

PRIORITY-BASED RIEMANN SOLVER FOR TRAFFIC FLOW ON NETWORKS*

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Abstract. In this article we introduce a new Riemann solver for traffic flow on networks. The Priority Riemann solver (\mathcal{PRS}) provides a solution at junctions by taking into consideration priorities for the incoming roads and maximization of through flux. We prove existence of solutions for the solver for junctions with up to two incoming and two outgoing roads and show numerically the comparison with previous Riemann solvers. Additionally, we introduce a second version of the solver that considers the priorities as softer constraints and illustrate numerically the differences between the two solvers.

Keywords. scalar conservation laws; traffic flow; Riemann solver.

AMS subject classifications. 90B20; 35L65.

1. Introduction

Conservation laws on network is now a mature field with a high number of contributions published in math journals every year. One of the reasons for such success and interest is the large set of possible applications, which include vehicular traffic [12], supply chains [1], irrigation channels [2], telecommunications [9] and others. For a complete account of recent results and references we refer the reader to the survey [4], while here we focus on the vehicular traffic case.

The theory for the scalar case is quite developed (see [7, 13, 15]), with most results based on the concept of Riemann solver. The latter is the network equivalent to the classical Riemann solver for conservation laws on the real line and provides a solution to Riemann problems at junctions, i.e., Cauchy problems with constant initial data on each road.

To better explain the models proposed and analyzed in the literature, and our contribution, we take advantage of a simple, yet real and important, situation appearing in vehicular traffic: a single junction with two incoming and two outgoing roads. The latter will allow us to illustrate the different mathematical theories at work. To fix the ideas we parametrize the incoming roads as $I_1 = I_2 =]-\infty, 0]$ and the outgoing ones as $I_3 = I_4 = [0, +\infty[$, so the junction J corresponds to coordinate $x=0$ in all roads. The dynamics on roads is given by the conservation laws (following the classical LWR model [17, 18]):

$$\partial_t \rho_l + \partial_x f_l(\rho_l) = 0, \quad t \geq 0, x \in I_l, \quad (1.1)$$

where $l \in \{1, 2, 3, 4\}$, $\rho_l(t, x) \in [0, \rho_{\max}]$, is the car density, $v_l = v_l(\rho_l)$ is the average velocity and $f_l = f_l(\rho_l) = \rho_l v_l(\rho_l)$ is the flux. A Riemann problem at J is a Cauchy problem for (1.1) with constant initial datum $\rho_{0,l}$, $l = 1, \dots, 4$, on each road. As it happens for conservation laws on the real line (see [3]), a map assigning a solution to every

*This research was supported by the NSF grant CNS #1446715, by KI-Net “Kinetic description of emerging challenges in multiscale problems of natural sciences” - NSF grant # 1107444 and by the INRIA associated team Optimal RERoute Strategies for Traffic management (ORESTE).

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Riemann problem is called a *Riemann solver*. Such solutions are formed by simple waves: traveling discontinuities called shocks and continuous opening waves called rarefactions. Moreover, for every fixed Riemann solvers, one can construct approximate solutions via the Wave-Front Tracking algorithm [16]. Under suitable assumptions, one can also prove uniqueness and continuous dependence from initial data, i.e. well-posedness of Cauchy problems [3].

Notice that solving Riemann problems at J is equivalent to assign boundary values $\tilde{\rho}_l$, $l=1, \dots, 4$, and solve four Initial-Boundary Value Problems (IBVP) on the roads I_1, \dots, I_4 . Unfortunately (see [3]), solutions to IBVP may not attain the boundary value $\tilde{\rho}_i$ and, for vehicular traffic, such solutions may violate conservation of the number of cars through the junction (see [12]). To ensure that the boundary value is attained by the solution, one needs to prescribe a sign to wave speeds: for incoming roads the wave $(\rho_{0,i}, \tilde{\rho}_i)$, $i=1, 2$, must have negative speed, while for outgoing ones the wave $(\tilde{\rho}_i, \rho_{0,j})$, $j=3, 4$, must have positive speed. Under this assumption, one can instead assign the boundary values of flows $f_l(\tilde{\rho}_l)$ to determine uniquely the boundary values of densities $\tilde{\rho}_l$ (see [12] for details). Moreover, the wave speed signs imply limitations on possible flows, see Proposition 2.1. We call admissible a Riemann solver satisfying the wave speed sign conditions.

The admissible Riemann solver proposed in [7] was based on two main modeling assumptions:

- (A) Traffic from incoming roads distribute to outgoing roads according to fixed traffic coefficients;
- (B) Traffic flow through the junction is maximized.

To illustrate (A) and (B), let use focus on our simple junction J . The rule (A) is captured assigning a 2×2 matrix $A = (\alpha_{ji})$, where $i=1, 2$ ranges over incoming roads and $j=3, 4$ over outgoing ones. More precisely $\alpha_{ji} \in]0, 1[$ represents the percentage of traffic going from road i to road j and it holds $\sum_j \alpha_{ji} = 1$, $i=1, 2$. If γ_l represents the flow to or from road l , then rule (A) corresponds to set $(\gamma_3, \gamma_4)^t = A \cdot (\gamma_1, \gamma_2)^t$.

The only rule (A) does not isolate a unique solution. An easy example is as follows. If $\rho_{0,1} = \rho_{0,2} = \rho_{\max}$ and $\rho_{0,3} = \rho_{0,4} = 0$, then the constant-in-time solution trivially satisfies rule (A), since all flows to and from the junction are vanishing (under the hypothesis that $f_l(0) = 0 = f_l(\rho_{\max})$). Rule (B) can be mathematically expressed as the maximization of the flow from incoming roads: $\gamma_1 + \gamma_2$. If $\alpha_{31} \neq \alpha_{32}$, then the only solution satisfying rule (B) is formed by rarefaction waves on all roads and is depicted in Figure 1.1. Using this solver, solutions to Cauchy problems were defined for arbitrary networks, see [7, 11, 12]. Moreover, a general existence theorem was proved in [13] for every Riemann solver satisfying three general properties, called (P1)-(P3). Roughly speaking, (P1) asks for the solver to depend only on the flow limitations (and not on the specific densities), (P2) is a bound on the increase of the total variation of the flow due to interactions of waves with junctions, while (P3) asks the flow through the junction to decrease if the flow from one road decreases. Due to finite speed of propagation of waves, one can focus on a network with a single junction. Then the existence proof is based on estimates on the total variation in space of the flow on the whole network in terms of the total variation in time of the flow Γ through the junction, see Definition (2.5). In turn, the latter is bounded thanks to the general property (P3).

To better capture the possibility that one incoming road has a priority over the other, one can introduce a priority vector $P = (p_1, p_2)$ (with $p_i > 0$ and $\sum_i p_i = 1$) and require the incoming flow vector (γ_1, γ_2) to be parallel to P , [10, 19]. However, such

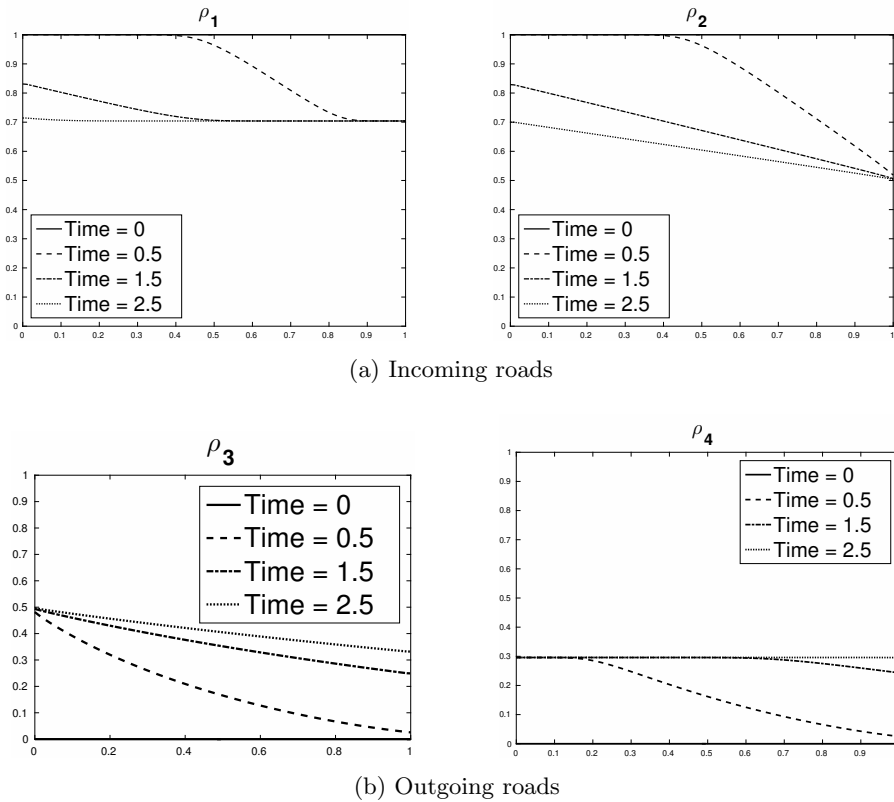


Fig. 1.1: Solution of the problem using the Riemann solver proposed in [7].

rule would not guarantee flow maximization, thus we define our Riemann Solver so that the road with highest priority will use the maximal possible flow (respecting the incoming and outgoing maximal flows given by Proposition 2.1 and priority vector), then the second road will use the remaining capacity of the junction (if there is enough demand). Let us illustrate what this means for the 2×2 junction J . As shown in Figure 1.2, let us suppose that we have two incoming roads 1 and 2 and that the flux of cars from each one of the incoming roads would reach either road 3 or road 4. The priority parameter tells us exactly which ones of the cars in the intersection has the right of way with respect to the other. Our construction can be generalized to junctions with an arbitrary number of n incoming and m outgoing roads. Again, the road with the highest priority will use the maximal flow taking into account also outgoing roads constraints. If some room is left for additional flow, then the road with the second highest priority will use the left space and so on. A precise definition of this new Riemann solver, that we call *Priority Riemann Solver*, is based on an $m \times n$ traffic distribution matrix A (Definition 3.1), a priority vector $P = (p_1, \dots, p_n)$ (with $p_i > 0$ and $\sum_i p_i = 1$) and requires a recursion method, which is described in Algorithm 1. We also model special situations in which some outgoing roads do not absorb traffic from some incoming ones and propose an alternative solver with softer priorities, see Algorithm 2. During the writing of this manuscript we discovered that our priority-based Riemann solver may be obtained as limit of solvers defined by Dynamic Traffic Assignment based on junctions with queues [5].

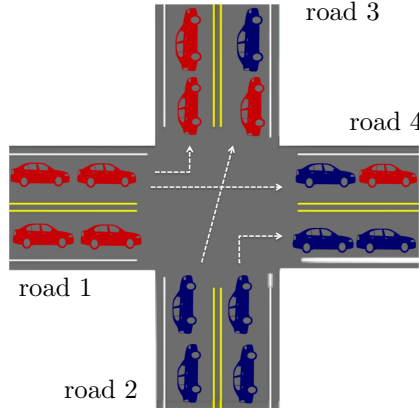


Fig. 1.2: Illustration of a 2×2 intersection.

The general existence theorem of [13] can not be applied to our case. Indeed the property (P3) is not satisfied by the Priority Riemann Solver (see the Appendix: case A2 with flux increase corresponding to Figure A.2a). Therefore, we achieve existence via a new set of general properties. Property (P1) is the same as in [13], while we modify (P2) and (P3) by using estimates involving not only Γ but also the maximal flow along the priority vector P in the set of admissible flows, see Definition (3.3). We apply the general theory to the Priority Riemann Solver by proving that the new properties (P1)-(P3) are satisfied for junctions with at most two incoming and two outgoing roads. Then, to illustrate the properties of the Priority Riemann Solver and the one with soft priorities, and compare with existing Riemann Solvers, we implement numerical simulations via Godunov scheme.

