COMMUN. MATH. SCI.
Vol. 16, No. 2, pp. 363–391

QUASINEUTRAL LIMIT FOR THE COMPRESSIBLE QUANTUM NAVIER–STOKES–MAXWELL EQUATIONS*

MIN LI†, XUEKE PU‡, AND SHU WANG§

Abstract. In this paper, we study the quasi-neutral limit of the full quantum Navier–Stokes–Maxwell equation as the Debye length tends to zero. We justify rigorously the quasi-neutral limit by establishing rigorous uniform estimates on the error functions with respect to the Debye length and by using the formal asymptotic expansion and singular perturbation methods combined with curl-div decomposition of the gradient. The key difficulty is to deal with the quantum effects, which do play important roles in establishing *a priori* estimates.

Keywords. quantum Navier–Stokes–Maxwell system; quasineutral limit; uniform energy estimates; curl-div decomposition; singular perturbation methods.

AMS subject classifications. 35B40; 35B45; 35C20; 76Y05.

1. Introduction

In this paper, we consider the quasineutral limit of the following full quantum Navier–Stokes–Maxwell system consisting of the mass conservation, momentum and energy equations coupled to the Maxwell equations, describing the transport of charged particles with viscosity in a plasma [17]

$$\begin{cases} \dfrac{\partial n}{\partial t}+\operatorname{div}(nu)=0, & (1.1a)\\ m[\dfrac{\partial (nu)}{\partial t}+\operatorname{div}(nu\otimes u)]-\operatorname{div}P=-ne(E+u\times B)+\operatorname{div}S, & (1.1b)\\ \dfrac{\partial W}{\partial t}+\operatorname{div}(uW-uP+q)=-enu(E+u\times B)+\operatorname{div}(uS), & (1.1c)\\ \partial_t B+\nabla\times E=0,\quad c^{-2}\partial_t E-\nabla\times B=\mu_0 enu, & (1.1d)\\ \operatorname{div}B=0,\quad \operatorname{div}E=\varepsilon_0^{-1}e(n_i-n), & (1.1e)\end{cases}$$

where ε_0,μ_0,c are the vacuum permittivity, permeability and light speed, respectively, which satisfy $\varepsilon_0\mu_0c^2=1$. In the above system, n, u, E and B are functions of time t and position $x\in\mathbb{T}^3$, the three dimensional torus, representing the electron density, velocity, electric field and magnetic field, respectively. Moreover, the parameters m, e and n_i stand for the electron mass, the electron charge and the ion density. The ions in plasma are taken to be motionless and hence the density n_i is taken to be constant in time and space. Moreover, $P=(P_{ij})_{3\times3}$ is the stress tensor, W is the energy density and q is the heat flux. It is customary that the heat flux is assumed to obey the Fourier law $q=-\kappa\nabla T$, where T is the temperature. To include quantum effects, the stress tensor

*Received: April 28, 2017; accepted (in revised form): November 22, 2017. Communicated by Song Jiang. Corresponding to Shu Wang. This work is partially supported by the NSF of China (Grant No. 11471057 and 11771031).

†School of Mathematics and Statistics, Chongqing University, Chongqing 401331, P.R. China (530294442@qq.com).

‡School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, P.R. China (puxueke@gmail.com).

§School of Mathematics and Information Sciences, Guangzhou University, Guangzhou 510006, P.R. China (wangshu1968@aliyun.com).

P and W are defined in terms of n, u and T by

$$P_{ij} = -k_B nT\delta_{ij} + \frac{\hbar^2 n}{12m}\frac{\partial^2}{\partial_{x_i}\partial_{x_j}}\log n,$$

and

$$W = \frac{3}{2}k_B nT + \frac{1}{2}nm|u|^2 - \frac{\hbar^2 n}{24m}\Delta\log n,$$

respectively, where k_B is the Boltmmann constant and $\hbar > 0$ is the Planck constant which is very small compared to macro quantities. For the scaled Planck constant in a device with some physical parameters, see [12, 21]. More general electromagnetic quantum hydrodynamic models for multi-species plasma can be derived starting with a Eigner function for each species by moment expansion method [17]. More general models taking into consideration quantum effects can be found [1, 4–8, 12, 17, 21, 23, 38]. The viscous stress tensor S in system (1.1) is given by

$$S = \mu(\nabla u + (\nabla u)^\top) + \nu(\mathrm{div}u)I,$$

to include the viscosity, where $\mu > 0$ and $2\mu + 3\nu \geq 0$. We also remark that the quantum stress tensor is closely related to the quantum Bohm potential [1, 38]

$$Q(n) = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{n}}{\sqrt{n}}$$

through the formula

$$-\frac{\partial}{\partial x_i}P_{ij} = \frac{\partial}{\partial x_i}(nT) + \frac{n}{3}\frac{\partial Q}{\partial x_i}.$$

By simple calculations, it gives that

$$-\nabla Q(n) = \frac{\hbar^2}{4nm}\mathrm{div}(n\nabla^2\log n) = \frac{\hbar^2}{4m}(\nabla\Delta logn + \frac{1}{2}\nabla|\nabla logn|^2).$$

Moreover, we obtain the following relation

$$\nabla\Delta logn + \frac{1}{2}\nabla|\nabla logn|^2 = \frac{\Delta\nabla n}{n} - \{\frac{(\Delta n\nabla n + \nabla n\cdot\nabla^2 n)}{(n)^2} - \frac{(\nabla n\cdot\nabla n)\nabla n}{(n)^3}\}.$$

By these computations, the full quantum Navier–Stokes–Maxwell system (1.1) can be rewritten as

$$\begin{cases} \partial_t n + \nabla\cdot(nu) = 0, & (1.2a)\\ m[\partial_t(nu) + \nabla\cdot(nu\otimes u)] + k_B\nabla(nT) - \frac{\hbar^2}{12m}\mathrm{div}\{n(\nabla\otimes\nabla)\log n\} = -ne(E + u\times B) & \\ \qquad +\mu\Delta u + (\mu+\nu)\nabla\mathrm{div}u, & (1.2b)\\ k_B[\partial_t(nT) + \mathrm{div}(nuT)] + k_B\frac{2}{3}nT\nabla\cdot u - \frac{2}{3}\nabla\cdot(\kappa\nabla T) + \frac{\hbar^2}{36m}\nabla\cdot(n\Delta u) & \\ \qquad -\frac{2}{3}\Big\{\frac{\mu}{2}|\nabla u + (\nabla u)^\top|^2 + \nu(\mathrm{div}u)^2\Big\} = 0, & (1.2c)\\ \partial_t B + \nabla\times E = 0,\ \varepsilon_0\mu_0\partial_t E - \nabla\times B = e\mu_0 nu, & (1.2d)\\ \mathrm{div}B = 0,\ \varepsilon_0\mathrm{div}E = e(n_i - n). & (1.2e) \end{cases}$$

Here we have used the following computation owing to the mass conservation equation (1.1a) and the process $\frac{2}{3}\big((1.1c)-u\cdot(1.1b)\big)$

$$\begin{aligned}
&-\partial_t(n\Delta\log n)-\operatorname{div}(nu\Delta\log n+2nu\cdot\nabla^2\log n)+2\operatorname{div}(n\nabla^2\log n)\cdot u\\
=&n\Delta\frac{\operatorname{div}(nu)}{n}-nu\cdot\nabla\Delta\log n-2n\nabla u:\nabla^2\log n\\
=&\operatorname{div}(n\Delta u)-\nabla n\cdot\Delta u+n\nabla\Delta\log n\cdot u+n\nabla\log n\cdot\Delta u+2n\nabla u:\nabla^2\log n\\
&\quad-nu\cdot\nabla\Delta\log n-2n\nabla u:\nabla^2\log n=\operatorname{div}(n\Delta u).
\end{aligned}$$

In recent years, the asymptotic analysis of hydrodynamical models has attracted much attention, and there have been many interesting results. Without quantum effect and Maxwell equation, Cordier and Grenier [2] studied the quasi-neutral limit of the isentropic Euler–Poisson equation for ions by the pseudo-differential energy estimates. The result was recently generalized to the pressureless case for the cold ions by [35]. In the past two decades, the asymptotic limit for Euler/Navier–Stokes–Poisson systems with well/ill-prepared initial data was extensively studied and many interesting results were obtained. In [14, 15], Gasser and Marcati studied the combined relaxation and vanishing Debye length limit in the unipolar and bipolar case for semiconductors in one space dimension. Wang and Jiang [37] investigated the combined quasineutral and inviscid limit for the case of general initial data by using the relative entropy theory and weak convergence method. For more quasineutral limit of Euler/Navier–Stokes–Poisson systems, we refer the interested reader to [9, 10, 32, 33] and their references and to [20] for the study on general initial data. In the case with quantum effect, Li and Lin [27] obtained the convergence from the isentropic quantum hydrodynamic model for semiconductors to the incompressible Euler equations with general initial data. Remarkably, some efforts were made on the well-posedness and the vanishing capillarity limit of the solutions for the compressible fluid models of Korteweg type. More precisely, in [18, 19], Hattori and Li justified the existence of local and global solutions for the multidimensional isothermal Navier–Stokes–Korteweg equations. Furthermore, Jüngel, *et al* in [22] established the combined incompressible and vanishing capillarity limit based on a modulated energy method.

However, there are relatively less results on the compressible Euler/Navier–Stokes–Maxwell systems. Quasineutral limit and other related limit problems were studied by Peng, Wang *et al* in [28–31]. To the best of our knowledge, there are few mathematical studies for the full quantum Navier–Stokes or the full quantum Navier–Stokes–Maxwell system. Recently, Pu and Guo studied the global existence of smooth solutions and semiclassical limit in [34]. For two-fluid Euler-Maxwell system in 3D, Guo, Ionescu and Pausader [16] proved existence of global smooth solutions.

In the present paper, we are interested in the quasi-neutral limit for system (1.1) with well-prepared initial data, as Debye length goes to zero. The case for ill-prepared initial data will be discussed in the future. After some scalings in Subsection 1.1, the system (1.1) can be rewritten as (1.4). We show that smooth solutions to system (1.4) converge to solutions of the electron magnetohydrodynamics equation (1.5) on a time interval independent of the Debye length parameter ε, as Debye length goes to zero. The scalings are chosen according to the principle of least degeneracy in [3]. The main result is stated in Theorem 1.3.

In what follows, we apply a weighted energy combined with singular perturbation methods, curl-div decomposition of the gradient and elaborate energy method to get the estimates for the remainder terms uniformly in ε and $\hbar$. The weighted energy norm we finally adopt is the triple norm of definition (2.1) incorporating the quantum parameter $\hbar > 0$. In this paper, straightforward Sobolev energy estimates can not be derived directly since we need to obtain the estimate not just for $N_R, \sqrt{\varepsilon}E_R$ but also for $N_R/\sqrt{\varepsilon}$ with lower order derivatives. Hence, we introduce the wave-type equations of the Maxwell equations and the equations of vorticity and divergence to derive higher order estimates.

The main difficulties in dealing with the quasi-neutral limit are to control the higher order derivatives, and to deal with the energy equation which includes quantum effect through the energy density W and to overcome the oscillatory behavior of the electric field. Another difficulty comes from the fact that the system cannot be written in a symmetric form, which makes the estimates very delicate. To overcome these difficulties, we introduce a weighted energy triple norm, which incorporates the delicate structure of the linearized system. Then, using this triple norm, combined with the uniform *a priori* estimate and applying vector analysis techniques, we finally close the high order energy estimates for the full nonlinear system.

In the rest of the introduction, we first scale the quantum Navier–Stokes–Maxwell system according to the principle of least degeneracy. Then we give the formal expansions of the solutions (in ε) and derive the electron magnetohydrodynamics equation (1.5) and the remainder equation (1.8). We show the local existence in Theorem 1.2 and the main result is stated in Theorem 1.3. Note that we can obtain weak convergence for the electric field, since the electric filed in initial data (1.13) is bounded uniformly in ε and we can pass to the limit since all terms involving E^ε in system (1.4) are linear. Section 2 gives some basic uniform estimates in a series lemmas. In Section 3, we give a proof of Theorem 1.3 by using non-linear Gronwall's type inequality in [11]. Finally, the Appendix gives some preliminaries that are useful to the required estimates.

Throughout this paper, we let α be a multi-index and $\partial^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$ and $H^s(\mathbb{T}^3)$ be the standard Sobolev's space in the torus $\mathbb{T}^3$. We sometimes abuse the notation $\partial^{\alpha+1}$ to stand for $\partial^{\alpha+\beta}$ with $|\beta| = 1$ for some multi-index β. Moreover, we let $a \lesssim b$ mean $a \leq Cb$, where C is a constant and $[A,B] = AB - BA$ denote the commutator of A and B.

1.1. Scaling of the quantum Navier–Stokes–Maxwell system. To scale the system (1.2) to dimensionless units, we use scaling units $x_0, t_0, n_0, u_0, T_0, E_0, B_0, \rho_0, j_0$ for space, time, density, velocity, temperature, electric field, magnetic field, charge density and current density, respectively. By making the following hypotheses [3]:

$$x_0 = u_0 t_0, m u_0^2 = k_B T_0 = e E_0 x_0 = n_0^{-1} P_0, n_0 = n_i, \rho_0 = e n_0, j_0 = e n_0 u_0,$$

we reduce the number of dimensionless parameters. Then there are only six dimensionless parameters,

$$\alpha = \frac{u_0^2}{c^2}, \beta = \frac{u_0 B_0}{E_0}, \gamma = \frac{1}{m^2 x_0^2 u_0^2}, \iota = \frac{1}{m n_0 u_0 x_0}, \tilde{\iota} = \frac{1}{n_0 u_0 x_0 k_B}, \varepsilon = \frac{\varepsilon_0 k_B T_0}{e^2 n_0 x_0^2}.$$

Denote

$$(\tilde{n}, \tilde{u}, \tilde{T}, \tilde{B}, \tilde{E}) = (\frac{n}{n_0}, \frac{u}{u_0}, \frac{T}{T_0}, \frac{B}{B_0}, \frac{E}{E_0}).$$

We keep the same notations for the dimensionless variables as for the physical variables.

In this scaling, the full quantum Navier–Stokes–Maxwell system is rewritten as:

$$\begin{cases} \partial_t n+\nabla\cdot(nu)=0, & \text{(1.3a)}\\ \partial_t(nu)+\nabla\cdot(nu\otimes u)+\nabla(nT)-\gamma\frac{\hbar^2}{12}\mathrm{div}\{n(\nabla\otimes\nabla)\log n\}=-n(E+\beta u\times B) & \\ \qquad +\iota\mu\Delta u+\iota(\mu+\nu)\nabla\mathrm{div}u, & \text{(1.3b)}\\ \partial_t(nT)+\mathrm{div}(nuT)+\frac{2}{3}nT\nabla\cdot u-\frac{2}{3}\tilde{\iota}\nabla\cdot(\kappa\nabla T)+\gamma\frac{\hbar^2}{36}\nabla\cdot(n\Delta u) & \\ \qquad -\iota\frac{2}{3}\Big\{\frac{\mu}{2}|\nabla u+(\nabla u)^\top|^2+\nu(\mathrm{div}u)^2\Big\}=0, & \text{(1.3c)}\\ \beta\partial_t B+\nabla\times E=0,\ \varepsilon(\alpha\partial_t E-\beta\nabla\times B)=\alpha nu, & \text{(1.3d)}\\ \mathrm{div}B=0,\ \varepsilon\mathrm{div}E=1-n. & \text{(1.3e)} \end{cases}$$

By using the principle of the least degeneracy [3], we choose the scaling that produces the limit system with the largest number of terms. Interested in the limit $\varepsilon\to 0$, we observe that whatever the choice of α there holds $\varepsilon\alpha\partial_t E\ll\alpha nu$. So we can only retain αnu. The principle of least degeneracy thus impose $\varepsilon\beta=\alpha$, since the same order can result in the reservation of the two terms $\varepsilon\beta\nabla\times B$ and αnu. Moreover, it leads us to choose $\beta,\gamma,\iota,\tilde{\iota}=1$, because, either $\beta,\gamma,\iota,\tilde{\iota}\gg 1$ or $\beta,\gamma,\iota,\tilde{\iota}\ll 1$ will lead to reduced number of terms. So we choose $\alpha=\varepsilon$ and $\beta,\gamma,\iota,\tilde{\iota}=1$, which leads the final form of the scaled full quantum Navier–Stokes–Maxwell system:

$$\begin{cases} \partial_t n^\varepsilon+\nabla\cdot(n^\varepsilon u^\varepsilon)=0, & \text{(1.4a)}\\ \partial_t(n^\varepsilon u^\varepsilon)+\nabla\cdot(n^\varepsilon u^\varepsilon\otimes u^\varepsilon)+\nabla(n^\varepsilon T^\varepsilon)-\frac{\hbar^2}{12}\mathrm{div}\{n^\varepsilon(\nabla\otimes\nabla)\log n^\varepsilon\} & \\ \qquad =-n^\varepsilon(E^\varepsilon+u^\varepsilon\times B^\varepsilon)+\mu\Delta u^\varepsilon+(\mu+\nu)\nabla\mathrm{div}u^\varepsilon, & \text{(1.4b)}\\ \partial_t(n^\varepsilon T^\varepsilon)+\mathrm{div}(n^\varepsilon u^\varepsilon T^\varepsilon)+\frac{2}{3}n^\varepsilon T^\varepsilon\nabla\cdot u^\varepsilon-\frac{2}{3}\nabla\cdot(\kappa\nabla T^\varepsilon)+\frac{\hbar^2}{36}\nabla\cdot(n^\varepsilon\Delta u^\varepsilon) & \\ \qquad -\frac{2}{3}\Big\{\frac{\mu}{2}|\nabla u^\varepsilon+(\nabla u^\varepsilon)^\top|^2+\nu(\mathrm{div}u^\varepsilon)^2\Big\}=0, & \text{(1.4c)}\\ \partial_t B^\varepsilon+\nabla\times E^\varepsilon=0,\ \varepsilon\partial_t E^\varepsilon-\nabla\times B^\varepsilon=n^\varepsilon u^\varepsilon, & \text{(1.4d)}\\ \mathrm{div}B^\varepsilon=0,\ \varepsilon\mathrm{div}E^\varepsilon=1-n^\varepsilon, & \text{(1.4e)} \end{cases}$$

where ε is a singular perturbation parameter. This implies that the magnetic field will not vanish in the limiting process. The typical values of the scaled Debye length ε go from $10^{-3}m$ to $10^{-8}m$ in plasmas, which are very small compared to characteristic observation length. Hence it is interesting to study the limit as ε goes to zero. Moreover the Planck constant $\hbar=6.63\times 10^{-34}\ J\cdot s$ is also very small compared to macro quantities and $\hbar\to 0$ is usually called the semiclassical limit. In this paper, we consider the main limit $\varepsilon\to 0$, since the semiclassical limit $\hbar\to 0$ has been considered in [34], which shows the convergence of the solutions of the quantum hydrodynamic equations to the classical hydrodynamic equations. Furthermore, combined the semiclassical limit $\hbar\to 0$ and the quasineutral limit $\varepsilon\to 0$ leads also to the electron magnetohydrodynamics equations (1.5) owing to $\mathrm{div}u^0=1$ and $n^0=1$.

