COMMUN. MATH. SCI.
Vol. 16, No. 1, pp. 239–250

STRONG WELL-POSEDNESS FOR THE PHASE-FIELD NAVIER–STOKES EQUATIONS IN THE MAXIMAL REGULARITY CLASS*

NAOTO KAJIWARA†

Abstract. In this paper we study the dynamics of vesicle membranes in incompressible viscous fluids. We prove existence and uniqueness of the local strong solution for this model coupling of the Navier–Stokes equations with a phase field equation in an L_p-L_q setting. We transform the equation into a quasi-linear parabolic evolution equation and use the general theory proved by Prüss et al. [16,23,25]. Since the operator and the nonlinear term are analytic, we have that the solution is real analytic in time and space. At last it is shown that the variational strict stable solution is exponentially stable, provided the product of the viscosity coefficient and the mobility constant is large.

Keywords. phase field; Navier–Stokes; well-posedness; stability; vesicle membrane; fluid vesicle interaction; bending elastic energy.

AMS subject classifications. 35Q30; 35Q35; 35D35; 76D03; 76D05; 76T10.

1. Introduction

It is important to understand the deformation of vesicle membranes in a liquid in many biological and physiological applications. The vesicle contains a liquid and is surrounded by another liquid. The function of the vesicle is to store and/or transport substances. They are not only essential to the function of cells but also interesting since they change their shapes such as spheres, discocytes, stomatocytes, tori and double tori. These are concrete examples of minimizers of different surface energies, such as the bending elasticity (Willmore, mean curvature square) energy in the calculus of variations and its different variations like the general Helfrich energy [2,14,27,34]. We consider that the equilibrium configurations of vesicle membranes can be characterized the minimizer of the following Helfrich bending elastic energy of the surface:

$$E_{\text{elastic}} = \int_{\Gamma} \frac{k}{2}(H - c_0)^2 dS,$$

where Γ is the surface of the vesicle membrane, H is the mean curvature of Γ, c_0 is the spontaneous curvature that describes certain physical/chemical difference between the inside and the outside of the membrane, and k is the bending modulus (bending rigidity) that depends on the local heterogeneous concentration of the species. Here we assumed the evolution of the vesicle membrane does not change its topology so that the energy is simplified. For details, see [9,27,34].

The model which represent the deformation of the vesicle in a liquid was first constructed in [8]. They derived the system of the equations via an energetic variational approach. In this phase field Navier–Stokes equations, the description of the membrane is given the terms of a phase field function φ. The labeling function φ takes value $+1$ inside the vesicle membrane and -1 outside, and the thin transition layer of width is characterized by a small parameter ε. The zero level set of φ ($\{x \,|\, \varphi(x) = 0\}$) represents the surface of the vesicle membrane. The fluid is modeled by the incompressible

*Received: April 15, 2017; accepted (in revised form): November 22, 2017. Communicated by Chun Liu.

†Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba Meguro-ku Tokyo 153-8914, Japan (kajiwara@ms.u-tokyo.ac.jp).

Navier–Stokes equations in the whole domain containing the inside and outside of the vesicle.

The phase field approximation of the Helfrich bending elasticity energy is given by a modified Willmore energy [10]:

$$E_\varepsilon(\varphi)=\int_\Omega \frac{k}{2\varepsilon}\left(\varepsilon\Delta\varphi+(\frac{1}{\varepsilon}\varphi+c_0\sqrt{2})(1-\varphi^2)\right)^2 dx.$$

The convergence from $E_\varepsilon(\varphi)$ to E_{elastic} was studied in [7,30]. In this paper, for the sake of simplicity, we assume that k is a positive constant and $c_0=0$. Since the vesicle preserves its volume and surface area, we use the penalty formulation about its energy [27]. These two constraint functionals for the vesicle volume and surface area are given by

$$A(\varphi)=\int_\Omega \varphi dx, \qquad B(\varphi)=\int_\Omega \frac{\varepsilon}{2}|\nabla\varphi|^2+\frac{1}{4\varepsilon}(\varphi^2-1)^2dx,$$

respectively. The modified energy is formulated in the following form:

$$E(\varphi)=E_\varepsilon(\varphi)+\frac{1}{2}M_1[A(\varphi)-A(\varphi_0)]^2+\frac{1}{2}M_2[B(\varphi)-B(\varphi_0)]^2,$$

where M_1 and M_2 are two penalty constants and φ_0 is the initial phase function.

We consider the phase field Navier–Stokes equations derived from above energy [8, 32]. Let $\Omega\subset\mathbb{R}^n$ be a bounded C^4-domain, the function u be the unknown velocity field, the function p be the pressure and φ be the phase field function. We denote by ν the fluid viscosity and γ the mobility coefficient, which are positive constants. The model is

$$\text{(PFNS)}\quad \begin{cases} \partial_t u+(u\cdot\nabla)u=\nu\Delta u-\nabla p+\frac{\delta E(\varphi)}{\delta\varphi}\nabla\varphi & \text{in } [0,T]\times\Omega,\\ \operatorname{div} u=0 & \text{in } [0,T]\times\Omega,\\ \partial_t\varphi+(u\cdot\nabla)\varphi=-\gamma\frac{\delta E(\varphi)}{\delta\varphi} & \text{in } [0,T]\times\Omega,\\ u=0 & \text{on } [0,T]\times\partial\Omega,\\ \varphi=-1,\quad \Delta\varphi=0 & \text{on } [0,T]\times\partial\Omega,\\ u(0)=u_0,\quad \varphi(0)=\varphi_0 & \text{in } \Omega, \end{cases}$$

where

$$\begin{aligned}\frac{\delta E(\varphi)}{\delta\varphi}&=kg(\varphi)+M_1[A(\varphi)-A(\varphi_0)]+M_2[B(\varphi)-B(\varphi_0)]f(\varphi)\\ &=k\left(\varepsilon\Delta^2\varphi-\frac{1}{\varepsilon}\Delta(\varphi^3)-\frac{3}{\varepsilon}\varphi^2\Delta\varphi+\frac{2}{\varepsilon}\Delta\varphi+\frac{1}{\varepsilon^3}(3\varphi^5-4\varphi^3+\varphi)\right)\\ &\qquad+M_1[A(\varphi)-A(\varphi_0)]+M_2[B(\varphi)-B(\varphi_0)]f(\varphi)\\ &=: W(\varphi)\\ &=: k\varepsilon\Delta^2\varphi+L(\varphi)\end{aligned}$$

and

$$\begin{aligned} f(\varphi)&=-\varepsilon\Delta\varphi+\frac{1}{\varepsilon}(\varphi^2-1)\varphi,\\ g(\varphi)&=-\Delta f(\varphi)+\frac{1}{\varepsilon^2}(3\varphi^2-1)f(\varphi).\end{aligned}$$

We note that the term $\frac{\delta E(\varphi)}{\delta\varphi}=W(\varphi)$ is the so-called chemical potential and $E_\varepsilon(\varphi)=\frac{k}{2\varepsilon}\int_\Omega|f(\varphi)|^2dx$. In this paper as usual we consider the Dirichlet type for the phase field function φ and the no-slip boundary condition for the velocity field u.