The paper is organized as follows. In Section 2 we introduce the basic definitions of the theory of conservation laws on networks, then in Section 3 we define our Priority Riemann Solver and prove existence of solutions to Cauchy problems in Section 4. In Section 5, an alternative definition of the Riemann Solver with softer priorities is described and lastly, in Section 6, we propose a numerical discretization and show some numerical simulations comparing our Solvers to existing ones. The Appendix A collects the proof of the main theorem of the paper.

2. Basics

In this section we recall the basic definitions and results of the theory of conservation laws on networks, based on the concept of Riemann solver at junctions. Due to finite propagation speed of waves, to achieve existence results for Cauchy problems it is not restrictive to focus on a single junction. For details on how to extend the results to a general network, we refer the reader to [12, 13].

Fix a junction J with n incoming roads I_1, \dots, I_n and m outgoing roads I_{n+1}, \dots, I_{n+m} , where $I_i =]-\infty, 0]$ ($i \in \{1, \dots, n\}$) and $I_j = [0, +\infty[$ ($j \in \{n+1, \dots, n+m\}$). The traffic on each road I_l ($l \in \{1, \dots, n+m\}$) is modeled using the celebrated Lighthill–Whitham–Richards model (briefly LWR, see [17, 18]):

$$\partial_t \rho_l + \partial_x f(\rho_l) = 0, \quad t \geq 0, x \in I_l, \quad (2.1)$$

where $\rho_l(t, x) \in [0, \rho_{\max}]$, is the car density, $v_l = v_l(\rho_l)$ is the average velocity and $f_l = f_l(\rho_l) = \rho_l v_l(\rho_l)$ is the flux. For simplicity, throughout the paper we assume $\rho_{\max} = 1$ and $f_l = f$ for all $l = 1, \dots, n+m$.

We make the following assumptions on the flux function f :

- (H) $f : [0, 1] \rightarrow \mathbb{R}$ is a Lipschitz continuous and concave function satisfying
 - (1) $f(0) = f(1) = 0$;
 - (2) there exists a unique $\rho_{cr} \in]0, 1[$ such that f is strictly increasing in $[0, \rho_{cr}[$ and strictly decreasing in $] \rho_{cr}, 1]$.

As usual, entropic solutions and weak solutions at junctions are given by:

DEFINITION 2.1. A function $\rho_l \in C([0, +\infty[; L^1_{loc}(I_l))$ is an entropy-admissible solution to equation (2.1) in the arc I_l if, for every $k \in [0, 1]$ and every $\varphi : [0, +\infty[\times I_l \rightarrow \mathbb{R}$ smooth, positive and with compact support in $]0, +\infty[\times (I_l \setminus \{0\})$, it holds

$$\int_0^{+\infty} \int_{I_l} (|\rho_l - k| \partial_t \varphi + \text{sgn}(\rho_l - k)(f(\rho_l) - f(k)) \partial_x \varphi) dx dt \geq 0. \tag{2.2}$$

DEFINITION 2.2. A collection of functions $\rho_l \in C([0, +\infty[; L^1_{loc}(I_l))$, ($l \in \{1, \dots, n+m\}$) is a weak solution at J if

- (1) for every $l \in \{1, \dots, n+m\}$, the function ρ_l is an entropy-admissible solution to equation (2.1) in the road I_l ;
- (2) for every $l \in \{1, \dots, n+m\}$ and for a.e. $t > 0$, the function $x \mapsto \rho_l(t, x)$ has a version with bounded total variation;
- (3) for a.e. $t > 0$, it holds

$$\sum_{i=1}^n f(\rho_i(t, 0-)) = \sum_{j=n+1}^{n+m} f(\rho_j(t, 0+)), \tag{2.3}$$

where ρ_l stands for the version with bounded total variation of 2.

A Riemann problem at the junction J is a Cauchy problem with constant initial data on each road. More precisely, given $\rho_{1,0}, \dots, \rho_{n+m,0} \in [0, 1]$, the corresponding Riemann problem at J is given by

$$\begin{cases} \partial_t \rho_l + \partial_x f(\rho_l) = 0, \\ \rho_l(0, \cdot) = \rho_{0,l}, \end{cases} \quad l \in \{1, \dots, n+m\}. \tag{2.4}$$

For a collection of functions $\rho_l \in C([0, +\infty[; L^1_{loc}(I_l))$ ($l \in \{1, \dots, n+m\}$) such that, for every $l \in \{1, \dots, n+m\}$ and a.e. $t > 0$, the map $x \mapsto \rho_l(t, x)$ has a version with bounded total variation, we define the functionals

$$\Gamma(t) := \sum_{i=1}^n f(\rho_i(t, 0-)) \tag{2.5}$$

and

$$\text{TV}_f(t) := \sum_{l=1}^{n+m} \text{TV}(f(\rho_l(t, \cdot))). \tag{2.6}$$

Notice that Γ is the flux through the junction, i.e. the total number of cars crossing the junction J per unit of time, while TV_f is the total variation of the flux on the whole network. From the flux bounds we easily derive:

$$0 \leq \Gamma(t) \leq n f(\rho_{cr}). \tag{2.7}$$

A Riemann solver at J is defined by:

DEFINITION 2.3. A Riemann solver \mathcal{RS} is a function

$$\mathcal{RS}: \quad [0,1]^{n+m} \longrightarrow [0,1]^{n+m}$$

$$(\rho_{1,0}, \dots, \rho_{n+m,0}) \longmapsto (\bar{\rho}_1, \dots, \bar{\rho}_{n+m})$$

satisfying the following properties

$$(1) \quad \sum_{i=1}^n f(\bar{\rho}_i) = \sum_{j=n+1}^{n+m} f(\bar{\rho}_j);$$

(2) for every $i \in \{1, \dots, n\}$, the classical Riemann problem

$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R}, t > 0, \\ \rho(0, x) = \begin{cases} \rho_{i,0}, & \text{if } x < 0, \\ \bar{\rho}_i, & \text{if } x > 0, \end{cases} \end{cases}$$

is solved with waves with negative speed;

(3) for every $j \in \{n+1, \dots, n+m\}$, the classical Riemann problem

$$\begin{cases} \rho_t + f(\rho)_x = 0, & x \in \mathbb{R}, t > 0, \\ \rho(0, x) = \begin{cases} \bar{\rho}_j, & \text{if } x < 0, \\ \rho_{j,0}, & \text{if } x > 0, \end{cases} \end{cases}$$

is solved with waves with positive speed.

Moreover, the Riemann solver \mathcal{RS} must satisfy the consistency condition if

$$\mathcal{RS}(\mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0})) = \mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0})$$

for every $(\rho_{1,0}, \dots, \rho_{n+m,0}) \in [0,1]^{n+m}$.

For future use, we now provide some definitions for the LWR model and for Riemann problems at junctions, for more details see [12].

DEFINITION 2.4. We say that $(\rho_{1,0}, \dots, \rho_{n+m,0})$ is an equilibrium for the Riemann solver \mathcal{RS} if

$$\mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0}) = (\rho_{1,0}, \dots, \rho_{n+m,0}).$$

DEFINITION 2.5. We say that a datum $\rho_i \in [0,1]$ in an incoming road is a good datum if $\rho_i \in [\rho_{\text{cr}}, 1]$ and a bad datum otherwise.

We say that a datum $\rho_j \in [0,1]$ in an outgoing road is a good datum if $\rho_j \in [0, \rho_{\text{cr}}]$ and a bad datum otherwise.

We also define the following function:

DEFINITION 2.6. Let $\tau: [0,1] \rightarrow [0,1]$ be the map such that:

(1) $f(\tau(\rho)) = f(\rho)$ for every $\rho \in [0,1]$;

(2) $\tau(\rho) \neq \rho$ for every $\rho \in [0,1] \setminus \{\rho_{\text{cr}}\}$.

Clearly, the function τ is well defined and satisfies

$$0 \leq \rho \leq \rho_{\text{cr}} \iff \rho_{\text{cr}} \leq \tau(\rho) \leq 1, \quad \rho_{\text{cr}} \leq \rho \leq 1 \iff 0 \leq \tau(\rho) \leq \rho_{\text{cr}}.$$

Given initial data (of Riemann type) $\rho_{1,0}, \dots, \rho_{n+m,0} \in [0,1]$ we define:

(1) for every $i \in \{1, \dots, n\}$

$$\gamma_i^{max} = \begin{cases} f(\rho_{i,0}), & \text{if } 0 \leq \rho_{i,0} \leq \rho_{cr}, \\ f(\rho_{cr}), & \text{if } \rho_{cr} \leq \rho_{i,0} \leq 1; \end{cases} \quad (2.8)$$

(2) for every $j \in \{n+1, \dots, n+m\}$

$$\gamma_j^{max} = \begin{cases} f(\rho_{cr}), & \text{if } 0 \leq \rho_{j,0} \leq \rho_{cr}, \\ f(\rho_{j,0}), & \text{if } \rho_{cr} \leq \rho_{j,0} \leq 1; \end{cases} \quad (2.9)$$

(3) for every $l \in \{1, \dots, n+m\}$

$$\Omega_l = [0, \gamma_l^{max}]. \quad (2.10)$$

Moreover, we have the following result (see [13]):

PROPOSITION 2.1. *It holds:*

- (1) For every $i \in \{1, \dots, n\}$, an element $\bar{\gamma}$ belongs to Ω_i if and only if there exists $\bar{\rho}_i \in [0, 1]$ such that $f(\bar{\rho}_i) = \bar{\gamma}$ and point 2 of Definition 2.3 is satisfied.
- (2) For every $j \in \{n+1, \dots, n+m\}$, an element $\bar{\gamma}$ belongs to Ω_j if and only if there exists $\bar{\rho}_j \in [0, 1]$ such that $f(\bar{\rho}_j) = \bar{\gamma}$ and point 3 of Definition 2.3 is satisfied.

3. Definition of the priority Riemann solver

In this section we define a new Riemann solver based on priorities. For this purpose, we first fix a matrix A belonging to the set of matrices:

$$\mathcal{A} := \left\{ A = \{a_{ji}\}_{\substack{i=1, \dots, n \\ j=n+1, \dots, n+m}} : \begin{array}{l} 0 \leq a_{ji} \leq 1 \forall i, j, \\ \sum_{j=n+1}^{n+m} a_{ji} = 1 \forall i \end{array} \right\} \quad (3.1)$$

and a priority vector $P = (p_1, \dots, p_n) \in \mathbb{R}^n$, with $p_i > 0$, $\sum_i p_i = 1$, indicating priorities among incoming roads.

Consider the closed, convex and non-empty set

$$\Omega = \left\{ (\gamma_1, \dots, \gamma_n) \in \prod_{i=1}^n \Omega_i : A \cdot (\gamma_1, \dots, \gamma_n)^T \in \prod_{j=n+1}^{n+m} \Omega_j \right\}, \quad (3.2)$$

and define:

$$\bar{h} = \sup\{h \in \mathbb{R}^+ : hP \in \Omega\}. \quad (3.3)$$

Given Riemann data $(\rho_{1,0}, \dots, \rho_{m+n,0})$, we define a vector $Q = (\bar{\gamma}_1, \dots, \bar{\gamma}_n)$ of incoming fluxes by a recursive procedure. First we explain the procedure in steps and then provide a pseudo-code in Algorithm 1.

- **STEP 1.** For every $i \in \{1, \dots, n\}$ define

$$h_i = \max\{h : hp_i \leq \gamma_i^{max}\} = \frac{\gamma_i^{max}}{p_i},$$

and for every $j \in \{n+1, \dots, n+m\}$ define

$$h_j = \max\left\{h : (A \cdot (hP))_j = h \left(\sum_i a_{ji} p_i \right) \leq \gamma_j^{max} \right\} = \frac{\gamma_j^{max}}{\sum_i a_{ji} p_i}.$$

In other words, h_i is the maximal t so that hP verifies the flux constraint for the i -th road, similarly for h_j .

