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# FAR-FIELD REGULARITY FOR THE SUPERCRITICAL QUASI-GEOSTROPHIC EQUATION*

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**Abstract.** We address the far field regularity for solutions of the surface quasi-geostrophic equation

$$\theta_t + u\cdot\nabla\theta + \Lambda^{2\alpha}\theta = 0$$
$$u = \mathcal{R}^\perp\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$$

in the supercritical range $0<\alpha<1/2$ with $\alpha$ sufficiently close to $1/2$. We prove that if the datum is sufficiently regular, then the set of space-time singularities is compact in $\mathbb{R}^2\times\mathbb{R}$. The proof depends on a new spatial decay result on solutions in the supercritical range.

**Keywords.** quasi-geostrophic equation; eventual regularity; supercritical; weighted decay.

**AMS subject classifications.** 35Q35;76D03; 35R11.

## 1. Introduction

In this paper, we address regularity and weighted decay for solutions of the supercritical 2D Surface Quasi-Geostrophic (SQG) equation, given by

$$\theta_t + u\cdot\nabla\theta + \Lambda^{2\alpha}\theta = 0 \tag{1.1}$$
$$u = \mathcal{R}^\perp\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \tag{1.2}$$

with the initial condition

$$\theta(0) = \theta_0, \tag{1.3}$$

where $\mathcal{R}_i$ is the $i$-th Riesz transform and $\alpha\in(0,1]$, and $\Lambda=\sqrt{-\triangle}$ is the square root of the negative Laplacian. Note that the nonlocal operator $\Lambda^\beta$ is defined by

$$(\Lambda^\beta f)\hat{}(\xi) = |\xi|^\beta \hat{f}(\xi),$$

where $\hat{f}(\xi) = (2\pi)^{-1}\int f(x)e^{-i\xi\cdot x}\,dx$ is the Fourier transform of $f$. The scalar function $\theta$ in system (1.1)–(1.3) represents the potential temperature, and $u$ stands for the velocity. In particular, when $\alpha=1/2$, the SQG equation describes the temperature distribution on the 2D boundary of a rapidly rotating fluid with small Rossby and Ekman numbers. Besides its physical significance, the SQG equation also serves as a model for the vorticity evolution of the 3D Navier–Stokes equations.

The equations (1.1)–(1.2) dissipate the $L^p$ norm for every $p\in[2,\infty]$ as shown by Resnick in [31], who, based on this fact, established the existence of weak solutions for all $\alpha>0$ (cf. also [8]). Since $L^\infty$ is the critical space for $\alpha=1/2$, the equations (1.1)–(1.2) are called subcritical for $1/2<\alpha<1$, critical for $\alpha=1/2$, and supercritical for $0<\alpha<1/2$. Recently, this equation has received a considerable interest, especially in the critical and subctritical regimes. In particular, well-posedness for $\alpha=1/2$ was shown in [3,28], with

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different alternative proofs presented in [6, 7, 27] (cf. also [4, 5, 8, 9, 11, 25, 26, 40, 41] for other results on the regularity of solutions). However, the existence of smooth solutions for the supercritical SQG equation, even in the case of compactly supported smooth data, is still open.

There are several interesting regularity results available when $0<\alpha<1/2$. As pointed out above, with minimal assumptions on the initial data ($\theta_0\in L^p$) there exists a global weak solution [31]. The uniqueness of weak solutions is not known, however the weak-strong uniqueness was proven in [8]. In [9] it was proven that the solutions are smooth provided they are Hölder with the exponent larger than $1-2\alpha$ (cf. also [17–19]) Also, for a logarithmically supercritical SQG equation, the global regular solution exists for any sufficiently smooth data [14, 15].

A remarkable result on regularity was obtained by Silvestre in [37], who proved that for $\alpha<1/2$ sufficiently close to $1/2$, the solutions of the SQG equation become regular (i.e., Hölder continuous with an appropriate exponent) for $t$ sufficiently large. This is so called the "eventual regularity". There is a previous analogous theorem for the 3D Navier–Stokes equation, but in the 3D NSE case it is easy to obtain as it is a direct consequence of the decay [20, 21, 24, 30, 32, 33, 39] (due to the decay, the critical $L^3$ norm is eventually small, leading to regularity after the time this happens). This argument is not available for the SQG equation. Instead, the proof in [37] depends on a de Giorgi iteration and on the Caffarelli-Silvestre representation of the fractional operator. The restriction for $\alpha$ to be sufficiently close to $1/2$ was removed in [13] as well as in [12] with a different proof.

For the 3D Navier–Stokes equations, Caffarelli, Kohn, and Nirenberg proved in [1] that if the initial datum is sufficiently nice (suitable smoothness and decay), then the singularity set is a compact subset in space-time. The main purpose of the present paper is to prove this for the super-critical quasi-geostrophic equation. The main difficulty is that there are no partial regularity theorems available for the SQG equation. In fact, it is open whether the Lebesgue measure of the singularities in space-time is zero. The main difference between the SQG equation and the 3D Navier–Stokes system is that the critical norm for the 3D NSE is $L^3$, while for the SQG equation is actually a Hölder norm. Note, however, that Giusti's lemma (cf. [1, p. 807] or [22, p. 106]), which is the main ingredient in the partial-regularity theory, relies on finiteness and criticality in a Lebesgue space $L^q$ with some $q<\infty$. This is available in the 3D NSE since the $L^3_{x,t}$ norm of a weak solution is bounded by the energy, while no such bounds are available for the Hölder norm in the case of the SQG equation.

In the first part of the paper, we establish the spatial decay for weak solutions of the SQG equation for all $\alpha>0$. Previously, the time decay $\|\theta\|_{L^2}\le C(1+t)^{-1/2\alpha}$ for $\alpha\in(0,1)$ was obtained by Constantin and Wu in [8] (cf. also [34]). The spatial decay was established in the subcritical range by M. and T. Schonbek in [35] with the weight $x$, i.e., using the norms $\||x|\theta\|_{L^2}$, while the authors of the present paper showed in [29] the decay for the norms $\||x|^r\theta\|_{L^2}$ for all $r\in(0,1+\alpha)$, also only in the subcritical range $\alpha\in(1/2,1)$. Note however that we are able to obtain the spatial decay only for the $L^q$ with $q<\infty$, which poses a difficulty for the iterative part of the proof. The next part of the proof relies on a de Giorgi iteration argument and consists of two steps. In the first, the iteration is done using solely the smallness of the solution at the spatial infinity, while the second uses a Lagrangian variable as in [37] to eliminate the small scales and iteratively conclude the Hölder regularity.

It remains open if our theorem can be extended to the full range of parameters of dissipation, i.e., whether compactness of the space-time singular set holds for equation

(1.1) with arbitrary $\alpha>0$.

The paper is organized as follows. In Section 2, we introduce the notation and state both main theorems, on the spatial decay and the large $x$ regularity. The next section contains the proof of the weighted decay. Section 4 provides a proof of large $x$ regularity, divided into three steps. In the first step, we use the Lagrangian change of variable to eliminate the small scales and obtain smallness in a cube. The second step contains of a finite number of iteration, where the Lagrangian change is not used. After a certain number of steps, the smallness in the cube is such that the Silvestre-type iteration may be performed. As in [37], all iterations are based on a $L^q$ regularity lemma due to Constantin and Wu [10].