The well-posedness of the system has been studied in [6,17,31]. In [6], they proved the existence of the global weak solution by using the Galerkin method. With a better regularity assumption on the weak solutions, as in the case of the conventional Navier–Stokes equations [28], they proved the uniqueness result. In [17], they proved the local in time existence and uniqueness of the strong solution in an L_2 framework. The idea was to rewrite (PFNS) as a semi-linear equation for the Stokes equation coupled with a parabolic equation whose operator is bi-Laplace operator. However, to estimate the nonlinear term $\Delta^2\varphi\nabla\varphi$ they needed the higher regularity class for the function φ compared with the usual parabolic equation. More precisely, they proved that if u_0, φ_0 satisfy $u_0\in H_0^1(\Omega)$, $\operatorname{div}u_0=0$, $\varphi_0+1\in H^{2+\frac{3}{8}}(\Omega)\cap H_0^1(\Omega)$ then (PFNS) has a unique strong solution

$$\begin{cases} u\in L_2(0,T;H^2(\Omega))\cap H^1(0,T;L_2(\Omega)) \\ \varphi\in L_2(0,T;H^{4+\frac{3}{8}}(\Omega))\cap H^1(0,T;H^{\frac{3}{8}}(\Omega)) \end{cases}$$

for some $T=T(\|u_0\|_{H^1},\|\varphi_0\|_{H^{2+\frac{3}{8}}})>0$. In [31] they considered the (PFNS) in a periodic box and proved existence/uniqueness of strong solutions and some regularity criteria. Moreover they investigated the stability of the system near local minimizers of the elastic bending energy by using Łojasiewicz-Simon-type inequality.

The main purpose of this paper is to study the existence, uniqueness and regularity of the strong solution to (PFNS) in an L_p-L_q framework as well as their exponential stability in the n dimension, while papers [6,17,31] are L_2 setting and $n=3$. The main idea is to consider (PFNS) as a quasi-linear equation not as a semi-linear equation. We consider the quasi-linear operator $A(z)(z={}^T(u,\phi))$ given by

$$A(z)=\begin{pmatrix} \nu\mathcal{A} & -k\varepsilon\mathbb{P}\mathcal{B}(\phi) \\ 0 & \gamma k\varepsilon\mathcal{D} \end{pmatrix},$$

where $\mathcal{A}$ denotes the Stokes operator, $\mathcal{D}$ the bi-Laplace operator, $\mathbb{P}$ the Helmholtz projection, and $\mathcal{B}$ is given by $\mathcal{B}(\phi)h:=\Delta^2h\nabla\phi$. For the quasi-linear parabolic equation, we use maximal regularity in a weighted L_p spaces and well-posed result proved by Prüss et al. [16,23,25]. This quasi-linear approach has already used to analyze nematic liquid crystal flows [15] and viscoelastic Poiseuille-type flows [12] as pioneering works. We employ time weight L_p spaces:

$$\begin{aligned} L_{p,\mu}(0,T;X)&:=\{z:(0,T)\to X\,|\,t^{1-\mu}z\in L_p(0,T;X)\},\\ H^1_{p,\mu}(0,T;X)&:=\{z\in L_{p,\mu}(0,T;X)\cap W^1_1(0,T;X)\,|\,\dot z\in L_{p,\mu}(0,T;X)\}, \end{aligned}$$

for $p\in(1,\infty)$, $\mu\in(1/p,1]$ and a Banach space X. The merit of the time weight is to observe that the class of initial data can be taken larger and the solution regularizes instantly in time. Furthermore, we prove the stationary solution $(0,\varphi^*)$ is exponentially stable even under including fluid effect if the product of the coefficients $\nu\gamma$ is sufficiently large and φ^* is the variational strict stable solution, i.e. φ^* satisfies $W(\varphi^*)=0$ and $(\frac{\delta^2E(\varphi^*)}{\delta\varphi^2}\psi,\psi)\ge c\|\psi\|^2_{L^2}$ for some $c>0$ and for any $\psi\in H_2^4(\Omega)$ satisfying $\psi|_{\partial\Omega}=-1,\Delta\psi|_{\partial\Omega}=0$.

Let us state the main results.

THEOREM 1.1 (Local existence and uniqueness of strong solutions). *Let* $p,q\in(1,\infty),\mu\in(1/p,1]$ *be* $\frac{1}{2}+\frac{1}{p}+\frac{n}{2q}<\mu\le 1$ *and assume that*

$$\begin{cases}(u_0,\varphi_0)\in B^{2(\mu-1/p)}_{q,p}(\Omega)\times B^{4(\mu-1/p)}_{q,p}(\Omega) & \\ \operatorname{div} u_0=0 & \text{in } \Omega,\\ u_0=0 & \text{on } \partial\Omega,\\ \varphi_0=-1,\quad \Delta\varphi_0=0 & \text{on } \partial\Omega.\end{cases}$$

Then there exists $T=T\left(\|u_0\|_{B^{2(\mu-1/p)}_{q,p}},\|\varphi_0\|_{B^{4(\mu-1/p)}_{q,p}}\right)>0$ *such that the equations* (PFNS) *have a unique strong solution*

$$\begin{cases}u\in H^1_{p,\mu}(0,T;L_{q,\sigma}(\Omega))\cap L_{p,\mu}(0,T;(H^2_q(\Omega))^n),\\ \varphi\in H^1_{p,\mu}(0,T;L_q(\Omega))\cap L_{p,\mu}(0,T;H^4_q(\Omega)),\\ \nabla p\in L_{p,\mu}(0,T;(L_q(\Omega))^n),\end{cases}$$

where the interval $[0,T)$ *is a maximal time interval of existence. Moreover the solution depends continuously on* u_0 *and* φ_0.

REMARK 1.1. The condition $\frac{1}{2}+\frac{1}{p}+\frac{n}{2q}<\mu\le 1$ comes from the embedding exponent such that $B^{2(\mu-1/p)}_{q,p}(\Omega)\hookrightarrow C^1(\overline{\Omega})$. Note that this condition implies that the embedding $B^{4(\mu-1/p)}_{q,p}(\Omega)\hookrightarrow C^2(\overline{\Omega})$.