Set $\bar{h} = \min_{i,j} \{h_i, h_j\}$.

We distinguish two cases:

- CASE 1. If there exists j such that $h_j = \bar{h}$, then we set $Q = \bar{h}P$ and we are done.
- CASE 2. Otherwise, let $I_1 = \{i \in \{1, \dots, n\} : h_i = \bar{h}\}$ (by assumption $I_1 \neq \emptyset$). We set $Q_i = \bar{h}p_i$ for $i \in I_1$ and we go to next step.
- **STEP S.** In step $S-1$ we defined a set I_{S-1} and, by induction, all components of Q are fixed for $i \in J_S = I_1 \cup \dots \cup I_{S-1}$. We let $|J_S| < n$ denote the cardinality of J_S and denote by J_S^c the complement of J_S in $\{1, \dots, n\}$. We now define h_i for $i \in J_S^c$ by:

$$h_i = \max\{h : hp_i \leq \gamma_i^{max}\} = \frac{\gamma_i^{max}}{p_i},$$

and for every $j \in \{n+1, \dots, n+m\}$ define

$$h_j = \max \left\{ h : \sum_{i \in J_S} a_{ji} Q_i + h \left(\sum_{i \in J_S^c} a_{ji} p_i \right) \leq \gamma_j^{max} \right\}.$$

We then proceed similarly to STEP 1, setting $\bar{h} = \min_{i,j} \{h_i, h_j\}$ and distinguishing two cases:

- CASE 1. If there exists j such that $h_j = \bar{h}$, then we set $Q_i = \bar{h}P_i$ for $i \in J_S^c$ and we are done.
- CASE 2. Otherwise, let $I_S = \{i \in J_S^c : h_i = \bar{h}\}$ (by assumption $I_S \neq \emptyset$). We set $Q_i = \bar{h}p_i$ for $i \in I_S$. If $J_S \cup I_S = \{1, \dots, n\}$ then we stop, otherwise we go to next step.

Algorithm 1 Recursive definition of \mathcal{PRS}

Set $J = \emptyset$ and $J^c = \{1, \dots, n\} \setminus J$.

while $|J| < n$ **do**

$\forall i \in J^c \rightarrow h_i = \max\{h : hp_i \leq \gamma_i^{max}\} = \frac{\gamma_i^{max}}{p_i}$,

$\forall j \in \{n+1, \dots, n+m\} \rightarrow h_j = \sup\{h : \sum_{i \in J} a_{ji} Q_i + h(\sum_{i \in J^c} a_{ji} p_i) \leq \gamma_j^{max}\}$.

Set $\bar{h} = \min_{i,j} \{h_i, h_j\}$.

if $\exists j$ s.t. $h_j = \bar{h}$ **then**

Set $Q = \bar{h}P$ and $J = \{1, \dots, n\}$.

else

Set $I = \{i \in J^c : h_i = \bar{h}\}$ and $Q_i = \bar{h}p_i$ for $i \in I$.

Set $J = J \cup I$.

end if

end while

We are now ready to define the Priority Riemann Solver (briefly \mathcal{PRS}).

DEFINITION 3.1. Let $Q = (\bar{\gamma}_1, \dots, \bar{\gamma}_n)$ be the vector of incoming fluxes defined by Algorithm 1, then the vector of outgoing fluxes is given by $A \cdot Q^T = (\bar{\gamma}_{n+1}, \dots, \bar{\gamma}_{n+m})^T$. For every $i \in \{1, \dots, n\}$, set $\bar{\rho}_i$ equal either to $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or to the solution to

$f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \rho_{\text{cr}}$. For every $j \in \{n+1, \dots, n+m\}$, set $\bar{\rho}_j$ equal either to $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or to the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \rho_{\text{cr}}$. Finally, $\mathcal{PRS}: [0, 1]^{n+m} \rightarrow [0, 1]^{n+m}$ is given by

$$\mathcal{PRS}(\rho_{1,0}, \dots, \rho_{n+m,0}) = (\bar{\rho}_1, \dots, \bar{\rho}_n, \bar{\rho}_{n+1}, \dots, \bar{\rho}_{n+m}). \tag{3.4}$$

4. Existence result for Cauchy problems

Given initial data of bounded variation $\rho_{0,1}(\cdot), \dots, \rho_{0,n}(\cdot): [0, +\infty[\rightarrow [0, 1]$ and $\rho_{0,n+1}(\cdot), \dots, \rho_{0,n+m}(\cdot):]-\infty, 0] \rightarrow [0, 1]$ the corresponding Cauchy problem is defined by:

$$\begin{cases} \partial_t \rho_l + \partial_x f(\rho_l) = 0, \\ \rho_l(0, x) = \rho_{0,l}(x), \end{cases} \quad l \in \{1, \dots, n+m\}. \tag{4.1}$$

To solve Cauchy problems one can construct approximate solutions via Wave Front Tracking (WFT). In simple words, one first approximate the initial data by piecewise constant functions, then solve the corresponding Riemann problems within roads and at junctions approximating rarefaction waves by a fan of rarefaction shocks and solve new Riemann problems when waves interact with each other or with the junction. We refer the reader to [12] for details. Notice that all waves in a WFT approximate solution are shocks, i.e. traveling discontinuities. For every wave we will usually indicate by ρ_l , respectively ρ_r , the left limit, respectively right limit, of the approximate solution at the discontinuity point. To prove convergence of WFT approximations, one needs to estimate the number of waves, the number of wave interactions and provide estimates on the total variation of approximate solutions. The general theory of [13] is based on three properties which guarantee such estimates. Along the same idea we define three general properties (P1)-(P3) which will ensure existence of solutions.

The first property requires that equilibria are determined only by bad data values (and coincides with (P1) in [13]), more precisely:

DEFINITION 4.1. *We say that a Riemann solver \mathcal{RS} has the property (P1) if the following condition holds. Given $(\rho_{1,0}, \dots, \rho_{n+m,0})$ and $(\rho'_{1,0}, \dots, \rho'_{n+m,0})$ two initial data such that $\rho_{l,0} = \rho'_{l,0}$ whenever either $\rho_{l,0}$ or $\rho'_{l,0}$ is a bad datum, then*

$$\mathcal{RS}(\rho_{1,0}, \dots, \rho_{n+m,0}) = \mathcal{RS}(\rho'_{1,0}, \dots, \rho'_{n+m,0}). \tag{4.2}$$

The second property requires for bounds in the increase of the flux variation for waves interacting with J . More precisely the latter is bounded in terms of the strength of the interacting wave as well as the sum of the changes in the incoming fluxes and in \bar{h} (see definition (3.3)). Moreover, the increase in \bar{h} is bounded by the strength of the interacting wave.

DEFINITION 4.2. *We say that a Riemann solver \mathcal{RS} has the property (P2) if there exists a constant $C \geq 1$ such that the following condition holds. For every equilibrium $(\rho_{1,0}, \dots, \rho_{n+m,0})$ of \mathcal{RS} and for every wave $(\rho_i, \rho_{i,0})$ for $i = 1, \dots, n$ (respectively (ρ_j, ρ_j) for $j = n+1, \dots, n+m$) interacting with J at time $\bar{t} > 0$ and producing waves in the arcs according to \mathcal{RS} , we have*

$$\begin{aligned} & \text{TV}_f(\bar{t}+) - \text{TV}_f(\bar{t}-) \\ & \leq C \min \{ |f(\rho_{l,0}) - f(\rho_l)|, |\Gamma(\bar{t}+) - \Gamma(\bar{t}-)| + |\bar{h}(\bar{t}+) - \bar{h}(\bar{t}-)| \} \end{aligned} \tag{4.3}$$

and

$$\bar{h}(\bar{t}+) - \bar{h}(\bar{t}-) \leq C |f(\rho_{l,0}) - f(\rho_l)|. \quad (4.4)$$

Finally, we state the third property: if a wave interacts with J and provokes a flux decrease then \bar{h} decreases and the increase of Γ is bounded by the change in \bar{h} .

DEFINITION 4.3. *We say that a Riemann solver \mathcal{RS} has the property (P3) if the following holds. For every equilibrium $(\rho_{1,0}, \dots, \rho_{n+m,0})$ of \mathcal{RS} and for every wave $(\rho_i, \rho_{i,0})$ with $f(\rho_i) < f(\rho_{i,0})$ for $i=1, \dots, n$ (respectively (ρ_j, ρ_j) with $f(\rho_j) < f(\rho_{j,0})$ for $j=n+1, \dots, n+m$) interacting with J at time $\bar{t} > 0$ and producing waves in the arcs according to \mathcal{RS} , we have*

$$\Gamma(\bar{t}+) - \Gamma(\bar{t}-) \leq C |\bar{h}(\bar{t}+) - \bar{h}(\bar{t}-)|, \quad (4.5)$$

$$\bar{h}(\bar{t}+) \leq \bar{h}(\bar{t}-). \quad (4.6)$$

THEOREM 4.1. *If a Riemann solver satisfies properties (P1)-(P3), then every Cauchy problem with initial data of bounded variation admits a weak solution.*

In order to prove Theorem 4.1, we need first to provide some definition and results. We start by giving the following:

DEFINITION 4.4. *A wave along a WFT approximate solution generated at time $t=0$ inside a road is called original, while the ones generated by J are called not original. If two original waves interact, then the resulting wave is still called original, while if an original wave interacts with a not original wave then the resulting wave is not original.*

We now have the following result:

PROPOSITION 4.1. *Let (ρ_l, ρ_r) be a wave generated on an incoming road I from the junction at time \bar{s} . Assume that there exists a time $\bar{t} > \bar{s}$ at which the wave interacts with J (after interacting with waves inside I) and call $\bar{\rho}_l$, respectively $\bar{\rho}_r$, its left, respectively, right limit at \bar{t} . If I is an incoming road then we have $\bar{\rho}_r \geq \rho_{\text{cr}}$ and $f(\bar{\rho}_l) < f(\bar{\rho}_r)$. If I is an outgoing road then we have $\bar{\rho}_l \leq \rho_{\text{cr}}$ and $f(\bar{\rho}_r) < f(\bar{\rho}_l)$.*

Proof. We prove the result for an incoming road, the other case being similar. First notice that (ρ_l, ρ_r) must have negative speed, thus if $\rho_l < \rho_{\text{cr}}$ then $\rho_r > \tau(\rho_r) \geq \rho_{\text{cr}}$, while if $\rho_l \geq \rho_{\text{cr}}$ then $\rho_r \geq \rho_{\text{cr}}$. Therefore in both cases we have $\rho_r \geq \rho_{\text{cr}}$. If the wave interacts with waves coming from the left then the value of ρ_r does not change. If the wave interacts with a wave $(\rho_r, \hat{\rho})$ coming from the right, then the wave was generated from the junction J (or obtained by interactions of waves generated from J) and thus must satisfy $\hat{\rho} \geq \rho_{\text{cr}}$. Finally, $\bar{\rho}_r \geq \rho_{\text{cr}}$. Then, since $(\bar{\rho}_l, \bar{\rho}_r)$ must have positive speed, we deduce that $\bar{\rho}_l < \rho_{\text{cr}}$ and $f(\bar{\rho}_l) < f(\bar{\rho}_r)$, thus we conclude. \square

From Proposition 4.1 we have the following:

COROLLARY 4.1. *If a wave (ρ_l, ρ_r) interacts with J from an incoming road and satisfy $f(\rho_l) > f(\rho_r)$ then it is an original wave. If a wave (ρ_l, ρ_r) interacts with J from an outgoing road and satisfy $f(\rho_r) > f(\rho_l)$ then it is an original wave.*

Proof. [Theorem 4.1] We will prove that the variation of the flux is bounded along WFT approximate solutions. Then, a bound on the number of waves is obtained using property (P1) following the same proof of Proposition 10 of [13].