## 2. The notation and the main results

We consider a solution of the SQG equation

$$\theta_t+u\cdot\nabla\theta+\Lambda^{2\alpha}\theta=0 \tag{2.1}$$

$$u=\mathcal{R}^{\perp}\theta=(-\mathcal{R}_2\theta,\mathcal{R}_1\theta), \tag{2.2}$$

with the initial condition

$$\theta(0)=\theta_0, \tag{2.3}$$

where $\alpha\in(0,1/2)$ is in the supercritical range. Above $\mathcal{R}_1$ and $\mathcal{R}_2$ denote the Riesz transforms

$$\mathcal{R}_j f(x)=c_0\,\text{P.V.}\int\frac{y_j f(x-y)}{|y|^3}\,dy, \qquad j=1,2, \tag{2.4}$$

where $c_0=\pi^{-3/2}\Gamma(3/2)$. Our main result asserts that if $\alpha$ is slightly below $1/2$ and if the initial datum belongs to a sufficiently regular space, then the singular set is a compact subset of $\mathbb{R}^2\times(0,\infty)$. Recall that a point $(x_0,t_0)$ is regular if there exists a neighborhood where the solution is Hölder continuous with the Hölder exponent at least $1-2\alpha$.

For the Navier–Stokes equations, the smallness of a weak solution in a certain Lebesgue space implies regularity [36], while such results are not available for the SQG equation. However, smallness still plays a role in the regularity as we are going to take advantage of the following weighted decay result of independent interest. Denote

$$\phi(x,t)=\Big(|x|^2+(1+t)^{1/\alpha}\Big)^{1/2}, \tag{2.5}$$

for $x\in\mathbb{R}^2$ and $t\geq 0$.

Theorem 2.1 (Weighted decay). *Let $0\leq\alpha\leq 1/2$ and $p\geq 2$. Assume that $\theta_0\in L^1\cap L^\infty$ and $(|x|^r+1)\theta_0\in L^p(\mathbb{R}^2)$ for some $r$ such that $0<rp<\alpha$. Then there exists a global weak solution to the SQG system* (2.1)–(2.3) *such that*

$$\|\phi^r\theta\|_{L^p}\leq C, \tag{2.6}$$

*where the constant $C$ depends on $\alpha$, $r$, $p$, and the initial data.*

Interpolating the bound (2.6) with $p=2$ and

$$\|\theta\|_{L^p}\leq\frac{C}{(1+t)^{(p-1)/\alpha p}}, \qquad 2\leq p\leq\infty \tag{2.7}$$

from [8, 10] with $p=\infty$, we obtain

$$\||x|^r\theta\|_{L^p} \le \||x|^{pr/2}\theta\|_{L^2}^{2/p}\|\theta\|_{L^\infty}^{1-2/q} \le \frac{C}{(1+t)^{(1-2/p)/\alpha}}, \qquad p\in[2,\infty), \qquad rp<\alpha. \quad (2.8)$$

Then, interpolating between estimates (2.7) and (2.8), we get

$$\||x|^r\theta\|_{L^p} \le \frac{C}{(1+t)^\gamma} \quad (2.9)$$

where

$$\gamma = \frac{rp}{\alpha^2}\left(1-\frac{2}{p}\right)+\left(1-\frac{rp}{\alpha}\right)\frac{p}{p-1}-\epsilon, \qquad p\in[2,\infty], \qquad rp<\alpha, \quad (2.10)$$

where $\epsilon>0$ is arbitrary.

Next, we state the main result on the regularity for large $|x|$ of solutions of a slightly supercritical SQG equation.

THEOREM 2.2 (Regularity). *Let $0<\alpha<1/2$ with $1/2-\alpha<1/C$, where $C$ is a sufficiently large constant. Assume that $\theta_0\in L^1(\mathbb{R}^2)\cap H^{2-2\alpha}(\mathbb{R}^2)$ and $(|x|^2+1)^{\alpha^2/8}\theta_0\in L^{2/\alpha}(\mathbb{R}^2)$. Let $\theta$ be a solution to the supercritical SQG equation with the initial datum $\theta_0$ as in Theorem 2.1. Then there exists $\delta>1-2\alpha$ independent of the initial datum and $R>0$ such that*

$$|\theta(x,t)-\theta(y,t)| \le C|x-y|^\delta, \qquad |x|,|y|\ge R,$$

*where $C$ is a constant.*

The proof of Theorem 2.2 is given in Section 4 below.

REMARK 2.1. By [16], the solution is regular on an interval $(0,T_1)$, where $T_1$ depends on the initial data. Also, by [12, 13, 37], it is regular on the interval $(T_2,\infty)$ for some $T_2\ge T_1$. Theorem 2.2 shows that if there is any singularity, it must occur in the cylinder $\{x\in\mathbb{R}^2 : |x|\le R\}\times[T_1,T_2]$.

## 3. Weighted decay

This section is devoted to the proof of the statement on weighted decay of solutions.

*Proof.* **(Proof of Theorem 2.1.)** The proof consists of a priori estimates, which can be justified easily by adding an additional dissipation term with a vanishing viscosity parameter.

Multiplying equation (2.1) by $\phi^{rp}|\theta|^{p-2}\theta$, where $\phi$ is given in definition (2.5), and integrating over $\mathbb{R}^2$, we obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|\phi^r\theta\|_{L^p}^p + \int(\Lambda^{2\alpha}\theta)|\theta|^{p-2}\theta\phi^{rp}\,dx \\ &\le \int\partial_t(\phi^{rp})\theta|\theta|^{p-2}\theta\,dx - \int(u\cdot\nabla\theta)|\theta|^{p-2}\theta\phi^{rp}\,dx = I_1+I_2. \end{aligned}$$

Since $rp<\alpha<1$, we have

$$I_1 \le C\|\theta\|_{L^p}^p.$$

For the term $I_2$, we integrate by parts to get

$$I_2 = \frac{1}{p}\int u|\theta|^p\cdot\nabla(\phi^{rp})\,dx.$$

Therefore, by Hölder's and the Calderón–Zygmund inequalities, we arrive at

$$I_2 \leq C\|\theta\|_{L^\infty}\|\theta\|_{L^p}^p \leq \frac{C}{(1+t)^{(p-1)/\alpha p}}\|\theta\|_{L^p}^p \leq C\|\theta\|_{L^p}^p,$$

where we used estimate (2.7). In order to estimate the diffusion term, we use a pointwise inequality for the nonlocal differential operator $\Lambda^{2\alpha}$ to obtain

$$\int (\Lambda^{2\alpha}\theta)|\theta|^{p-2}\theta\phi^{rp}\,dx \geq \frac{1}{C}\int \Lambda^{2\alpha}(|\theta|^{p/2})|\theta|^{p/2}\phi^{rp}\,dx$$