THEOREM 1.2. *The solution* ${}^T(u,\varphi)$ *in Theorem 1.1 satisfies for each* $j\in\mathbb{N}$,

$$t^j\left[\frac{d}{dt}\right]^j\begin{pmatrix}u\\ \varphi\end{pmatrix}\in H^1_{p,\mu}(0,T;L_{q,\sigma}(\Omega)\times L_q(\Omega))\cap L_{p,\mu}(0,T;(H^2_q(\Omega))^n\times H^4_q(\Omega)).$$

Moreover, the solution ${}^T(u,\varphi)$ *is real analytic from* $(0,T)$ *to* $(H^2_q(\Omega))^n\times H^4_q(\Omega)$.

REMARK 1.2. By the scaling techniques in time and space, the maximal regularity and the implicit function theorem, it is proved that ${}^T(u,\varphi)$ is real analytic in $(0,T)\times\Omega$. See [22] for parabolic equations, and see [24] for Navier–Stokes equations.

At last we study the stability of the solution near the local minimizers of the elastic bending energy.

THEOREM 1.3. *Let* $p,q\in(1,\infty),\mu\in(1/p,1]$ *satisfy the assumption in Theorem 1.1 and* ${}^T(0,\varphi^*)\in\{0\}\times H^4_q(\Omega)$ *be the variational strictly stable solution of (PFNS) i.e.* φ^* *satisfies* $W(\varphi^*)=0$ *and* $(\frac{\delta^2E(\varphi^*)}{\delta\varphi^2}\psi,\psi)\ge c\|\psi\|^2_{L^2}$ *for some* $c>0$ *and for any* $\psi\in H^4_2(\Omega)$ *satisfying* $\psi|_{\partial\Omega}=-1,\Delta\psi|_{\partial\Omega}=0$. *Then there are* $\varepsilon>0,C>0$ *such that for each* ${}^T(u_0,\varphi_0)\in B^{2(\mu-1/p)}_{q,p}(\Omega)\times B^{4(\mu-1/p)}_{q,p}(\Omega)$ *satisfying* $\|u_0\|_{B^{2(\mu-1/p)}_{q,p}}+\|\varphi_0-\varphi^*\|_{B^{4(\mu-1/p)}_{q,p}}<\varepsilon$ *and for any* ν *and* γ *satisfying* $\nu\gamma>C$, *there exists a unique global solution*

$$\begin{cases}u\in H^1_{p,\mu,loc}(\mathbb{R}_+;L_{q,\sigma}(\Omega))\cap L_{p,\mu,loc}(\mathbb{R}_+;(H^2_q(\Omega))^n),\\ \varphi\in H^1_{p,\mu,loc}(\mathbb{R}_+;L_q(\Omega))\cap L_{p,\mu,loc}(\mathbb{R}_+;H^4_q(\Omega)).\end{cases}$$

Furthermore, there is a $\beta>0$ *such that*

$$\begin{cases}e^{\beta t}u\in H^1_{p,\mu}(\mathbb{R}_+;L_{q,\sigma}(\Omega))\cap L_{p,\mu}(\mathbb{R}_+;(H^2_q(\Omega))^n)\cap C_0(\mathbb{R}_+;(B^{2(\mu-1/p)}_{q,p}(\Omega))^n),\\ e^{\beta t}(\varphi-\varphi^*)\in H^1_{p,\mu}(\mathbb{R}_+;L_q(\Omega))\cap L_{p,\mu}(\mathbb{R}_+;H^4_q(\Omega))\cap C_0(\mathbb{R}_+;B^{4(\mu-1/p)}_{q,p}(\Omega)).\end{cases}$$

In particular, the equilibrium ${}^{T}(0,\varphi^*)$ *is exponentially stable in* $B^{2(\mu-1/p)}_{q,p}(\Omega)\times B^{4(\mu-1/p)}_{q,p}(\Omega)$.

In this paper we are not able to guarantee the existence of the variational strict stable solution because of the two penalty terms. In [31] they dealt with the case of $c=0$ and other stability result.

2. General theory for quasi-linear evolution equations

We explain the theory of the quasi-linear parabolic equation via the maximal regularity. For details we refer the papers [16, 23]. See also [25].

Let X_0 and X_1 be Banach spaces such that $X_1 \overset{d}{\hookrightarrow} X_0$, i.e. X_1 is continuously and densely embedded in X_0. Let $T>0$ or $T=\infty$. For a closed linear operator A in X_0, we say A has the property of the maximal $L_{p,\mu}$-regularity, for short $A\in\mathcal{MR}_{p,\mu}(X_1,X_0)$, if for each $f\in L_{p,\mu}(0,T;X_0)$ there exists a unique solution $u\in H^1_{p,\mu}(0,T;X_0)\cap L_{p,\mu}(0,T;X_1)$ of the linear problem $\dot{u}+Au=f$ $(t\in(0,T))$ with initial value $u(0)=0$. For classical case $\mu=1$, denote $A\in\mathcal{MR}_p(X_1,X_0)$. In [16, 23] it was proved that

$$A\in\mathcal{MR}_{p,\mu}(X_1,X_0)\Leftrightarrow A\in\mathcal{MR}_p(X_1,X_0) \qquad \forall p\in(1,\infty),\mu\in(1/p,1],$$

and, concerning nontrivial initial data, if $A\in\mathcal{MR}_p(X_1,X_0)$ then

$$\begin{aligned}&A\in\mathcal{MR}_{p,\mu}(X_1,X_0)\Leftrightarrow \forall f\in L_{p,\mu}(0,T;X_0)\ \forall u_0\in X_{\gamma,\mu},\\ &\exists! u\in H^1_{p,\mu}(0,T;X_0)\cap L_{p,\mu}(0,T;X_1) \text{ s.t. } \dot{u}+Au=f\ (t\in(0,T)), u(0)=u_0,\end{aligned}$$

where $X_{\gamma,\mu}:=(X_0,X_1)_{\mu-1/p,p}$ is the trace space for $p\in(1,\infty)$ and $\mu\in(1/p,1]$. For trace spaces, see also [21]. The case $\mu=1$, denote $X_\gamma:=X_{\gamma,1}$.

We consider the following quasi-linear parabolic equation (QL):

$$\text{(QL)} \qquad \begin{cases}\dot{z}(t)+A(z(t))z(t)=F(z(t)) & t\in(0,T),\\ \qquad\qquad z(0)=z_0.\end{cases}$$

Here we impose regularity assumptions

$$(A_-)\ A\in\mathrm{Lip}_{loc}(X_{\gamma,\mu};\mathcal{L}(X_1,X_0)), \qquad (F_-)\ F\in\mathrm{Lip}_{loc}(X_{\gamma,\mu};X_0)$$

and $z_0\in X_{\gamma,\mu}$ for $p\in(1,\infty),\mu\in(1/p,1]$. By $\mathcal{L}(X_1,X_0)$ we denote the set of all bounded linear operator from X_1 to X_0. We now give existence and uniqueness results for (QL). Local in time existence and uniqueness of (QL) was shown by Clément and Li [3] in the case $\mu=1$ and by Köhne, Prüss and Wilke [16] for the case $\mu\in(1/p,1]$.