We now prove that the total variation of the flux $\text{TV}(f)$ remains uniformly bounded in time along WFT approximate solutions. The main idea is to first bound the total variation in time of \bar{h} and then of Γ . This in turn will provide the desired estimate. Let us indicate with PV the positive variation of a function and with NV the negative one. Then:

$$\text{TV}(\bar{h}) = \text{PV}(\bar{h}) + \text{NV}(\bar{h})$$

and

$$\text{PV}(\bar{h}) = \text{PV}^O(\bar{h}) + \text{PV}^R(\bar{h}),$$

where PV^O is the variation due to interactions of original waves with the junction and PV^R the one due to returning waves (i.e. not original).

From property (P2) we get:

$$\text{PV}^O(\bar{h}) \leq C \text{TV}(f(\rho_0)) \leq C \max_{\rho \in [0,1]} |f'(\rho)| \text{TV}(\rho_0)$$

and from property (P3) and Corollary 4.1 it follows $\text{PV}^R(\bar{h}) = 0$. Then $\text{PV}(\bar{h})$ is bounded and, since $\bar{h} \leq f^{\max} / \max_i p_i$, also $\text{TV}(\bar{h})$ is bounded. Similarly, for Γ we can write

$$\text{TV}(\Gamma) = \text{PV}(\Gamma) + \text{NV}(\Gamma)$$

and

$$\text{PV}(\Gamma) = \text{PV}^O(\Gamma) + \text{PV}^R(\Gamma).$$

Following the proof of [13, Lemma 12], (P2) implies

$$\text{PV}^O(\Gamma) \leq (C+2) \text{TV}(f(\rho_0)) \leq (C+2) \max_{\rho \in [0,1]} |f'(\rho)| \text{TV}(\rho_0).$$

From property (P3) we have that

$$\text{PV}^R(\Gamma) \leq C \text{TV}(\bar{h}),$$

which we just proved to be bounded. Therefore $\text{TV}(\Gamma)$ is also bounded.

Now, define Int the set of times at which a wave interacts with the junction J and, for $s \in \text{Int}$ let us indicate by $\Delta \text{TV}_f(s)$ the change due to the interaction. By property (P2) we have

$$\begin{aligned} \text{TV}(f(t)) &\leq \text{TV}(f(\rho_0)) + \sum_{s \in \text{Int}, s \leq t} \Delta \text{TV}_f(s) \\ &\leq \max_{\rho \in [0,1]} |f'(\rho)| \text{TV}(\rho_0) + C(\text{TV}(\Gamma) + \text{TV}(\bar{h})). \end{aligned}$$

Once $\text{TV}(f)$ is bounded, one can obtain a bound on $\text{TV}(\rho)$ as in [13] and conclude by passing to the limit in WFT approximate solutions. \square

PROPOSITION 4.2. *The Priority Riemann Solver \mathcal{PRS} satisfies properties (P1)-(P3) for junctions with $n \leq 2$, $m \leq 2$ and $0 < a_{ji} < 1$ for all i, j .*

The technical proof is deferred to Section A.

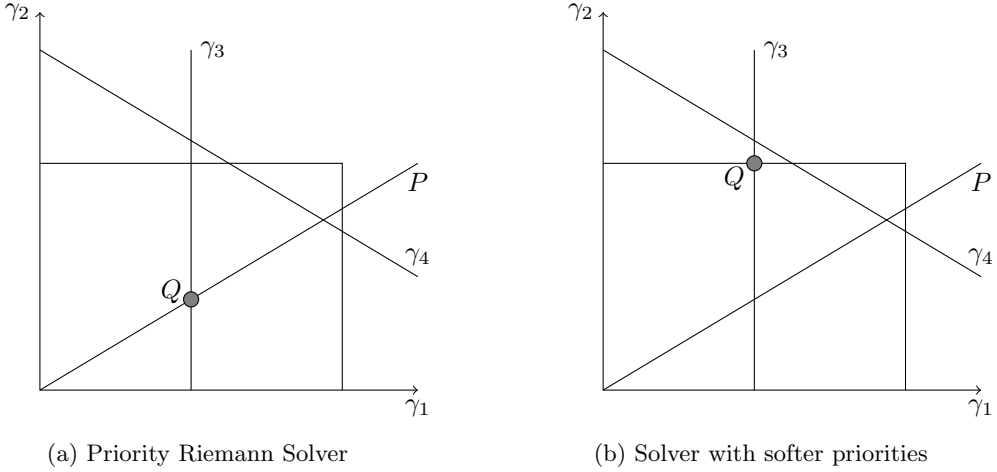


Fig. 5.1: Different solution for the two solvers: PRS and SPRS. The distribution matrix A satisfies $a_{13}=0$, i.e. no cars enter road 3 from road 1. Correspondingly the constraint on the flux γ_3 is represented by a vertical line. The PRS selects the point Q in (a), while SPRS the point Q in (b). Since priorities are softer the flux through the junction of the SPRS solution is higher than that of the PRS.

5. Solver with softer priorities

In this section we define a different version of the Riemann solver that uses priorities as softer constraints. In particular, this solver will differ from the solver PRS defined in Section 3 when one of the entries of the matrix A , as in definition (3.1), vanishes, see Figure 5.1. Notice that the softer priority of the SPRS will allow some flow from road 2 to pass through the junction, when the maximal flow from road 1 is already reached. This reflects the situation where the physical geometry of the junction allows for traffic from road 2 to road 4 (no traffic goes from 2 to 3) even if the traffic from road 1 to road 3 is maximal and has higher priority. For this purpose, we consider a matrix A that may have $a_{ji}=0$ for some i, j and a priority vector $P=(p_1, \dots, p_n) \in \mathbb{R}^n$, with $p_i > 0$, $\sum_i p_i = 1$. Then the Riemann solver with softer priorities (briefly SPRS) can be defined by the following recursive algorithm:

Algorithm 2 Recursive definition of SPRS

```

Set  $J = \emptyset$  and  $J^c = \{1, \dots, n\} \setminus J$ .
while  $|J| < n$  do
   $\forall i \in J^c \rightarrow h_i = \max\{h : hp_i \leq \gamma_i^{max}\} = \frac{\gamma_i^{max}}{p_i}$ ,
   $\forall j \in \{n+1, \dots, n+m\} \rightarrow h_j = \sup\{h : \sum_{i \in J} a_{ji}Q_i + h(\sum_{i \in J^c} a_{ji}p_i) \leq \gamma_j^{max}\}$ .
  Set  $\bar{h} = \min_{ij} \{h_i, h_j\}$ .
  if  $\exists j$  s.t.  $h_j = \bar{h}$  then
    Set  $I = \{i \in J^c : a_{ji} \neq 0\}$  and  $Q_i = \bar{h}p_i$  for  $i \in I$ .
  else
    Set  $I = \{i \in J^c : h_i = \bar{h}\}$  and  $Q_i = \bar{h}p_i$  for  $i \in I$ .
  end if
  Set  $J = J \cup I$ .
end while

```

We are now ready to define the Softer Priority Riemann Solver.

DEFINITION 5.1. *Let $Q = (\bar{\gamma}_1, \dots, \bar{\gamma}_n)$ be the vector of incoming fluxes defined by Algorithm 2, then the vector of outgoing fluxes is given by $A \cdot Q^T = (\bar{\gamma}_{n+1}, \dots, \bar{\gamma}_{n+m})^T$. For every $i \in \{1, \dots, n\}$, set $\bar{\rho}_i$ equal either to $\rho_{i,0}$ if $f(\rho_{i,0}) = \bar{\gamma}_i$, or to the solution to $f(\rho) = \bar{\gamma}_i$ such that $\bar{\rho}_i \geq \rho_{cr}$. For every $j \in \{n+1, \dots, n+m\}$, set $\bar{\rho}_j$ equal either to $\rho_{j,0}$ if $f(\rho_{j,0}) = \bar{\gamma}_j$, or to the solution to $f(\rho) = \bar{\gamma}_j$ such that $\bar{\rho}_j \leq \rho_{cr}$. Finally, $SPRS: [0, 1]^{n+m} \rightarrow [0, 1]^{n+m}$ is given by*

$$SPRS(\rho_{1,0}, \dots, \rho_{n+m,0}) = (\bar{\rho}_1, \dots, \bar{\rho}_n, \bar{\rho}_{n+1}, \dots, \bar{\rho}_{n+m}). \tag{5.1}$$

6. Numerical scheme and numerical simulations

To illustrate the PRS and $SPRS$ dynamics we provide some simulations based on the well-known Godunov scheme [14] on networks (see [12]), which is based on solutions to Riemann problems.

Define a numerical grid on $[0, T] \times \mathbb{R}$ given by:

- Δx is the fixed space grid size;
- Δt^ν , $\nu \in \mathbb{N}$, is the time grid size satisfying the CFL condition [8]:

$$\Delta t^\nu \max_{j \in \mathbb{Z}} |f'(u_j^\nu)| \leq \frac{1}{2} \Delta x \tag{6.1}$$

- $(t^\nu, x_j) = (t^{\nu-1} + \Delta t^\nu, j \Delta x)$ for $\nu \in \mathbb{N}$ and $j \in \mathbb{Z}$ are the grid points.

Consider a scalar conservation laws equipped with initial data:

$$\begin{aligned} \partial_t u + \partial_x f(u) &= 0, & x \in \mathbb{R}, t \in [0, T], \\ u(0, x) &= u_0(x), & x \in \mathbb{R}. \end{aligned} \tag{6.2}$$

An approximate solution of the problem is constructed first by taking a piecewise constant approximation of the initial data

$$u_j^0 = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} u_0(x) dx, \quad j \in \mathbb{Z}, \tag{6.3}$$

and then defining u_j^ν recursively from u_j^0 as follows. Under the CFL (6.1) the waves generated by different Riemann problem at the cell interfaces do not interact and the scheme can be written as follows

$$u_j^{\nu+1} = u_j^\nu - \frac{\Delta t^\nu}{\Delta x} (G(u_j^\nu, u_{j+1}^\nu) - G(u_{j-1}^\nu, u_j^\nu)), \tag{6.4}$$

where the numerical flux G is given by

$$G(u, v) = \begin{cases} \min_{z \in [u, v]} f(z) & \text{if } u \leq v \\ \max_{z \in [v, u]} f(z) & \text{if } v \leq u. \end{cases} \tag{6.5}$$

To impose boundary conditions and conditions at junctions we use the classical approach introduced in [6].

Boundary conditions.

Each road is divided into M cells, numbered from 1 to M . Boundary conditions are imposed using ghost cells. For an incoming road I_i we define:

$$u_{i,1}^{\nu+1} = u_{i,1}^\nu - \frac{\Delta t^\nu}{\Delta x} (G(u_{i,1}^\nu, u_{i,2}^\nu) - G(u_{i,0}^\nu, u_{i,1}^\nu)) \tag{6.6}$$

where $u_{i,0}^\nu$ is the value of the density at the boundary.

The outgoing boundary for I_j is treated in the same way by defining:

$$u_{j,M}^{\nu+1} = u_{j,M}^\nu - \frac{\Delta t^\nu}{\Delta x} (G(u_{j,M}^\nu, u_{j,M+1}^\nu) - G(u_{j,M-1}^\nu, u_{j,M}^\nu)) \tag{6.7}$$

with $u_{i,M+1}^\nu$ the value of the density at the outgoing boundary.

Conditions at the junction.