1.2. Formal expansions. Now, setting formally $\varepsilon=0$ in the system (1.4), we derive the so-called electron magnetohydrodynamics (e-MHD) equations [13, 25]:

$$(\mathcal{S}_0)\begin{cases} \partial_t u^0 + u^0\cdot\nabla u^0 + \nabla T^0 + E^0 + u^0\times B^0 - \mu\Delta u^0 = 0, & (1.5a)\\ \partial_t T^0 + u^0\cdot\nabla T^0 - \dfrac{2\kappa}{3}\Delta T^0 - \dfrac{\mu}{3}|\nabla u^0 + (\nabla u^0)^\top|^2 = 0, & (1.5b)\\ \partial_t B^0 + \nabla\times E^0 = 0,\ -\nabla\times B^0 = u^0, & (1.5c)\\ \mathrm{div}B^0 = 0, \quad n^0 = 1, & (1.5d)\end{cases}$$

which is a simplified description of the phenomena at fast time scales of electrons. At such fast time scales, electron motion has a dominant role in plasma dynamics whereas the ions provide merely a immobile background. The solutions of the e-MHD system are in the form of electron vortex currents together with associated magnetic fields.

Formally, as $\varepsilon\to 0$, the solutions of system (1.4) should converge to those of (1.5). Next, passing the fact $n^0=1$ to the limit in the mass conservation equation (1.4a), we get the incompressible e-MHD system. We will not write out the incompressible condition since it can be obtained by the equation $u^0=-\nabla\times B^0$. Noticing $\mathrm{div}B^0=0$ implies that there exists some magnetic potential A^0 such that $\nabla\times A^0 = B^0$.

Introducing the general vorticity $\omega^0=\nabla\times(u^0-A^0)$, we can rewrite the limit system (1.5) in following form owing to (A.14) $\sim$ (A.16) in the Appendix:

$$(\mathcal{S}_1)\begin{cases} \partial_t \omega^0 + u^0\cdot\nabla\omega^0 - \omega^0\cdot\nabla u^0 - \mu\Delta^2 B^0 = 0, & (1.6a)\\ \partial_t T^0 + u^0\cdot\nabla T^0 - \dfrac{2\kappa}{3}\Delta T^0 - \dfrac{\mu}{3}|\nabla u^0 + (\nabla u^0)^\top|^2 = 0, & (1.6b)\\ u^0 = -\nabla\times B^0,\ \omega^0 = \Delta B^0 - B^0, & (1.6c)\\ (u^0,T^0,B^0)(t=0) = (u_0^0,T_0^0,B_0^0). & (1.6d)\end{cases}$$

Substituting (1.6c) into (1.6a), we have

$$\partial_t(\Delta B^0 - B^0) - \nabla\times B^0\cdot\nabla(\Delta B^0 - B^0) + (\Delta B^0 - B^0)\cdot\nabla\nabla\times B^0 - \mu\Delta^2 B^0 = 0.$$

Then the existence result of the solution (u^0,T^0,B^0) to the system (1.6) is the same as the incompressible Navier–Stokes equation [24, 26] together with the parabolic theory for the temperature equation.

THEOREM 1.1. *Let $\tilde{s}\geq 8$. Then for any given initial data $(u_0^0,T_0^0,B_0^0)\in H^{\tilde{s}}(\mathbb{T}^3)$, satisfying*

$$-\nabla\times B_0^0 = u_0^0, \quad \mathrm{div}B_0^0 = 0,$$

there exists some $0<\tau_*\leq+\infty$, maximal time of existence such that the e-MHD system (1.5) admits a unique solution (u^0,T^0,B^0,E^0) such that $(u^0,T^0,B^0,E^0)\in C^i([0,\tau_*],H^{\tilde{s}-i}(\mathbb{T}^3)\times H^{\tilde{s}-i}(\mathbb{T}^3)\times H^{\tilde{s}+1-i}(\mathbb{T}^3)\times H^{\tilde{s}-1-i}(\mathbb{T}^3))$ for $i=0,1$.

1.3. Derivation of the error equations. To make the above formal derivation rigorous, we denote this approximate solution by

$$\begin{cases} n^\varepsilon = 1 - \varepsilon\mathrm{div}E^0 + N_R, & (1.7a)\\ (u^\varepsilon,T^\varepsilon,E^\varepsilon,B^\varepsilon) = (u^0,T^0,E^0,B^0) + (U_R,T_R,E_R,B_R), & (1.7b)\end{cases}$$

where $(n^\varepsilon,u^\varepsilon,T^\varepsilon,E^\varepsilon,B^\varepsilon)$ is the solution to system (1.4), (u^0,T^0,E^0,B^0) satisfies (1.5) and (N_R,U_R,T_R,E_R,B_R) is the remainder. Then the remainder system satisfied by

(N_R,U_R,T_R,E_R,B_R) can be obtained after some careful computations,

$$
\begin{cases}
\partial_t N_R+\mathrm{div}(n^\varepsilon U_R+N_R u^0)+\varepsilon\Re_1=0, & (1.8a)\\
\partial_t U_R+U_R\cdot\nabla u^0+u^\varepsilon\cdot\nabla U_R+E_R+\nabla T_R+\dfrac{T^\varepsilon\nabla N_R}{n^\varepsilon}-\varepsilon\dfrac{T^\varepsilon\nabla\mathrm{div}E^0}{n^\varepsilon} & \\
\quad -\dfrac{\hbar^2}{12}\nabla\Delta logn^\varepsilon-\dfrac{\hbar^2}{24}\nabla|\nabla logn^\varepsilon|^2-\mu(\dfrac{\Delta U_R}{n^\varepsilon}-\dfrac{N_R\Delta u_0}{n^\varepsilon})-(\mu+\nu)\dfrac{\nabla\mathrm{div}U_R}{n^\varepsilon} & \\
\quad +u^\varepsilon\times B_R+U_R\times B^0-\mu\dfrac{\varepsilon\mathrm{div}E^0\Delta u_0}{n^\varepsilon}=0, & (1.8b)\\
\partial_t T_R+u^\varepsilon\cdot\nabla T_R+\dfrac{2}{3}T^\varepsilon\mathrm{div}U_R-\dfrac{2\kappa}{3}\dfrac{\Delta T_R}{n^\varepsilon}+\dfrac{\hbar^2}{36}(\mathrm{div}\Delta U_R-\varepsilon\dfrac{\nabla\mathrm{div}E^0\cdot\Delta U_R}{n^\varepsilon} & \\
\quad +\dfrac{\nabla N_R\cdot\Delta U_R}{n^\varepsilon})-\dfrac{\mu}{3}\Big(\dfrac{2(\nabla u^0+(\nabla u^0)^\top)(\nabla U_R+(\nabla U_R)^\top)}{n^\varepsilon}+\dfrac{(\nabla U_R+(\nabla U_R)^\top)^2}{n^\varepsilon}\Big) & \\
\quad -\dfrac{2\nu}{3}\dfrac{(\mathrm{div}U_R)^2}{n^\varepsilon}+\Re_3=0, & (1.8c)\\
\varepsilon\partial_t E_R-\nabla\times B_R=n^\varepsilon U_R+N_R u^0-\varepsilon\Re_4,\ \partial_t B_R+\nabla\times E_R=0, & (1.8d)\\
\varepsilon\mathrm{div}E_R=-N_R,\ \mathrm{div}B_R=0, & (1.8e)
\end{cases}
$$

where R_i $(i=1,3,4)$ are given by

$$
\begin{cases}
\Re_1=-\partial_t\mathrm{div}E^0-\mathrm{div}(u^0\mathrm{div}E^0), & (1.9a)\\
\Re_3=U_R\cdot\nabla T^0+\dfrac{2\kappa}{3}\{\dfrac{N_R\Delta T_0}{n^\varepsilon}-\varepsilon\dfrac{\mathrm{div}E^0\Delta T^0}{n^\varepsilon}\}+\dfrac{\hbar^2}{36}\dfrac{\Delta u^0\cdot\nabla N_R}{n^\varepsilon}-\dfrac{\hbar^2}{36}\dfrac{\varepsilon\nabla\mathrm{div}E^0\cdot\Delta u^0}{n^\varepsilon} & \\
\qquad +\dfrac{\mu}{3}\Big\{\dfrac{N_R|\nabla u^0+(\nabla u^0)^\top|^2}{n^\varepsilon}-\varepsilon\dfrac{\mathrm{div}E^0|\nabla u^0+(\nabla u^0)^\top|^2}{n^\varepsilon}\Big\}, & (1.9b)\\
\Re_4=\partial_t E^0+u^0\mathrm{div}E^0. & (1.9c)
\end{cases}
$$

1.4. Local-in-time existence theory of smooth solution. In this subsection, we are in the position to discuss the existence of the local-in-time solution to system (1.8). The proof is based on the dual argument, the energy estimates and the iteration techniques.

THEOREM 1.2. *Let $(u^0,T^0,B^0,E^0)\in H^{\tilde{s}}(\mathbb{T}^3)$ be a solution constructed in Theorem 1.1 with initial data $(u^0_0,T^0_0,B^0_0)\in H^{\tilde{s}}(\mathbb{T}^3)$ satisfying $-\nabla\times B^0_0=u^0,\mathrm{div}B^0_0=0$ and $(N_{R0},U_{R0},T_{R0},B_{R0},E_{R0})\in H^5(\mathbb{T}^3)\times H^4(\mathbb{T}^3)\times H^3(\mathbb{T}^3)\times H^4(\mathbb{T}^3)\times H^4(\mathbb{T}^3)$. For all fixed ε, there exist a positive constant $\tau_\varepsilon\le+\infty$ and a unique solution $(N_R,U_R,T_R,B_R,E_R)\in C^i([0,\tau_\varepsilon),H^{5-2i}(\mathbb{T}^3)\times H^{4-2i}(\mathbb{T}^3)\times H^{3-2i}(\mathbb{T}^3)\times H^{4-i}(\mathbb{T}^3)\times H^{4-i}(\mathbb{T}^3))$, $i=0,1$, to (1.8) with initial data $(N_{R0},U_{R0},T_{R0},B_{R0},E_{R0})$.*

To prove Theorem 1.2, we introduce the following linear remainder system for the unknowns $(\hat{N}_R,\hat{U}_R,\hat{T}_R,\hat{E}_R,\hat{B}_R)$

$$
\begin{cases}
\partial_t\hat{N}_R+n^\varepsilon\mathrm{div}\hat{U}_R+u^\varepsilon\cdot\nabla\hat{N}_R+\ell_1=0, & (1.10a)\\
\partial_t\hat{U}_R+u^\varepsilon\cdot\nabla\hat{U}_R+\dfrac{T^\varepsilon\nabla\hat{N}_R}{n^\varepsilon}-\dfrac{\hbar^2}{12}\dfrac{\nabla\Delta\hat{N}_R}{n^\varepsilon}-\mu\dfrac{\Delta\hat{U}_R}{n^\varepsilon}-(\mu+\nu)\dfrac{\nabla\mathrm{div}\hat{U}_R}{n^\varepsilon} & \\
\quad +\ell_2=0, & (1.10b)\\
\partial_t\hat{T}_R+u^\varepsilon\cdot\nabla\hat{T}_R+\dfrac{2}{3}\hat{T}_R\mathrm{div}U_R-\dfrac{2\kappa}{3}\dfrac{\Delta\hat{T}_R}{n^\varepsilon}+\ell_3=0, & (1.10c)\\
\varepsilon\partial_t\hat{E}_R-\nabla\times\hat{B}_R=\ell_4,\ \partial_t\hat{B}_R+\nabla\times\hat{E}_R=0,\ \mathrm{div}\hat{B}_R=0, & (1.10d)
\end{cases}
$$

where $(n^\varepsilon, u^\varepsilon, T^\varepsilon)$ is given in definition (1.7) and

$$\begin{cases} \ell_1 = -\varepsilon U_R\cdot\nabla \mathrm{div} E^0 + \varepsilon \Re_1, & (1.11a)\\ \ell_2 = U_R\cdot\nabla u^0 + \nabla \hat{T}_R + E_R - \varepsilon\dfrac{T^\varepsilon \nabla \mathrm{div} E^0}{n^\varepsilon} + \dfrac{\hbar^2}{12}\Big(\varepsilon\dfrac{\nabla\Delta \mathrm{div} E^0}{n^\varepsilon} + \dfrac{\Delta n^\varepsilon \nabla n^\varepsilon + \nabla n^\varepsilon\cdot\nabla^2 n^\varepsilon}{(n^\varepsilon)^2} \\ \qquad - \dfrac{|\nabla n^\varepsilon|^2\nabla n^\varepsilon}{(n^\varepsilon)^3}\Big) + \mu\dfrac{N_R\Delta u_0}{n^\varepsilon} + u^\varepsilon\times B_R + U_R\times B^0 - \mu\dfrac{\varepsilon \mathrm{div} E^0 \Delta u_0}{n^\varepsilon}, & (1.11b)\\ \ell_3 = \dfrac{2}{3}T^0 \mathrm{div} U_R + \dfrac{\hbar^2}{36}(\mathrm{div}\Delta U_R - \varepsilon\dfrac{\nabla \mathrm{div} E^0\cdot\Delta U_R}{n^\varepsilon} + \dfrac{\nabla N_R\cdot \Delta U_R}{n^\varepsilon}) - \dfrac{2\nu}{3}\dfrac{(\mathrm{div} U_R)^2}{n^\varepsilon} \\ \qquad -\dfrac{\mu}{3}\Big(\dfrac{2(\nabla u^0 + (\nabla u^0)^\top)(\nabla U_R + (\nabla U_R)^\top)}{n^\varepsilon} + \dfrac{(\nabla U_R + (\nabla U_R)^\top)^2}{n^\varepsilon}\Big) + \Re_3, & (1.11c)\\ \ell_4 = n^\varepsilon U_R + N_R u^0 - \varepsilon\Re_4,\ \varepsilon \mathrm{div} E_R = -N_R. & (1.11d) \end{cases}$$

We supplement the system (1.10) with the following initial conditions

$$(\hat{N}_R, \hat{U}_R, \hat{T}_R, \hat{B}_R, \hat{E}_R)(0,x) = (N_{R0}, U_{R0}, T_{R0}, B_{R0}, E_{R0}), \quad x\in\mathbb{T}^3.$$

LEMMA 1.1. *Under the same assumption in Theorem 1.2, for all fixed ε, there exist a positive constant $\tau_\varepsilon \le +\infty$ such that, if the functions $(N_R, U_R, T_R, E_R, B_R)$ in the coefficients in system* (1.10) *satisfy*

$$(N_R, E_R, B_R)\in C([0,\tau_\varepsilon), H^5(\mathbb{T}^3)\times H^4(\mathbb{T}^3)\times H^4(\mathbb{T}^3))$$

and

$$(U_R, T_R)\in C([0,\tau_\varepsilon), H^4(\mathbb{T}^3)\times H^3(\mathbb{T}^3))\cap L^2(0,\tau_\varepsilon; H^5(\mathbb{T}^3)\times H^4(\mathbb{T}^3)),$$

then the linear system (1.10) *admits a unique solution $(\hat{N}_R, \hat{U}_R, \hat{T}_R, \hat{E}_R, \hat{B}_R)$ in the same set.*

Proof. Firstly, the parabolic equation (1.10c) has a solution $\hat{T}_R \in C([0,\tau_\varepsilon), H^3(\mathbb{T}^3))\cap L^2(0,\tau_\varepsilon; H^4(\mathbb{T}^3))$ for any given functions $(N_R, E_R, B_R)\in C([0,\tau_\varepsilon), H^5(\mathbb{T}^3)\times H^4(\mathbb{T}^3)\times H^4(\mathbb{T}^3))$ and $(U_R, T_R)\in C([0,\tau_\varepsilon), H^4(\mathbb{T}^3)\times H^3(\mathbb{T}^3))\cap L^2(0,\tau_\varepsilon; H^5(\mathbb{T}^3)\times H^4(\mathbb{T}^3))$ by Galerkin method and the standard theory of the ordinary differential equation. Next, the existence of local solutions to the equations (1.10a) and (1.10b) can be proved in a similar fashion as Section 3 in [18] by the dual argument. Note that the unknown function $\hat{T}_R$ in ℓ_2 can be controlled by the heat conductivity term in the parabolic equation (1.10c). Applying Faedo–Galerkin method again, we can obtain the estimate for the electric field $\hat{E}_R$ and magnetic field $\hat{B}_R$ in equation (1.10d). This complete the proof of Lemma 1.1. □

By Lemma 1.1, we can show the local existence theory in Theorem 1.2.