Proposition 2.1. *Let* $1<p<\infty$, $\mu\in(1/p,1]$, $z_0\in X_{\gamma,\mu}$, *and suppose that the assumption* $(A_-),(F_-)$ *and* $A(z_0)\in\mathcal{MR}_p(X_1,X_0)$ *are satisfied. Then, there exists* $a>0$, *such that* (QL) *admits a unique solution* z *on* $J=[0,a]$ *in the regularity class*

$$z\in H^1_{p,\mu}(J;X_0)\cap L_{p,\mu}(J;X_1)\hookrightarrow C(J;X_{\gamma,\mu})\cap C((0,a];X_\gamma).$$

The solution depends continuously on z_0, *and can be extended to a maximal interval of existence* $J(z_0)=[0,t^+(z_0))$.

Smoothing effects often appear in parabolic problems. The parameter trick method by Angenent [1] is well known. A similar method has already been used in the study of Navier–Stokes equation in [19, 20]. We state the regularity of the solution of (QL) in terms of the regularity of A and F.

We use the following notation to state two propositions:

$$(A_k)\ A\in C^k(X_{\gamma,\mu};\mathcal{L}(X_1,X_0)), \qquad (F_k)\ F\in C^k(X_{\gamma,\mu};X_0)$$

for $k\in\mathbb{N}\cup\{\infty,\omega\}$, where the index ω refers to real analyticity.

Let us recall the definition of the real analytic between Banach spaces [33]. Suppose X,Y are two Banach spaces. We say the operator $T:X\to Y$ is analytic if for any $x_0\in X$ there exists a small neighborhood of x_0 such that

$$T(x_0+h)-T(x_0)=\sum_{n\geq 1}T_n(x_0)(h,\cdots,h) \qquad \forall h\in X,\ \|h\|_X<r\ll 1.$$

Here $T_n(x_0)$ is a continuous symmetrical n-linear operator on $X^n\to Y$ and satisfies

$$\sum_{n\geq 1}\|T_n(x_0)\|_{\mathcal{L}(X^n,Y)}\|h\|_X^n<\infty.$$

PROPOSITION 2.2. *Let $1<p<\infty$, $\mu\in(1/p,1]$, $z_0\in X_{\gamma,\mu}$, $k\in\mathbb{N}\cup\{\infty,\omega\}$ and suppose that the assumption $(A_k),(F_k)$ and $A(z_0)\in\mathcal{MR}_p(X_1,X_0)$ are satisfied. Let z be the solution in Proposition 2.1 and assume $A(z(t))\in\mathcal{MR}_p(X_1,X_0)$ for all $t\in J$. Then*

$$t^j\left[\frac{d}{dt}\right]^j z\in H^1_{p,\mu}(J;X_0)\cap L_{p,\mu}(J;X_1), \qquad j\leq k$$

Furthermore, if $k=\infty$ then $z\in C^\infty(J;X_1)$, and if $k=\omega$ then $z\in C^\omega(J;X_1)$.

If we impose the Fréchet differentiability of A and F, then the solution exists globally provided the initial data is close to an equilibrium point, see [22] for the case $\mu=1$, and [16] for the case $\mu\in(1/p,1]$. Let $\mathcal{E}:=\{z\in X_1\,|\,A(z)z=F(z)\}$ be the equilibria of (QL) and A_0 be the linearization of (QL) at $z^*\in\mathcal{E}$, i.e.

$$A_0w=A(z^*)w+(A'(z^*)w)z^*-F'(z^*)w, \qquad w\in X_1.$$

We denote the spectrum of the operator A by $\sigma(A)$ and denote the resolvent set of the operator A by $\rho(A)$.

PROPOSITION 2.3. *Let $1<p<\infty$, $\mu\in(1/p,1]$ and $z^*\in\mathcal{E}$ be $A(z^*)\in\mathcal{MR}_p(X_1,X_0)$ on $\mathbb{R}_+$ and the assumptions (A_1) and (F_1) are satisfied. Suppose that $\sigma(A_0)\subset\{\lambda\in\mathbb{C}\,|$ $\mathrm{Re}\,\lambda>0\}$. Then there is $\varepsilon>0$ such that for each $z_0\in B_\varepsilon(z^*)\subset X_{\gamma,\mu}$ there exists a unique global solution $z\in H^1_{p,\mu,loc}(\mathbb{R}_+;X_0)\cap L_{p,\mu,loc}(\mathbb{R}_+;X_1)$ of (QL). Furthermore, there is a $\beta>0$ such that*

$$e^{\beta t}(z-z^*)\in H^1_{p,\mu}(\mathbb{R}_+;X_0)\cap L_{p,\mu}(\mathbb{R}_+;X_1)\cap C_0(\mathbb{R}_+;X_{\gamma,\mu}).$$

In particular, the equilibrium z^* is exponentially stable in $X_{\gamma,\mu}$.

In this proposition, the constant $\varepsilon>0$ depends only on the maximal regularity constant of A_0 and the local Lipschitz constants of A and F. Here $C_0(\mathbb{R}_+;X_{\gamma,\mu})$ is the space of $X_{\gamma,\mu}$-valued continuous function vanishing at the time-infinity.

3. Quasilinear approach for the phase field Navier–Stokes equations

3.1. Quasilinear formulation. In this section we transform (PFNS) into quasi-linear evolution equations for the unknown $z={}^T(u,\phi)$. Let $1<q<\infty$ and $\Omega\subset\mathbb{R}^n$ be a bounded C^4-domain. We choose the Banach space

$$X_0:=L_{q,\sigma}(\Omega)\times L_q(\Omega).$$

As usual, $L_{q,\sigma}(\Omega)$ is the subspace of $(L_q(\Omega))^n$ consisting of solenoidal vector fields. We denote by $\mathbb{P}:(L_q(\Omega))^n\to L_{q,\sigma}(\Omega)$ the Helmholtz projection and the Stokes operator $\mathcal{A}_q:D(\mathcal{A}_q)\to L_{q,\sigma}(\Omega)$, where $D(\mathcal{A}_q)=\{u\in(H^2_q(\Omega))^n\cap L_{q,\sigma}(\Omega)\,|\,u=0 \text{ a.e. on } \partial\Omega\}$, $\mathcal{A}_q=-\mathbb{P}\Delta$. The maximal L_q-regularity result for the Stokes operator $\mathcal{A}_q$ is well-known; see e.g. [11, 13]. The bi-Laplace operator $\mathcal{D}_q$ in $L_q(\Omega)$ is defined by $\mathcal{D}_q=\Delta^2$ with the domain $D(\mathcal{D}_q)=\{\phi\in H^4_q(\Omega)\,|\,\phi=\Delta\phi=0 \text{ a.e. on } \partial\Omega\}$. The maximal L_q-regularity result for bi-Laplace operator $\mathcal{D}_q$ is also well-known; see e.g. [4]. We choose the Banach space

$$X_1:=D(\mathcal{A}_q)\times D(\mathcal{D}_q),$$

equipped with its canonical norms. Then $X_1\overset{d}{\hookrightarrow}X_0$.