For I_i with $i \in \{1, \dots, n\}$ that is connected at the junction at the right endpoint we set:

$$u_{i,M}^{\nu+1} = u_{i,M}^\nu - \frac{\Delta t^\nu}{\Delta x} (Q_i^\nu - G(u_{i,M-1}^\nu, u_{i,M}^\nu)) \quad \text{for } i \in \{1, \dots, n\}, \tag{6.8}$$

while for the outgoing roads, connected at the junction with the left endpoint we have:

$$u_{j,1}^{\nu+1} = u_{j,1}^\nu - \frac{\Delta t^\nu}{\Delta x} (G(u_{j,1}^\nu, u_{j,2}^\nu) - Q_j^\nu) \quad \text{for } j \in \{n+1, \dots, n+m\}, \tag{6.9}$$

where Q_i^ν, Q_j^ν are the incoming and outgoing fluxes given by the Riemann solvers at junction corresponding to the initial data

$$(u_{1,M}^\nu, \dots, u_{n,M}^\nu, u_{n+1,1}^\nu, \dots, u_{n+m,1}^\nu)$$

(see Algorithms 1, 2).

6.1. Numerical results. For the simulations, we set the length of each road equal to 1 and incoming roads are parametrized by the interval $I_i = [-1, 0]$ while outgoing roads are given by $I_j = [0, 1]$, with the junction placed at $x = 0$. Moreover, we fix $f(\rho) = \rho(1 - \rho)$, thus $\rho_{cr} = 0.5$.

(1) **Case I:** Comparison \mathcal{PRS} vs. \mathcal{SPRS} .

This case illustrates the different dynamics given by the two Riemann solvers proposed in this article.

We consider a junction with 2 incoming roads (I_1, I_2) and 2 outgoing roads (I_3, I_4). We fix the matrix A and the priority vector P as follows:

$$A = \begin{bmatrix} 0.6 & 0 \\ 0.4 & 1 \end{bmatrix} \quad P = [0.7 \ 0.3].$$

We consider the following initial data:

$$\rho_{1,0} = 0.6, \quad \rho_{2,0} = 0.2, \quad \rho_{3,0} = 0.85, \quad \rho_{4,0} = 0.2.$$

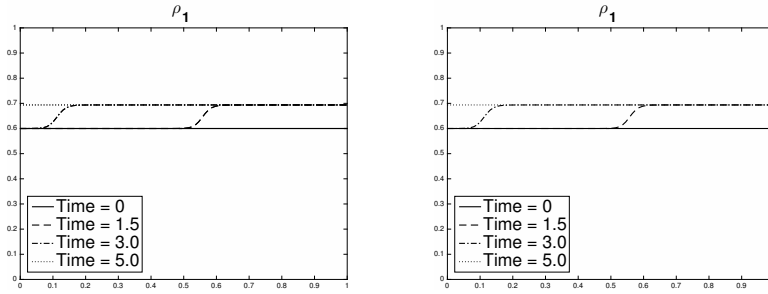
The different results of the simulations (see Figures 6.1) can be seen in particular in road 2 ad 4. We observe that \mathcal{SPRS} allows more flux through the junction than \mathcal{PRS} , for which we observe the formation of a big shock moving backwards on road 2.

(2) **Case II:** Comparison \mathcal{PRS} vs. \mathcal{RS}_{CGP} .

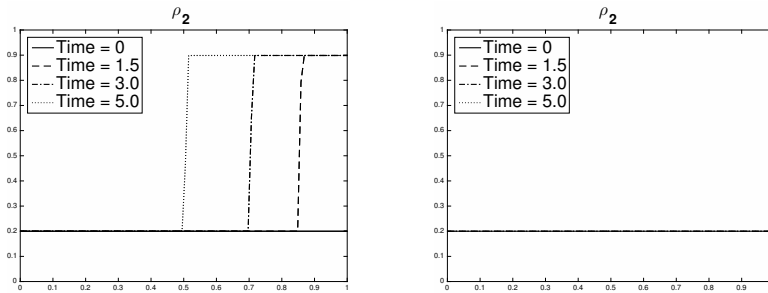
We propose here a comparison between the \mathcal{PRS} with the Riemann solver proposed by Coclite, Garavello and Piccoli in [7] and briefly referred to as \mathcal{RS}_{CGP} .

We consider a 2×2 junction and we fix the matrix A and the priority vector P as follows:

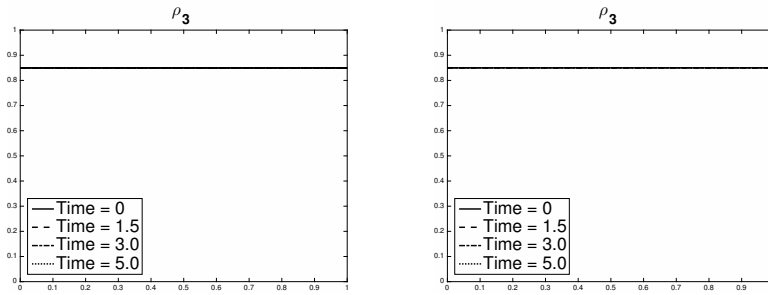
$$A = \begin{bmatrix} 0.5 & 0.6 \\ 0.5 & 0.4 \end{bmatrix} \quad P = [0.7 \ 0.3].$$



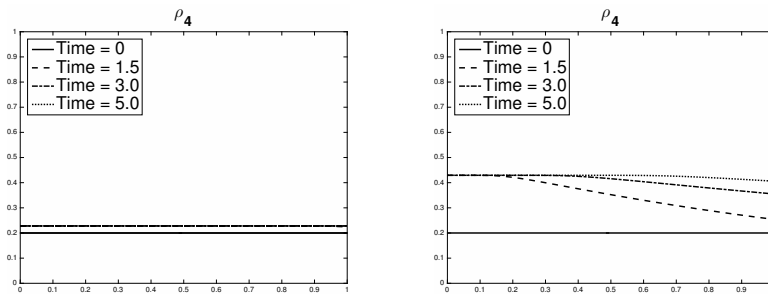
(a) Road 1



(b) Road 2

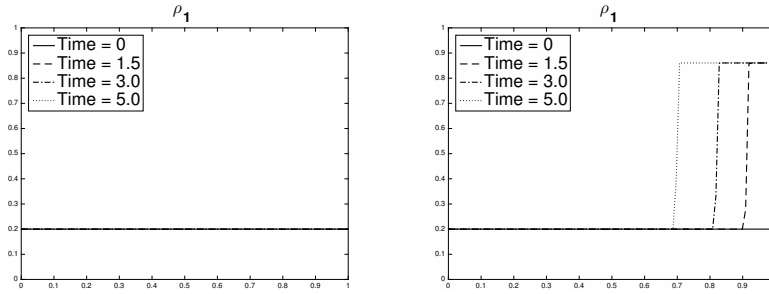


(c) Road 3

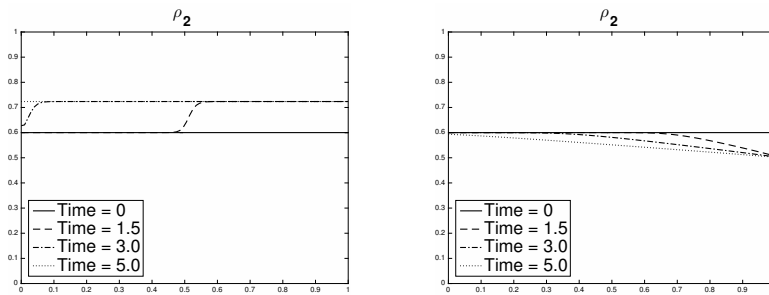


(d) Road 4

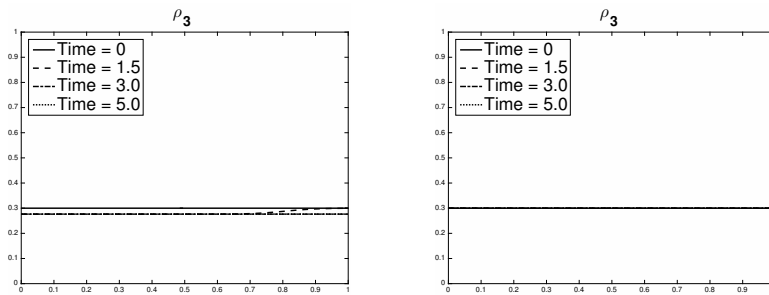
Fig. 6.1: Case I : Solution of the problem using PRS on the left and SPRS on the right.



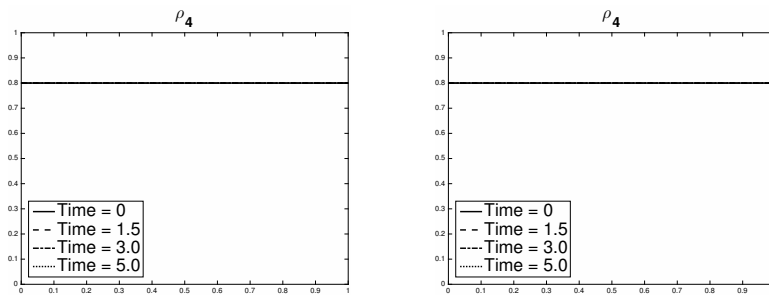
(a) Road 1



(b) Road 2



(c) Road 3



(d) Road 4

Fig. 6.2: Case II: Solution of the problem using PRS on the left and RS_CGP on the right.

We consider the following initial data:

$$\rho_{1,0} = 0.2, \quad \rho_{2,0} = 0.6, \quad \rho_{3,0} = 0.3, \quad \rho_{4,0} = 0.8.$$

The simulations (see Figures 6.2) show clearly the different solutions of the Riemann solvers. In particular, $\mathcal{RS}_{\text{CGP}}$ creates a big shock in the incoming road 1 decreasing its flux. This wave does not appear in our Riemann solver \mathcal{PRS} .

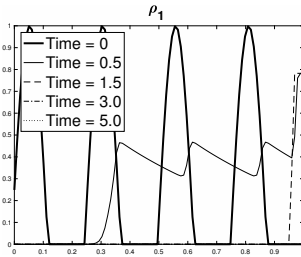
(3) **Case III:** 3×2 junction.

We fix the matrix A and the priority vector P as follows:

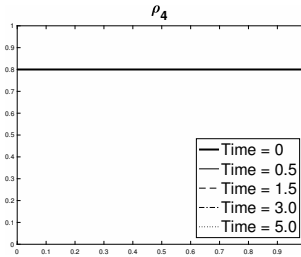
$$A = \begin{bmatrix} 0.5 & 0.6 & 0.2 \\ 0.5 & 0.4 & 0.8 \end{bmatrix} \quad P = [0.5 \ 0.3 \ 0.2].$$

We consider the following initial data:

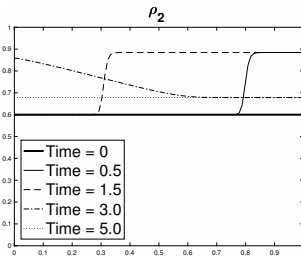
$$\begin{aligned} \rho_{1,0}(x) &= \max(\sin(8\pi x), 0), & \rho_{2,0} &= 0.6, & \rho_{3,0} &= 0.3, \\ & & \rho_{4,0} &= 0.8, & \rho_{5,0} &= 0.2. \end{aligned}$$



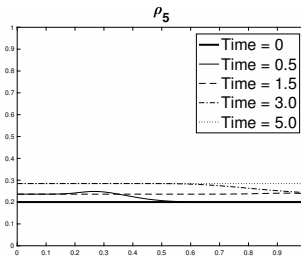
(a) Road 1



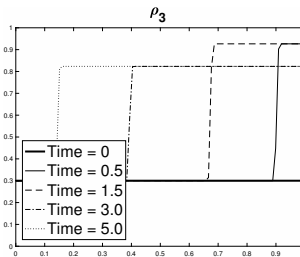
(b) Road 4



(c) Road 2



(d) Road 5



(e) Road 3

Fig. 6.3: Case III: Solution of the problem using \mathcal{PRS}

This situation displays multiple waves arising in different roads. Due to the lower priorities given to roads 2 and 3 we can see that queues are created in the two incoming roads, see Figure 6.3. Note also that this case cannot be handled by $\mathcal{RS}_{\text{CGP}}$ since $n > m$.