[11, 23]. Combining the above estimates, we get

$$\frac{1}{p}\frac{d}{dt}\|\phi^r\theta\|_{L^p}^p + \delta\int \Lambda^{2\alpha}(|\theta|^{p/2})|\theta|^{p/2}\phi^{rp}\,dx \leq C\|\theta\|_{L^p}^p, \tag{3.1}$$

where $\delta > 0$ is a sufficiently small constant. The inequality (3.1) may be rewritten as

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|\phi^r\theta\|_{L^p}^p + \delta\int (\Lambda^\alpha(|\theta|^{p/2}))^2\phi^{rp}\,dx \\
\leq{}& \delta\int (\Lambda^\alpha(|\theta|^{p/2}))^2\phi^{rp}\,dx - \delta\int \Lambda^\alpha(|\theta|^{p/2})\Lambda^\alpha\left(|\theta|^{p/2}\phi^{rp}\right)dx \\
&\qquad + \delta\int \Lambda^\alpha(|\theta|^{p/2})|\theta|^{p/2}\Lambda^\alpha(\phi^{rp})\,dx - \delta\int \Lambda^\alpha(|\theta|^{p/2})|\theta|^{p/2}\Lambda^\alpha(\phi^{rp})\,dx \\
&\qquad + C\|\theta\|_{L^p}^p \\
={}& \delta\int \Lambda^\alpha(|\theta|^{p/2})\left(\Lambda^\alpha(|\theta|^{p/2})\phi^{rp} - \Lambda^\alpha\left(|\theta|^{p/2}\phi^{rp}\right) + |\theta|^{p/2}\Lambda^\alpha(\phi^{rp})\right)dx \\
&\qquad - \delta\int \Lambda^\alpha(|\theta|^{p/2})|\theta|^{p/2}\Lambda^\alpha(\phi^{rp})\,dx + C\|\theta\|_{L^p}^p \\
={}& \delta A - \delta\int \Lambda^\alpha(|\theta|^{p/2})|\theta|^{p/2}\Lambda^\alpha(\phi^{rp})\,dx + C\|\theta\|_{L^p}^p,
\end{aligned} \tag{3.2}$$

where $A$ denotes the commutator term

$$A = -\int \Lambda^\alpha(|\theta|^{p/2})\left(\Lambda^\alpha\left(|\theta|^{p/2}\phi^{rp}\right) - \Lambda^\alpha(|\theta|^{p/2})\phi^{rp} - |\theta|^{p/2}\Lambda^\alpha(\phi^{rp})\right)dx.$$

By a Kato–Ponce type commutator estimate, we deduce

$$A \leq C\|\Lambda^\alpha(|\theta|^{p/2})\|_{L^2}\|\Lambda^{s_1}(|\theta|^{p/2})\|_{L^\rho}\|\Lambda^{s_2}\phi^{rp}\|_{L^\mu},$$

where $s_1 = (\alpha - rp)/2$, $s_2 = (\alpha + rp)/2$, $\rho = 8/(4-\alpha+rp)$, and $\mu = 8/(\alpha - rp)$. Note that $\mu(s_2 - rp) > 2$ implies

$$\|\Lambda^{s_2}(\phi^{rp})\|_{L^\mu} < \infty.$$

Applying the Gagliardo–Nirenberg inequality to $\|\Lambda^{s_1}|(\theta|^{p/2})\|_{L^\rho}$ gives

$$\|\Lambda^{s_1}(|\theta|^{p/2})\|_{L^\rho} \leq C\||\theta|^{p/2}\|_{L^2}^{1-\beta}\|\Lambda^\alpha(|\theta|^{p/2})\|_{L^2}^\beta = C\|\theta\|_{L^p}^{p(1-\beta)/2}\|\Lambda^\alpha(|\theta|^{p/2})\|_{L^2}^\beta,$$

where $\beta = 3(\alpha - rp)/4\alpha$. Hence, we may bound

$$A \leq C\|\theta\|_{L^p}^{p(1-\beta)/2}\|\Lambda^\alpha(|\theta|^{p/2})\|_{L^2}^{1+\beta} \leq C\|\theta\|_{L^p}^p + \frac{1}{C}\|\Lambda^\alpha(|\theta|^{p/2})\|_{L^2}^2,$$

where $C$ is sufficiently large. As for the second term on the far right side of inequality (3.2), Hölder's and Young's inequalities imply

$$-\delta\int\Lambda^\alpha|\theta|^{p/2}|\theta|^{p/2}\Lambda^\alpha(\phi^{rp})\,dx\le C\|\Lambda^\alpha(|\theta|^{p/2})\|_{L^2}\||\theta|^{p/2}\|_{L^2}\|\Lambda^\alpha\phi^{rp}\|_{L^\infty}$$
$$\le C\|\Lambda^\alpha(|\theta|^{p/2})\|_{L^2}\|\theta\|_{L^p}^{p/2}\|\Lambda^\alpha\phi^{rp}\|_{L^\infty}\le C\|\theta\|_{L^p}^{p/2}+\frac{1}{4}\|\Lambda^\alpha|\theta|^{p/2}\|_{L^2}^2,$$

where we used

$$\|\Lambda^\alpha(\phi^{rp})\|_{L^\infty}\le C$$

due to $\alpha>rp$. Summarizing, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\phi^r\theta\|_{L^p}^p+\frac{1}{C}\int(\Lambda^\alpha(|\theta|^{p/2}))^2\phi^{rp}\,dx\le C\|\theta\|_{L^p}^p.$$

Using estimate (2.7), we get

$$\frac{d}{dt}\|\phi^r\theta\|_{L^p}^p\le C(1+t)^{-(p-1)/\alpha},$$

from where

$$\|\phi^r\theta\|_{L^p}\le C,\qquad t\ge 0$$

since $(p-1)/\alpha>1$, and the proof is concluded. □

## 4. Regularity of solutions for large $x$

In this section, we provide the proof of Theorem 2.2.

**4.1. Extension device.** In this paper, we use the extension of a function $\theta$ to the upper half plane introduced by Caffarelli and Silvestre in [2]. Namely, given $\theta(x,t)$, where $x\in\mathbb{R}^2$ and $t\ge 0$, denote by $\bar\theta(x,z,t)$ the function defined in $\mathbb{R}^2\times\mathbb{R}^+\times\mathbb{R}^+$ which is the solution of

$$\operatorname{div}(z^\epsilon\nabla\bar\theta)=0,$$
$$\bar\theta(x,0,t)=\theta(x,t),$$

where

$$\epsilon=1-2\alpha$$

and $\nabla$ is the gradient in both variables $x$ and $z$. Then we have

$$\Lambda^{2\alpha}\theta(x,t)=\lim_{z\to 0}z^\epsilon\partial_z\bar\theta(x,z,t).$$

The advantage of the above extension is that, by adding a variable $z$, it allows us to represent a nonlocal operator using a local one.

**4.2. Notation.** Throughout this section, we use the notation

$$B_r(x)=\{y\in\mathbb{R}^2:|y-x|<r\}$$
$$B_r=\{y\in\mathbb{R}^2:|y|<r\}$$
$$B_r^*=B_r\times[0,r)$$

$$Q_r = B_r \times [0,r) \times (1-r^{2\alpha},1].$$

For a function $\theta$ defined in $\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+$, we also write

$$\theta_\rho(x,z,t) = \frac{1}{\rho^\delta}\theta\big(\rho x, \rho z, \rho^{2\alpha}(t-1)+1\big), \tag{4.1}$$

where $\delta \in (0,1)$ is to be determined in conditions (4.22) and (4.23) below.