For we treat (PFNS) as the quasi-linear equation, we define the operator

$$A(z):=\begin{pmatrix}\nu\mathcal{A}_q & -k\varepsilon\mathbb{P}\mathcal{B}_q(\phi)\\ 0 & \gamma k\varepsilon\mathcal{D}_q\end{pmatrix},$$

where the operator $\mathcal{B}_q$ is given by $\mathcal{B}_q(\phi)h:=\Delta^2h\nabla\phi$. Let $\phi=\varphi+1$ and apply the Helmholtz projection $\mathbb{P}$ to the first equation in (PFNS), then we are able to rewrite (PFNS) of the form:

$$(*)\begin{cases}\dfrac{d}{dt}z+A(z)z=F(z):=\begin{pmatrix}-\mathbb{P}((u\cdot\nabla)u)+\mathbb{P}(L(\phi-1)\nabla\phi)\\ -(u\cdot\nabla)\phi-\gamma L(\phi-1)\end{pmatrix} & t\in(0,T),\\ z(0)=z_0:=\begin{pmatrix}u_0\\ \phi_0\end{pmatrix}:=\begin{pmatrix}u_0\\ \varphi_0+1\end{pmatrix}.\end{cases}$$

We show the $A(z)$ has the property of maximal regularity for each $z\in X_{\gamma,\mu}$ and assumptions (A_ω) and (F_ω). We have that $\mathcal{B}_q(\phi):D(\mathcal{D}_q)\to(L_q(\Omega))^n$ is bounded for each $\phi\in C^1(\overline{\Omega})$ and the map $\phi\to\mathbb{P}\mathcal{B}_q(\phi)$ is real analytic. So $A(z)\in C^\omega(L_{q,\sigma}(\Omega)\times C^1(\overline{\Omega}),\mathcal{L}(X_1,X_0))$. By the tri-diagonal structure of $A(z)$ and by the regularity of $\mathcal{B}_q$ one can easily see that $A(z)\in\mathcal{MR}_p(X_1,X_0)$ for each $z={}^T(u,\phi)\in L_{q,\sigma}(\Omega)\times C^1(\overline{\Omega})$. Indeed, from $\begin{pmatrix}\nu\mathcal{A}_q & 0\\ 0 & \gamma k\varepsilon\mathcal{D}_q\end{pmatrix}\in\mathcal{MR}_p(X_1,X_0)$, for any ${}^T(f,g)\in L_p(0,T;X_0)$, we can take ${}^T(\tilde{u},\tilde{\phi})\in H^1_p(0,T;X_0)\cap L_p(0,T;X_1)$ which is the solution of

$$\begin{cases}\dfrac{d}{dt}\begin{pmatrix}\tilde{u}\\ \tilde{\phi}\end{pmatrix}+\begin{pmatrix}\nu\mathcal{A}_q & 0\\ 0 & \gamma k\varepsilon\mathcal{D}_q\end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{\phi}\end{pmatrix}=\begin{pmatrix}f+k\varepsilon\mathbb{P}\mathcal{B}_q(\phi)\overline{\phi}\\ g\end{pmatrix} & t\in(0,T),\\ \begin{pmatrix}\tilde{u}\\ \tilde{\phi}\end{pmatrix}(0)=\begin{pmatrix}0\\ 0\end{pmatrix},\end{cases}$$

where $\overline{\phi}$ is the solution of

$$\begin{cases}\dfrac{d}{dt}\overline{\phi}+k\gamma\varepsilon\mathcal{D}_q\overline{\phi}=g & t\in(0,T),\\ \overline{\phi}(0)=0.\end{cases}$$

Note that $k\varepsilon\mathbb{P}\mathcal{B}_q(\phi)\overline{\phi}\in L_p(0,T;L_{q,\sigma}(\Omega))$ for each $\phi\in C^1(\overline{\Omega})$ and $\tilde{\phi}=\overline{\phi}$. This implies that for any ${}^T(f,g)\in L_p(0,T;X_0)$, we can take ${}^T(\tilde{u},\tilde{\phi})\in H_p^1(0,T;X_0)\cap L_p(0,T;X_1)$ which is the solution of

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix}\tilde{u}\\ \tilde{\phi}\end{pmatrix}+\begin{pmatrix}\nu\mathcal{A}_q & -k\varepsilon\mathbb{P}\mathcal{B}_q(\phi)\\ 0 & \gamma k\varepsilon\mathcal{D}_q\end{pmatrix}\begin{pmatrix}\tilde{u}\\ \tilde{\phi}\end{pmatrix}=\begin{pmatrix}f\\ g\end{pmatrix} & t\in(0,T),\\ \begin{pmatrix}\tilde{u}\\ \tilde{\phi}\end{pmatrix}(0)=\begin{pmatrix}0\\ 0\end{pmatrix}. \end{cases}$$

So $A(z)\in\mathcal{MR}_p(X_1,X_0)$ for each $z={}^T(u,\phi)\in L_{q,\sigma}(\Omega)\times C^1(\overline{\Omega})$.