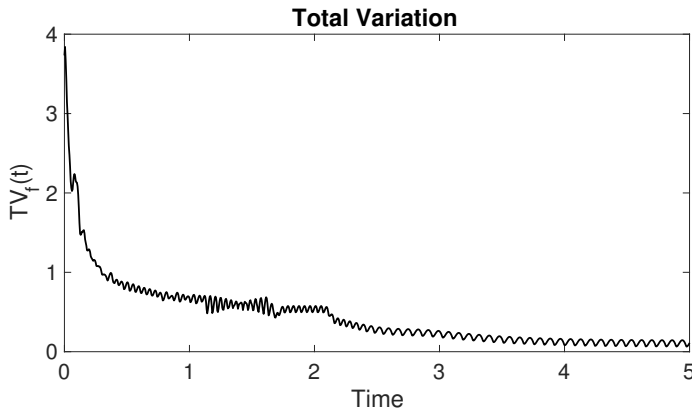


Fig. 6.4: *Case III: Time evolution of TV_f*

Figure 6.4 shows the time evolution of TV_f . We remark that the oscillatory behavior is due to the viscosity present in the Godunov scheme and to the lack of a maximum principle for $f(\rho)$ across ρ_{cr} .

Appendix A. Proof of Proposition 4.2. The construction of \mathcal{PRS} depends only on the matrix A , the priority vector P and the sets Ω_l . The latter, in turn, depends only on bad data, thus property (P1) holds true.

We prove properties (P2) and (P3) for the case $n = m = 2$ and distinguish between three different generic situations for the initial equilibrium: demand constrained, demand/supply constrained and supply constrained (where demand indicates flow from incoming road and supply flow to outgoing ones). In the first situation the incoming roads act as constraint in the definition of the set Ω (see definition (3.2)) and the equilibrium corresponds to the point Q_0 as in Figure A.1a. The second case corresponds to one incoming and one outgoing road acting as constraint and to the point Q_0 as in Figure A.4a. Finally, the third case corresponds to outgoing roads acting as constraint and to the point Q_0 as in Figure A.8a.

- **Case A: Demand constrained.** By symmetries, it is not restrictive to assume that the priority line hP , $h > 0$, intersects the constraint $\gamma_2^{\text{inc}} = \gamma_{2,0}$. We have to distinguish several subcases:

Case A1: The incoming wave is $(\rho_1, \rho_{1,0})$ (on road 1). Since $\gamma_{1,0}$ is an active constraint, $\rho_{1,0} \leq \rho_{\text{cr}}$ and $\rho_1 \leq \rho_{\text{cr}}$. We distinguish the two situations:

- If $f(\rho_1) > f(\rho_{1,0})$ we define $\gamma_{1,1} = \frac{\gamma_3 - a_{32}\gamma_2}{a_{31}}$ and $\gamma_{1,0} \leq \gamma_{1,1} \leq \gamma_1$ (see Figure A.1a). We get:

$$\begin{aligned} \text{TV}(f)^+ &= |\gamma_1 - \gamma_{1,1}| + a_{31}|\gamma_{1,1} - \gamma_{1,0}| + a_{41}|\gamma_{1,1} - \gamma_{1,0}|, \\ \Delta\text{TV}(f) &= (a_{31} + a_{41} - 1)|\gamma_{1,1} - \gamma_{1,0}|, \end{aligned}$$

$$\begin{aligned} \Delta\Gamma &= \gamma_{1,1} - \gamma_{1,0} > 0, \\ \Delta\bar{h} &= 0. \end{aligned}$$

Hence, property (P2) holds and property (P3) doesn't need to be verified.

– If $f(\rho_1) > f(\rho_{1,0})$, we define $\gamma_{1,1} = \frac{p_1}{p_2}\gamma_2$, hence $\gamma_{1,0} \geq \gamma_{1,1} \geq \gamma_1$ (see Figure A.1b). We have:

$$\begin{aligned} \text{TV}(f)^- &= |\gamma_1 - \gamma_{1,0}|, \\ \text{TV}(f)^+ &= a_{31}|\gamma_1 - \gamma_{1,0}| + a_{41}|\gamma_1 - \gamma_{1,0}|, \\ \Delta\text{TV}(f) &= (a_{31} + a_{41} - 1)|\gamma_1 - \gamma_{1,0}|, \\ \Delta\Gamma &= (\gamma_1 - \gamma_{1,0}) < 0, \\ \Delta\bar{h} &= \frac{1}{p_1}(\gamma_1 - \gamma_{1,1}) < 0. \end{aligned}$$

Hence, properties (P2) and (P3) hold.

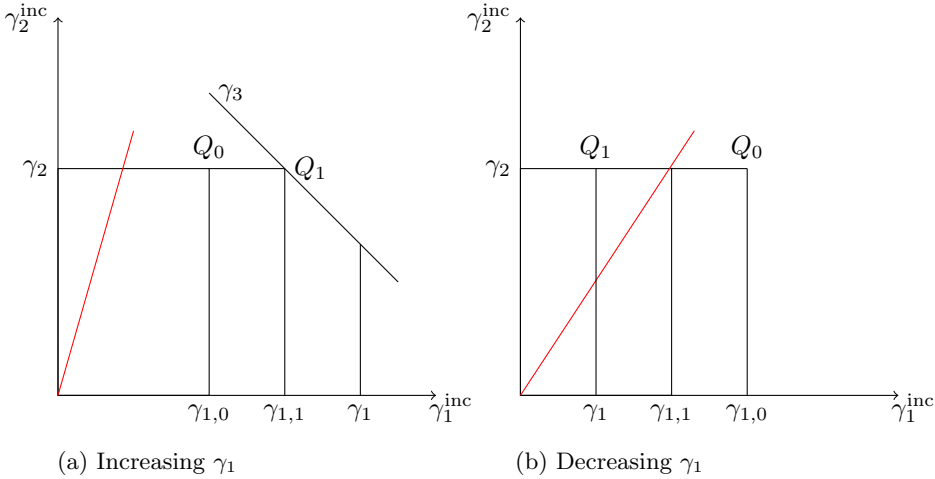


Fig. A.1: *Case A1*

Case A2: The incoming wave is $(\rho_2, \rho_{2,0})$ (on road 2). Since $\gamma_{2,0}$ is an active constraint, $\rho_{2,0} \leq \rho_{cr}$ and $\rho_2 \leq \rho_{cr}$.

–If $f(\rho_2) > f(\rho_{2,0})$, we define

$$\gamma_{2,1} = \frac{\gamma_3 - a_{31}\gamma_1}{a_{32}} \quad \text{and} \quad \gamma_{2,2} = \frac{p_2}{a_{31}p_1 + a_{32}p_2} \gamma_3^{\max}$$

so that $\gamma_{2,0} \leq \gamma_{2,1} \leq \gamma_{2,2} \leq \gamma_2$, see Figure A.2a. Note that this case is the same as in A1 except for the case in the drawing. In this case we have:

$$\begin{aligned} \text{TV}(f)^- &= |\gamma_2 - \gamma_{2,0}| = |\gamma_2 - \gamma_{2,2}| + |\gamma_{2,2} - \gamma_{2,1}| + |\gamma_{2,1} - \gamma_{2,0}|, \\ \text{TV}(f)^+ &= \frac{a_{32}}{a_{31}}|\gamma_{2,2} - \gamma_{2,1}| + |\gamma_2 - \gamma_{2,2}| + a_{32}|\gamma_{2,1} - \gamma_{2,0}| \end{aligned}$$

$$\begin{aligned}
& + \left| a_{42} |\gamma_{2,1} - \gamma_{2,0}| + \left(a_{42} - a_{41} \frac{a_{32}}{a_{31}} \right) |\gamma_{2,2} - \gamma_{2,1}| \right|, \\
\Delta \text{TV}(f) &= \left(\frac{a_{32}}{a_{31}} - 1 \right) |\gamma_{2,2} - \gamma_{2,1}| + (a_{32} - 1) |\gamma_{2,1} - \gamma_{2,0}| \\
& + \left| a_{42} |\gamma_{2,1} - \gamma_{2,0}| + \left(a_{42} - a_{41} \frac{a_{32}}{a_{31}} \right) |\gamma_{2,2} - \gamma_{2,1}| \right|, \\
\Delta \Gamma &= (\gamma_{2,1} - \gamma_{2,0}) + \left(1 - \frac{a_{32}}{a_{31}} \right) (\gamma_{2,2} - \gamma_{2,1}), \\
\Delta \bar{h} &= \frac{1}{p_2} (\gamma_{2,2} - \gamma_{2,0})
\end{aligned}$$

Hence, property (P2) holds while property (P3) doesn't need to be checked.

– If $f(\rho_2) < f(\rho_{2,0})$ one has $\gamma_2 < \gamma_{2,0}$, see Figure A.2b. Therefore:

$$\begin{aligned}
\text{TV}(f)^- &= |\gamma_2 - \gamma_{2,0}|, \\
\text{TV}(f)^+ &= (a_{32} + a_{42}) |\gamma_2 - \gamma_{2,0}|, \\
\Delta \text{TV}(f) &= 0 \quad (\text{recall that } a_{32} + a_{42} = 1), \\
\Delta \Gamma &= (\gamma_2 - \gamma_{2,0}) < 0, \\
\Delta \bar{h} &= \frac{1}{p_2} (\gamma_2 - \gamma_{2,0}) < 0.
\end{aligned}$$

Hence, properties (P2) and (P3) hold.

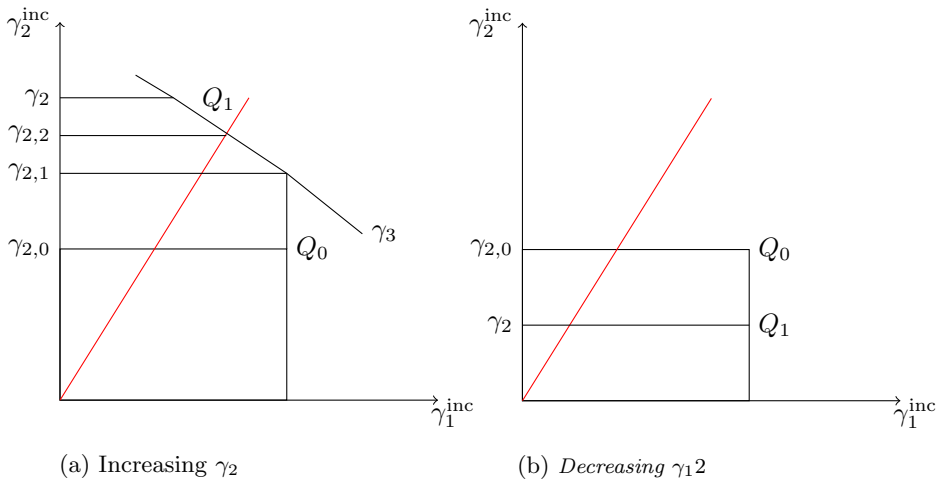


Fig. A.2: Case A2

Case A3: The incoming wave is $(\rho_{3,0}, \rho_3)$ (on road 3, the case of road 4 being similar).