Proof. **(The end of the proof of Theorem 1.2.)** We define the approximation sequence by $\{(N_R^k, U_R^k, T_R^k, E_R^k, B_R^k)\}_{k=0}^\infty$. By Lemma 1.1, the approximation sequence $\{(N_R^k, U_R^k, T_R^k, E_R^k, B_R^k)\}_{k=0}^\infty$ is well defined. Consider now the iteration scheme

$$(N_R^0, U_R^0, T_R^0, E_R^0, B_R^0) = (N_{R0}, U_{R0}, T_{R0}, E_{R0}, B_{R0}),$$

$$(N_R^{k+1}, U_R^{k+1}, T_R^{k+1}, E_R^{k+1}, B_R^{k+1}) = \Psi(N_R^k, U_R^k, T_R^k, E_R^k, B_R^k),$$

where the generator Ψ maps the known vector (N_R,U_R,T_R,E_R,B_R) into solution $(\hat{N}_R,\hat{U}_R,\hat{T}_R,\hat{E}_R,\hat{B}_R)$ of the linear system (1.10). Next, applying the standard energy method to the system satisfied by the difference $(N_R^{k+1}-N_R^k,U_R^{k+1}-U_R^k,T_R^{k+1}-T_R^k,E_R^{k+1}-E_R^k,B_R^{k+1}-B_R^k)$, we can obtain the convergence of the approximating sequence $\{(N_R^k,U_R^k,T_R^k,E_R^k,B_R^k)\}_{k=0}^{\infty}$. Further, the limit function is the desired solution to the system (1.8). ∎

The main result of this paper is stated in the following.

1.5. Main results.

THEOREM 1.3. *Let $\tilde{s}\geq 8$, $(u^0,T^0,B^0,E^0)\in H^{\tilde{s}}(\mathbb{T}^3)$ be a solution to the limit system* (1.5)*, and (N_R,U_R,T_R,B_R,E_R) be a solution to remainder system* (1.8)*. Assume that the initial data $(n^\varepsilon,u^\varepsilon,T^\varepsilon,B^\varepsilon,E^\varepsilon)(t=0)=(n_0^\varepsilon,u_0^\varepsilon,T_0^\varepsilon,B_0^\varepsilon,E_0^\varepsilon)$ of the solutions to compressible quantum Navier–Stokes–Maxwell system* (1.4) *satisfy*

$$\varepsilon \operatorname{div} E_0^\varepsilon = 1-n_0^\varepsilon, \quad \operatorname{div} B_0^\varepsilon = 0, \tag{1.12}$$

and

$$\|(n_0^\varepsilon-1,u_0^\varepsilon-u_0^0,T_0^\varepsilon-T_0^0,B_0^\varepsilon-B_0^0,\sqrt{\varepsilon}E_0^\varepsilon)\|_{H^3(\mathbb{T}^3)}^2\leq C\varepsilon, \tag{1.13}$$

where C is a constant independent of ε. Then for every $0<\tau_0<\tau_*$ with τ_* given in Theorem 1.1, there exists $\varepsilon_0=\varepsilon_0(\tau_0)$ such that if $0<\varepsilon<\varepsilon_0$ there is a maximal time interval $[0,\tau_\varepsilon]$ with $\liminf_{\varepsilon\to 0}\tau_\varepsilon\geq\tau_0$ such that the quantum Navier–Stokes–Maxwell system (1.4) *has a classical smooth solution $(n^\varepsilon,u^\varepsilon,T^\varepsilon,B^\varepsilon,E^\varepsilon)$ satisfy*

$$\sup_{t\in[0,\tau_0]}\|(n^\varepsilon-1,u^\varepsilon-u^0,T^\varepsilon-T^0,B^\varepsilon-B^0,\sqrt{\varepsilon}E^\varepsilon)(t)\|_{H^3(\mathbb{T}^3)}^2\leq C(\tau_0)\varepsilon. \tag{1.14}$$

REMARK 1.1. The profile $(n^\varepsilon,u^\varepsilon,T^\varepsilon,B^\varepsilon)$ converges strongly to $(1,u^0,T^0,B^0)$ in $L^\infty(0,\tau_0;H^3(\mathbb{T}^3))$. Note that E^ε is only bounded uniformly in ε for all time $t\in[0,\tau_0]$, so E^ε converges in the weak sense. However, it follows from the convergence of the other terms $(n^\varepsilon,u^\varepsilon,T^\varepsilon,B^\varepsilon)$ in (1.4b) and the uniqueness of solutions to the limit system (1.5) that, E^ε converges to E^0 in $W^{-1,\infty}(0,\tau_0;H^2(\mathbb{T}^3))$.

2. Uniform energy estimates

To give some uniform energy estimates of system (1.8), we introduce a weighted energy:

$$\begin{aligned}|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)|||_3 &= \|(N_R,U_R,T_R,B_R)\|_{H^3}^2+\|(\frac{N_R}{\sqrt{\varepsilon}},\sqrt{\varepsilon}\nabla\times E_R,\varepsilon\partial_t E_R)\|_{H^2}^2\\ &+\|(\hbar\frac{N_R}{\sqrt{\varepsilon}},\hbar\nabla N_R,\hbar\sqrt{\varepsilon}\nabla\times E_R,\hbar\varepsilon\partial_t E_R,\hbar\nabla U_R,\hbar\nabla B_R)\|_{H^3}^2+\|\hbar^2\Delta N_R\|_{H^3}^2,\end{aligned} \tag{2.1}$$

where we have used curl-div estimates (A.12) and ω_R, given in definition (2.9), to obtain, for $2\leq k\leq 3$,

$$\|\nabla U_R\|_{H^k}\lesssim\|\omega_R\|_{H^k}+\|(\operatorname{div}U_R,B_R)\|_{H^k}. \tag{2.2}$$

Next, by the local existence results in Theorem 1.2, there exists $\tau_\varepsilon>0$ such that on $(0,\tau_\varepsilon)$ there exists a smooth solution (N_R,U_R,T_R,E_R,B_R) to system (1.8) satisfying

$$\sup_{(0,\tau_\varepsilon)}\|N_R\|_{L^\infty}\leq\sup_{(0,\tau_\varepsilon)}\|N_R\|_{H^2}\leq C(\tau_\varepsilon)\sqrt{\varepsilon}.$$

Hence, recalling $n^\varepsilon = 1 - \varepsilon \mathrm{div} E^0 + N_R$, we immediately obtain n^ε is bounded from above and below

$$1/2 < n^\varepsilon < 3/2, \tag{2.3}$$

where $0 < \varepsilon < \min\{\frac{1}{4(\|E^0\|_{H^3} + C(\tau_\varepsilon))^2}, 1\}$ such that $|-\varepsilon \mathrm{div} E^0 + N_R| \le (\|E^0\|_{H^3} + C(\tau_\varepsilon))\sqrt{\varepsilon} \le \frac{1}{2}$.

The rest is to show that for any given $0 < \tau_0 < \tau_*$, there is some $\varepsilon_0 = \varepsilon_0(\tau_0) > 0$ such that the existence time $\tau_\varepsilon \ge \tau_0$ for any $0 < \varepsilon < \varepsilon_0$. To prove Theorem 1.3, we need to derive the uniform *a priori* estimates with respect to ε for the remainder system (1.8) by energy methods in several steps. To this end, we demonstrate some useful lemmas in the Appendix at first. Then by elaborate energy estimates and vector analysis, we derive some estimates through Lemma 2.1 to Lemma 2.6 that will be used for the proof of Proposition 2.1. Moreover, for the estimate of gradient, we need to obtain some estimates of vorticity and divergence.

For any $0 < \tau_1 < 1$, let $\tau = \min\{\tau_1, \tau_\varepsilon\}$.

PROPOSITION 2.1. *Let* $(N_R, U_R, T_R, E_R, B_R)$ *be a solution to system* (1.8). *For all* $t \in (0, \tau)$, *there holds*

$$\begin{aligned}
&|||(N_R, U_R, T_R, B_R, \sqrt{\varepsilon} E_R)(t)|||_3 + \int_0^t \Big(\kappa \|\nabla T_R\|_{H^3}^2 + \mu \|\nabla U_R\|_{H^3}^2 + (\mu+\nu)\|\mathrm{div} U_R\|_{H^3}^2 \\
&+ \hbar^2 \mu \|\Delta U_R\|_{H^3}^2 + \hbar^2(\mu+\nu)\|\nabla \mathrm{div} U_R\|_{H^3}^2\Big) \lesssim |||(N_R, U_R, T_R, B_R, \sqrt{\varepsilon} E_R)(0)|||_3 \\
&+ \int_0^t \Big((1 + |||(N_R, U_R, T_R, B_R, \sqrt{\varepsilon} E_R)|||_3^5)|||(N_R, U_R, T_R, B_R, \sqrt{\varepsilon} E_R)|||_3 + \varepsilon\Big).
\end{aligned}$$

This proposition is proved as a direct sequence of the following lemmas.

In the following, we first give the L^2-estimates.

2.1. L^2-estimates.

LEMMA 2.1. *Under the assumptions in Proposition 2.1, we obtain*

$$\begin{aligned}
&\frac{d}{dt}\|(U_R, N_R, \hbar \nabla N_R, \sqrt{\varepsilon} E_R, B_R)\|_{L^2}^2 + \mu \|\nabla U_R\|_{L^2}^2 + (\mu+\lambda)\|\mathrm{div} U_R\|_{L^2}^2 \\
&\lesssim (1 + \|(U_R, T_R, N_R)\|_{H^3}^2)\|(U_R, N_R, \sqrt{\varepsilon} E_R, B_R, \hbar \nabla N_R, \hbar \nabla U_R, \hbar \mathrm{div} U_R)\|_{L^2}^2 \\
&\quad + \frac{\kappa}{32}\|\nabla T_R\|_{L^2}^2 + (1 + \|U_R\|_{H^3}^2)\|\frac{N_R}{\sqrt{\varepsilon}}\|_{L^2}^2 + \varepsilon.
\end{aligned} \tag{2.4}$$

Proof. Taking inner product with U_R in equation (1.8b), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_R\|_{L^2}^2 + \int E_R \cdot U_R = R_{1,1} - \int \frac{T^\varepsilon \nabla N_R}{n^\varepsilon} \cdot U_R + \frac{\hbar^2}{12}\int \nabla \Delta log n^\varepsilon \cdot U_R \\
&\quad + \mu \int \frac{\Delta U_R}{n^\varepsilon} \cdot U_R + (\mu+\lambda)\int \frac{\nabla \mathrm{div} U_R}{n^\varepsilon} \cdot U_R = \sum_{i=1}^5 R_{1,i},
\end{aligned}$$

where

$$
\begin{aligned}
R_{1,1}=&-\int U_R\cdot\nabla u^0\cdot U_R+\frac{1}{2}\int \mathrm{div}u^\varepsilon|U_R|^2-\int\nabla T_R\cdot U_R+\varepsilon\int\frac{T^\varepsilon\nabla\mathrm{div}E^0}{n^\varepsilon}\cdot U_R\\
&-\frac{\hbar^2}{24}\int\frac{\nabla n^\varepsilon}{n^\varepsilon}\cdot\frac{\nabla n^\varepsilon}{n^\varepsilon}\mathrm{div}U_R-\mu\int\frac{N_R\Delta u_0}{n^\varepsilon}\cdot U_R-\int u^\varepsilon\times B_R\cdot U_R\\
&-\int U_R\times B^0\cdot U_R+\mu\int\frac{\varepsilon\mathrm{div}E^0\Delta u_0}{n^\varepsilon}\cdot U_R.
\end{aligned}
$$

Obviously, using Young's inequality, Hölder's inequality and Sobolev embedding $H^2\hookrightarrow L^\infty$, we deduce

$$
R_{1,1}\lesssim\frac{\kappa}{32}\|\nabla T_R\|_{L^2}^2+(1+\|(N_R,T_R,U_R)\|_{H^3}^2)\|(N_R,U_R,T_R,B_R,\hbar\nabla N_R,\hbar\mathrm{div}U_R)\|_{L^2}^2+\varepsilon.
$$

Owing to equation (1.8a), we have

$$
n^\varepsilon\mathrm{div}U_R=-\partial_t N_R-u^\varepsilon\cdot\nabla N_R+\varepsilon U_R\cdot\nabla\mathrm{div}E^0-\varepsilon\Re_1, \tag{2.5}
$$

which enables us to rewrite $R_{1,2}$ as

$$
\begin{aligned}
R_{1,2}=&-\int\frac{T^\varepsilon\nabla N_R}{n^\varepsilon}\cdot U_R=\int N_R\nabla(\frac{T^\varepsilon}{n^\varepsilon})\cdot U_R+\int\frac{T^\varepsilon N_R\mathrm{div}U_R}{n^\varepsilon}\\
=&\int N_R\nabla(\frac{T^\varepsilon}{n^\varepsilon})\cdot U_R+\int\frac{T_RN_R\mathrm{div}U_R}{n^\varepsilon}-\frac{1}{2}\frac{d}{dt}\int\frac{T^0|N_R|^2}{(n^\varepsilon)^2}+\frac{1}{2}\int\partial_t\frac{T^0}{(n^\varepsilon)^2}|N_R|^2\\
&+\frac{1}{2}\int\mathrm{div}(\frac{T^0u^\varepsilon}{(n^\varepsilon)^2})|N_R|^2+\int\frac{T^0N_R}{(n^\varepsilon)^2}(\varepsilon U_R\cdot\nabla\mathrm{div}E^0-\varepsilon\Re_1)\\
\lesssim&-\frac{1}{2}\frac{d}{dt}\int\frac{T^0|N_R|^2}{(n^\varepsilon)^2}+(1+\|(T_R,U_R,N_R)\|_{H^3}^2)\|(N_R,U_R)\|_{L^2}^2+\frac{\mu+\nu}{32}\|\mathrm{div}U_R\|_{L^2}^2+\varepsilon,
\end{aligned}
$$

thanks to Young's inequality, Hölder's inequality, integration by parts, and estimate (A.8) in the Appendix.

We now treat $R_{1,3}$. It requires much efforts since it involves higher order terms. We apply integration by parts and equation (2.5) to decompose

$$
\begin{aligned}
R_{1,3}=&\frac{\hbar^2}{12}\int\nabla\Delta logn^\varepsilon\cdot U_R=-\frac{\hbar^2}{12}\int\mathrm{div}(\frac{\nabla n^\varepsilon}{n^\varepsilon})\mathrm{div}U_R\\
=&-\frac{\hbar^2}{12}\int\frac{\Delta n^\varepsilon\mathrm{div}U_R}{n^\varepsilon}-\frac{\hbar^2}{12}\int\nabla\frac{1}{n^\varepsilon}\cdot\nabla n^\varepsilon\mathrm{div}U_R\\
=&-\frac{\hbar^2}{24}\frac{d}{dt}\int\frac{|\nabla N_R|^2}{(n^\varepsilon)^2}+\frac{\hbar^2}{12}\int\frac{\Delta N_R(u^\varepsilon\cdot\nabla N_R)}{(n^\varepsilon)^2}+\Big\{\varepsilon\frac{\hbar^2}{12}\int\frac{\Delta\mathrm{div}E^0\mathrm{div}U_R}{n^\varepsilon}\\
&+\frac{\hbar^2}{6}\int\frac{\nabla n^\varepsilon\cdot\nabla N_R\partial_tN_R}{(n^\varepsilon)^3}-\frac{\hbar^2}{12}\int\frac{|\nabla N_R|^2}{(n^\varepsilon)^3}\partial_tn^\varepsilon-\frac{\hbar^2}{12}\int\nabla\frac{1}{n^\varepsilon}\cdot\nabla n^\varepsilon\mathrm{div}U_R\\
&+\varepsilon\frac{\hbar^2}{12}\int\frac{\Delta N_R(-U_R\cdot\nabla\mathrm{div}E^0+\Re_1)}{(n^\varepsilon)^2}\Big\}\\
=&-\frac{\hbar^2}{24}\frac{d}{dt}\int\frac{|\nabla N_R|^2}{(n^\varepsilon)^2}+R_{1,3,1}+R_{1,3,2}.
\end{aligned}
$$

By integration by parts, Hölder's inequality, Sobolev embedding $H^2 \hookrightarrow L^\infty$ and estimate (A.10) in the Appendix, $R_{1,3,1}$ can be treated as

$$\begin{aligned}R_{1,3,1} =& \frac{\hbar^2}{6}\int \frac{\nabla n^\varepsilon \cdot \nabla N_R (u^\varepsilon \cdot \nabla N_R)}{(n^\varepsilon)^3} - \frac{\hbar^2}{12}\int \frac{\nabla N_R \cdot \nabla (u^\varepsilon \cdot \nabla N_R)}{(n^\varepsilon)^2}\\ =& \frac{\hbar^2}{6}\int \frac{\nabla n^\varepsilon \cdot \nabla N_R (u^\varepsilon \cdot \nabla N_R)}{(n^\varepsilon)^3} - \frac{\hbar^2}{12}\int \frac{\nabla N_R \cdot \nabla u^\varepsilon \cdot \nabla N_R}{(n^\varepsilon)^2}\\ &+ \frac{\hbar^2}{24}\int \mathrm{div}\Big(\frac{u^\varepsilon}{(n^\varepsilon)^2}\Big)|\nabla N_R|^2\\ \lesssim& (1+\|(N_R,U_R)\|_{H^3}^2)\|\hbar\nabla N_R\|_{L^2}^2.\end{aligned}$$

In what follows, using integration by parts, Young's inequality, and estimate (A.8) in the Appendix together with the fact that E^0, u^0 are known smooth solutions of the limit system (1.5) by Theorem 1.1 and $\Re_1$ only consists of E^0 and u^0, we derive

$$R_{1,3,2} \lesssim (1+\|(U_R,N_R)\|_{H^3}^2)\|(U_R,\hbar\nabla N_R,\hbar\mathrm{div}U_R,\hbar\nabla U_R)\|_{L^2}^2+\varepsilon.$$

Therefore, combining all estimates for $R_{1,3}$, we obtain

$$R_{1,3} \lesssim -\frac{\hbar^2}{24}\frac{d}{dt}\int \frac{|\nabla N_R|^2}{(n^\varepsilon)^2} + (1+\|(U_R,N_R)\|_{H^3}^2)\|(U_R,\hbar\nabla N_R,\hbar\nabla U_R)\|_{L^2}^2+\varepsilon.$$

For the term $R_{1,4}$, with the aid of Young's inequality, the bounds (2.3), and integration by parts, we have

$$\begin{aligned}R_{1,4} =& \mu\int \frac{\Delta U_R}{n^\varepsilon}\cdot U_R = -\mu\int \frac{|\nabla U_R|^2}{n^\varepsilon} + \mu\int \frac{\nabla n^\varepsilon \cdot \nabla U_R \cdot U_R}{(n^\varepsilon)^2}\\ \lesssim& -\mu\int \frac{|\nabla U_R|^2}{n^\varepsilon} + \frac{\mu}{32}\|\nabla U_R\|_{L^2}^2 + (1+\|N_R\|_{H^3}^2)\|U_R\|_{L^2}^2.\end{aligned}$$

Similar to $R_{1,4}$, we obtain

$$R_{1,5} \lesssim -(\mu+\lambda)\int \frac{|\mathrm{div}U_R|^2}{n^\varepsilon} + \frac{(\mu+\nu)}{32}\|\mathrm{div}U_R\|_{L^2}^2 + (1+\|N_R\|_{H^3}^2)\|U_R\|_{L^2}^2.$$

On the other hand, multiplying the first equation by E_R and the second equation by B_R in equation (1.8d), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|(\sqrt{\varepsilon}E_R,B_R)\|_{L^2}^2 - \int \nabla\times B_R\cdot E_R + \int \nabla \times E_R \cdot B_R - \int E_R\cdot U_R\\ =& -\int \varepsilon \mathrm{div}E^0 U_R\cdot E_R + \int N_R u^\varepsilon \cdot E_R - \varepsilon\int \Re_4\cdot E_R\\ \lesssim& \varepsilon\|E_R\|_{L^2}^2 + \varepsilon\|U_R\|_{L^2}^2 + (1+\|U_R\|_{H^3}^2)\|\frac{N_R}{\sqrt{\varepsilon}}\|_{L^2}^2+\varepsilon,\end{aligned}$$

thanks to $\int(\nabla\times B_R\cdot E_R - \nabla\times E_R\cdot B_R) = \int \mathrm{div}(B_R\times E_R) = 0$ and Young's inequality.