The nonlinear term F is also real analytic from $(C^1(\overline{\Omega}))^n\times C^2(\overline{\Omega})$ into X_0. If we get $X_{\gamma,\mu}\hookrightarrow(C^1(\overline{\Omega}))^n\times C^2(\overline{\Omega})$, then (A_ω) and (F_ω) hold. The space $X_{\gamma,\mu}$ is given by

$$X_{\gamma,\mu}=(X_0,X_1)_{\mu-1/p,p}=D_{\mathcal{A}_q}(\mu-1/p,p)\times D_{\mathcal{D}_q}(\mu-1/p,p),$$

provided $p\in(1,\infty)$ and $\mu\in(1/p,1]$; see [18, 21]. Here $D_{\mathcal{A}_q}(\theta,p)$ is the real interpolation $(L_{q,\sigma}(\Omega),D(\mathcal{A}_q))_{\theta,p}$ and $D_{\mathcal{D}_q}(\theta,p)=(L_q(\Omega),D(\mathcal{D}_q))_{\theta,p}$. We need to consider two embedding exponent, one is $D_{\mathcal{A}_q}(\mu-1/p,p)\hookrightarrow(C^1(\overline{\Omega}))^n$ and the other is $D_{\mathcal{D}_q}(\mu-1/p,p)\hookrightarrow C^2(\overline{\Omega})$:

$$\frac{1}{2}+\frac{1}{p}+\frac{n}{2q}<\mu\le 1\Rightarrow D_{\mathcal{A}_q}(\mu-1/p,p)\hookrightarrow(C^1(\overline{\Omega}))^n.$$
$$\frac{1}{2}+\frac{1}{p}+\frac{n}{4q}<\mu\le 1\Rightarrow D_{\mathcal{D}_q}(\mu-1/p,p)\hookrightarrow C^2(\overline{\Omega}).$$

Note that $\frac{1}{2}+\frac{1}{p}+\frac{n}{4q}<\frac{1}{2}+\frac{1}{p}+\frac{n}{2q}$.

Under the condition $\frac{1}{2}+\frac{1}{p}+\frac{n}{2q}<\mu\le 1$, we can characterize the interpolation spaces by Besov spaces:

$$u\in D_{\mathcal{A}_q}(\mu-1/p,p)\Leftrightarrow u\in(B_{q,p}^{2(\mu-1/p)}(\Omega))^n\cap L_{q,\sigma}(\Omega),\quad u=0,\ \text{a.e. on }\partial\Omega,$$
$$\phi\in D_{\mathcal{D}_q}(\mu-1/p,p)\Leftrightarrow\phi\in B_{q,p}^{4(\mu-1/p)}(\Omega),\quad \phi=\Delta\phi=0,\ \text{a.e. on }\partial\Omega.$$

For Besov spaces, see [29].

We are ready to prove well-posedness results in Section 1. The proof is based on propositions in Section 2.

Proof. (**Proof of Theorem 1.1.**) We transformed (PFNS) into the quasi-linear parabolic equation $(*)$. The condition $z_0\in X_{\gamma,\mu}$ is equivalent to the conditions in Theorem 1.1 and $\tilde{A}(z_0)\in\mathcal{MR}_p(X_1,X_0)$. So we are able to apply Proposition 2.1. ☐

Proof. (**Proof of Theorem 1.2.**) We have already checked the conditions $(A_\omega),(F_\omega)$. Since the solution $z(t)\in X_{\gamma,\mu}$ for all $t\in J$, assumptions in Proposition 2.2 are satisfied. ☐

3.2. Spectral analysis of the linearized operator. In order to prove the stability result of Theorem 1.3, we calculate the linearized operator near the local minimizers of the elastic bending energy. The equilibria $\mathcal{E}$ is the set

$$\mathcal{E}=\{z^*={}^T(0,\phi^*)\in X_1\,|\,W(\phi^*-1)=0\}.$$

The linearized operator A_0 at z^* is given by $A_0w=A(z^*)z+(A'(z^*)w)z^*-F'(z^*)w$ for $w={}^T(w_1,w_2)\in X_1$. By direct calculation

$$A(z^*)w=\begin{pmatrix}-\nu\mathbb{P}\Delta & -k\varepsilon\mathbb{P}(\Delta^2\cdot\nabla\phi^*)\\ 0 & \gamma k\varepsilon\Delta^2\end{pmatrix}w$$

$$(A'(z^*)w)z^* = \begin{pmatrix} 0 & -k\varepsilon\mathbb{P}(\Delta^2\cdot\nabla w_2) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \phi^* \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -k\varepsilon\mathbb{P}(\Delta^2\phi^*\nabla\cdot) \\ 0 & 0 \end{pmatrix} w$$

$$F'(z^*)w = \begin{pmatrix} 0 & -k\varepsilon\mathbb{P}(\Delta^2\phi^*\nabla\cdot)+\mathbb{P}(G(\phi^*)\cdot\nabla\phi^*) \\ -C(\phi^*) & -\gamma G(\phi^*) \end{pmatrix} w,$$

where the linear operator $C(\phi^*)$ and $G(\phi^*)$ is defined as follows:

$$C(\phi^*)w_1 = (w_1\cdot\nabla)\phi^*$$

$$G(\phi^*)w_2 = G_1(\phi^*)w_2 + M_1\int_\Omega w_2 dx + M_2\left\{f(\phi^*-1)\int_\Omega f(\phi^*-1)w_2 dx + G_2(\phi^*)w_2\right\}$$

and

$$\begin{aligned} &G_1(\phi^*)w_2 \\ &= -\frac{k}{\varepsilon}\left\{\left(6(\phi^*-1)^2-2\right)\Delta w_2 + 6\nabla((\phi^*-1)^2)\cdot\nabla w_2\right. \\ &\quad\left. + \left(3\Delta((\phi^*-1)^2)+6(\phi^*-1)\Delta\phi^* + (15(\phi^*-1)^4-12(\phi^*-1)^2+1)\right)w_2\right\} \\ &G_2(\phi^*)w_2 = [B(\phi^*-1)-B(\phi_0-1)]\left\{-\varepsilon\Delta w_2 + \frac{1}{\varepsilon}\left(3(\phi^*-1)^2-1\right)w_2\right\}. \end{aligned}$$

Therefore the linearized operator A_0 is

$$A_0 = \begin{pmatrix} -\nu\mathbb{P}\Delta & -\mathbb{P}\left((k\varepsilon\Delta^2+G(\phi^*))\cdot\nabla\phi^*)\right) \\ C(\phi^*) & \gamma(k\varepsilon\Delta^2+G(\phi^*)) \end{pmatrix}.$$

Since $\frac{\delta^2 E(\phi^*)}{\delta\phi^2} = k\varepsilon\Delta^2 + G(\phi^*)$, the realization of this operator A_0 in L^q spaces can be rewritten as

$$A_0 = \begin{pmatrix} \nu\mathcal{A}_q & -\mathbb{P}\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\cdot\nabla\phi^*\right) \\ \mathcal{C}_q & \gamma\frac{\delta^2 E(\phi^*)}{\delta\phi^2} \end{pmatrix}, \quad D(A_0) = D(\mathcal{A}_q)\times D(\mathcal{D}_q),$$

where $\mathcal{C}_q = C(\phi^*)$ with the domain $D(\mathcal{C}_q) = L_{q,\sigma}(\Omega)$.