**4.3. Oscillation lemma.** The following oscillation lemma due to Constantin and Wu is essential in the iteration argument below.

LEMMA 4.1 ( [10, Proposition 3.2]). *Let $\theta$ be a solution of the equations*

$$\begin{aligned}
&\partial_t\theta(x,0,t) + u\cdot\nabla_x\theta(x,0,t) + \lim_{z\to 0} z^\epsilon \partial_z\theta(x,z,t) = 0\\
&\operatorname{div}(z^\epsilon \nabla\theta) = 0,
\end{aligned}$$

*for an arbitrary divergence free vector field $u$ with $\|u\|_{L^\infty([0,1];L^{2/\alpha}(B_1))} \le K$ for some $K>0$. Then*

$$\operatorname{osc}_{Q_{1/2}}\theta \le (1-\eta)\operatorname{osc}_{Q_1}\theta$$

*holds for some $0<\eta<1$ depending on $K$ and $\epsilon$.*

**4.4. Proof of the far-field regularity.**

*Proof.* **(Proof of Theorem 2.2.)** By the local well-posedness (cf. Remark 2.1 above), there exists $T_1>0$ depending on the initial datum $\theta_0$ such that $u$ is regular on the interval $[0,T_1]$. Also, by the eventual regularity results (cf. [12,13,37]), the solution is regular for $t \ge T_2$ for some $T_2>0$ depending on the initial data. Thus we only need to consider the regularity for $t\in[T_1,T_2]$. We thus assume that $t\in[T_1,T_2]$ and prove that it is also regular for $|x|$ sufficiently large. Let $\bar\theta$ be the extension of $\theta$ as described above, satisfying

$$\begin{aligned}
&\partial_t\bar\theta(x,0,t) + u\cdot\nabla_x\bar\theta(x,0,t) + \lim_{z\to 0} z^\epsilon \partial_z\bar\theta(x,z,t) = 0\\
&\operatorname{div}(z^\epsilon \nabla\bar\theta) = 0,
\end{aligned}$$

where $u = \mathcal{R}^\perp\theta$. Fix any time $T\ge T_1$, and introduce a normalization of $\bar\theta(x,z,t)$ by

$$N(\theta)(x,z,t) = \bar\theta\left(T^{1/2\alpha}x, T^{1/2\alpha}z, Tt\right). \tag{4.2}$$

Then $N(\theta)$ satisfies the equations

$$\partial_t N(\theta)(x,0,t) + MN(u)\cdot\nabla_x N(\theta)(x,0,t) + \lim_{z\to 0} z^\epsilon\partial_z N(\theta)(x,z,t) = 0, \tag{4.3}$$

$$\operatorname{div}(z^\epsilon\nabla N(\theta)) = 0, \tag{4.4}$$

where $N(u) = u(T^{1/2\alpha}x, Tt)$ and

$$M = T^{1-1/2\alpha}.$$

Since $\theta$ is a solution as in Theorem 2.1, we have

$$\left\|\phi^{\alpha^2/4}\theta\right\|_{L^{2/\alpha}} \le C,$$

from where in particular

$$\||x|^{\alpha^2/8}\theta\|_{L^{2/\alpha}} \le \frac{C}{(1+t)^{\alpha^2/16}}, \qquad t \ge 0.$$

Now, recall the weighted bound on the Riesz transforms

$$\|(\mathcal{R}_j f)(x)|x|^\beta\|_{L^p} \le A_{p,\beta}\|f(x)|x|^\beta\|_{L^p}, \qquad -\frac{2}{p} < \beta < \frac{2}{p}, \qquad 1<p<\infty, \qquad j=1,2$$

due to Stein [38], stated here in $\mathbb{R}^2$. Using this inequality, we have

$$\||x|^{\alpha^2/8}u\|_{L^{2/\alpha}} \le \frac{C}{(1+t)^{\alpha^2/16}}, \qquad t \ge 0.$$

Hence, for $\epsilon_0 > 0$ sufficiently small and $r > 0$ sufficiently large, to be determined below, there exists $R > 0$ such that

$$\|\theta\|_{L^\infty L^{2/\alpha}([0,\infty)\times B_r(x))} \le \epsilon_0, \qquad |x| \ge R$$

and

$$\|u\|_{L^\infty L^{2/\alpha}([0,\infty)\times B_r(x))} \le \epsilon_0, \qquad |x| \ge R.$$

We now assume that $x$ is sufficiently large. For simplicity of notation, we translate so that $x=0$ and, based on the considerations above, assume that

$$\|\theta\|_{L^\infty L^{2/\alpha}([0,\infty)\times B_r)} \le \epsilon_0$$

and

$$\|u\|_{L^\infty L^{2/\alpha}([0,\infty)\times B_r)} \le \epsilon_0,$$

where $\epsilon_0 > 0$ may be assumed arbitrarily small and $r > 0$ arbitrarily large. After the scaling specified in definition (4.2), we have

$$\|N(\theta)|_{z=0}\|_{L^\infty L^{2/\alpha}([0,\infty)\times B_{\bar r})} \le \epsilon_0 T^{1/2} = \epsilon_1, \tag{4.5}$$

where $\bar r = r/T^{1/2\alpha}$. Since $N(\theta) \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+)$, we may also assume that

$$\operatorname{osc}_{\mathbb{R}^2\times[0,\infty)} \theta \le Q \tag{4.6}$$

for some constant $Q > 0$.

***Step 1: The initial step of the induction.*** As in [37], we perform a change of variable to follow the flow. Let

$$\tilde u(x,t) = c_0 \int_{\mathbb{R}^2\setminus B_{R_1}} \frac{N(\theta)(x-y,0,t)y^\perp}{|y|^3}\,dy,$$

where $R_1$ is a positive number to be determined and $c_0$ is as in definition (2.4). By the Cauchy-Schwarz inequality, we have

$$|\tilde u(x,t)| \le C\frac{\|N(\theta)|_{z=0}\|_{L^2}}{R_1}. \tag{4.7}$$

Choose $R_1 \geq 1$ sufficiently large so that

$$\frac{1}{R_1}\|N(\theta)(x,0,0)\|_{L^2(\mathbb{R}^2)} \leq \epsilon_1. \tag{4.8}$$

Then by the maximum principle for equations (4.4)–(4.3), we obtain from condition (4.8)

$$\frac{1}{R_1}\|N(\theta)(x,0,t)\|_{L^2(\mathbb{R}^2)} \leq \epsilon_1, \qquad t \geq 0$$

and by inequality (4.7) we have a uniform bound

$$|\tilde{u}(x,t)| \leq C\epsilon_1, \qquad (x,t) \in \mathbb{R}^2 \times [0,\infty).$$