Now, putting all the estimates together, we complete the proof of Lemma 2.1, thanks to the bounds (2.3). ∎

Note that an extra singular term $\|\frac{N_R}{\sqrt{\varepsilon}}\|_{L^2}$ appears, which makes the higher order estimates more difficult, since the order of derivatives of the term $\frac{N_R}{\sqrt{\varepsilon}}$ is not the same as others. However, it is also vital that the extra singular term has no effect on the temperature equation, which enables us to derive the following higher order estimates.

2.2. High order estimates for the temperature.

LEMMA 2.2. *Let $0\le k\le 3$ be an integer, (N_R,U_R,T_R,E_R,B_R) be a solution to system (1.8), and α be a multi-index with $|\alpha|=k$, then for any $t\in(0,\tau)$, we obtain*

$$\begin{aligned}&\frac{d}{dt}\|\partial^\alpha T_R\|_{L^2}^2+\kappa\|\partial^\alpha\nabla T_R\|_{L^2}^2\\ &\lesssim(1+\|(U_R,N_R,T_R)\|_{H^3}^8)\|(N_R,U_R,T_R,\hbar\nabla N_R)\|_{H^3}^2+\frac{\hbar^2\mu}{32}\|\Delta U_R\|_{H^3}^2\\ &\quad+\frac{(\mu+\nu)}{16}\|\text{div}U_R\|_{H^3}^2+\frac{\mu}{8}\|\nabla U_R\|_{H^3}^2+2\hbar^4\|\Delta U_R\|_{H^3}^2+\varepsilon.\end{aligned}\tag{2.6}$$

Proof. Applying ∂^α to equation (1.8c) and taking inner product with $\partial^\alpha T_R$ yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha T_R\|_{L^2}^2+\frac{2\kappa}{3}\int\frac{|\partial^\alpha\nabla T_R|^2}{n^\varepsilon}\\ =&\frac{2\kappa}{3}\int[\partial^\alpha,\frac{1}{n^\varepsilon}]\Delta T_R\partial^\alpha T_R+\frac{2\kappa}{3}\int\frac{\nabla n^\varepsilon\cdot\partial^\alpha\nabla T_R\partial^\alpha T_R}{(n^\varepsilon)^2}-\int[\partial^\alpha,u^\varepsilon]\cdot\nabla T_R\partial^\alpha T_R\\ &-\int u^\varepsilon\cdot\partial^\alpha\nabla T_R\partial^\alpha T_R-\frac{2}{3}\int[\partial^\alpha,T^\varepsilon]\text{div}U_R\partial^\alpha T_R-\frac{2}{3}\int T^\varepsilon\partial^\alpha\text{div}U_R\partial^\alpha T_R\\ &+\frac{\hbar^2}{36}\int\partial^\alpha\Delta U_R\cdot\partial^\alpha\nabla T_R-\frac{\hbar^2}{36}\int\partial^\alpha\Big(\frac{\nabla N_R\cdot\Delta U_R}{n^\varepsilon}-\varepsilon\frac{\nabla\text{div}E^0\cdot\Delta U_R}{n^\varepsilon}\Big)\partial^\alpha T_R\\ &+\frac{\mu}{3}\int\partial^\alpha\Big(\frac{2(\nabla u^0+(\nabla u^0)^\top)(\nabla U_R+(\nabla U_R)^\top)}{n^\varepsilon}+\frac{(\nabla U_R+(\nabla U_R)^\top)^2}{n^\varepsilon}\Big)\partial^\alpha T_R\\ &+\frac{2\nu}{3}\int\partial^\alpha\Big(\frac{(\text{div}U_R)^2}{n^\varepsilon}\Big)\partial^\alpha T_R-\int\partial^\alpha\Re_3\partial^\alpha T_R\\ \lesssim&-\frac{\hbar^2}{36}\int\partial^\alpha\Big(\frac{\nabla N_R\cdot\Delta U_R}{n^\varepsilon}\Big)\partial^\alpha T_R+\frac{\mu}{3}\int\partial^\alpha\Big(\frac{(\nabla U_R+(\nabla U_R)^\top)^2}{n^\varepsilon}\Big)\partial^\alpha T_R\\ &+\frac{2\nu}{3}\int\partial^\alpha\Big(\frac{(\text{div}U_R)^2}{n^\varepsilon}\Big)\partial^\alpha T_R+(1+\|(U_R,N_R,T_R)\|_{H^3}^6)\|(N_R,U_R,T_R,\hbar\nabla N_R)\|_{H^3}^2\\ &+\frac{\kappa}{8}\|\nabla T_R\|_{H^3}^2+\frac{(\mu+\nu)}{32}\|\text{div}U_R\|_{H^3}^2+\frac{\mu}{32}\|\nabla U_R\|_{H^3}^2+\hbar^4\|\Delta U_R\|_{H^3}^2+\varepsilon,\end{aligned}$$

thanks to integration by parts, definition (1.9b), the bounds (2.3), together with estimates (A.1) and (A.10) in the Appendix. Moreover, recalling the definition of $\Re_3$ in (1.9b), we note that $\Re_3$ depends on (E^0,u^0,T^0), (N_R,U_R) and the first derivation of N_R. We only deal with the first term on the RHS. It can be estimated that

$$\begin{aligned}&-\frac{\hbar^2}{36}\int\partial^\alpha\Big(\frac{\nabla N_R\cdot\Delta U_R}{n^\varepsilon}\Big)\partial^\alpha T_R\\ &\lesssim\hbar^2\|T_R\|_{H^k}\Big(\|\frac{\nabla N_R}{n^\varepsilon}\|_{L^\infty}\|\Delta U_R\|_{H^k}+(\|\nabla N_R\|_{H^k}+\|\frac{1}{n^\varepsilon}\|_{H^3}\|\nabla N_R\|_{L^\infty})\|\Delta U_R\|_{L^\infty}\Big)\\ &\lesssim\frac{\hbar^2\mu}{32}\|\Delta U_R\|_{H^3}^2+\frac{\mu}{16}\|\nabla U_R\|_{H^3}^2+(1+\|(U_R,N_R,T_R)\|_{H^3}^8)\|(T_R,\hbar\nabla N_R)\|_{H^3}^2.\end{aligned}$$

The second and the third integrals on the RHS can be estimated similarly. Now, putting all estimates together, we complete the proof of Lemma 2.2. ∎

The above method can not be applied directly to the higher order Sobolev energy estimates for velocity field, since the order of derivatives of the term $\frac{N_R}{\sqrt{\varepsilon}}$ is less than U_R. So we need to establish the following equations of vorticity and divergence.

2.3. Vorticity and divergence equations. Taking curl to equation (1.8b) and using the equation $\partial_t B_R + \nabla\times E_R = 0$, we have

$$\begin{aligned}&\partial_t\big(\nabla\times(U_R-A_R)\big)+\nabla\times(u^\varepsilon\cdot\nabla U_R+U_R\cdot\nabla u^0)+\nabla\times(\frac{T^\varepsilon\nabla N_R}{n^\varepsilon})\\&-\mu\nabla\times(\frac{\Delta U_R}{n^\varepsilon})-(\mu+\nu)\nabla\times(\frac{\nabla\mathrm{div}U_R}{n^\varepsilon})+\nabla\times(-\varepsilon\frac{T^\varepsilon\nabla\mathrm{div}E^0}{n^\varepsilon}+\mu\frac{N_R\Delta u^0}{n^\varepsilon}\\&-\mu\frac{\varepsilon\mathrm{div}E^0\Delta u_0}{n^\varepsilon})+\nabla\times(u^\varepsilon\times B_R+U_R\times B^0)=0.\qquad(2.7)\end{aligned}$$

Here we have used $B_R=\nabla\times A_R$, for some vector field A_R, since $\mathrm{div}B_R=0$. Thanks to identities (A.13) and (A.14) in the Appendix, we derive

$$\begin{aligned}\nabla\times(u^\varepsilon\cdot\nabla U_R)=&\nabla\times(u^0\cdot\nabla U_R)+\nabla\times(U_R\cdot\nabla U_R)\\=&\nabla\times\big((\nabla\times U_R)\times u^\varepsilon\big)+\nabla\times(\nabla\times u^0\times U_R)-\nabla\times(\nabla u^0\cdot U_R).\qquad(2.8)\end{aligned}$$

In the following, we denote

$$\omega_R=\nabla\times(U_R-A_R).\qquad(2.9)$$

Putting the above result (2.8) into equation (2.7), and using identity (A.16) in the Appendix, we have

$$\begin{aligned}&\partial_t\omega_R+u^\varepsilon\cdot\nabla\omega_R-\omega_R\cdot\nabla u^\varepsilon+\omega_R\mathrm{div}u^\varepsilon-\mu\nabla\times(\frac{\Delta U_R}{n^\varepsilon})+(\mu+\nu)(\frac{\nabla n^\varepsilon\times\nabla\mathrm{div}U_R}{(n^\varepsilon)^2})\\&+\nabla\frac{T^\varepsilon}{n^\varepsilon}\times\nabla N_R+F_1=0,\qquad(2.10)\end{aligned}$$

where F_1 depends at most on the first derivatives of T_R, N_R and U_R,

$$F_1=\nabla\times(-\varepsilon\frac{T^\varepsilon\nabla\mathrm{div}E^0}{n^\varepsilon}+\mu\frac{N_R\Delta u^0}{n^\varepsilon}-\mu\frac{\varepsilon\mathrm{div}E^0\Delta u_0}{n^\varepsilon})+\nabla\times(\nabla\times u^0\times U_R+U_R\times B^0).\qquad(2.11)$$

Taking div to equation (1.8b) and using the equation $\varepsilon\mathrm{div}E_R=-N_R$ in equation (1.8e), we obtain

$$\begin{aligned}&\partial_t\mathrm{div}U_R+\mathrm{div}(u^\varepsilon\cdot\nabla U_R)-\frac{N_R}{\varepsilon}+\Delta T_R+\mathrm{div}(\frac{T^\varepsilon\nabla N_R}{n^\varepsilon})-\frac{\hbar^2}{12}(\Delta\Delta logn^\varepsilon+\frac{1}{2}\Delta|\nabla logn^\varepsilon|^2)\\&-\mu\mathrm{div}(\frac{\Delta U_R}{n^\varepsilon})-(\mu+\nu)\mathrm{div}(\frac{\nabla\mathrm{div}U_R}{n^\varepsilon})+F_2=0,\qquad(2.12)\end{aligned}$$

where

$$F_2=\mathrm{div}(-\varepsilon\frac{T^\varepsilon\nabla\mathrm{div}E^0}{n^\varepsilon}+\mu\frac{N_R\Delta u^0}{n^\varepsilon}-\mu\frac{\varepsilon\mathrm{div}E^0\Delta u^0}{n^\varepsilon}+u^\varepsilon\times B_R+U_R\cdot\nabla u^0+U_R\times B^0).\qquad(2.13)$$

Using the vorticity and divergency equations, we can derive following lemmas. In what follows, we sometimes abuse the notation $\alpha+1$ to stand for $\alpha+\beta$ for a multi-index with $|\beta|=1$.

2.4. High order estimates for the velocity field. To close the inequality in Lemma 2.2, we need to get suitable controls no only for $\|(N_R,U_R,\hbar\nabla N_R)\|_{H^3}^2$ but also for $\|\hbar^2\Delta U_R\|_{H^3}^2$. However, the control for H^5-norm of U_R depends on the higher order term with quantum effect in the divergency equation (2.12) and hence can not be closed until Lemma 2.4. Moreover, by the curl-div decomposition formula of the gradient in the Appendix, we derive the estimates for (U_R,B_R) at third order and the estimates for $(\hbar U_R,\hbar B_R)$ at fourth order. Finally, we derive the estimates for the two terms $\|\nabla U_R\|_{H^3}$ and $\hbar\|\Delta U_R\|_{H^3}$, owing to $\nabla \mathrm{div} U_R=\Delta U_R+\nabla\times(\nabla\times U_R)$.

LEMMA 2.3. *Let α be a muti-index with $|\alpha|=k$ for any integer number $0\le k\le 2$. (N_R,U_R,T_R,E_R,B_R) be a solution to system (1.8). For all $t\in(0,\tau)$, we have*

$$\begin{aligned}&\frac{d}{dt}\|(\partial^\alpha\omega_R,\hbar\partial^{\alpha+1}\omega_R)\|_{L^2}^2+\mu\|\partial^\alpha\nabla(\nabla\times U_R)\|_{L^2}^2+\hbar^2\mu\|\partial^{\alpha+1}\nabla(\nabla\times U_R)\|_{L^2}^2\\ \lesssim&\frac{\kappa}{32}\|\nabla T_R\|_{H^3}^2+\frac{\mu}{32}\|\nabla U_R\|_{H^3}^2+\frac{(\mu+\lambda)}{16}\|\nabla \mathrm{div} U_R\|_{H^2}^2+\frac{(\mu+\lambda)\hbar^2}{32}\|\mathrm{div} U_R\|_{H^4}^2\\ &+(1+\|\omega_R\|_{H^2}^2+\|(U_R,T_R,N_R,\hbar\nabla N_R)\|_{H^3}^6)\big(\|\omega_R\|_{H^2}^2\\ &+\|(B_R,U_R,T_R,N_R,\hbar\omega_R,\hbar\nabla U_R,\hbar\mathrm{div} U_R,\hbar\nabla N_R,\hbar\nabla B_R)\|_{H^3}^2\big)+\varepsilon.\end{aligned}$$

Proof. Taking ∂^α to equation (2.10) and taking inner product with $\partial^\alpha\omega_R$, and using integration by parts and the commutator, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\omega_R\|_{L^2}^2=&\mu\int\partial^\alpha\nabla\times(\frac{\Delta U_R}{n^\varepsilon})\partial^\alpha\omega_R-\int[\partial^\alpha,u^\varepsilon]\cdot\nabla\omega_R\cdot\partial^\alpha\omega_R+\frac{1}{2}\int \mathrm{div} u^\varepsilon|\partial^\alpha\omega_R|^2\\ &+\int\partial^\alpha(\omega_R\cdot\nabla u^\varepsilon)\cdot\partial^\alpha\omega_R-\int\partial^\alpha(\omega_R\mathrm{div} u^\varepsilon)\cdot\partial^\alpha\omega_R-\int\partial^\alpha F_1\partial^\alpha\omega_R\\ &-(\mu+\nu)\int\partial^\alpha(\frac{\nabla n^\varepsilon\times\nabla\mathrm{div} U_R}{(n^\varepsilon)^2})\partial^\alpha\omega_R-\int\partial^\alpha(\nabla\frac{T^\varepsilon}{n^\varepsilon}\times\nabla N_R)\partial^\alpha\omega_R.\end{aligned}\tag{2.14}$$

For the first term on the RHS of equation (2.14), from the bounds (2.3), the definition of ω_R in (2.9), integration by parts, Young's inequality, and estimates (A.1), (A.6) and (A.10) in the Appendix, we have

$$\begin{aligned}&\mu\int\partial^\alpha\nabla\times(\frac{\Delta U_R}{n^\varepsilon})\cdot\partial^\alpha\omega_R\\ =&\mu\int[\partial^\alpha\nabla\times,\frac{1}{n^\varepsilon}]\Delta U_R\cdot\partial^\alpha\omega_R+\mu\int\frac{\partial^\alpha\Delta\nabla\times U_R}{n^\varepsilon}\cdot\partial^\alpha(\nabla\times U_R-B_R)\\ =&-\mu\int\frac{|\partial^\alpha\nabla(\nabla\times U_R)|^2}{n^\varepsilon}-\mu\int\nabla\frac{1}{n^\varepsilon}\cdot\partial^\alpha\nabla(\nabla\times U_R)\cdot\partial^\alpha(\nabla\times U_R-B_R)\\ &+\mu\int\frac{\partial^\alpha\nabla(\nabla\times U_R):\partial^\alpha\nabla B_R}{n^\varepsilon}+\mu\int[\partial^\alpha\nabla\times,\frac{1}{n^\varepsilon}]\Delta U_R\cdot\partial^\alpha\omega_R\\ \lesssim&-\mu\int\frac{|\partial^\alpha\nabla(\nabla\times U_R)|^2}{n^\varepsilon}+\frac{\mu}{32}\|\nabla(\nabla\times U_R)\|_{H^k}^2+\frac{\mu}{32}\|\nabla U_R\|_{H^3}^2\\ &+(1+\|N_R\|_{H^3}^6)\|(\omega_R,\nabla B_R)\|_{H^k}^2.\end{aligned}$$