From now we show that $\sigma(A_0)\subset\{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda>0\}$. Denote $\Sigma_{\eta,M} := \{\lambda = re^{i\theta}\in\mathbb{C}\setminus\{0\}\,|\,r\geq M, \eta<|\theta|\}$ for some $M\geq 0$ and $\eta\in(0,\pi/2)$, and denote $\mathbb{C}_+ := \{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda>0\}$ and $\overline{\mathbb{C}}_- := \{\lambda\in\mathbb{C}\,|\,\mathrm{Re}\lambda\leq 0\}$.

LEMMA 3.1. *Assume that there exists $c>0$ such that for all $\psi\in D(\mathcal{D}_2)$, $\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\psi,\psi\right)\geq c\|\psi\|_{L^2}^2$. Then $\sigma\left(\frac{\delta^2 E(\phi^*)}{\delta\phi^2}\right)\subset\mathbb{C}_+$.*

Proof. We consider that $G(\phi^*)$ is a lower order perturbation of $k\varepsilon\Delta^2$. Then for any $0<\eta<\pi/2$, there exists $M>0$ such that $\Sigma_{\eta,M}\subset\rho(\frac{\delta^2 E(\phi^*)}{\delta\phi^2})$. Fix $\lambda_0\in\rho(\frac{\delta^2 E(\phi^*)}{\delta\phi^2})$. From compactness of the operator $(\lambda_0-\frac{\delta^2 E(\phi^*)}{\delta\phi^2})^{-1}$ and Fredholm theory, we have the injection of the operator $\lambda-\frac{\delta^2 E(\phi^*)}{\delta\phi^2}$ implies $\lambda\in\rho(\frac{\delta^2 E(\phi^*)}{\delta\phi^2})$. Let $\psi\in D(\mathcal{D}_q)$ satisfy $(\lambda-\frac{\delta^2 E(\phi^*)}{\delta\phi^2})\psi=0$. We may assume $\psi\in D(\mathcal{D}_2)$. In fact, the boundedness of Ω implies that

$D(\mathcal{D}_q)\subset D(\mathcal{D}_2)$, when $2\le q<\infty$. On the other hand, when $1<q<2$, by the Sobolev embedding theorem and bootstrap argument for the equation $(\lambda+\lambda_0-\frac{\delta^2E(\phi^*)}{\delta\phi^2})\psi=\lambda_0\psi$, we see that $\psi\in D(\mathcal{D}_2)$. From $(\lambda-\frac{\delta^2E(\phi^*)}{\delta\phi^2})\psi=0$, we see

$$\mathrm{Re}\lambda\|\psi\|_2-\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\psi,\psi\right)=0$$

and, from the assumption on $\frac{\delta^2E(\phi^*)}{\delta\phi^2}$ we see $\psi=0$ when $\lambda\in\overline{\mathbb{C}}_-$. □

REMARK 3.1. We have the following resolvent estimate for the operator $\frac{\delta^2E(\phi^*)}{\delta\phi^2}$:

$$\left\|\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q(\Omega))}\le C_\eta\quad(\forall\lambda\in\Sigma_{\eta,0}).$$

THEOREM 3.1. *Assume that there exists $c>0$ such that for all $\psi\in D(\mathcal{D}_2)$, $\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\psi,\psi\right)\ge c\|\psi\|_{L^2}^2$. Then there exists $C=C_{\phi^*}>0$ such that if ν and γ satisfy $\nu\gamma>C$ then $\sigma(A_0)\subset\mathbb{C}_+$.*

Proof. By the similar method of Lemma 3.1, it suffices that $(\lambda-A_0)z=0$ ($z={}^T(u,\phi)\in X_1$) implies $z=0$ for $\lambda\in\overline{\mathbb{C}}_-$. The second equation of this resolvent equation

$$(\lambda-\nu\mathcal{A}_q)u+\mathbb{P}\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\phi\nabla\phi^*\right)=0$$
$$-\mathcal{C}_qu+\left(\lambda-\gamma\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)\phi=0$$

derives

$$-\mathbb{P}\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_qu\nabla\phi^*\right)+\mathbb{P}\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\phi\nabla\phi^*\right)=0.$$

By subtracting the first equation,

$$\left(\lambda-\nu\mathcal{A}_q+\mathbb{P}\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)\right)u=0.$$

We use the perturbation theory of the generator of analytic semigroups. The calculation

$$\begin{aligned}&\left\|\mathbb{P}\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)u\right\|_{L_{q,\sigma}}\\
&\le\|\nabla\phi^*\|_\infty\left\|\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_qu\right\|_q\\
&\le\|\nabla\phi^*\|_\infty\sup_{\lambda\in\overline{\mathbb{C}}_-}\left\|\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q)}\frac{1}{\gamma}\|(u\cdot\nabla)\phi^*\|_q\\
&\le\|\nabla\phi^*\|_\infty^2\sup_{\lambda\in\overline{\mathbb{C}}_-}\left\|\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q)}\frac{1}{\gamma}\|u\|_q\end{aligned}$$

$$\leq C\|\nabla\phi^*\|_\infty^2 \sup_{\lambda\in\overline{\mathbb{C}}_-}\left\|\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\right\|_{\mathcal{L}(L^q)}\frac{1}{\nu\gamma}\|\nu\mathcal{A}_q u\|_q$$
$$\leq C_{\phi^*}\frac{1}{\nu\gamma}\|\nu\mathcal{A}_q u\|_q$$

implies that if $\nu\gamma$ is sufficiently large, then $\mathbb{P}\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)$ is small perturbation of $\nu\mathcal{A}_q$. So from $\lambda\in\overline{\mathbb{C}}_-\subset\rho(\nu\mathcal{A}_q)$, we have $\lambda\in\rho\left(\nu\mathcal{A}_q-\mathbb{P}\left(\frac{\delta^2E(\phi^*)}{\delta\phi^2}\left(\lambda-\gamma\frac{\delta^2E(\phi^*)}{\delta\phi^2}\right)^{-1}\mathcal{C}_q\cdot\nabla\phi^*\right)\right)$ and then $u=0$. By lemma 3.1, $\phi=0$. It concludes that $\sigma(A_0)\subset\mathbb{C}_+$. ∎

Proof. (**Proof of Theorem 1.3.**) Since $\phi^*=\varphi^*-1$, the proof is straightly based on Proposition 2.3. ∎

Acknowledgements. The author would like to thank Professor Matthias Hieber for giving him the method of Section 2 in intensive lectures at the University of Tokyo. This work was supported by JSPS-DFG Japanese-German Graduate Externship. This work was done while the author is a research assistant of the university of Tokyo supported by JSPS through scientific grant Kiban S (26220702).