We introduce the Lagrangian path $V\colon [0,1] \to \mathbb{R}^2$ as the solution of the ODE

$$\begin{aligned} &V'(t) = -M\tilde{u}(V,t)\\ &V(1) = 0. \end{aligned}$$

Letting

$$\tilde{\theta} = N(\theta)(x+V(t),z,t),$$

we obtain from condition (4.6)

$$\operatorname{osc}_{\mathbb{R}^2\times[0,\infty)\times[0,\infty)} \tilde{\theta} \leq Q.$$

Note that the equations for $\tilde{\theta}$ read

$$\begin{aligned} &\partial_t\tilde{\theta}(x,0,t) + M\big(u(x,t) - \tilde{u}(V(t),t)\big)\cdot\nabla\tilde{\theta}(x,0,t) + \lim_{z\to 0} z^\epsilon \partial_z\tilde{\theta}(x,z,t) = 0\\ &\operatorname{div}(z^\epsilon\nabla\tilde{\theta}) = 0. \end{aligned}$$

Choose $\rho > 0$ sufficiently small so that

$$CM\epsilon_1\rho^{2\alpha} + \rho \leq \frac{1}{2} \tag{4.9}$$

where $C > 0$ is sufficiently large. Then we have $(x+V(t),z,t) \in Q_{1/2}$ for any $(x,z,t) \in Q_\rho$. We choose

$$m \in \Big[\inf_{\mathbb{R}^2\times[0,\infty)\times[0,\infty)} \tilde{\theta}, \sup_{\mathbb{R}^2\times[0,\infty)\times[0,\infty)} \tilde{\theta}\Big]$$

such that $|\tilde{\theta} - m| \leq Q/2$, i.e., $m = (\inf_{\mathbb{R}^2\times[0,\infty)\times[0,\infty)} \tilde{\theta} + \sup_{\mathbb{R}^2\times[0,\infty)\times[0,\infty)} \tilde{\theta})/2$, and define

$$\theta_1 = (\tilde{\theta} - m)_\rho$$

(cf. the notation (4.1)). Then one may readily check that

$$|\theta_1| \leq \frac{Q}{2\rho^\delta} \quad \text{on } \mathbb{R}^2 \times \{0\} \times (0,\infty)$$

and

$$\operatorname{osc}_{Q_1} \theta_1 \le \frac{Q}{\rho^\delta}.$$

A direct computation shows that $\theta_1$ satisfies

$$\begin{aligned}&\partial_t \theta_1(x,0,t) + \rho^{\delta-\epsilon} M(u_1 - \bar{u}_1)\cdot\nabla_x \theta_1(x,0,t) + \lim_{z\to 0} z^\epsilon \partial_z \theta_1(x,z,t) = 0\\ &\operatorname{div}(z^\epsilon \nabla\theta_1) = 0,\end{aligned}$$

where $u_1 = \mathcal{R}^\perp \theta_1$ and

$$\bar{u}_1(t) = c_0 \int_{\mathbb{R}^2\setminus B_{R_1/\rho}} \frac{\theta_1(y,0,t) y^\perp}{|y|^3}\,dy.$$

Next, observe that

$$\begin{aligned}u_1 - \bar{u}_1 &= c_0 \int_{\mathbb{R}^2} \frac{\theta_1(x-y,0,t)y^\perp}{|y|^3}\,dy - c_0 \int_{\mathbb{R}^2\setminus B_{R_1/\rho}} \frac{\theta_1(y,0,t)y^\perp}{|y|^3}\,dy\\ &= \frac{c_0}{\rho^\delta}\Bigg(\int_{\mathbb{R}^2} \frac{\tilde{\theta}(y,0,\rho^\alpha(t-1)+1)(\rho x - y)^\perp}{|\rho x - y|^3}\,dy\\ &\qquad - \int_{\mathbb{R}^2\setminus B_{R_1}} \frac{\tilde{\theta}(y,0,\rho^\alpha(t-1)+1)y^\perp}{|y|^3}\,dy\Bigg).\end{aligned}$$

Decomposing the first integral over the regions $B_{R_1}$ and $\mathbb{R}^2\setminus B_{R_1}$, we get

$$\begin{aligned}u_1 - \bar{u}_1 &= \frac{c_0}{\rho^\delta}\int_{B_{R_1}} \frac{\tilde{\theta}(y,0,\rho^\alpha(t-1)+1)(\rho x-y)^\perp}{|\rho x-y|^3}\,dy\\ &\quad + \frac{c_0}{\rho^\delta}\int_{\mathbb{R}^2\setminus B_{R_1}} \tilde{\theta}(y,0,\rho^\alpha(t-1)+1)\left(\frac{(\rho x-y)^\perp}{|\rho x-y|^3} - \frac{y^\perp}{|y|^3}\right)dy\\ &= w_1 + w_2.\end{aligned}$$

Note that $w_1$ is a $\rho^{-\delta}$ multiple of the Riesz transform $(-\mathcal{R}_2, \mathcal{R}_1)$ of $1_{B_{R_1}}\tilde{\theta}$ evaluated at $\rho x$. Setting $\bar{r} = R_1 + 1$ in inequality (4.5), we obtain

$$\begin{aligned}\|w_1\|_{L^\infty([0,1];L^{2/\alpha}(B_1))} \le \|w_1\|_{L^\infty([0,1];L^{2/\alpha}(\mathbb{R}^2))} &\le \frac{C}{\rho^{\delta+\alpha}} \|\tilde{\theta}(y,0,t)\|_{L^\infty([0,1];L^{2/\alpha}(B_{R_1}))}\\ &\le \frac{C}{\rho^{\delta+\alpha}}\epsilon_1,\end{aligned}$$

where $C$ is a constant, in particular not depending on $\rho$. Since $\rho \le 1$ by condition (4.9), we may increase $R_1$ if necessary and obtain

$$\left|\frac{(\rho x-y)^\perp}{|\rho x-y|^3} - \frac{y^\perp}{|y|^3}\right| \le \frac{1}{|y|^3}, \qquad |x|\le 1, \qquad |y| \ge R_1.$$

Applying the Cauchy–Schwarz inequality, we arrive at

$$|w_2| \le \frac{C}{\rho^\delta}\frac{\|\tilde{\theta}|_{z=0}\|_{L^2(\mathbb{R}^2)}}{R_1^2} \le \frac{C}{\rho^\delta}\epsilon_1, \qquad |x|\le 1, \qquad t\in[0,1],$$

which implies

$$\|w_2\|_{L^\infty([0,1];L^{2/\alpha}(B_1))} \leq \frac{C\epsilon_1}{\rho^\delta}.$$

Combining the above estimates for $w_1$ and $w_2$, we obtain

$$\|M\rho^{\delta-\epsilon}(u_1-\bar{u}_1)\|_{L^\infty([0,1];L^{2/\alpha}(B_1))} \leq C\rho^{\delta-\epsilon}\rho^{-\delta-\alpha}\epsilon_1 = \frac{C\epsilon_1}{\rho^{1-\alpha}},$$

where $C$ depends on $T$ (which is considered fixed). By Lemma 4.1, applied with $K=1$, there exists $\eta>0$ such that

$$\operatorname{osc}_{Q_{1/2}}\theta_1 \leq (1-\eta)\frac{Q}{\rho^\delta}.$$

Note that in order to apply Lemma 4.1, we need

$$\frac{C\epsilon_1}{\rho^{1-\alpha}} \leq 1. \tag{4.10}$$

This completes the first step of the iteration.