– If $f(\rho_3) < f(\rho_{3,0})$ We define $\gamma_{3,1} = \left(a_{31} \frac{p_1}{p_2} + a_{32} \right) \gamma_2$ so that $\gamma_3 \leq \gamma_{3,1} \leq \gamma_{3,0}$

(see Figure A.3):

$$\begin{aligned} \text{TV}(f)^- &= |\gamma_3 - \gamma_{3,0}|, \\ \text{TV}(f)^+ &= \frac{a_{41} + 1}{a_{31}} |\gamma_{3,1} - \gamma_{3,0}| + \frac{(a_{41} + 1)p_1 + (a_{42} + 1)p_2}{a_{31}p_1 + a_{32}p_2} |\gamma_3 - \gamma_{3,1}|, \\ \Delta\text{TV}(f) &= \frac{a_{41}}{a_{31}} |\gamma_{3,1} - \gamma_{3,0}| + \left(\frac{(a_{41} + 1)p_1 + (a_{42} + 1)p_2}{a_{31}p_1 + a_{32}p_2} - 1 \right) |\gamma_3 - \gamma_{3,1}|, \\ \Delta\Gamma &= \frac{1}{a_{31}} (\gamma_{3,1} - \gamma_{3,0}) + \frac{a_{41}p_1 + a_{42}p_2}{a_{31}p_1 + a_{32}p_2} (\gamma_3 - \gamma_{3,1}) < 0, \\ \Delta\bar{h} &= \frac{1}{a_{31}p_1 + a_{32}p_2} (\gamma_3 - \gamma_{3,1}) < 0. \end{aligned}$$

Hence, properties (P2) and (P3) hold.

– If $f(\rho_3) > f(\rho_{3,0})$ and $\gamma_3 > \gamma_{3,0}$ then we stay demand constrained and nothing happens.

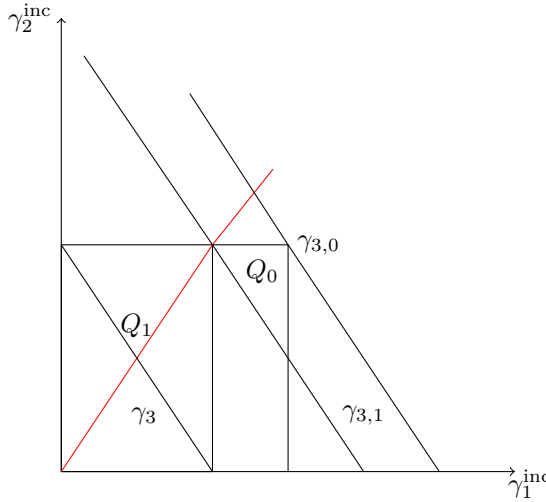


Fig. A.3: Case a3 - Decreasing γ_3

- **Case B: Supply constrained, priority line intersects a demand constraint.** Even in this case, it is not restrictive to assume that the priority line hP , $h > 0$, intersects the constraint $\gamma_1^{\text{inc}} = \gamma_{1,0}$. We split the proof in several subcases depending on the origin of the incoming wave:

Case B1: The incoming wave is $(\rho_1, \rho_{1,0})$ (on road 1). We define $\gamma_{1,1} = \frac{a_{42} - a_{32}}{a_{31}a_{42} - a_{32}a_{41}} \gamma_1$ (see Figure A.4a and A.4b). We distinguish the two situations:

– If $f(\rho_1) > f(\rho_{1,0})$ we get $\gamma_{1,0} \leq \gamma_{1,1} \leq \gamma_{1,2} \leq \gamma_1$, where $\gamma_{1,2} = \frac{p_1}{a_{41}p_1 + a_{42}p_2} \gamma_4$ (see Figure A.4a). Then we have:

$$\begin{aligned} \text{TV}(f)^- &= |\gamma_1 - \gamma_{1,0}|, \\ \text{TV}(f)^+ &= |\gamma_1 - \gamma_{1,2}| + \left(\frac{a_{41}}{a_{42}} + \left| a_{31} - a_{32} \frac{a_{41}}{a_{42}} \right| \right) |\gamma_{1,2} - \gamma_{1,1}| \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{a_{31}}{a_{32}} + \left| a_{41} - a_{42} \frac{a_{31}}{a_{32}} \right| \right) |\gamma_{1,1} - \gamma_{1,0}|, \\
\Delta \text{TV}(f) &= \left(\frac{a_{41}}{a_{42}} + \left| a_{31} - a_{32} \frac{a_{41}}{a_{42}} \right| - 1 \right) |\gamma_{1,2} - \gamma_{1,1}| \\
& + \left(\frac{a_{31}}{a_{32}} + \left| a_{41} - a_{42} \frac{a_{31}}{a_{32}} \right| - 1 \right) |\gamma_{1,1} - \gamma_{1,0}|, \\
\Delta \Gamma &= \left(1 - \frac{a_{31}}{a_{32}} \right) (\gamma_{1,1} - \gamma_{1,0}) + \left(1 - \frac{a_{41}}{a_{42}} \right) (\gamma_{1,2} - \gamma_{1,1}), \\
\Delta \bar{h} &= \frac{1}{p_1} (\gamma_{1,2} - \gamma_{1,0}).
\end{aligned}$$

Hence, property (P2) holds and property (P3) doesn't need to be checked.

– If $f(\rho_1) < f(\rho_{1,0})$ and $\gamma_{1,0} \geq \gamma_{1,1} \geq \gamma_{1,2} \geq \gamma_1$ (see Figure A.4b), we define $\gamma_{1,2} = \frac{\gamma_3 - a_{32}\gamma_2}{a_{31}}$ and we get:

$$\begin{aligned}
\text{TV}(f)^- &= |\gamma_1 - \gamma_{1,0}|, \\
\text{TV}(f)^+ &= (a_{31} + a_{41}) |\gamma_1 - \gamma_{1,2}| + \left(\frac{a_{31}}{a_{32}} + \left| a_{41} - a_{42} \frac{a_{31}}{a_{32}} \right| \right) |\gamma_{1,2} - \gamma_{1,1}| \\
& + \left(\frac{a_{41}}{a_{42}} + \left| a_{31} - a_{32} \frac{a_{41}}{a_{42}} \right| \right) |\gamma_{1,1} - \gamma_{1,0}|, \\
\Delta \text{TV}(f) &= \left(\frac{a_{31}}{a_{32}} + \left| a_{41} - a_{42} \frac{a_{31}}{a_{32}} \right| - 1 \right) |\gamma_{1,2} - \gamma_{1,1}| \\
& + \left(\frac{a_{41}}{a_{42}} + \left| a_{31} - a_{32} \frac{a_{41}}{a_{42}} \right| - 1 \right) |\gamma_{1,1} - \gamma_{1,0}|, \\
\Delta \Gamma &= (\gamma_1 - \gamma_{1,2}) + \left(1 - \frac{a_{31}}{a_{32}} \right) (\gamma_{1,2} - \gamma_{1,1}) + \left(1 - \frac{a_{41}}{a_{42}} \right) (\gamma_{1,1} - \gamma_{1,0}), \\
\Delta \bar{h} &= \frac{1}{p_1} (\gamma_1 - \gamma_{1,0}) < 0.
\end{aligned}$$

Hence, properties (P2) and (P3) hold.

Case B2: The incoming wave is $(\rho_2, \rho_{2,0})$ (on road 2).

– If $f(\rho_2) > f(\rho_{2,0})$: nothing happens (see Figure A.5a).

– If $f(\rho_2) < f(\rho_{2,0})$ (see Figure A.5b) we define $\gamma_{2,1} = \frac{p_1}{p_2} \gamma_1$ and we compute:

$$\begin{aligned}
\text{TV}(f)^- &= |\gamma_2 - \gamma_{2,0}|, \\
\text{TV}(f)^+ &= (a_{32} + a_{42}) |\gamma_2 - \gamma_{2,0}|, \\
\Delta \text{TV}(f) &= (a_{32} + a_{42} - 1) |\gamma_2 - \gamma_{2,0}| = 0, \\
\Delta \Gamma &= (\gamma_2 - \gamma_{2,0}) < 0, \\
\Delta \bar{h} &= \frac{1}{p_2} (\gamma_2 - \gamma_{2,1}) < 0.
\end{aligned}$$

Hence, properties (P2) and (P3) hold.

Case B3: The incoming wave is $(\rho_{3,0}, \rho_3)$ (on road 3).

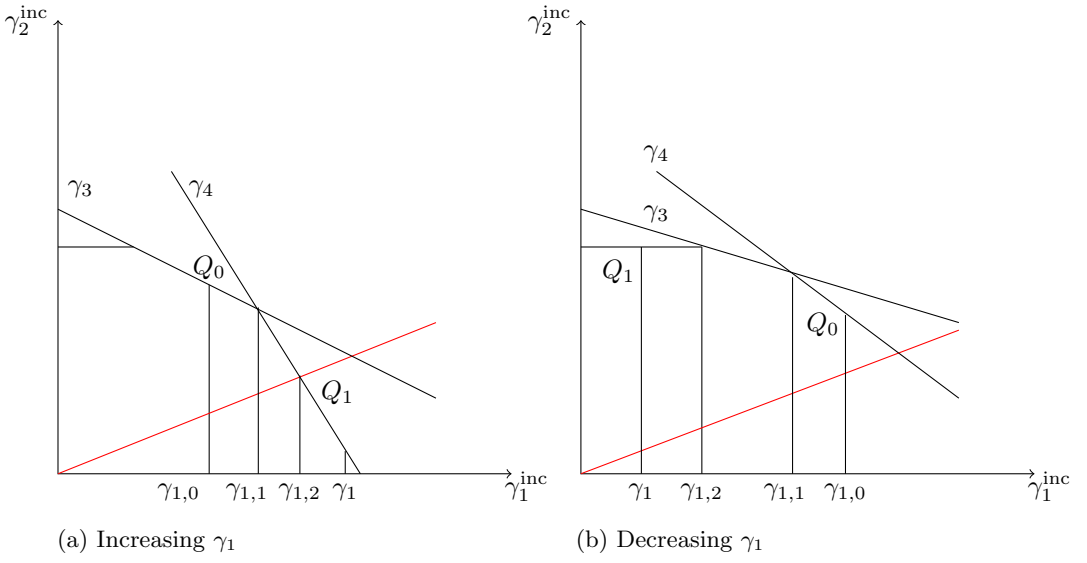


Fig. A.4: *Case B1*

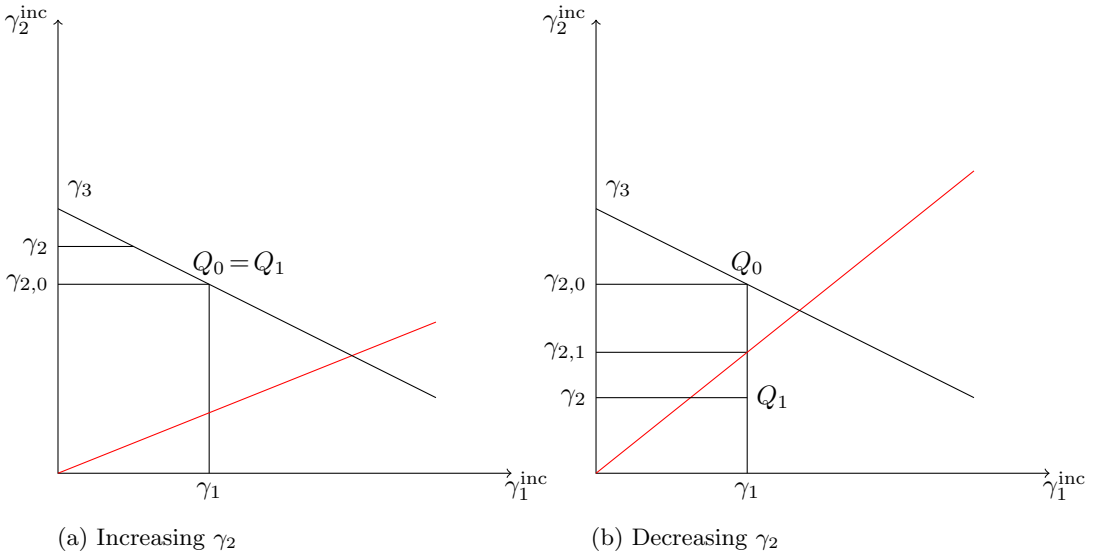


Fig. A.5: *Case B2*

– If $f(\rho_3) > f(\rho_{3,0})$ (see Figure A.6a), we define

$$\gamma_{3,1} = \min \left\{ \begin{array}{l} a_{31}\gamma_1 + a_{32}\gamma_2^{\max}, \\ a_{31}\gamma_1 + \frac{a_{32}}{a_{42}}(\gamma_4^{\max} - a_{41}\gamma_1). \end{array} \right.$$

We get:

$$\begin{aligned}
\mathrm{TV}(f)^- &= |\gamma_3 - \gamma_{3,0}|, \\
\mathrm{TV}(f)^+ &= \left(\frac{1}{a_{32}} + \frac{a_{42}}{a_{32}} \right) |\gamma_{3,1} - \gamma_{3,0}| + |\gamma_3 - \gamma_{3,1}|, \\
\Delta \mathrm{TV}(f) &= \left(\frac{1}{a_{32}} + \frac{a_{42}}{a_{32}} - 1 \right) |\gamma_{3,1} - \gamma_{3,0}|, \\
\Delta \Gamma &= \frac{1}{a_{32}} (\gamma_{3,1} - \gamma_{3,0}) > 0, \\
\Delta \bar{h} &= 0.
\end{aligned}$$

Hence, property (P2) holds and property (P3) does not need to be checked.

