right|.$$

Since $\|\eta(t)\|_s \leq \Phi(t)$,

$$|\Phi'(t)| \leq C_0 \left(2 + \sqrt{\log(1 + \Phi(t))}\right) \Phi(t) \leq 3C_0 (1 + \log(1 + \Phi(t)))(\Phi(t) + 1),$$

which implies

$$\left| \frac{d}{dt} \log(1 + \log(\Phi(t) + 1)) \right| \leq 3C_0.$$

Integration from zero to t leads to

$$\log(1 + \log(\Phi(t) + 1)) - \log(1 + \log(\|\phi\|_s + 1)) \leq 3C_0 |t|.$$

Taking exponential twice

$$\|\eta(t)\|_s \leq \Phi(t) \leq e^{C_1 e^{3C_0 |t|}},$$

with $C_1 = 1 + \log(\|\phi\|_s + 1)$, so neither C_0 nor C_1 depends on $t \in [-T,T]$. This proves that $\|\eta(t)\|_s$ is bounded on every interval of the form $[-T,T]$ so the solution can be extended to $\mathbb{R}$. ∎

4. Appendix: Auxiliary results

Here we prove some auxiliary results that support the main theorems considered in this work.

LEMMA 4.1. *The inequality*

$$|x| \leq x \coth(x) \leq |x| + 1,$$

is valid for all $x \in \mathbb{R}$, considering the limit value when $x = 0$.

Proof. Using the facts that

$$\coth|x| = 1 + \frac{2}{e^{2|x|} - 1}$$

and

$$0 < \frac{2|x|}{e^{2|x|} - 1} \leq 1,$$

for $x \neq 0$, we obtain

$$|x| < |x| \coth|x| = |x| + \frac{2|x|}{e^{2|x|} - 1} \leq 1 + |x|.$$

Since the function $x \coth(x)$ is even,

$$|x| < x \coth x \leq |x| + 1, \ \forall \ x \neq 0.$$

Finally, if $x=0$, we obtain the trivial inequality $0\leq 1\leq 1$ and the lemma is proved. □

LEMMA 4.2. *The norms* $\|\cdot\|_{\frac{1}{2}}$ *and* $|||\cdot|||_{\frac{1}{2}}$ *are equivalent.*

Proof. As proved in [13], Lemma 3.2.1, there exist $C^*=C^*(\rho_1,\rho_2,\beta)>0$ and $C^{**}=C^{**}(h,\rho_1,\rho_2,\beta)>0$ such that, $\forall\ k\in\mathbb{R}$,

$$C^*A(k)\leq\sqrt{1+k^2}\leq C^{**}A(k),$$

where $A(k)=1+\sqrt{\beta}\frac{\rho_2}{\rho_1}k\coth(hk)$. Then

$$C^*|||\cdot|||_{\frac{1}{2}}\leq\|\cdot\|_{\frac{1}{2}}\leq C^{**}|||\cdot|||_{\frac{1}{2}}.$$

□

LEMMA 4.3. (Λ,d) *is a complete metric space.*

Proof. Note that $\Lambda\subset C([-T,T];H^s)$ is the closed ball centered at Φ with radius $R>0$. Thus, it is enough to prove that $(C([-T,T];H^s),d)$ is a complete metric space. That d is a metric is straightforward, so we focus on completeness.

Given $\epsilon>0$, consider a Cauchy sequence $(f_n)_{n\in\mathbb{N}}\subset(C([-T,T];H^s),d)$, for which there exists $n_0=n_0(\epsilon)\in\mathbb{N}$ such that for all $m,n\geq n_0$,

$$d(f_n,f_m)=\sup_{t\in[-T,T]}\{\|f_n(t)-f_m(t)\|_s\}<\epsilon.$$

That is, for all $t\in[-T,T]$,

$$\|f_n(t)-f_m(t)\|_s<\epsilon,\qquad\forall\ m,n\geq n_0. \tag{4.1}$$

For each fixed $t\in[-T,T]$, $(f_n(t))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Hilbert space H^s. So there exists a limit $f_t\in H^s$:

$$f_n(t)\overset{n\to\infty}{\longrightarrow}f_t.$$

Define $f:[-T,T]\to H^s$ by $f(t)=f_t$, so that the sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f. To guarantee that $f\in C([-T,T];H^s)$ we take the limit when $m\to\infty$ in the inequality (4.1) and after taking the supremum,

$$\sup_{t\in[-T,T]}\{\|f_n(t)-f(t)\|_s\}\leq\epsilon,\qquad\forall\ n\geq n_0,$$

This means that $f_n\overset{n\to\infty}{\longrightarrow}f$ uniformly in $[-T,T]$, so f is continuous and (Λ,d) is complete.

□

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