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LARGE DEVIATION FOR THE STOCHASTIC 2D PRIMITIVE EQUATIONS WITH ADDITIVE LÉVY NOISE*

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Abstract. The two dimensional Primitive Equations with Lévy noise are studied in this paper. A number of exponential estimates of the solutions as well as the exponential convergence of the approximating solutions have been established, finally, a large deviation principle has been obtained.

Keywords. Primitive equation; Lévy noise; large deviation principle.

AMS subject classifications. 35Q99; 60H15; 76M35; 86A05; 86A10.

1. Introduction

As a fundamental and very important model in meteorology, by the buoyancy forces and stratification effects under the Boussinesq approximation and the vertical motion with the hydrostatic balance, the Primitive Equations (PEs) were derived from the Navier–Stokes equations, with rotation, coupled to thermodynamics and salinity diffusion-transport equations. For further physical background see [4] or [29], for example.

The mathematical study of the PEs originated in a series of articles by J. L. Lions, R. Temam and S. Wang in the early 1990s [26–28]. Existence of solutions and their uniqueness for the 3D deterministic PEs had been widely studied, such as [2, 19, 20, 23, 24, 36]. Especially in [2], Cao and Titi developed a delicate approach to proving that the L^6-norm of the fluctuation $\widetilde{v}$ of horizontal velocity is bounded and obtained the global well-posedness for the 3D viscous PEs. The existence of the attractor was obtained in [22]. In [25], existence and uniqueness for different physically relevant boundary conditions had been established with a third method (different from both [2, 23]) that directly treated the pressure terms in the equations. For general reference on the current research of the (deterministic) mathematical theory of the PEs, we can refer in [31].

In the context of fluids, complex phenomena related to turbulence may also be produced by stochastic perturbations. Stochastic solutions of the 2D PEs of the ocean and atmosphere with an additive noise had been study in [12]. Random attractor was obtained for 3D stochastic PEs with additive noise in [14] and [18]. The existence and uniqueness of solutions for 2D stochastic PEs with multiplicative noise had been obtained in [17]. There were other recent works on the stochastic 2D and 3D PEs with multiplicative noise [8, 9, 16], in both works a coupling with temperature and salinity equations as well as physically relevant boundary conditions were considered.

In fact, the climate systems often have variable structures subject to stochastic abrupt changes, which may result from abrupt phenomena such as changes in the in-

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terconnections and sudden environment changes, etc. As a consequence, these systems are very complex and their sample paths may not be continuous, which yields that PEs driven by a Brownian motion fail to cope with them, it is reasonable that we introduce Lévy noise in our system. Stochastic partial differential equations driven by jump processes gain attention recently due to its important applications in mathematical physics [30, 33] (and references therein). Meanwhile, small noise large deviations (i.e., of Freidlin–Wentzell type) for SPDEs have a long history and have been studied extensively, such as [3,11,13,35,38]. Specially, in account of stochastic equations driven by Lévy noise, large deviation principles were established in [7,32,37]. Until now, there is not much work on stochastic Primitive equation driven by Lévy noise. In [34], by a priori estimates, weak convergence method and monotonicity arguments, we studied 2D primitive equation with Lévy noise and proved the existence and uniqueness of the solutions in a fixed probability space, meanwhile, large deviation principle for 2D stochastic primitive equation driven by multiplicative Gaussian noise was obtained in [15].

In this paper, large deviation principle for 2D stochastic PEs driven by Lévy noise is studied. In [37], by a number of exponential estimates for energy of the solutions as well as the exponential convergence of the approximating solutions, the authors obtained the large deviation for 2D stochastic Navier–Stokes equations driven by Lévy noise. The main difference between Navier–Stokes equations and PEs is the nonlinear term $B(u,u)$. From Lemma 2.1, we know that the term $B(u,u)$ of PEs is more complicated, some exponential estimates of Navier–Stokes equations in [37] might not be suitable for PEs. In this paper, how to deal with the term $\partial_z u$ in estimates of $B(u,u)$ is crucial. By introducing $g_2(y)$ and Lemma 3.1, we derive exponential estimates for $\partial_z u$ (see Lemma 3.3). Moreover, we obtain the estimates of linear Equations (3.17) and (3.18) (see Lemma 3.5). With above two Lemmas, Lemma 3.2 and Lemma 3.4, by contraction principle in the theory of large deviations(see Theorem 4.2.23 in [10]), we establish the large deviation principle.

This paper is organized as follows. The mathematical formulation for the stochastic PEs is in §2. Then a number of exponential estimates, which will play an important role in the rest of the paper, are obtained in §3. A large deviation principle will be established in §4

2. Mathematical formulation

Firstly, we introduce some definitions and basic properties of Wiener processes and Lévy processes. For more details, one can see [5] or [30], for example.