REFERENCES

[1] Sigurd B. Angenent, *Nonlinear analytic semiflows*, Proc. Roy. Soc. Edinburgh Sect. A, 115(1-2):91–107, 1990.
[2] Peter B Canham, *The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell*, Journal of Theoretical Biology, 26(1):61IN777–76IN881, 1970.
[3] Philippe Clément and Shuanhu Li, *Abstract parabolic quasilinear equations and application to a groundwater flow problem*, Adv. Math. Sci. Appl., 3(Special Issue):17–32, 1993/94.
[4] Robert Denk, Matthias Hieber, and Jan Prüss, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc., 166(788):viii+114, 2003.
[5] Robert Denk, Matthias Hieber, and Jan Prüss, *Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data*, Mathematische Zeitschrift, 257(1):193–224, 2007.
[6] Qiang Du, Manlin Li, and Chun Liu, *Analysis of a phase field Navier-Stokes vesicle-fluid interaction model*, Discrete Contin. Dyn. Syst. Ser. B, 8(3):539–556 (electronic), 2007.
[7] Qiang Du, Chun Liu, Rolf Ryham, and Xiaoqiang Wang, *A phase field formulation of the Willmore problem*, Nonlinearity, 18(3):1249–1267, 2005.
[8] Qiang Du, Chun Liu, Rolf Ryham, and Xiaoqiang Wang, *Energetic variational approaches in modeling vesicle and fluid interactions*, Phys. D, 238(9-10):923–930, 2009.
[9] Qiang Du, Chun Liu, Rolf Ryham, Xiaoqiang Wang, et al. *Diffuse interface energies capturing the euler number: Relaxation and renomalization*, Commun. Math. Sci., 5(1):233–242, 2007.
[10] Qiang Du, Chun Liu, and Xiaoqiang Wang, *A phase field approach in the numerical study of the elastic bending energy for vesicle membranes*, J. Comput. Phys., 198(2):450–468, 2004.
[11] Matthias Geissert, Matthias Hess, Matthias Hieber, Céline Schwarz, and Kyriakos Stavrakidis, *Maximal L^p-L^q-estimates for the Stokes equation: a short proof of Solonnikov's theorem*, J. Math. Fluid Mech., 12(1):47–60, 2010.
[12] Y Giga, J Sauer, and K Schade, *Strong stability of 2d viscoelastic Poiseuille-type flows*, 2014.
[13] Yoshikazu Giga, *Domains of fractional powers of the Stokes operator in L_r spaces*, Arch. Rational Mech. Anal., 89(3):251–265, 1985.
[14] Wolfgang Helfrich, *Elastic properties of lipid bilayers: theory and possible experiments*, Zeitschrift für Naturforschung C, 28(11-12):693–703, 1973.
[15] Matthias Hieber, Manuel Nesensohn, Jan Prüss, and Katharina Schade, *Dynamics of nematic liquid crystal flows: the quasilinear approach*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33(2):397–408, 2016.
[16] Matthias Köhne, Jan Prüss, and Mathias Wilke, *On quasilinear parabolic evolution equations in weighted L_p-spaces*, J. Evol. Equ., 10(2):443–463, 2010.

[17] Yuning Liu, Takéo Takahashi, and Marius Tucsnak, *Strong solutions for a phase field Navier-Stokes vesicle-fluid interaction model*, J. Math. Fluid Mech., 14(1):177–195, 2012.

[18] Alessandra Lunardi, *Interpolation Theory*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, Second Edition, 2009.

[19] Kyûya Masuda, *On the analyticity and the unique continuation theorem for solutions of the Navier-Stokes equation*, Proc. Japan Acad., 43:827–832, 1967.

[20] Kyûya Masuda, *On the regularity of solutions of the nonstationary Navier-Stokes equations*, in Approximation methods for Navier-Stokes problems (Proc. Sympos., Univ. Paderborn, Paderborn, 1979), Lecture Notes in Math., Springer, Berlin, 771:360–370, 1980.

[21] Martin Meyries and Roland Schnaubelt, *Interpolation, embeddings and traces of anisotropic fractional Sobolev spaces with temporal weights*, J. Funct. Anal., 262(3):1200–1229, 2012.

[22] Jan Prüss, *Maximal regularity for evolution equations in L_p-spaces*, Conf. Semin. Mat. Univ. Bari, (285):1–39, 2002.

[23] Jan Prüss and Gieri Simonett, *Maximal regularity for evolution equations in weighted L_p-spaces*, Arch. Math. (Basel), 82(5):415–431, 2004.

[24] Jan Prüss and Gieri Simonett, *Analytic solutions for the two-phase Navier-Stokes equations with surface tension and gravity*, in Parabolic problems, Progr. Nonlinear Diff. Eqs. Appl., Birkhäuser/Springer Basel AG, Basel, 80:507–540, 2011.

[25] Jan Prüss and Gieri Simonett, *Moving Interfaces and Quasilinear Parabolic Evolution Equations*, Monographs in Mathematics, Birkhäuser/Springer, [Cham], 105, 2016.

[26] Jan Prüss, Gieri Simonett, and Rico Zacher, *On convergence of solutions to equilibria for quasilinear parabolic problems*, J. Diff. Eqs., 246(10):3902–3931, 2009.

[27] U Seifert and R Lipowsky, *Morphology of vesicles*, Handbook of Biological Physics, 1:403–464, 1995.

[28] Roger Temam, *Navier-Stokes equations*, AMS Chelsea Publishing, Providence, RI, Theory and Numerical Analysis, Reprint of the 1984 Edition, 2001.

[29] Hans Triebel, *Interpolation theory, function spaces, differential operators*, Johann Ambrosius Barth, Heidelberg, Second Edition, 1995.

[30] Xiaoqiang Wang, *Asymptotic analysis of phase field formulations of bending elasticity models*, SIAM J. Math. Anal., 39(5):1367–1401, 2008.

[31] Hao Wu and Xiang Xu, *Strong solutions, global regularity, and stability of a hydrodynamic system modeling vesicle and fluid interactions*, SIAM J. Math. Anal., 45(1):181–214, 2013.

[32] Pengtao Yue, James J. Feng, Chun Liu, and Jie Shen, *A diffuse-interface method for simulating two-phase flows of complex fluids*, J. Fluid Mech., 515:293–317, 2004.

[33] Eberhard Zeidler, *Nonlinear functional analysis and its applications. I*, Springer-Verlag, New York, 1986. Fixed-point theorems, Translated from the German by Peter R. Wadsack.

[34] Ou-Yang Zhong-Can, Liu Ji-Xing, and Xie Yu-Zhang, *Geometric methods in the elastic theory of membranes in liquid crystal phases*, World Scientific, 2, 1999.