– If $f(\rho_3) < f(\rho_{3,0})$ (see Figure A.6b), we define $\gamma_{3,1} = \left(a_{31} + a_{32} \frac{p_2}{p_1} \right) \gamma_1$ and we compute:

$$\begin{aligned}
\mathrm{TV}(f)^- &= |\gamma_3 - \gamma_{3,0}|, \\
\mathrm{TV}(f)^+ &= \frac{p_1}{a_{31}p_1 + a_{32}p_2} |\gamma_3 - \gamma_{3,1}| + \frac{1}{a_{32}} |\gamma_{3,1} - \gamma_{3,0}| + \frac{p_2}{a_{31}p_1 + a_{32}p_2} |\gamma_3 - \gamma_{3,1}| \\
&\quad + \frac{a_{42}}{a_{32}} |\gamma_{3,1} - \gamma_{3,0}| + \frac{a_{41}p_1 + a_{42}p_2}{a_{31}p_1 + a_{32}p_2} |\gamma_3 - \gamma_{3,1}|, \\
\Delta \mathrm{TV}(f) &= \left(\frac{(1 + a_{41})p_1 + (1 + a_{42})p_2}{a_{31}p_1 + a_{32}p_2} - 1 \right) |\gamma_3 - \gamma_{3,1}| + \left(\frac{1 + a_{42}}{a_{32}} - 1 \right) |\gamma_{3,1} - \gamma_{3,0}|, \\
\Delta \Gamma &= \frac{p_1 + p_2}{a_{31}p_1 + a_{32}p_2} (\gamma_3 - \gamma_{3,1}) + \frac{1}{a_{32}} (\gamma_{3,1} - \gamma_{3,0}) < 0, \\
\Delta \bar{h} &= \frac{1}{a_{31}p_1 + a_{32}p_2} (\gamma_3 - \gamma_{3,1}) < 0.
\end{aligned}$$

Hence, properties (P2) and (P3) hold.

Case B4: The incoming wave is $(\rho_{4,0}, \rho_4)$ (on road 4).

– If $f(\rho_4) > f(\rho_{4,0})$ nothing changes.

– If $f(\rho_4) < f(\rho_{4,0})$: the same as case **B3** decreasing exchanging the roles of γ_3 and γ_4 .

- **Case C: Supply constrained, priority line intersects a supply constraint.** It is not restrictive to assume that the priority line hP , $h > 0$, intersects the constraint $a_{31}\gamma_1^{\mathrm{inc}} + a_{32}\gamma_2^{\mathrm{inc}} = \gamma_{3,0}$. We distinguish the following subcases, depending on the origin of the incoming wave:

Case C1: The incoming wave is $(\rho_1, \rho_{1,0})$ (on road 1).

– If $f(\rho_1) > f(\rho_{1,0})$, nothing changes.

– If $f(\rho_1) < f(\rho_{1,0})$, the analysis is similar to case **B1** decreasing (see Figure A.7).

Case C2: The incoming wave is $(\rho_2, \rho_{2,0})$ (on road 2). This case is symmetric to **C1**.

Case C3: The incoming wave is $(\rho_{3,0}, \rho_3)$ (on road 3).

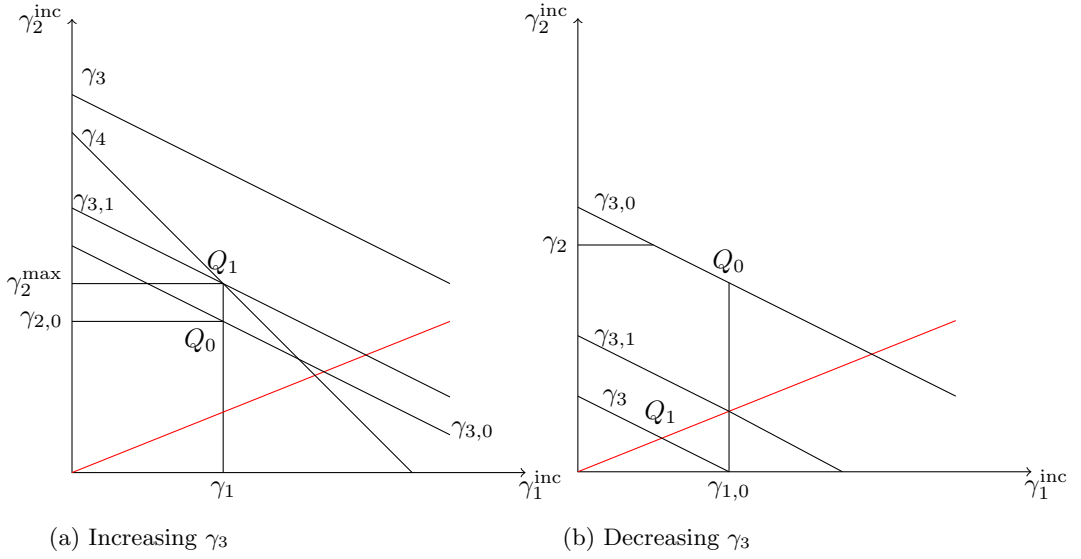


Fig. A.6: Case B3

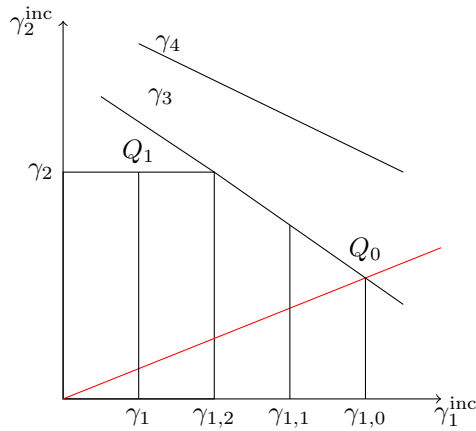


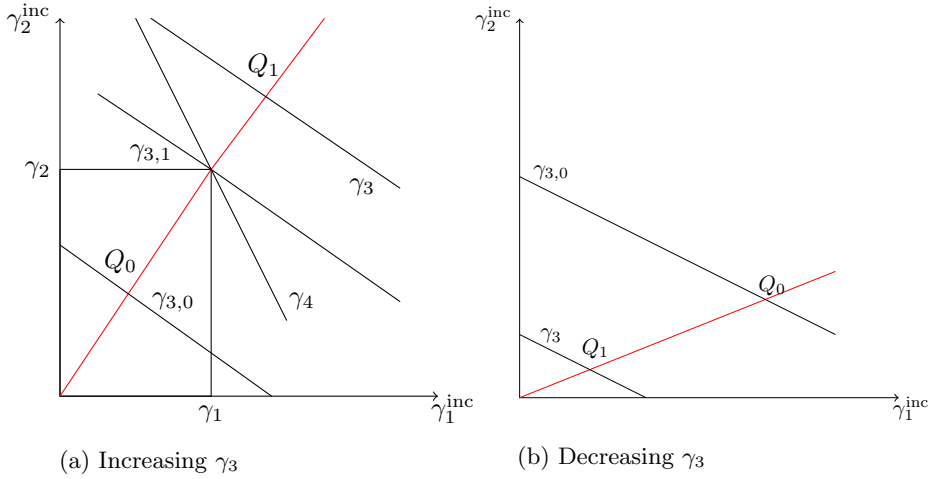
Fig. A.7: Case C1 - decreasing γ_1

– If $f(\rho_3) > f(\rho_{3,0})$ (see Figure A.8a), we define

$$\gamma_{3,1} = \min \begin{cases} \left(a_{31} + a_{32} \frac{p_2}{p_1} \right) \gamma_1^{\max}, \\ \left(a_{31} \frac{p_1}{p_2} + a_{32} \right) \gamma_2^{\max}, \\ \frac{a_{31}p_1 + a_{32}p_2}{a_{41}p_1 + a_{42}p_2} \gamma_4^{\max}, \end{cases}$$

and we compute:

$$\text{TV}(f)^- = |\gamma_3 - \gamma_{3,0}|,$$

Fig. A.8: *Case C3*

$$\begin{aligned} \text{TV}(f)^+ &= \frac{p_1 + p_2}{a_{31}p_1 + a_{32}p_2} |\gamma_{3,1} - \gamma_{3,0}| + |\gamma_3 - \gamma_{3,1}| + \frac{a_{41}p_1 + a_{42}p_2}{a_{31}p_1 + a_{32}p_2} |\gamma_{3,1} - \gamma_{3,0}|, \\ \Delta\text{TV}(f) &= \left(\frac{(a_{41} + 1)p_1 + (a_{42} + 1)p_2}{a_{31}p_1 + a_{32}p_2} - 1 \right) |\gamma_{3,1} - \gamma_{3,0}|, \\ \Delta\Gamma &= \frac{p_1 + p_2}{a_{31}p_1 + a_{32}p_2} (\gamma_{3,1} - \gamma_{3,0}) > 0, \\ \Delta\bar{h} &= \frac{1}{a_{31}p_1 + a_{32}p_2} (\gamma_{3,1} - \gamma_{3,0}) > 0. \end{aligned}$$

Hence, property (P2) holds and property (P3) does not count since we are increasing fluxes.

– If $f(\rho_3) < f(\rho_{3,0})$ (see Figure A.8b) we get:

$$\begin{aligned} \text{TV}(f)^- &= |\gamma_3 - \gamma_{3,0}|, \\ \text{TV}(f)^+ &= \frac{p_1 + p_2}{a_{31}p_1 + a_{32}p_2} |\gamma_3 - \gamma_{3,0}| + \frac{a_{41}p_1 + a_{42}p_2}{a_{31}p_1 + a_{32}p_2} |\gamma_3 - \gamma_{3,0}|, \\ \Delta\text{TV}(f) &= \left(\frac{(a_{41} + 1)p_1 + (a_{42} + 1)p_2}{a_{31}p_1 + a_{32}p_2} - 1 \right) |\gamma_3 - \gamma_{3,0}|, \\ \Delta\Gamma &= \frac{p_1 + p_2}{a_{31}p_1 + a_{32}p_2} (\gamma_3 - \gamma_{3,0}) < 0, \\ \Delta\bar{h} &= \frac{1}{a_{31}p_1 + a_{32}p_2} (\gamma_3 - \gamma_{3,0}) < 0. \end{aligned}$$

Hence, properties (P2) and (P3) hold.

Case C4: The incoming wave is $(\rho_{4,0}, \rho_4)$ (on road 4).

– If $f(\rho_4) > f(\rho_{4,0})$, nothing happens.

– If $f(\rho_4) < f(\rho_{4,0})$, the situation is similar to case **C3** with the roles of γ_3 and γ_4 reversed.

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