In this paper, $W(t)$ is a Wiener process defined on a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, taking value in a Hilbert space H, with linear symmetric positive covariant operator Q. We assume that Q is trace class (and hence compact [5]), i.e., $tr(Q)<\infty$. Let $(X,\mathcal{B}(X))$ be a measurable space and $\nu(d\zeta)$ be a $\sigma-$finite measure on it. Let $p=(p(t),t\in D_p)$ be a stationary $\mathcal{F}_t-$ Poisson point process on X with characteristic measure $\nu(d\zeta)$, where D_p is a countable subset of $[0,\infty)$ depending on random parameter ω (see [21]). Denote by $N(dt,d\zeta)$ the Poisson counting measure associated with p, i.e., $N(t,A)=\sum_{s\in D_p,s\leq t} I_A(p(s))$. Let $\widetilde{N}(dt,d\zeta)=N(dt,d\zeta)-dt\nu(d\zeta)$ be the compensated Poisson random measure. Denoted by $\widetilde{N}_n(dt,d\zeta)$ the compensated Poisson measure with the characteristic measure $n\nu$. We assume $W(t)$ and $N(dt,d\zeta)$ are independent.

Definition 2.1. *Let $I=[a,b]$ be an interval in $\mathbb{R}^+$. A mapping $g: I\to\mathbb{R}^d$ is said to be càdlàg if, for all $t\in[a,b]$, g has a left limit at t and right continuous at t. Let $D([0,T],H)$ be the space of all càdlàg paths from $[0,T]$ into H.*

Definition 2.2. *Let E and F be separable Banach spaces, let $F_t:=\mathcal{B}(\mathbb{R}^+\times E)\otimes\mathcal{F}_t$*

be the product $\sigma-$algebra generated by the semi-ring $\mathcal{B}(\mathbb{R}^+\times E)\times\mathcal{F}_t$ of the product sets $X\times F$, $X\in\mathcal{B}(\mathbb{R}^+\times E)$, $F\in\mathcal{F}_t$ (where $\mathcal{F}_t$ is the filtration of the process $p(t)$). Let $T>0$ and

$$\mathbb{H}(X)=\Big\{g:\mathbb{R}^+\times X\times\Omega\to F, such\ that\ g\ is\ F_T/\mathcal{B}(F)$$
$$measurable\ and\ g(t,\zeta,w)\ is\ \mathcal{F}_t-adapted\ for\ \forall\zeta\in X,\ \forall t\in(0,T]\Big\}.$$

Let $p\geq 1$,

$$\mathbb{H}^p_\nu([0,T]\times X;F)=\Big\{g\in\mathbb{H}(X):\int_0^T\int_X\mathbb{E}\|g(t,\zeta,w)\|^p_F\nu(d\zeta)dt<\infty\Big\}.$$

The two dimensional PEs can be formally derived from the full three dimensional system under the assumption of invariance with respect to the second horizontal variable y as in [17], then we arrive at the following stochastic evolution system:

$$\begin{aligned} du^n=&[\nu_1\Delta u^n-u^n\partial_x u^n-w^n\partial_z u^n-\partial_x p+b]dt \\ &+\frac{1}{\sqrt{n}}dW(t)+\frac{1}{n}\int_X f(x,z,t,\zeta,\omega)\widetilde{N}_n(dt,d\zeta), \end{aligned} \tag{2.1}$$

$$\partial_z p=0, \tag{2.2}$$

$$\partial_x u^n=-\partial_z w^n, \tag{2.3}$$

with velocity $u^n=u^n(t,x,z,\omega)$, pressure p, $(x,z)\in\mathcal{M}=[0,l]\times[-h,0]$ and $t>0$. Let $b(x,z,t,\omega)\in H$, in the sequel, to ease the notation, we will suppose that $b(x,z,t,\omega)=b(x,z,\omega)$; however, all the results have a straightforward extension to time-dependent $b(x,z,t,\omega)$. $W(t)$ is an $H-$valued Wiener process and f is a measure mapping form some measurable space X to H, the definition of space H will be given in the follow. Here Δ is the Laplacian operator, without loss of generality in this paper, we take ν_1 to be 1, noting equation (2.2), p does not depend on variable z. In above formulation, we have ignored the coupling with the temperature and salinity equations in order to focus main attention on the difficulties from nonlinear terms in equation (2.2).