***Step 2: The iterations 2 through*** $N$**.** Next, we provide iterations 2 through $N$, for a certain sufficiently large integer $N\geq 2$, to be determined below. In Step 1, we have obtained $\theta_1$ such that

$$|\theta_1| \leq \frac{Q}{2\rho^\delta} \quad \text{on } \mathbb{R}^2\times\{0\}\times(0,\infty) \tag{4.11}$$

$$\|M\rho^{\delta-\epsilon}(u_1-\bar{u}_1)\|_{L^\infty([0,1];L^{2/\alpha}(B_1))} \leq \frac{C\epsilon_1}{\rho^{1-\alpha}} \tag{4.12}$$

$$\operatorname{osc}_{Q_1}\theta_1 \leq \frac{Q}{\rho^\delta}$$

$$\operatorname{osc}_{Q_{1/2}}\theta_1 \leq (1-\eta)\frac{Q}{\rho^\delta}.$$

At this point, we use the following fact: If $\operatorname{osc}_{Q_1}\theta\leq M$ and $\operatorname{osc}_{Q_{1/2}}\theta\leq m\leq M$, then there exists $m_1\leq M-m$ such that $|\theta-m_1|\leq m/2$ on $Q_{1/2}$. Thus there is $m_1\in[\inf_{Q_1}\theta_1,\sup_{Q_1}\theta_1]$ such that

$$|\theta_1-m_1| \leq \frac{1}{2}(1-\eta)\frac{Q}{\rho^\delta} \quad \text{on } Q_{1/2}$$

and

$$|m_1| \leq \frac{Q}{\rho^\delta}-(1-\eta)\frac{Q}{\rho^\delta} = \eta\frac{Q}{\rho^\delta}. \tag{4.13}$$

Define

$$\theta_2 = (\theta_1-m_1)_\rho.$$

Then we have

$$|\theta_2| \leq \frac{1}{2\rho^\delta}(1-\eta)\frac{Q}{\rho^\delta} \quad \text{on } Q_1$$

and consequently

$$\operatorname{osc}_{Q_1}\theta_2 \le \frac{1}{\rho^\delta}(1-\eta)\frac{Q}{\rho^\delta}.$$

Using properties (4.11) and (4.13), we then deduce

$$\begin{aligned}|\theta_2| &\le \frac{1}{\rho^\delta}\left(\frac{Q}{2\rho^\delta}+m_1\right)\le \frac{1}{\rho^\delta}\left(\frac{Q}{2\rho^\delta}+\eta\frac{Q}{2\rho^\delta}\right)\\ &=\frac{1}{2}\frac{Q}{\rho^{2\delta}}(1+\eta)\le \frac{Q}{\rho^{2\delta}}(1-\eta)\le \frac{Q}{\rho^{2\delta}}(1-\eta)|x|^{2\delta}\end{aligned}$$

for $|x|\ge 1$ if we require $\eta\le 1/3$. Note that $\theta_2$ satisfies

$$\begin{aligned}&\partial_t\theta_2(x,0,t)+M\rho^{2(\delta-\epsilon)}(u_2-\bar{u}_2)\cdot\nabla_x\theta_2(x,0,t)+\lim_{z\to 0}z^\epsilon\partial_z\theta_2(x,z,t)=0\\ &\operatorname{div} z^\epsilon\nabla\theta_2=0,\end{aligned}$$

where $u_2=\mathcal{R}^\perp\theta_2$ and

$$\bar{u}_2(t)=c_0\int_{\mathbb{R}^2\setminus B_{R_1/\rho^2}}\frac{\theta_2(y,0,t)y^\perp}{|y|^3}\,dy.$$

Writing the Riesz transform in the integral form, we obtain

$$\begin{aligned}u_2(x,t)-\bar{u}_2(t)&=c_0\int_{\mathbb{R}^2}\frac{\theta_2(x-y,0,t)y^\perp}{|y|^3}\,dy-c_0\int_{\mathbb{R}^2\setminus B_{R_1/\rho^2}}\frac{\theta_2(y,0,t)y^\perp}{|y|^3}\,dy\\ &=\frac{c_0}{\rho^\delta}\int_{\mathbb{R}^2}\frac{\theta_1(\rho x-y,0,\rho^\delta(t-1)+1)y^\perp}{|y|^3}\,dy\\ &\quad-\frac{c_0}{\rho^\delta}\int_{\mathbb{R}^2\setminus B_{R_1/\rho}}\frac{\theta_1(y,0,\rho^\delta(t-1)+1)y^\perp}{|y|^3}\,dy=(u_1(x,t)-\bar{u}_1(t))_\rho.\end{aligned}$$

Therefore, combining the above calculation and inequality (4.12), we get

$$\begin{aligned}&\|M\rho^{2(\delta-\epsilon)}(u_2-\bar{u}_2)\|_{L^\infty([0,1];L^{2/\alpha}(B_1))}=\frac{M\rho^{\delta-\epsilon}}{\rho^{1-\alpha}}\|u_1-\bar{u}_1\|_{L^\infty([0,1];L^{2/\alpha}(B_\rho))}\\ &\le C\rho^{2(\alpha-1)}\epsilon_1.\end{aligned}$$

Applying Lemma 4.1, we then arrive at

$$\operatorname{osc}_{Q_{1/2}}\theta_2\le(1-\eta)\operatorname{osc}_{Q_1}\theta_2\le\frac{(1-\eta)^2Q}{\rho^{2\delta}}$$

provided

$$C\rho^{2(\alpha-1)}\epsilon_1\le 1. \tag{4.14}$$

Now, in order to use the induction, assume that we have constructed $\theta_k$, where $k\in\{2,3,\dots\}$, which satisfy the equations

$$\partial_t\theta_k(x,0,t)+M\rho^{k(\delta-\epsilon)}(u_k-\bar{u}_k)\cdot\nabla_x\theta_k(x,0,t)+\lim_{z\to 0}z^\epsilon\partial_z\theta_k(x,z,t)=0$$

$$\operatorname{div}(z^\epsilon \nabla\theta_k)=0,$$

where $u_k=\mathcal{R}^\perp\theta_k$ and

$$\bar u_k(x,t)=c_0\int_{\mathbb{R}^2\setminus B_{R_1/\rho^k}}\frac{\theta_k(y,0,t)y^\perp}{|y|^3}\,dy.$$

Furthermore, suppose that

$$|\theta_k|\le \frac{1}{2\rho^\delta}(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q \quad \text{on } Q_1;$$

$$|\theta_k|\le \frac{1}{\rho^\delta}(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q|x|^{2\delta} \quad \text{in } B_1^c\times\{0\}\times[0,\infty); \tag{4.15}$$

$$\|M\rho^{k(\delta-\epsilon)}(u_k-\bar u_k)\|_{L^\infty([0,1);L^{2/\alpha}(B_1))}\le \frac{C\epsilon_1}{\rho^{k(1-\alpha)}}; \tag{4.16}$$

$$\operatorname{osc}_{Q_1}\theta_k\le \frac{1}{\rho^\delta}(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q; \tag{4.17}$$

$$\operatorname{osc}_{Q_{1/2}}\theta_k\le (1-\eta)^k\frac{1}{\rho^{k\delta}}Q. \tag{4.18}$$