In particular, for the commutator term, we have, thanks to (A.1), (A.6) and (A.10) in the Appendix, that

$$\mu\int[\partial^\alpha\nabla\times,\frac{1}{n^\varepsilon}]\Delta U_R\cdot\partial^\alpha\omega_R$$

$$
\begin{aligned}
&\lesssim \mu(\|\nabla \frac{1}{n^\varepsilon}\|_{L^\infty}\|\Delta U_R\|_{H^k}+\|\frac{1}{n^\varepsilon}\|_{H^k}\|\nabla n^\varepsilon\|_{H^k}\|\Delta U_R\|_{L^\infty})\|\omega_R\|_{H^k}\\
&\lesssim \frac{\mu}{32}\|\nabla U_R\|_{H^3}^2+\frac{\mu}{32}\|\Delta U_R\|_{H^k}^2+(1+\|N_R\|_{H^3}^6)\|\omega_R\|_{H^k}^2.
\end{aligned}
$$

Recalling F_1 given by definition (2.11), and using Sobolev embedding $H^1\hookrightarrow L^3,L^6;H^2\hookrightarrow L^\infty$, Hölder's inequality, together with estimates (A.1), (A.2) and (A.4) in the Appendix, the other seven terms in equation (2.14) can be estimated by

$$
\frac{(\mu+\lambda)}{32}\|\nabla \mathrm{div} U_R\|_{H^2}^2+(1+\|(U_R,T_R,N_R)\|_{H^3}^6)\big(\|\omega_R\|_{H^2}^2+\|(U_R,T_R,N_R)\|_{H^3}^2\big)+\varepsilon.
$$

Putting all the above estimates together, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha \omega_R\|_{L^2}^2+\mu\int\frac{|\partial^\alpha\nabla(\nabla\times U_R)|^2}{n^\varepsilon}\\
&\lesssim \frac{\mu}{32}\|\nabla U_R\|_{H^3}^2+\frac{(\mu+\lambda)}{32}\|\nabla \mathrm{div} U_R\|_{H^2}^2+(1+\|(U_R,T_R,N_R)\|_{H^3}^6)\big(\|\omega_R\|_{H^2}^2\\
&\qquad+\|(U_R,T_R,N_R,B_R)\|_{H^3}^2\big)+\varepsilon.
\end{aligned}
$$

Moreover, applying the operator $\partial^{\alpha+1}$ to equation (2.10) and taking inner product with $\hbar^2\partial^{\alpha+1}\omega_R$, we obtain

$$
\begin{aligned}
&\frac{\hbar^2}{2}\frac{d}{dt}\|\partial^{\alpha+1}\omega_R\|_{L^2}^2+\hbar^2\mu\int\frac{|\partial^{\alpha+1}\nabla(\nabla\times U_R)|^2}{n^\varepsilon}\\
&\lesssim \frac{\mu\hbar^2}{32}\|\nabla\times U_R\|_{H^4}^2+\frac{\kappa}{32}\|\nabla T_R\|_{H^3}^2+\frac{(\mu+\lambda)}{32}\|\mathrm{div} U_R\|_{H^3}^2+\hbar^4\|\nabla(\nabla\times U_R)\|_{H^3}^2\\
&\qquad+\frac{(\mu+\lambda)\hbar^2}{32}\|\mathrm{div} U_R\|_{H^4}^2+(1+\|\omega_R\|_{H^2}^2+\|(U_R,T_R,N_R,\hbar\nabla N_R)\|_{H^3}^6)\\
&\qquad\|(B_R,U_R,T_R,N_R,\hbar\omega_R,\hbar\nabla U_R,\hbar\mathrm{div} U_R,\hbar\nabla N_R,\hbar\nabla B_R)\|_{H^3}^2+\varepsilon,
\end{aligned}
$$

thanks to definition (2.11), and estimates (A.2), (A.3) and (A.6) in the Appendix.

Now, putting all the estimates together then completes the proof of Lemma 2.3, thanks to $\hbar^4\ll\hbar^2$ for sufficiently small $\hbar>0$. ∎

LEMMA 2.4. *Let α be a multi-index with $|\alpha|=k$ for any integer number $0\le k\le 2$. (N_R,U_R,T_R,E_R,B_R) be a solution to system (1.8). For all $t\in(0,\tau)$, we obtain*

$$
\begin{aligned}
&\frac{d}{dt}\|\partial^\alpha(\mathrm{div} U_R,\nabla N_R,\hbar\Delta N_R,\frac{N_R}{\sqrt{\varepsilon}})\|_{L^2}^2+\frac{d}{dt}\|\partial^{\alpha+1}(\hbar\mathrm{div} U_R,\hbar\frac{N_R}{\sqrt{\varepsilon}},\hbar^2\Delta N_R)\|_{L^2}^2\\
&\quad+(2\mu+\nu)\|\partial^\alpha\nabla\mathrm{div} U_R\|_{L^2}^2+(2\mu+\nu)\hbar^2\|\partial^{\alpha+1}\nabla\mathrm{div} U_R\|_{L^2}^2\\
&\lesssim \frac{\kappa}{32}\|\nabla T_R\|_{H^3}^2+(1+\|(U_R,T_R,N_R)\|_{H^3}^{10})\Big(\|\frac{N_R}{\sqrt{\varepsilon}})\|_{H^2}^2\\
&\qquad+\|(U_R,N_R,T_R,B_R,\hbar\mathrm{div} U_R,\hbar\nabla U_R,\hbar\nabla N_R,\hbar^2\Delta N_R,\hbar\frac{N_R}{\sqrt{\varepsilon}})\|_{H^3}^2\Big)+\varepsilon.
\end{aligned}
$$

Proof. Applying the operator ∂^α to equation (2.12) and taking inner product with $\partial^\alpha\mathrm{div} U_R$, we derive

$$
\frac{1}{2}\frac{d}{dt}\|\partial^\alpha\mathrm{div} U_R\|_{L^2}^2=R_{3,1}+\int\partial^\alpha\frac{N_R}{\varepsilon}\partial^\alpha\mathrm{div} U_R-\int\mathrm{div}\partial^\alpha(\frac{T^0\nabla N_R}{n^\varepsilon})\partial^\alpha\mathrm{div} U_R
$$

$$+\frac{\hbar^2}{12}\int\partial^\alpha\Delta\Delta logn^\varepsilon\partial^\alpha\mathrm{div}U_R+\mu\int\partial^\alpha\mathrm{div}(\frac{\Delta U_R}{n^\varepsilon})\partial^\alpha\mathrm{div}U_R$$
$$+(\mu+\lambda)\int\partial^\alpha\mathrm{div}(\frac{\nabla\mathrm{div}U_R}{n^\varepsilon})\partial^\alpha\mathrm{div}U_R=\sum_{i=1}^{6}R_{3,i}, \tag{2.15}$$

where

$$R_{3,1}=-\int[\partial^\alpha\mathrm{div},u^\varepsilon]\nabla U_R\partial^\alpha\mathrm{div}U_R+\frac{1}{2}\int\mathrm{div}u^\varepsilon|\partial^\alpha\mathrm{div}U_R|^2+\int\partial^\alpha\nabla T_R\cdot\partial^\alpha\nabla\mathrm{div}U_R$$
$$+\int\partial^\alpha(\frac{T_R\nabla N_R}{n^\varepsilon})\cdot\partial^\alpha\nabla\mathrm{div}U_R-\frac{\hbar^2}{24}\int\partial^\alpha\nabla\Big(\frac{\nabla n^\varepsilon\cdot\nabla n^\varepsilon}{(n^\varepsilon)^2}\Big)\cdot\nabla\partial^\alpha\mathrm{div}U_R$$
$$-\int\partial^\alpha F_2\partial^\alpha\mathrm{div}U_R.$$

Using definition (2.13) and the estimates (A.1)−(A.5) in the Appendix, we have, with the aid of integration by part, Young's inequality and Sobolev embedding,

$$R_{3,1}\lesssim\frac{\mu+\nu}{8}\|\mathrm{div}U_R\|_{H^3}^2+(1+\|(U_R,N_R,T_R)\|_{H^3}^8)\|(N_R,U_R,B_R,T_R,\hbar\nabla N_R)\|_{H^3}^2+\varepsilon.$$

Owing to equation (2.5) and the commutator

$$n^\varepsilon\partial^\alpha\mathrm{div}U_R=\partial^\alpha(n^\varepsilon\mathrm{div}U_R)-[\partial^\alpha,n^\varepsilon]\mathrm{div}U_R, \tag{2.16}$$

$R_{3,2}$ can be decomposed and bounded by

$$R_{3,2}=-\frac{1}{2\varepsilon}\frac{d}{dt}\int\frac{|\partial^\alpha N_R|^2}{n^\varepsilon}+\frac{1}{2\varepsilon}\int\partial_t(\frac{1}{n^\varepsilon})|\partial^\alpha N_R|^2-\int\frac{\partial^\alpha N_R[\partial^\alpha,u^\varepsilon]\cdot\nabla N_R}{\varepsilon n^\varepsilon}$$
$$+\frac{1}{2\varepsilon}\int\mathrm{div}\frac{u^\varepsilon}{n^\varepsilon}|\partial^\alpha N_R|^2+\int\frac{\partial^\alpha N_R\partial^\alpha(\varepsilon U_R\cdot\nabla\mathrm{div}E^0-\varepsilon\Re_1)}{\varepsilon n^\varepsilon}$$
$$-\int\frac{\partial^\alpha N_R[\partial^\alpha,n^\varepsilon]\mathrm{div}U_R}{\varepsilon n^\varepsilon}$$
$$\lesssim-\frac{1}{2\varepsilon}\frac{d}{dt}\int\frac{|\partial^\alpha N_R|^2}{n^\varepsilon}+(1+\|(N_R,U_R)\|_{H^3}^2)(\|\frac{N_R}{\sqrt{\varepsilon}}\|_{H^k}^2+\|U_R\|_{H^2}^2)$$
$$+\frac{1}{\sqrt{\varepsilon}}(\|\nabla u^\varepsilon\|_{L^\infty}\|\nabla N_R\|_{H^{k-1}}+\|\nabla u^\varepsilon\|_{H^k}\|\nabla N_R\|_{L^3}+\|\nabla n^\varepsilon\|_{L^6}\|\mathrm{div}U_R\|_{H^k}$$
$$+\|\nabla n^\varepsilon\|_{H^{k-1}}\|\mathrm{div}U_R\|_{L^\infty})\|\frac{N_R}{\sqrt{\varepsilon}}\|_{H^k}+\varepsilon$$
$$\lesssim-\frac{1}{2\varepsilon}\frac{d}{dt}\int\frac{|\partial^\alpha N_R|^2}{n^\varepsilon}+(1+\|(N_R,U_R)\|_{H^3}^2)(\|\frac{N_R}{\sqrt{\varepsilon}}\|_{H^k}^2+\|U_R\|_{H^3}^2)+\varepsilon,$$

thanks to estimates (A.1) and (A.8) in the Appendix.

Now we deal with the estimate for $R_{3,3}$. Again, using the commutator and equation (2.5), we divide it into following

$$R_{3,3}=\int\partial^\alpha(\frac{T^0\nabla N_R}{n^\varepsilon})\cdot\partial^\alpha\nabla\mathrm{div}U_R$$
$$=\Big\{\int[\partial^\alpha,\frac{T^0}{n^\varepsilon}]\nabla N_R\cdot\partial^\alpha\nabla\mathrm{div}U_R-\int\frac{T^0\partial^\alpha\nabla N_R\cdot[\partial^\alpha\nabla,n^\varepsilon]\mathrm{div}U_R}{(n^\varepsilon)^2}\Big\}$$
$$-\int\frac{T^0\partial^\alpha\nabla N_R\cdot\nabla\partial^\alpha(\partial_tN_R+u^\varepsilon\cdot\nabla N_R-\varepsilon U_R\cdot\nabla\mathrm{div}E^0+\varepsilon\Re_1)}{(n^\varepsilon)^2}$$
$$=R_{3,3,1}+R_{3,3,2}.$$

Since T^0 is smooth, we obtain from the bounds (2.3), and estimates (A.1) and (A.10) in the Appendix that

$$\begin{aligned}R_{3,3,1}\lesssim&(\|\nabla\frac{T^0}{n^\varepsilon}\|_{L^\infty}\|N_R\|_{H^k}+\|\frac{T^0}{n^\varepsilon}\|_{H^k}\|\nabla N_R\|_{L^\infty})\|\nabla\mathrm{div}U_R\|_{H^k}+(\|\nabla n^\varepsilon\|_{L^\infty}\|\mathrm{div}U_R\|_{H^k}\\&+\|\nabla n^\varepsilon\|_{H^k}\|\mathrm{div}U_R\|_{L^\infty})\|\nabla N_R\|_{H^k}\\\lesssim&\frac{(\mu+\nu)}{16}\|\mathrm{div}U_R\|^2_{H^3}+(1+\|N_R\|^4_{H^3})\|(N_R,U_R)\|^2_{H^3}.\end{aligned}$$

With the aid of estimates (A.1) and (A.8) in the Appendix, we derive

$$\begin{aligned}R_{3,3,2}=&-\frac{1}{2}\frac{d}{dt}\int\frac{T^0|\partial^\alpha\nabla N_R|^2}{(n^\varepsilon)^2}+\frac{1}{2}\int\partial_t(\frac{T^0}{(n^\varepsilon)^2})|\partial^\alpha\nabla N_R|^2+\frac{1}{2}\int\mathrm{div}(\frac{T^0u^\varepsilon}{(n^\varepsilon)^2})|\partial^\alpha\nabla N_R|^2\\&-\int\frac{T^0\partial^\alpha\nabla N_R\cdot[\nabla\partial^\alpha,u^\varepsilon]\cdot\nabla N_R}{(n^\varepsilon)^2}+\int\frac{T^0\partial^\alpha\nabla N_R\cdot\nabla\partial^\alpha(\varepsilon U_R\cdot\nabla\mathrm{div}E^0-\varepsilon\Re_1)}{(n^\varepsilon)^2}\\\lesssim&-\frac{1}{2}\frac{d}{dt}\int\frac{T^0|\nabla\partial^\alpha N_R|^2}{(n^\varepsilon)^2}+(1+\|(N_R,U_R)\|^2_{H^3})\|(N_R,U_R)\|^2_{H^3}+\varepsilon.\end{aligned}$$

Combining all estimates for $R_{3,3}$, there holds

$$\begin{aligned}R_{3,3}\lesssim&-\frac{1}{2}\frac{d}{dt}\int\frac{T^0|\nabla\partial^\alpha N_R|^2}{(n^\varepsilon)^2}\\&+\frac{(\mu+\nu)}{16}\|\mathrm{div}U_R\|^2_{H^3}+(1+\|(N_R,U_R)\|^4_{H^3})\|(N_R,U_R)\|^2_{H^3}+\varepsilon.\end{aligned}$$

Now we turn to estimate $R_{3,4}$. In this term, we need to cope with higher order terms. To make it clear, we divide it into two parts. Using definition (1.7) and integration by parts to obtain

$$\begin{aligned}R_{3,4}=&\frac{\hbar^2}{12}\int\partial^\alpha\Delta\Delta logn^\varepsilon\partial^\alpha\mathrm{div}U_R=-\frac{\hbar^2}{12}\int\nabla\partial^\alpha\mathrm{div}(\frac{\nabla N_R-\varepsilon\nabla\mathrm{div}E^0}{n^\varepsilon})\cdot\nabla\partial^\alpha\mathrm{div}U_R\\=&\Big\{-\frac{\hbar^2}{12}\int[\partial^\alpha\nabla\mathrm{div},\frac{1}{n^\varepsilon}](\nabla N_R-\varepsilon\nabla\mathrm{div}E^0)\cdot\nabla\partial^\alpha\mathrm{div}U_R\\&+\frac{\hbar^2}{12}\int\partial^\alpha\Delta N_R\nabla\frac{1}{n^\varepsilon}\cdot\partial^\alpha\nabla\mathrm{div}U_R+\frac{\hbar^2\varepsilon}{12}\int\frac{\partial^\alpha\nabla\Delta\mathrm{div}E^0\cdot\nabla\partial^\alpha\mathrm{div}U_R}{n^\varepsilon}\Big\}\\&+\frac{\hbar^2}{12}\int\frac{\partial^\alpha\Delta N_R\Delta\partial^\alpha\mathrm{div}U_R}{n^\varepsilon}=R_{3,4,1}+R_{3,4,2}.\end{aligned}$$

For the term $R_{3,4,1}$, it is obvious by estimates (A.1) and (A.7) in the Appendix that

$$\begin{aligned}R_{3,4,1}\lesssim&\hbar^2\Big(\|\nabla\frac{1}{n^\varepsilon}\|_{L^\infty}(\|\Delta N_R\|_{H^k}+\varepsilon)+\|\Delta\frac{1}{n^\varepsilon}\|_{H^k}(\|\nabla N_R\|_{L^\infty}+\varepsilon)\Big)\|\nabla\mathrm{div}U_R\|_{H^k}\\&+\hbar^2\varepsilon\|\nabla\mathrm{div}U_R\|_{H^k}+\hbar^2\|\nabla\frac{1}{n^\varepsilon}\|_{L^\infty}\|\Delta N_R\|_{H^k}\|\nabla\mathrm{div}U_R\|_{H^k}\\\lesssim&\frac{3\hbar^2(\mu+\nu)}{32}\|\nabla\mathrm{div}U_R\|^2_{H^k}+(1+\|N_R\|^{10}_{H^3})(\hbar^2\|(\Delta N_R,\nabla N_R)\|^2_{H^k}+\|N_R\|^2_{H^3})+\varepsilon.\end{aligned}$$

Here we require $E^0\in H^6$. Owing to equation (2.5) and the commutator, we decompose

$R_{3,4,2}$ into following

$$
\begin{aligned}
R_{3,4,2} =& -\frac{\hbar^2}{24}\frac{d}{dt}\int \frac{|\partial^\alpha \Delta N_R|^2}{(n^\varepsilon)^2} + \frac{\hbar^2}{24}\int \partial_t \frac{1}{(n^\varepsilon)^2}|\partial^\alpha \Delta N_R|^2 - \frac{\hbar^2}{12}\int \frac{\partial^\alpha \Delta N_R[\partial^\alpha \Delta, u^\varepsilon]\cdot \nabla N_R}{(n^\varepsilon)^2} \\
&+ \frac{\hbar^2}{24}\int \mathrm{div}\left(\frac{u^\varepsilon}{(n^\varepsilon)^2}\right)|\partial^\alpha \Delta N_R|^2 + \frac{\hbar^2\varepsilon}{12}\int \frac{\partial^\alpha \Delta N_R[\partial^\alpha \Delta, U_R]\cdot \nabla \mathrm{div} E^0}{(n^\varepsilon)^2} \\
&+ \frac{\hbar^2\varepsilon}{12}\int \frac{\partial^\alpha \Delta N_R(U_R\cdot \partial^\alpha \nabla \Delta \mathrm{div} E^0 + \partial^\alpha \Delta \Re_1)}{(n^\varepsilon)^2} - \frac{\hbar^2}{12}\int \frac{\partial^\alpha \Delta N_R[\partial^\alpha \Delta, n^\varepsilon]\mathrm{div} U_R}{(n^\varepsilon)^2} \\
\lesssim& -\frac{\hbar^2}{24}\frac{d}{dt}\int \frac{|\partial^\alpha \Delta N_R|^2}{(n^\varepsilon)^2} + \hbar^2(1+\|(N_R,U_R)\|_{H^3}^2)\|\Delta N_R\|_{H^k}^2 + \hbar^2\varepsilon \|\Delta N_R\|_{H^k}(1 \\
&+ \|U_R\|_{H^3} + \|\Delta U_R\|_{H^k}) + \hbar^2 \|\Delta N_R\|_{H^k}\Big(\|(n^\varepsilon, u^\varepsilon)\|_{H^3}\|(\nabla \mathrm{div} U_R, \Delta N_R)\|_{H^k} \\
&+ \|(U_R,N_R)\|_{H^3}\|(\Delta n^\varepsilon, \Delta u^\varepsilon)\|_{H^k}\Big) \\
\lesssim& -\frac{\hbar^2}{24}\frac{d}{dt}\int \frac{|\partial^\alpha \Delta N_R|^2}{(n^\varepsilon)^2} + (1+\|(U_R,N_R)\|_{H^3}^2)(\hbar^2\|\Delta N_R\|_{H^k}^2 + \|(N_R,U_R)\|_{H^3}^2) \\
&+ \hbar^2\frac{\mu}{16}\|\Delta U_R\|_{H^k}^2 + \hbar^2\frac{(\mu+\nu)}{32}\|\nabla \mathrm{div} U_R\|_{H^k}^2 + \varepsilon,
\end{aligned}
$$

thanks to integration by parts and estimates (A.1) and (A.8) in the Appendix. In summary,

$$
\begin{aligned}
R_{3,4} \lesssim& -\frac{\hbar^2}{24}\frac{d}{dt}\int \frac{|\partial^\alpha \Delta N_R|^2}{(n^\varepsilon)^2} + (1+\|(U_R,N_R)\|_{H^3}^{10})(\hbar^2\|(\Delta N_R, \nabla N_R)\|_{H^k}^2 \\
&+ \|(N_R,U_R)\|_{H^3}^2) + \hbar^2\frac{\mu}{16}\|\Delta U_R\|_{H^k}^2 + \hbar^2\frac{\mu+\nu}{8}\|\nabla \mathrm{div} U_R\|_{H^k}^2 + \varepsilon.
\end{aligned}
$$

Recalling estimate (A.11) in the Appendix which implies $\|\hbar\partial^\alpha \Delta N_R\|_{L^2}^2 \ge \|\hbar\partial^\alpha \nabla^2 N_R\|_{L^2}^2$, we can derived the H^3-norm of $\hbar\nabla N_R$ due to $0\le k\le 2$.

For $R_{3,5}$, we have, by integration by parts twice and the commutator estimate,

$$
\begin{aligned}
R_{3,5} =& -\mu\int \partial^\alpha\left(\frac{\Delta U_R}{n^\varepsilon}\right)\cdot \nabla \partial^\alpha \mathrm{div} U_R \\
=& -\mu\int [\partial^\alpha, \frac{1}{n^\varepsilon}]\Delta U_R \cdot \nabla \partial^\alpha \mathrm{div} U_R - \mu\int \frac{\partial^\alpha \Delta U_R\cdot \partial^\alpha \nabla \mathrm{div} U_R}{n^\varepsilon} \\
=& -\mu\int [\partial^\alpha, \frac{1}{n^\varepsilon}]\Delta U_R \cdot \partial^\alpha \nabla \mathrm{div} U_R + \mu\int \nabla \frac{1}{n^\varepsilon}\cdot \partial^\alpha \Delta U_R \partial^\alpha \mathrm{div} U_R \\
&- \mu\int \nabla \frac{1}{n^\varepsilon}\cdot \partial^\alpha \nabla \mathrm{div} U_R \partial^\alpha \mathrm{div} U_R - \mu\int \frac{|\partial^\alpha \nabla \mathrm{div} U_R|^2}{n^\varepsilon} \\
\lesssim& -\mu\int \frac{|\partial^\alpha \nabla \mathrm{div} U_R|^2}{n^\varepsilon} + \frac{\mu}{16}\|\nabla \mathrm{div} U_R\|_{H^k}^2 \\
&+ \frac{\mu}{32}\|\Delta U_R\|_{H^k}^2 + (1+\|N_R\|_{H^3}^6)(\|U_R\|_{H^3}^2 + \|(\nabla U_R, \mathrm{div} U_R)\|_{H^k}^2).
\end{aligned}
$$

Similarly, we can obtain

$$
\begin{aligned}
R_{3,6} =& -(\mu+\nu)\int \frac{|\partial^\alpha \nabla \mathrm{div} U_R|^2}{n^\varepsilon} - (\mu+\nu)\int [\partial^\alpha, \frac{1}{n^\varepsilon}]\nabla \mathrm{div} U_R \cdot \partial^\alpha \nabla \mathrm{div} U_R \\
\lesssim& -(\mu+\nu)\int \frac{|\partial^\alpha \nabla \mathrm{div} U_R|^2}{n^\varepsilon} + \frac{(\mu+\nu)}{32}\|\nabla \mathrm{div} U_R\|_{H^k}^2 \\
&+ (1+\|N_R\|_{H^3}^6)(\|\mathrm{div} U_R\|_{H^k}^2 + \|U_R\|_{H^3}^2).
\end{aligned}
$$

Summarizing all the above estimates, we have from equation (2.15) that

$$\frac{d}{dt}\|\partial^\alpha(\mathrm{div}U_R,\nabla N_R,\hbar\Delta N_R,\frac{N_R}{\sqrt{\varepsilon}})\|_{L^2}^2+(2\mu+\lambda)\|\partial^\alpha\nabla\mathrm{div}U_R\|_{L^2}^2$$
$$\lesssim(1+\|(U_R,N_R)\|_{H^3}^{10})\Big(\|\frac{N_R}{\sqrt{\varepsilon}})\|_{H^2}^2+\|(U_R,N_R,T_R,B_R,\hbar\nabla U_R,\hbar\nabla N_R)\|_{H^3}^2\Big)+\varepsilon.$$

Moreover, applying the operator $\partial^{\alpha+1}$ to equation (2.12) and taking inner product with $\hbar^2\partial^{\alpha+1}\mathrm{div}U_R$, we derive

$$\begin{aligned}
&\frac{\hbar^2}{2}\frac{d}{dt}\|\partial^{\alpha+1}\mathrm{div}U_R\|_{L^2}^2\\
=&R_{4,1}+\hbar^2\int\partial^{\alpha+1}\frac{N_R}{\varepsilon}\partial^{\alpha+1}\mathrm{div}U_R-\hbar^2\int\mathrm{div}\partial^{\alpha+1}(\frac{T^0\nabla N_R}{n^\varepsilon})\partial^{\alpha+1}\mathrm{div}U_R\\
&+\frac{\hbar^4}{12}\int\partial^{\alpha+1}\Delta\Delta logn^\varepsilon\partial^{\alpha+1}\mathrm{div}U_R+\mu\hbar^2\int\partial^{\alpha+1}\mathrm{div}(\frac{\Delta U_R}{n^\varepsilon})\partial^{\alpha+1}\mathrm{div}U_R\\
&+(\mu+\lambda)\hbar^2\int\partial^{\alpha+1}\mathrm{div}(\frac{\nabla\mathrm{div}U_R}{n^\varepsilon})\partial^{\alpha+1}\mathrm{div}U_R=\sum_{i=1}^6R_{4,i},
\end{aligned}\tag{2.17}$$

where

$$\begin{aligned}
R_{4,1}=&-\hbar^2\int[\partial^{\alpha+1}\mathrm{div},u^\varepsilon]\nabla U_R\partial^{\alpha+1}\mathrm{div}U_R+\frac{\hbar^2}{2}\int\mathrm{div}u^\varepsilon|\partial^{\alpha+1}\mathrm{div}U_R|^2\\
&+\hbar^2\int\partial^{\alpha+1}\nabla T_R\cdot\partial^{\alpha+1}\nabla\mathrm{div}U_R+\hbar^2\int\partial^{\alpha+1}(\frac{T_R\nabla N_R}{n^\varepsilon})\partial^{\alpha+1}\nabla\mathrm{div}U_R\\
&-\frac{\hbar^4}{12}\int\partial^{\alpha+1}\nabla\Big(\frac{\nabla n^\varepsilon\cdot\nabla n^\varepsilon}{(n^\varepsilon)^2}\Big)\cdot\nabla\partial^{\alpha+1}\mathrm{div}U_R+\hbar^2\int\partial^\alpha F_2\partial^{\alpha+2}\mathrm{div}U_R.
\end{aligned}$$

For the first term $R_{4,1}$, to make it clear, we write the estimate for the higher order term with the aid of estimates (A.2), (A.4)–(A.7) and (A.11) in the Appendix that

$$\begin{aligned}
&-\frac{\hbar^4}{12}\int\partial^{\alpha+1}\nabla\Big(\frac{\nabla n^\varepsilon\cdot\nabla n^\varepsilon}{(n^\varepsilon)^2}\Big)\cdot\nabla\partial^{\alpha+1}\mathrm{div}U_R\\
\lesssim&\hbar^4\|\nabla\mathrm{div}U_R\|_{H^3}\big(\|\frac{(\nabla n^\varepsilon)^3}{(n^\varepsilon)^3}\|_{H^3}+\|\frac{\nabla n^\varepsilon\cdot\nabla^2n^\varepsilon}{(n^\varepsilon)^2}\|_{H^3}\big)\\
\lesssim&\hbar^4\|\nabla\mathrm{div}U_R\|_{H^3}^2+\hbar^4\big(\|\frac{1}{n^\varepsilon}\|_{H^3}\|n^\varepsilon\|_{H^3}^2\|\nabla n^\varepsilon\|_{H^3}+\|\nabla\frac{1}{n^\varepsilon}\|_{L^\infty}\|\nabla^2n^\varepsilon\|_{H^3}\\
&+\|\nabla\frac{1}{n^\varepsilon}\|_{H^4}\|\nabla^2n^\varepsilon\|_{L^6}\big)^2\\
\lesssim&\hbar^4\|\nabla\mathrm{div}U_R\|_{H^3}^2+(1+\|N_R\|_{H^3}^{10})\|(N_R,\hbar\nabla N_R,\hbar^2\Delta N_R)\|_{H^3}^2+\varepsilon.
\end{aligned}$$

For the other five terms in $R_{4,1}$, thanks to estimates (A.1), (A.2) and (A.10) in the Appendix, they can be bounded by

$$\begin{aligned}
&\frac{\hbar^2(\mu+\nu)}{32}\|\nabla\mathrm{div}U_R\|_{H^3}^2+2\hbar^4\|\nabla\mathrm{div}U_R\|_{H^3}^2+\frac{\kappa}{32}\|\nabla T_R\|_{H^3}^2\\
&+(1+\|(U_R,N_R,T_R)\|_{H^3}^8)\|(N_R,U_R,T_R,B_R,\hbar\nabla N_R,\hbar\nabla U_R,\hbar\mathrm{div}U_R)\|_{H^3}^2+\varepsilon.
\end{aligned}$$

Similar to estimates for $R_{3,2}$, $R_{4,2}$ can be bounded by

$$R_{4,2}\lesssim-\frac{\hbar^2}{2\varepsilon}\frac{d}{dt}\int\frac{|\partial^{\alpha+1}N_R|^2}{n^\varepsilon}+(1+\|(N_R,U_R)\|_{H^3}^2)(\|\frac{\hbar N_R}{\sqrt{\varepsilon}}\|_{H^3}^2+\|U_R\|_{H^3}^2)+\varepsilon.$$

Now we will estimate the third term, which we use different method from $R_{3,3}$. Thanks to integration by parts, Young's inequality and estimate (A.2) in the Appendix, we obtain

$$\begin{aligned} R_{4,3} \lesssim & \hbar^2 \|\nabla \mathrm{div} U_R\|_{H^3} \|\frac{T^0}{n^\varepsilon}\|_{H^3} \|\nabla N_R\|_{H^3} \\ \lesssim & \hbar^2 \frac{\mu+\nu}{32} \|\nabla \mathrm{div} U_R\|_{H^3}^2 + (1+\|N_R\|_{H^3}^6)\|\hbar \nabla N_R\|_{H^3}^2. \end{aligned}$$

For the last three terms in equation (2.17), similar to the estimates for $R_{3,4} \sim R_{3,6}$, we obtain

$$\begin{aligned} & \frac{d}{dt}\|\hbar^2 \partial^{\alpha+1} \Delta N_R\|_{L^2}^2 + (2\mu+\nu)\hbar^2 \|\partial^{\alpha+1} \nabla \mathrm{div} U_R\|_{L^2}^2 \lesssim \hbar^4 \|\Delta U_R\|_{H^3}^2 \\ & \quad + (1+\|(U_R,N_R)\|_{H^3}^{10}) \|(U_R,N_R,T_R,B_R,\hbar \nabla U_R, \hbar \mathrm{div} U_R, \hbar \nabla N_R, \hbar^2 \Delta N_R)\|_{H^3}^2 + \varepsilon. \end{aligned}$$

Here we require $E^0 \in H^7$.

Now, putting all estimates together, and letting $\hbar \ll 1$, we complete the proof of Lemma 2.4. ∎

2.5. High order energy estimates for the electric-magnetic field.

LEMMA 2.5. *Let α be a muti-index with $|\alpha| = k$ for any integer number $0 \le k \le 2$. $(N_R, U_R, T_R, E_R, B_R)$ be a solution to system (1.8). For all $t \in (0,\tau)$, we have*

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|(\varepsilon \partial^\alpha \partial_t E_R, \sqrt{\varepsilon}\partial^\alpha \nabla \times E_R, \sqrt{\varepsilon}\partial^\alpha E_R, \hbar \varepsilon \partial^{\alpha+1}\partial_t E_R, \hbar\sqrt{\varepsilon}\partial^{\alpha+1}\nabla \times E_R, \hbar \sqrt{\varepsilon}\partial^{\alpha+1} E_R)\|_{L^2}^2 \\ & \lesssim (1+\|\frac{N_R}{\sqrt{\varepsilon}}\|_{H^2}^2 + \|(U_R,T_R,N_R,\hbar \nabla N_R, \frac{\hbar N_R}{\sqrt{\varepsilon}})\|_{H^3}^{10}) \Big(\|(\sqrt{\varepsilon}E_R, \varepsilon \partial_t E_R)\|_{H^2}^2 \\ & \quad + \|(N_R,U_R,T_R,B_R,\hbar \nabla N_R, \hbar \nabla U_R, \hbar \sqrt{\varepsilon} E_R, \hbar \varepsilon \partial_t E_R, \hbar \mathrm{div} U_R, \hbar^2 \Delta N_R)\|_{H^3}^2 \Big) \\ & \quad + \frac{\kappa}{32}\|\nabla T_R\|_{H^3}^2 + \frac{2\mu+\nu}{32}\|\nabla U_R\|_{H^3}^2 + \frac{\hbar^2 \mu}{32}\|\Delta U_R\|_{H^3}^2 + \frac{\hbar^2(\mu+\nu)}{32}\|\nabla \mathrm{div} U_R\|_{H^3}^2 + \varepsilon. \end{aligned}$$

Proof. It follows from equation (1.8d) that E_R satisfies the following wave-type equation:

$$\begin{aligned} & \varepsilon \partial_{tt} E_R + \nabla \times (\nabla \times E_R) + E_R - \varepsilon \mathrm{div} E^0 E_R + N_R E_R + n^\varepsilon \nabla T_R - \mu \Delta U_R - (\mu+\nu)\nabla \mathrm{div} U_R \\ & \quad - \frac{\hbar^2}{12}\{\Delta \nabla n^\varepsilon + \frac{(\Delta n^\varepsilon \nabla n^\varepsilon + \nabla n^\varepsilon \cdot \nabla^2 n^\varepsilon)}{n^\varepsilon} - \frac{(\nabla n^\varepsilon \cdot \nabla n^\varepsilon)\nabla n^\varepsilon}{(n^\varepsilon)^2}\} = F_3, \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} F_3 = & \partial_t n^\varepsilon U_R + \{-n^\varepsilon u^\varepsilon \cdot \nabla U_R - n^\varepsilon U_R \cdot \nabla u^0 - T^\varepsilon \nabla N_R + \varepsilon T^\varepsilon \nabla \mathrm{div} E^0 - \mu N_R \Delta u^0 \\ & - n^\varepsilon (u^\varepsilon \times B_R + U_R \times B^0) + \mu \varepsilon \mathrm{div} E^0 \Delta u^0\} + \partial_t N_R u^0 + N_R \partial_t u^0 - \varepsilon \partial_t \Re_4. \end{aligned} \tag{2.19}$$