Now, choose $m_k$ so that

$$\begin{aligned}
|m_k|&\le \frac12\left(\frac{1}{\rho^\delta}(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q-(1-\eta)^k\frac{1}{\rho^{k\delta}}Q\right);\\
&=\frac12\frac{1}{\rho^\delta}\eta(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q;\\
|\theta_k-m_k|&\le (1-\eta)^k\frac{1}{\rho^{k\delta}}Q/2 \quad \text{on } Q_{1/2}; \qquad (4.19)\\
|\theta_k-m_k|&\le \rho^{-\delta}(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q-(1-\eta)^k\frac{1}{\rho^{k\delta}}Q+\frac12(1-\eta)^k\frac{1}{\rho^{k\delta}}Q\\
&\le (1-\eta)^k\frac{1}{\rho^{k\delta}}Q \quad \text{on } Q_1,
\end{aligned}$$

which is possible by inequalities (4.17) and (4.18). The next iterate

$$\theta_{k+1}=(\theta_k-m_k)_\rho$$

then satisfies

$$|\theta_{k+1}|\le \frac12\frac{1}{\rho^\delta}(1-\eta)^k\frac{1}{\rho^{k\delta}}Q \quad \text{on } Q_1,$$

and consequently

$$\operatorname{osc}_{Q_1}\theta_{k+1}\le \frac{1}{\rho^\delta}(1-\eta)^k\frac{1}{\rho^{k\delta}}Q. \tag{4.20}$$

The equations satisfied by $\theta_{k+1}$ read

$$\partial_t\theta_{k+1}(x,0,t)+M\rho^{(k+1)(\delta-\epsilon)}(u_{k+1}-\bar u_{k+1})\cdot\nabla_x\theta_{k+1}(x,0,t)+\lim_{z\to0}z^\epsilon\partial_z\theta_{k+1}(x,z,t)=0$$

$$\operatorname{div}(z^\epsilon\nabla\theta_{k+1})=0,$$

where $u_{k+1}=\mathcal{R}^{\perp}\theta_{k+1}$ and

$$\bar{u}_{k+1}(x,t)=c_0\int_{\mathbb{R}^2\setminus B_{R_1/\rho^{k+1}}}\frac{\theta_{k+1}(y,0,t)y^{\perp}}{|y|^3}\,dy.$$

As in the above computation, we have

$$u_{k+1}(x,t)-\bar{u}_{k+1}(t)=(u_k(x,t)-\bar{u}_k(t))_{\rho}.$$

A straightforward calculation shows that

$$\|M\rho^{(k+1)(\delta-\epsilon)}(u_{k+1}-\bar{u}_{k+1})\|_{L^{\infty}([0,1);L^{2/\alpha}(B_1))}\leq\frac{C\epsilon_1}{\rho^{(k+1)(1-\alpha)}},$$

where we used inequality (4.16). Applying Lemma 4.1 combined with (4.20) gives

$$\operatorname{osc}_{Q_{1/2}}\theta_{k+1}\leq(1-\eta)^{k+1}\frac{1}{\rho^{(k+1)\delta}}Q$$

provided

$$\frac{C\epsilon_1}{\rho^{(k+1)(1-\alpha)}}\leq 1. \tag{4.21}$$

It remains to be checked that the condition (4.15) holds for $\theta_{k+1}$. In fact for $1\leq|x|\leq 1/\rho$, we have in view of condition (4.19)

$$|\theta_{k+1}|\leq\frac{1}{\rho^{\delta}}(1-\eta)^k\frac{1}{\rho^{k\delta}}Q\leq\frac{1}{\rho^{\delta}}(1-\eta)^k\frac{1}{\rho^{k\delta}}Q|x|^{2\delta}.$$

On the other hand, for $|x|\geq 1/\rho$, by conditions (4.15) and (4.19), we obtain

$$\begin{aligned}|\theta_{k+1}|&\leq\frac{1}{\rho^{\delta}}\left(\frac{1}{\rho^{\delta}}(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q|\rho x|^{2\delta}+\frac{1}{2}\frac{1}{\rho^{\delta}}\eta(1-\eta)^{k-1}\frac{1}{\rho^{(k-1)\delta}}Q\right)\\&\leq\rho^{-\delta}(1-\eta)^{(k-1)}\rho^{-k\delta}Q|x|^{2\delta}\left(\rho^{2\delta}+\frac{\eta\rho^{2\delta}}{2}\right)\\&=\frac{1}{\rho^{\delta}}(1-\eta)^k\frac{1}{\rho^{k\delta}}Q|x|^{2\delta}\rho^{2\delta}\left(1+\frac{\eta}{2}\right)(1+\eta)^{-1}.\end{aligned}$$

Then we have

$$|\theta_{k+1}|\leq\frac{1}{\rho^{\delta}}(1-\eta)^k\frac{1}{\rho^{k\delta}}Q|x|^{2\delta}$$

if we choose the constant $\delta$ so that

$$(1-\eta)\frac{1}{\rho^{\delta}}<1 \tag{4.22}$$

and

$$\rho^{2\delta}\left(1+\frac{\eta}{2}\right)(1+\eta)^{-1}\leq 1. \tag{4.23}$$

Note that the above requirement may be satisfied by choosing a sufficiently small $\delta$, namely

$$\rho^{\delta}=(1+\mu)(1-\eta)$$

for some $\mu > 0$. Then

$$\rho^{2\delta}\left(1+\frac{\eta}{2}\right)(1+\eta)^{-1} = (1+\mu)^2(1-\eta)\left(1+\frac{\eta}{2}\right) = (1+\mu)^2\left(1-\frac{\eta}{2}-\frac{\eta^2}{2}\right) \le 1$$

provided $\mu$ is sufficiently small. Now we choose $N$ sufficiently large so that

$$\frac{1}{\rho^\delta}(1-\eta)^{N-1}\frac{1}{\rho^{(N-1)\delta}}Q \le 2. \tag{4.24}$$

After $N$ iterations, we obtain

$$\begin{aligned}
&|\theta_N| \le 1 \quad \text{on } Q_1;\\
&|\theta_N| \le 2|x|^{2\delta} \quad \text{on } B_1^c \times [0,1] \times [0,\infty);\\
&\operatorname{osc}_{Q_1} \theta_N \le 2;\\
&\operatorname{osc}_{Q_{1/2}} \theta_N \le 2(1-\eta).
\end{aligned}$$

***Step 3: Concluding iterations.*** The $(N+1)$-th step is slightly different from the previous ones. Denote

$$\begin{aligned}
\tilde{u}_{N1} &= c_0 \int_{\mathbb{R}^2 \setminus B_2} \frac{\theta_N(x-y,0,t)y^\perp}{|y|^3}\,dy;\\
\tilde{u}_{N2} &= c_0 \int_{\mathbb{R}^2 \setminus B_{R_1/\rho^N}} \frac{\theta_N(y,0,t)y^\perp}{|y|^3}\,dy.
\end{aligned}$$