Taking ∂^α to equation (2.18) and taking inner product with $\varepsilon\partial^\alpha\partial_t E_R$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\varepsilon\partial^\alpha\partial_t E_R,\sqrt{\varepsilon}\partial^\alpha\nabla\times E_R,\sqrt{\varepsilon}\partial^\alpha E_R)\|_{L^2}^2\\
=&\frac{\hbar^2\varepsilon}{12}\int\partial^\alpha\{\Delta\nabla n^\varepsilon+\frac{(\Delta n^\varepsilon\nabla n^\varepsilon+\nabla n^\varepsilon\cdot\nabla^2 n^\varepsilon)}{n^\varepsilon}-\frac{(\nabla n^\varepsilon\cdot\nabla n^\varepsilon)\nabla n^\varepsilon}{(n^\varepsilon)^2}\}\cdot\partial^\alpha\partial_t E_R\\
&-\varepsilon\Big\{\int[\partial^\alpha,N_R]E_R\cdot\partial^\alpha\partial_t E_R+\int N_R\partial^\alpha E_R\cdot\partial^\alpha\partial_t E_R\Big\}\\
&-\varepsilon\Big\{-\varepsilon\int\partial^\alpha(\mathrm{div}E^0E_R)\cdot\partial^\alpha\partial_t E_R+\int[\partial^\alpha,n^\varepsilon]\nabla T_R\cdot\partial^\alpha\partial_t E_R-\int\partial^\alpha F_3\cdot\partial^\alpha\partial_t E_R\\
&+\int n^\varepsilon\partial^\alpha\nabla T_R\cdot\partial^\alpha\partial_t E_R-\int\partial^\alpha\Big(\mu\Delta U_R+(\mu+\nu)\nabla\mathrm{div}U_R\Big)\cdot\partial^\alpha\partial_t E_R\Big\}=\sum_{i=1}^3 R_{5,i}.
\end{aligned}\tag{2.20}$$

For the first term $R_{5,1}$ on the RHS, to make it easy to read, we divide it naturally into two parts as follows. The first part can be bounded by using the first equation in (1.8e) and estimate (A.9) in the Appendix

$$\begin{aligned}
R_{5,1,1}=&\frac{\hbar^2\varepsilon}{12}\int\partial^\alpha\Delta\nabla n^\varepsilon\cdot\partial^\alpha\partial_t E_R\\
=&-\frac{\hbar^2\varepsilon}{12}\int\partial^\alpha\Delta n^\varepsilon\partial^\alpha\partial_t\mathrm{div}E_R=\frac{\hbar^2}{12}\int\partial^\alpha(-\varepsilon\Delta\mathrm{div}E^0+\Delta N_R)\partial^\alpha\partial_t N_R\\
\lesssim&\hbar^2\|(\partial_t N_R,\Delta N_R)\|_{H^k}^2+\hbar^2\varepsilon\lesssim(1+\|(N_R,U_R)\|_{H^3}^2)\|(N_R,U_R,\hbar\nabla N_R)\|_{H^3}^2+\hbar^2\varepsilon.
\end{aligned}$$

Then the second part can be bounded by combining the bounds (2.3) with estimates (A.1), (A.4), (A.5) and (A.11) in the Appendix

$$\begin{aligned}
R_{5,1,2}=&\frac{\hbar^2\varepsilon}{12}\int\partial^\alpha\{\frac{(\Delta n^\varepsilon\nabla n^\varepsilon+\nabla n^\varepsilon\cdot\nabla^2 n^\varepsilon)}{n^\varepsilon}-\frac{(\nabla n^\varepsilon\cdot\nabla n^\varepsilon)\nabla n^\varepsilon}{(n^\varepsilon)^2}\}\cdot\partial^\alpha\partial_t E_R\\
\lesssim&\hbar^2\varepsilon\|\partial_t E_R\|_{H^k}\Big(\|(\frac{\Delta n^\varepsilon}{n^\varepsilon},\frac{\nabla^2 n^\varepsilon}{n^\varepsilon})\|_{L^\infty}\|\nabla n^\varepsilon\|_{H^k}+(\|\frac{1}{n^\varepsilon}\|_{L^\infty}\|(\Delta n^\varepsilon,\nabla^2 n^\varepsilon)\|_{H^k}\\
&+\|(\Delta n^\varepsilon,\nabla^2 n^\varepsilon)\|_{L^\infty}\|\frac{1}{n^\varepsilon}\|_{H^k})\|\nabla n^\varepsilon\|_{L^\infty}+(\|\frac{1}{n^\varepsilon}\|_{L^\infty}^2\|n^\varepsilon\|_{H^3}^2\|\nabla n^\varepsilon\|_{H^k}\\
&+\|\nabla n^\varepsilon\|_{L^\infty}^3\|\frac{1}{(n^\varepsilon)^2}\|_{H^k})\Big)\\
\lesssim&\varepsilon^2\|\partial_t E_R\|_{H^k}^2+(1+\|(N_R,\hbar\nabla N_R)\|_{H^3}^{10})\|(N_R,\hbar\nabla N_R)\|_{H^3}^2+\varepsilon.
\end{aligned}$$

By integration by parts and the commutator estimate, $R_{5,2}$ can be bounded by

$$\begin{aligned}
R_{5,2}\lesssim&\varepsilon(\|\nabla N_R\|_{L^6}\|E_R\|_{H^k}+\|N_R\|_{H^k}\|E_R\|_{L^\infty})\|\partial_t E_R\|_{H^k}\\
&+\varepsilon\|N_R\|_{L^\infty}\|E_R\|_{H^k}\|\partial_t E_R\|_{H^k}\\
\lesssim&\varepsilon\|E_R\|_{H^k}^2+\varepsilon\|E_R\|_{H^2}^2+\frac{1}{\varepsilon}(\|N_R\|_{H^2}^2+\|N_R\|_{H^k}^2)\varepsilon^2\|\partial_t E_R\|_{H^k}^2.
\end{aligned}$$

Next, recalling F_3 given by definition (2.19), $\Re_4$ given by definition (1.9c) and using integration by parts, Young's inequality, and estimates (A.1), (A.2) and (A.9) in the Appendix, $R_{5,3}$ in equation (2.20) can be bounded by

$$\begin{aligned}
R_{5,3}\lesssim&\frac{\mu}{32}\|\Delta U_R\|_{H^k}^2+\frac{\kappa}{32}\|\nabla T_R\|_{H^k}^2+\frac{\mu+\nu}{32}\|\nabla\mathrm{div}U_R\|_{H^k}^2+\varepsilon^2\|\partial_t E_R\|_{H^k}^2+\varepsilon\|E_R\|_{H^k}^2\\
&+\varepsilon\|E_R\|_{H^2}^2+(1+\|(N_R,U_R,T_R)\|_{H^3}^4)\|(N_R,U_R,T_R,B_R)\|_{H^3}^2+\varepsilon.
\end{aligned}$$

Hence, putting all the above estimates together, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\varepsilon\partial^\alpha\partial_t E_R,\sqrt{\varepsilon}\partial^\alpha\nabla\times E_R,\sqrt{\varepsilon}\partial^\alpha E_R)\|_{L^2}^2\\
\lesssim&(1+\|\frac{N_R}{\sqrt{\varepsilon}}\|_{H^2}^2+\|(U_R,T_R,N_R,\hbar\nabla N_R)\|_{H^3}^{10})\big(\|(N_R,U_R,T_R,B_R,\hbar\nabla N_R)\|_{H^3}^2\\
&\quad+\|(\sqrt{\varepsilon}E_R,\varepsilon\partial_t E_R)\|_{H^2}^2\big)+\frac{2\mu+\nu}{32}\|\nabla U_R\|_{H^3}^2+\varepsilon.
\end{aligned}$$

Moreover, applying the operator $\partial^{\alpha+1}$ to equation (2.18) and taking inner product with $\hbar^2\varepsilon\partial^{\alpha+1}E_R$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\hbar\varepsilon\partial^{\alpha+1}\partial_t E_R,\hbar\sqrt{\varepsilon}\partial^{\alpha+1}\nabla\times E_R,\hbar\sqrt{\varepsilon}\partial^{\alpha+1}E_R)\|_{L^2}^2\\
=&\frac{\hbar^4\varepsilon}{12}\int\partial^{\alpha+1}\{\Delta\nabla n^\varepsilon+\frac{(\Delta n^\varepsilon\nabla n^\varepsilon+\nabla n^\varepsilon\cdot\nabla^2 n^\varepsilon)}{n^\varepsilon}-\frac{(\nabla n^\varepsilon\cdot\nabla n^\varepsilon)\nabla n^\varepsilon}{(n^\varepsilon)^2}\}\cdot\partial^{\alpha+1}\partial_t E_R\\
&-\hbar^2\varepsilon\int\partial^{\alpha+1}(N_R E_R)\cdot\partial^{\alpha+1}\partial_t E_R-\Big(-\hbar^2\varepsilon^2\int\partial^{\alpha+1}(\mathrm{div}E^0 E_R)\cdot\partial^{\alpha+1}\partial_t E_R\\
&+\hbar^2\varepsilon\int[\partial^{\alpha+1},n^\varepsilon]\nabla T_R\cdot\partial^{\alpha+1}\partial_t E_R+\hbar^2\varepsilon\int n^\varepsilon\partial^{\alpha+1}\nabla T_R\cdot\partial^{\alpha+1}\partial_t E_R\\
&-\hbar^2\varepsilon\int\partial^{\alpha+1}F_3\cdot\partial^{\alpha+1}\partial_t E_R-\hbar^2\varepsilon\int\partial^{\alpha+1}\big(\mu\Delta U_R+(\mu+\nu)\nabla\mathrm{div}U_R\big)\cdot\partial^{\alpha+1}\partial_t E_R\Big)\\
=&\sum_{i=1}^3 R_{6,i}.
\end{aligned}\tag{2.21}$$

For the first term on the RHS, by recalling the first equation in (1.8e) and the definition (1.7), substituting $\alpha+1$ for α in $R_{5,1}$, we derive

$$\begin{aligned}
R_{6,1}=&\frac{\hbar^4}{12}\int\partial^{\alpha+1}(-\varepsilon\Delta\mathrm{div}E^0+\Delta N_R)\partial^{\alpha+1}\partial_t N_R+\frac{\hbar^4\varepsilon}{12}\int\partial^{\alpha+1}\{\frac{(\Delta n^\varepsilon\nabla n^\varepsilon+\nabla n^\varepsilon\cdot\nabla^2 n^\varepsilon)}{n^\varepsilon}\\
&-\frac{(\nabla n^\varepsilon\cdot\nabla n^\varepsilon)\nabla n^\varepsilon}{(n^\varepsilon)^2}\}\cdot\partial^{\alpha+1}\partial_t E_R\\
\lesssim&\hbar^2\varepsilon^2\|\partial_t E_R\|_{H^3}^2+\hbar^2\varepsilon\\
&+(1+\|(N_R,\hbar\nabla N_R,U_R)\|_{H^3}^{10})\|(N_R,U_R,\hbar\mathrm{div}U_R,\hbar\nabla N_R,\hbar^2\Delta N_R)\|_{H^3}^2.
\end{aligned}$$

For the second term $R_{6,2}$, using the commutator estimates, Hölder's inequality and estimate (A.2) in the Appendix, we derive

$$\begin{aligned}
R_{6,2}=&-\hbar^2\varepsilon\int[\partial^{\alpha+1},N_R]E_R\cdot\partial^{\alpha+1}\partial_t E_R-\hbar^2\varepsilon\int N_R\partial^{\alpha+1}E_R\cdot\partial^{\alpha+1}\partial_t E_R\\
\lesssim&\hbar^2\varepsilon\|N_R\|_{H^3}\|E_R\|_{H^2}\|\partial_t E_R\|_{H^3}+\hbar^2\varepsilon\|N_R\|_{L^\infty}\|E_R\|_{H^3}\|\partial_t E_R\|_{H^3}\\
\lesssim&\varepsilon\|E_R\|_{H^2}^2+\frac{1}{\varepsilon}\|\hbar N_R\|_{H^3}^2\varepsilon^2\hbar^2\|\partial_t E_R\|_{H^3}^2+\hbar^2\varepsilon\|E_R\|_{H^3}^2+\frac{1}{\varepsilon}\|N_R\|_{H^2}^2\hbar^2\varepsilon^2\|\partial_t E_R\|_{H^3}^2\\
\lesssim&(1+\|\frac{N_R}{\sqrt{\varepsilon}}\|_{H^2}^2+\|\frac{\hbar N_R}{\sqrt{\varepsilon}}\|_{H^3}^2)(\|\sqrt{\varepsilon}E_R\|_{H^2}^2+\|(\hbar\sqrt{\varepsilon}E_R,\hbar\varepsilon\partial_t E_R)\|_{H^3}^2).
\end{aligned}$$

Similar to $R_{5,3}$, we can derive

$$R_{6,3}\lesssim\frac{\mu\hbar^2}{32}\|\Delta U_R\|_{H^3}^2+\frac{\kappa}{32}\|\nabla T_R\|_{H^3}^2+\frac{(\mu+\nu)\hbar^2}{32}\|\mathrm{div}U_R\|_{H^4}^2+\|\hbar\varepsilon\partial_t E_R\|_{H^3}^2$$

$$+(1+\|(N_R,U_R,T_R)\|_{H^3}^4)\|(N_R,U_R,T_R,B_R,\hbar\nabla U_R,\hbar\mathrm{div}U_R,\hbar\nabla N_R)\|_{H^3}^2$$
$$+\|\hbar\sqrt{\varepsilon}E_R\|_{H^3}^2+\varepsilon.$$

Putting all estimates at the $(k+1)$-th order together, we obtain

$$\begin{aligned}&\frac{\hbar^2}{2}\frac{d}{dt}\|(\varepsilon\partial^{\alpha+1}\partial_t E_R,\sqrt{\varepsilon}\partial^{\alpha+1}\nabla\times E_R,\sqrt{\varepsilon}\partial^{\alpha+1}E_R)\|_{L^2}^2\\ &\lesssim(1+\|\frac{N_R}{\sqrt{\varepsilon}}\|_{H^2}^2+\|(N_R,U_R,T_R,\hbar\nabla N_R,\frac{\hbar N_R}{\sqrt{\varepsilon}})\|_{H^3}^{10})\Big(\|\sqrt{\varepsilon}E_R\|_{H^2}^2\\ &\quad+\|(N_R,U_R,T_R,B_R,\hbar\nabla U_R,\hbar\mathrm{div}U_R,\hbar\nabla N_R,\hbar\sqrt{\varepsilon}E_R,\varepsilon\hbar\partial_t E_R,\hbar^2\Delta N_R)\|_{H^3}^2\Big)\\ &\quad+\frac{\kappa}{32}\|\nabla T_R\|_{H^3}^2+\frac{\hbar^2(\mu+\nu)}{32}\|\nabla\mathrm{div}U_R\|_{H^3}^2+\frac{\hbar^2\mu}{32}\|\Delta U_R\|_{H^3}^2+\varepsilon.\end{aligned}$$

We complete the proof of Lemma 2.5 by combining all the estimates at the k-th order and the $(k+1)$-th order. □

LEMMA 2.6. *Let α be a multi-index with $|\alpha|=k$ for any integer number $0\le k\le 2$. (N_R,U_R,T_R,B_R) be a solution to system (1.8). For all $t\in(0,\tau)$, we obtain*

$$\begin{aligned}\|(\partial^\alpha\nabla B_R,\hbar\partial^{\alpha+1}\nabla B_R)\|_{L^2}^2\lesssim&\,\varepsilon^2\|\partial_t E_R\|_{H^2}^2+\hbar^2\varepsilon^2\|\partial_t E_R\|_{H^3}^2\\ &+(1+\|U_R\|_{H^2}^2)\|(U_R,N_R)\|_{H^3}^2+\varepsilon.\end{aligned}\tag{2.22}$$

Proof. From the first equation in (1.8d), we obtain

$$\|\partial^\alpha\nabla\times B_R\|_{L^2}\lesssim\|\varepsilon\partial_t E_R\|_{H^k}+(1+\|U_R\|_{H^2})\|(U_R,N_R)\|_{H^2}+\varepsilon.$$

The case $|\alpha|=k+1$ can be proved similarly. By the curl-div decomposition formula of the gradient for the magnetic field and using the second equation in (1.8e), we can obtain estimate (2.22). □

Then integrating these estimates from Lemma 2.1 to Lemma 2.6 over $(0,t)\subseteq(0,\tau)$, summing them up for all multi-index α, and recalling estimate (A.11) in the Appendix which implies $\|\hbar\Delta N_R\|_{H^2}^2\ge\|\hbar\nabla N_R\|_{H^3}^2$, we obtain Proposition 2.1.

3. Proof of Theorem 1.3

Proof. From Proposition 2.1, we have

$$\begin{aligned}&|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(t)|||_3\lesssim|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(0)|||_3\\ &\quad+\int_0^t\Big((1+|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)|||_3^5)|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)|||_3+\varepsilon\Big).\end{aligned}\tag{3.1}$$

Thanks to the Theorem 1.1 and the assumption $\|\sqrt{\varepsilon}E_0^\varepsilon\|_{H^3}^2\lesssim\varepsilon$, we have

$$\|\frac{N_R}{\sqrt{\varepsilon}}(0)\|_{H^2}^2\le\|\sqrt{\varepsilon}\mathrm{div}E_0^\varepsilon\|_{H^2}^2\le\|\sqrt{\varepsilon}E_0^\varepsilon\|_{H^3}^2\le C\varepsilon.$$

That is to say $|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(0)|||_3\le C\varepsilon$ for sufficiently small ε and a constant $C>0$ independent of ε. Applying the non-linear Gronwall-type inequality [11] to estimate (3.1), it follows from

$$|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(0)|||_3\le C\varepsilon$$

that there exist a ε_0 sufficiently small such that for any $\varepsilon<\varepsilon_0$ and $0<t<\tau$,

$$|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(t)|||_3\le C\varepsilon.$$

We claim there exist constants $0<\tau_1<1$ and $\varepsilon_0>0$, such that $\tau_\varepsilon\ge\tau_1$ for any $0<\varepsilon\le\varepsilon_0$. Otherwise, for any sufficiently small positive constants τ_1 and ε_0, there exists $0<\varepsilon\le\varepsilon_0$ such that $\tau_\varepsilon<\tau_1$ which implies $\tau=\tau_\varepsilon$. By the local existence results in Theorem 1.2, for any given constant $C>0$, we have

$$\lim_{t\to\tau_\varepsilon}|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(t)|||_3\ge 6C\varepsilon.$$

Thanks to the fact $|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(0)|||_3\le C\varepsilon$, there exists $0<\tau_2<\tau_\varepsilon<\tau_1$ such that

$$|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(\tau_2)|||_3=4C\varepsilon.$$

Choosing $0<\tau_3\le\tau_2$ satisfying

$$|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(\tau_3)|||_3=4C\varepsilon, \tag{3.2}$$

and

$$|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(t)|||_3<4C\varepsilon,\ t\in[0,\tau_3). \tag{3.3}$$

Thanks to estimate (3.1) and condition (3.2), we have for sufficiently small positive constant $\tau_1<1$,

$$|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(\tau_3)|||_3\le C\varepsilon\tau_1 4(1+(4C\varepsilon)^5)+2C\varepsilon\le 3C\varepsilon,$$

which contradicts condition (3.2). Repeating the process on $[\tau_1,2\tau_1]$, ..., we can extend $\tau_\varepsilon\ge\tau_0$ for any $0<\tau_0<\tau_*$, and derive

$$\sup_{t\in[0,\tau_0]}|||(N_R,U_R,T_R,B_R,\sqrt{\varepsilon}E_R)(t)|||_3\le C\varepsilon,$$

which implies $(n^\varepsilon,u^\varepsilon,T^\varepsilon,B^\varepsilon)$ converges strongly to $(1,u^0,T^0,B^0)$ in $L^\infty(0,\tau_0;H^3)$, and the convergence of E^ε to E^0 in $W^{-1,\infty}(0,\tau_0;H^2)$.

The proof of Theorem 1.3 is complete. □

Appendix A. The following basic Moser type calculus inequalities will be frequently used.

LEMMA A.1. *Let α be any multi-index with $|\alpha|=k$, $k\ge1$ and $p\in(1,\infty)$. Then there holds*

$$\begin{aligned}&\|\partial^\alpha(fg)\|_{L^p}\lesssim\|f\|_{L^{p^1}}\|g\|_{H^{k,p^2}}+\|f\|_{H^{k,p^3}}\|g\|_{L^{p^4}}\\&\|[\partial^\alpha,f]g\|_{L^p}\lesssim\|\nabla f\|_{L^{p^1}}\|g\|_{H^{k-1,p^2}}+\|f\|_{H^{k,p^3}}\|g\|_{L^{p^4}},\end{aligned} \tag{A.1}$$

where $f,g\in\mathbb{S}$, the Schwartz class and $p_2,p_3\in(1,\infty)$ such that $\frac{1}{p}=\frac{1}{p^1}+\frac{1}{p^2}=\frac{1}{p^3}+\frac{1}{p^4}$.

Let $k\ge2$. Due to Sobolev embedding and estimate (A.1), we can obtain

$$\|\partial^\alpha(fg)\|_{L^2}\lesssim\|f\|_{L^\infty}\|g\|_{H^k}+\|f\|_{H^k}\|g\|_{L^\infty}\lesssim\|f\|_{H^k}\|g\|_{H^k}, \tag{A.2}$$

and

$$\|[\partial^\alpha,f]g\|_{L^2}\lesssim\|\nabla f\|_{L^\infty}\|g\|_{H^{k-1}}+\|f\|_{H^k}\|g\|_{L^\infty}\lesssim\|f\|_{H^k}\|g\|_{H^{k-1}}. \tag{A.3}$$

Hence, for any $m\geq 1,k\geq 1$, with the aid of the bounds (2.3) and estimate (A.2), we can deduce the following inequality

$$\begin{aligned}\|(1/n^\varepsilon)^m\|_{H^k}\lesssim&\|(\frac{1}{n^\varepsilon})^{m-1}\|_{L^\infty}\|\frac{1}{n^\varepsilon}\|_{H^k}+\|(\frac{1}{n^\varepsilon})^{m-1}\|_{H^k}\|\frac{1}{n^\varepsilon}\|_{L^\infty}\\ \lesssim&\|\frac{1}{n^\varepsilon}\|_{H^k}+\|(\frac{1}{n^\varepsilon})^{m-1}\|_{H^k}\lesssim\cdots\cdots\lesssim\|\frac{1}{n^\varepsilon}\|_{H^k},\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}\|(\nabla n^\varepsilon)^m\|_{H^k}\lesssim&\|(\nabla n^\varepsilon)^{m-1}\|_{H^k}\|\nabla n^\varepsilon\|_{L^\infty}+\|\nabla n^\varepsilon\|_{L^\infty}^{m-1}\|\nabla n^\varepsilon\|_{H^k}\\ \lesssim&(\|(\nabla n^\varepsilon)^{m-2}\|_{H^k}\|\nabla n^\varepsilon\|_{L^\infty}+\|\nabla n^\varepsilon\|_{L^\infty}^{m-2}\|\nabla n^\varepsilon\|_{H^k})\|\nabla n^\varepsilon\|_{L^\infty}\\ &+\|\nabla n^\varepsilon\|_{L^\infty}^{m-1}\|\nabla n^\varepsilon\|_{H^k}\lesssim\cdots\cdots\lesssim\|n^\varepsilon\|_{H^3}^{m-1}\|\nabla n^\varepsilon\|_{H^k}.\end{aligned} \tag{A.5}$$

Due to estimates (A.2), (A.4) and (A.5), assuming $2\leq k\leq 3$, we compute

$$\|\nabla(\frac{1}{n^\varepsilon})\|_{H^k}\lesssim\|\frac{1}{n^\varepsilon}\|_{H^k}\|\nabla n^\varepsilon\|_{H^k}, \tag{A.6}$$

and

$$\|\Delta(\frac{1}{n^\varepsilon})\|_{H^k}\lesssim\|\frac{\Delta n^\varepsilon}{(n^\varepsilon)^2}\|_{H^k}+\|\frac{|\nabla n^\varepsilon|^2}{(n^\varepsilon)^3}\|_{H^k}\lesssim\|\frac{1}{n^\varepsilon}\|_{H^k}\|\Delta n^\varepsilon\|_{H^k}+\|\frac{1}{n^\varepsilon}\|_{H^k}\|n^\varepsilon\|_{H^3}\|\nabla n^\varepsilon\|_{H^k}. \tag{A.7}$$

LEMMA A.2. *Let $0\leq k\leq 3$ be an integer, (N_R,U_R,T_R,B_R,E_R) be a solution to system* (1.8)*, and α be a multi-index with $|\alpha|=k$, then we obtain*

$$\begin{aligned}&\|\partial_t N_R\|_{L^2}\lesssim(1+\|U_R\|_{H^2})\|(U_R,\text{div}U_R,\nabla N_R)\|_{L^2}+\varepsilon,\\ &\|\partial_t N_R\|_{L^\infty}\lesssim 1+\|(N_R,U_R)\|_{H^3}^2,\end{aligned} \tag{A.8}$$

and

$$\|\partial_t N_R\|_{H^k}\lesssim(1+\|(N_R,U_R)\|_{H^3})\big(\|(N_R,U_R,\text{div}U_R,\nabla N_R)\|_{H^k}+\|(N_R,U_R)\|_{H^3}\big)+\varepsilon. \tag{A.9}$$

Proof. By Sobolev embedding, Hölder's inequality, equation (1.8a) and the bounds (2.3), we derive

$$\|\partial_t N_R\|_{L^2}\lesssim(1+\|U_R\|_{H^2})\|(U_R,\text{div}U_R,\nabla N_R)\|_{L^2}+\varepsilon,$$

and

$$\|\partial_t N_R\|_{L^\infty}\lesssim 1+\|(N_R,U_R)\|_{H^3}^2.$$

Using estimate (A.1) in the Appendix we can bound

$$\begin{aligned}\|\partial_t N_R\|_{H^k}\leq&(1+\|U_R\|_{L^\infty})\|\nabla N_R\|_{H^k}+\|u^\varepsilon\|_{H^k}\|\nabla N_R\|_{L^\infty}+\|n^\varepsilon\|_{L^\infty}\|\text{div}U_R\|_{H^k}\\ &+\|n^\varepsilon\|_{H^k}\|\text{div}U_R\|_{L^\infty}+\varepsilon\|U_R\|_{H^3}+\varepsilon\end{aligned}$$

$$\lesssim (1+\|(N_R,U_R)\|_{H^3})\big(\|(N_R,U_R,\mathrm{div}U_R,\nabla N_R)\|_{H^k}+\|(N_R,U_R)\|_{H^3}\big)+\varepsilon.$$

□

Lemma A.3. *Under the same condition in Lemma A.2, we have the following estimate:*

$$\begin{aligned}\|\partial^\alpha(\frac{1}{n^\varepsilon})\|_{L^\infty} &\lesssim 1+\|N_R\|_{H^{2+k}}^k\\ \|\partial^\alpha(\frac{1}{n^\varepsilon})\|_{L^2} &\lesssim 1+\|N_R\|_{H^k}^k.\end{aligned} \tag{A.10}$$

Proof. Take $k=2$ for example. In what follows, we take L^2-norm, and apply Hölder's inequality, Sobolev embedding $H^1\hookrightarrow L^3,L^6$ to obtain

$$\begin{aligned}\|\partial^2(\frac{1}{n^\varepsilon})\|_{L^2} &\lesssim \varepsilon^2+4\varepsilon\|\partial N_R\|_{L^2}+2\|\partial N_R\|_{L^3}\|\partial N_R\|_{L^6}+\varepsilon+\|\partial^2 N_R\|_{L^2}\\ &\lesssim 1+\|N_R\|_{H^2}^2.\end{aligned}$$

Similarly, taking L^∞-norm yields

$$\|\partial^2(\frac{1}{n^\varepsilon})\|_{L^\infty}\lesssim 1+\|N_R\|_{H^4}^2.$$

The case of $k\neq 2$ can be proved similarly, thus we omit them. □

Lemma A.4. *Let $0\le k\le 3$ be an integer, $f\in\mathbb{S}$, the Schwartz class, then we obtain*

$$\|\nabla^2 f\|_{H^k}\lesssim \|\Delta f\|_{H^k}, \tag{A.11}$$

and

$$\|\nabla f\|_{H^k}\lesssim \|\nabla\times f\|_{H^k}+\|\mathrm{div} f\|_{H^k}, \tag{A.12}$$

The Lemma can be proved by introducing Riesz operator R_j defined by $\widehat{(R_j f)}=\frac{i\xi_j}{|\xi|}\hat{f}$ and $\nabla\mathrm{div} f=\Delta f+\nabla\times(\nabla\times f)$, where R_iR_j is bounded from L^p to L^p with $1<p<\infty$. For more details, one can see [36].

Lemma A.5. *Let f,g be the vector function in 3D. The following vector analysis formulas will be repeatedly used*

$$f\cdot\nabla g=(\nabla\times g)\times f+(\nabla\times f)\times g+\nabla(f\cdot g)-\nabla f\cdot g, \tag{A.13}$$

$$\nabla\times(f\cdot\nabla f)=\nabla\times(\nabla\times f\times f), \tag{A.14}$$

$$\nabla\times(\nabla\times f)=\nabla(\mathrm{div} f)-\Delta f, \tag{A.15}$$

and

$$\nabla\times(f\times g)=f\mathrm{div} g-g\mathrm{div} f+(g\cdot\nabla)f-(f\cdot\nabla)g. \tag{A.16}$$

REFERENCES

[1] D. Bohm, *A suggested interpretation of the quantum theory in terms of "hidden" valuables: I; II*, Phys. Rev., 85:166–179; 180–193, 1952.

[2] S. Cordier and E. Grenier, *Quasineutral limit of an Euler–Poisson system arising from plasma physics*, Commun. Part. Diff. Eqs., 23:1099–1113, 2000.

[3] P. Degond, F. Deluzet, and D. Savelief, *Numerical approximation of the Euler-Maxwell model in the quasineutral limit*, J. Comput. Phys., 231:1917–1946, 2012.

[4] P. Degond, S. Gallego, and F. Méhats, *On quantum hydrodynamic and quantum energy transport models*, Commun. Math. Sci., 5:887–908, 2007.

[5] P. Degond, S. Gallego, and F. Méhats, *Isothermal quantum hydrodynamics: derivation, asymptotic analysis, and simulation*, Multiscale Model. Simul., 6:246–272, 2007.

[6] P. Degond, S. Gallego, F. Méhats, and C. Ringhofer, *Quantum hydrodynamic and diffusion models derived from the entropy principle*, Quantum Transport, 1946:111–168, 2008.

[7] P. Degond, F. Méhats, and C. Ringhofer, *Quantum energy-transport and drift-diffusion models*, J. Stat. Phys., 118:625–667, 2005.

[8] P. Degond and C. Ringhofer, *Quantum moment hydrodynamics and the entropy principle*, J. Stat. Phys., 112:587–628, 2003.

[9] D. Donatelli and P. Marcati, *A quasineutral type limit for the Navier–Stokes–Poisson system with large data*, Nonlinearity, 21:135–148, 2008.

[10] D. Donatelli and P. Marcati, *Analysis of oscillations and defect measures for the quasineutral limit in plasma physics*, Arch. Ration. Mech. Anal., 206:159–188, 2012.

[11] S.S. Dragomir, *Some Gronwall Type Inequalities and Application*, New York, Nova Science Publishers, 2003.

[12] C.L. Gardner, *The quantum hydrodynamic model for semiconductor devices*, SIAM J. Appl. Math., 54(2):409–427, 1994.

[13] A.V. Gordeev, A.S. Kingsep, and L.I. Gordeev, *Electron magnetohydrodynamics*, Physics Reports, 243(5):215–315, 1994.

[14] I. Gasser and P. Marcati, *The combined relaxation and vanishing Debye length limit in the hydrodynamic model for semiconductors*, Math. Methods Appl. Sci., 24:81–92, 2001.

[15] I. Gasser and P. Marcati, *A vanishing Debye length limit in a hydrodynamic model for semiconductors*, Hyperbolic Problems: Theory, Numerics, Applications, 140:409–414, 2001.

[16] Y. Guo, A.D. Ionescu, and B. Pausader, *Global solutions of the Euler-Maxwell two-fluid system in 3D*, Annal. Math., 183(2):377–498, 2016.

[17] F. Haas, *Quantum Plasmas: An Hydrodynamic Approach*, Springer, New York, 2011.

[18] H. Hattori and D. Li, *Solutions for two-dimensional system for materials of Korteweg type*, SIAM J. Math. Anal., 25(1):85–98, 1994.

[19] H. Hattori and D. Li, *Global solutions of a high dimensional system for Korteweg materials*, J. Math. Anal. Appl., 198:84–97, 1996.

[20] Q. Ju, F. Li, and H. Li, *The quasineutral limit of compressible Navier–Stokes–Poisson system with heat conductivity and general initial data*, J. Diff. Eqs., 247:203–224, 2009.

[21] A. Jüngel, *A note on current-voltage characteristics from the quantum hydrodynamic equations for semiconductors*, Appl. Math. Letters, 10(4):29–34, 1997.

[22] A. Jüngel, C.K. Lin, and K.C. Wu, *An asymptotic limit of a Navier–Stokes system with capillary effects*, Comm. Math. Phys., 329(2):725–744, 2014.

[23] A. Jüngel and J.P. Milišić, *Full compressible Navier–Stokes equations for quantum fluids: Derivation and numerical solutions*, Kinetic and Related Models, 4(3):785–807, 2011.

[24] T. Kato, *Nonstationary flows of viscous and ideal fluids in* $\mathbb{R}^3$, J. Funct. Anal., 9:296–305, 1972.

[25] A.S. Kingsep, K.V. Chukbar, and V.V. Yankov, *Reviews of Plasma Physics*, Consultant Bureau, New York, 1990.

[26] P.L. Lions, *Mathematical Topics in Fluid Mechanics, vol. 1: Incompressible Models*, Oxford Lecture Ser. Math. Appl., The Clarendon Press/Oxford University Press, New York, 3, 1996.

[27] H. Li and C. Lin, *Zero Debye length asymptotic of the quantum hydrodynamic model for semiconductors*, Comm. Math. Phys., 256(1):195–212, 2005.

[28] Y.J. Peng and S.Wang, *Rigorous derivation of incompressible e-MHD equations from compressible Euler-Maxwell equations*, SIAM J. Math. Anal., 40:540–565, 2008.

[29] Y.J. Peng and S. Wang, *Convergence of compressible Euler-Maxwell equations to incompressible Euler equations*, Comm. Part. Diff. Eqs., 33:349–476, 2008.

[30] Y.J. Peng and S.Wang, *Asymptotic expansions in two-fluid compressible Euler-Maxwell equations with small parameters*, Discrete Contin. Dyn. Syst., 23:415–433, 2009.

[31] Y.J. Peng, S. Wang, and Q. Gu, *Relaxation limit and global existence of smooth solutions of compressible Euler-Maxwell equations*, SIAM J. Math. Anal., 43(2):944–970, 2011.

[32] Y.J. Peng and Y.G. Wang, *Convergence of compressible Euler–Poisson equations to incompressible type Euler equations*, Asymptot. Anal., 41:141–160, 2005.
[33] Y.J. Peng, Y. Wang, and W. Yong, *Quasi-neutral limit of the non-isentropic Euler–Poisson system*, Proceedings of the Royal Society of Edinburgh, 136:1013–1026, 2006.
[34] X. Pu and B. Guo, *Global existence and semiclassical limit for quantum hydrodynamic equations with viscosity and heat conducyion*, Kinet. Relat. Models, 9(1):165–191, 2016.
[35] X. Pu and B. Guo, *Quasineutral limit of the pressureless Euler–Poisson equation for ions*, Quart. Appl. Math., 74(2):245–273, 2016.
[36] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
[37] S. Wang and S. Jiang, *The convergence of Navier–Stokes–Poisson system to the incompressible Euler equations*, Comm. Part. Diff. Eqs., 31:571–591, 2006.
[38] E. Wigner, *On the quantum correction for thermodynamic equilibrium*, Phys. Rev., 40:749–759, 1932.