A straightforward computation shows

$$\tilde{u}_{N1} = \frac{c_0}{\rho^\delta}\int_{\mathbb{R}^2 \setminus B_{2\rho}} \frac{\theta_{N-1}(\rho x - y, 0, \rho^{2\alpha}(t-1)+1)y^\perp}{|y|^3}\,dy$$

and by unfolding the definition of $\theta_{N-1}$

$$\tilde{u}_{N1} = \frac{c_0}{\rho^{N\delta}}\int_{\mathbb{R}^2 \setminus B_{2\rho^N}} \frac{\tilde{\theta}(\rho^N x - y, 0, \rho^{2N\alpha}(t-1)+1)y^\perp}{|y|^3}\,dy.$$

Similarly, we get

$$\tilde{u}_{N2} = \frac{c_0}{\rho^{N\delta}}\int_{\mathbb{R}^2 \setminus B_{R_1}} \frac{\tilde{\theta}(y, 0, \rho^{2N\alpha}(t-1)+1)y^\perp}{|y|^3}\,dy.$$

Furthermore, we have the estimate

$$\begin{aligned}
|\tilde{u}_{N1}(x)| &\le \frac{c_0}{\rho^{N\delta}}\int_{B_{R_1}\setminus B_{2\rho^N}} \frac{|\tilde{\theta}(\rho^N x - y, 0, \rho^{2N\alpha}(t-1)+1)|}{|y|^2}\,dy\\
&\quad + \frac{c_0}{\rho^{N\delta}}\int_{\mathbb{R}^2\setminus B_{R_1}} \frac{|\tilde{\theta}(\rho^N x - y, 0, \rho^{2N\alpha}(t-1)+1)|}{|y|^2}\,dy\\
&\le C\frac{1}{\rho^{N(\delta+\alpha)}}\|\tilde{\theta}(y,0,t)\|_{L^\infty([0,1];L^{2/\alpha}(B_{R_1+\rho^N}))} + C\frac{1}{\rho^{N\delta}}\frac{\|\theta\|_{L^2}}{R_1}
\end{aligned}$$

$$\leq C\frac{1}{\rho^{N(\delta+\alpha)}}\epsilon_1 + C\frac{1}{\rho^{N\delta}}\epsilon_1 \leq C\frac{1}{\rho^{N(\delta+\alpha)}}\epsilon_1, \qquad x\in\mathbb{R}^2.$$

Also, by Hölder's inequality, we obtain

$$|\tilde{u}_{N2}(x)| \leq C\frac{1}{\rho^{N\delta}}\frac{\|\theta\|_{L^2}}{R_1} \leq C\frac{1}{\rho^{N\delta}}\epsilon_1, \qquad x\in\mathbb{R}^2.$$

We construct the Lagrangian path $V_N:[0,1]\to\mathbb{R}^2$ to be the solution of the ODE

$$\begin{aligned} &V_N'(t) = -M\rho^{N(\delta-\epsilon)}(\tilde{u}_{N1}(V_N,t)+\tilde{u}_{N2}(t)) \\ &V_N(1)=0. \end{aligned}$$

Choose $\rho$ and $\epsilon_1$ small enough such that

$$M\rho^{N(\delta-\epsilon)}C\frac{1}{\rho^{N(\delta+\alpha)}}\epsilon_1 + \rho = CM\frac{1}{\rho^{N(1-\alpha)}}\epsilon_1 + \rho \leq \frac{1}{2},$$

which implies $(x+V(t),z,t)\in Q_{1/2}$ if $(x,z,t)\in Q_\rho$. Denoting $\tilde{\theta}_N = \theta_N(x+V_N(t),z,t)$, we have

$$\operatorname{osc}_{Q_\rho}\tilde{\theta}_N \leq 2(1-\eta) \leq 2\rho^\delta.$$

We choose $m_N\in[-1+\rho^\delta, 1-\rho^\delta]$ such that $|\tilde{\theta}_N - m_N|\leq \rho^\delta$ in $Q_\rho$ and define

$$\theta_{N+1} = (\tilde{\theta}_N - m_N)_\rho.$$

It is not difficult to check

$$\begin{aligned} &|\theta_{N+1}|\leq 1 \quad \text{on } Q_1; \\ &\operatorname{osc}_{Q_1}\theta_1 \leq 2. \end{aligned}$$

The equations satisfied by $\theta_{N+1}$ are given by

$$\begin{aligned} &\partial_t\theta_{N+1}(x,0,t) + M\rho^{(\delta-\epsilon)(N+1)}(u_{N+1}-\bar{u}_{N+1})\cdot\nabla_x\theta_{N+1}(x,0,t) \\ &\qquad + \lim_{z\to 0} z^\epsilon\partial_z\theta_{N+1}(x,z,t) = 0 \\ &\operatorname{div}(z^\epsilon\nabla\theta_{N+1}) = 0, \end{aligned}$$

where $u_{N+1} = \mathcal{R}^\perp\theta_{N+1}$ and

$$\bar{u}_{N+1}(t) = c_0\int_{\mathbb{R}^2/B_{2/\rho}} \frac{\theta_{N+1}(y,0,t)y^\perp}{|y|^3}\,dy.$$

The same computation as in [37] shows that

$$|\theta_{N+1}| \leq 2|x|^{2\delta}$$

for $|x|\geq 1$. From here on, we perform exactly the same iteration as in [37] and get the Hölder continuity.

At last, we conclude the proof by illuminating how the constants are selected to make sure that there is no circular argument. We first choose $\rho$ sufficiently small such that it satisfies the conditions

$$C\rho^{2\alpha} + \rho \leq \frac{1}{2}; \tag{4.25}$$

$$-C\rho^{2\alpha}\log\rho + C\rho^{1+2\alpha} + \rho \le \frac{1}{2}; \tag{4.26}$$

$$\rho \le \frac{1}{16}. \tag{4.27}$$

Note that the first condition (4.25) comes from condition (4.9) since we could restrict $\epsilon_1$ so that $M\epsilon_1 \le 1$ (note that $T_1 \le T \le T_2$.) Conditions (4.26) and (4.27) are from [37]. The value of $\eta$ is determined by Lemma 4.1 with $K$ a constant (while we use $K=1$, the proof in [37], which we need in the last step, uses $K=C$, where $C$ is a constant). Once $\rho$ and $\eta$ are chosen, we determine $\delta > \epsilon$ such that conditions (4.22) and (4.23) hold. To have $\delta > \epsilon$, we need $\epsilon$ to be sufficiently close to 0, i.e., $\alpha$ needs to be sufficiently close to 1/2. After $\delta$ is fixed, we use the condition (4.24) gives the value of $N$. Finally, we choose $\epsilon_1$ small enough, by taking sufficiently large $\bar{r}$ and $R$, so that conditions (4.10), (4.14), and (4.21) hold (the last condition for $k = 1, 2, \ldots, N-1$). □

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