We partition the boundary into the top $\Gamma_u=\{z=0\}$, the bottom $\Gamma_b=\{z=-h\}$ and the sides $\Gamma_s=\{x=0\}\cup\{x=l\}$. In this paper, we consider the following boundary conditions:

$$\begin{aligned} &\text{on}\,\Gamma_u:\partial_z u^n=0,\ w^n=0,\\ &\text{on}\,\Gamma_b:\partial_z u^n=0,\ w^n=0,\\ &\text{on}\,\Gamma_s:\ \ u^n=0. \end{aligned}$$

Due to equation (2.1), we have that

$$w^n(x,z,t)=-\int_{-h}^z\partial_x u^n(x,\xi,t)d\xi. \tag{2.4}$$

We define the function spaces H and V as follows (see [17]):

$$H=\Big\{v\in L^2(\mathcal{M})\,|\int_{-h}^0 vdz=0\ \Big\}, \tag{2.5}$$

$$V=\Big\{v\in H^1(\mathcal{M})\,|\int_{-h}^{0}vdz=0,\ v|_{\Gamma_s}=0\Big\}. \tag{2.6}$$

These spaces are endowed with the L^2 and H^1 norms which we respectively denote by $|\cdot|$ and $\|\cdot\|$. The inner products and norms on H,V are given by(due to boundary conditions)

$$(v,v_1)=\int_{\mathcal{M}}vv_1dxdz,\ ((v,v_1))=\int_{\mathcal{M}}\nabla v\nabla v_1dxdz,$$

and

$$|v|=(v,v)^{\frac{1}{2}},\ \|v\|=((v,v))^{\frac{1}{2}},$$

where $v_1,v\in V$. Let V' be the dual space of V. We have the dense and continuous embeddings $V\hookrightarrow H=H'\hookrightarrow V'$ and denote by $\langle u,\psi\rangle$ the duality between $u\in V$ and $\psi\in V'$. For $u,v\in V$, we have

$$\begin{aligned}\langle -\Delta u,v\rangle&=-\int_{\mathcal{M}}u_{xx}vdxdz-\int_{\mathcal{M}}u_{zz}vdxdz\\&=\int_{\mathcal{M}}(u_xv_x+u_zv_z)dxdz-\oint_{\mathcal{M}}u_xvn_xds-\oint_{\mathcal{M}}u_zvn_zds,\end{aligned}$$

where $n=(n_x,n_z)$ is an outer unit normal vector, for example, $n=(0,1)$ on the top $\Gamma_u=\{z=0\}$ and $n=(0,-1)$ on the bottom $\Gamma_b=\{z=-h\}$. By boundary conditions in this paper, we obtain $\oint_{\mathcal{M}}u_xun_xds=\oint_{\mathcal{M}}u_zun_zds=0$, thus, $\langle -\Delta u,v\rangle=((u,v))$. Consider a Stokes-type unbounded linear operator $A:V\to V'$ with $D(A)=V\cap H^2(\mathcal{M})$ and define

$$\langle Au,v\rangle=((u,v)),\,\forall u,v\in V.$$

The Laplace operator A is self-adjoint, positive, with compact self-adjoint inverses, and it maps V to V'. Next we address the nonlinear term. In accordance with equatlity (2.4) we take

$$\mathcal{W}(v):=-\int_{-h}^{z}\partial_xv(x,\widetilde{z})d\widetilde{z}, \tag{2.7}$$

and let

$$B(u,v):=u\partial_xv+\mathcal{W}(u)\partial_zv, \tag{2.8}$$

where $u,v\in V$ and denoting $B(u,u)=B(u)$.

Define the bilinear operator $B(u,v):V\times V\to V'$ according to

$$\langle B(u,v),w\rangle=\mathfrak{b}(u,v,w),$$

where

$$\mathfrak{b}(u,v,w)=\int_{\mathcal{M}}(u\partial_xvw+\mathcal{W}(u)\partial_zvw)d\mathcal{M}.$$

In the sequel, when no confusion arises, we denote by C a constant which may change from one line to the next one.

LEMMA 2.1 (Estimates for $\mathfrak{b}$ and B (see [15, 17])). *The trilinear forms $\mathfrak{b}$ and B have the following properties. There exists a constant $C>0$ such that*

$$|\mathfrak{b}(u,v,w)| \le C\Big(|u|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}\|v\||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}} \\ +|\partial_x u||\partial_z v||w|^{\frac{1}{2}}\|w\|^{\frac{1}{2}}\Big), \qquad u,v,w\in V, \tag{2.9}$$

$$\mathfrak{b}(u,v,v)=0 \qquad u,v,w\in V, \tag{2.10}$$

$$\langle \mathfrak{b}(u,u),\partial_{zz}u\rangle = 0 \qquad u\in D(A). \tag{2.11}$$

Throughout this paper, we assume that, for $a>0$,

$$\int_X |f(\zeta)|^2 \exp(a|f(\zeta)|)\nu(d\zeta) < +\infty, \tag{2.12}$$

$$\int_X |\partial_z f(\zeta)|^2 \exp(a|\partial_z f(\zeta)|)\nu(d\zeta) < +\infty, \tag{2.13}$$

where $|\cdot|$ denotes L^2 norm.

Note that the above formulation is equivalent to projecting equations (2.2)-(2.1) from $L^2(\mathcal{M})$ into the space $H(\mathcal{M})$ and thus the pressure term $p(x,t)$ is absent. With these notations, the above PEs can be rewritten as

$$du^n + [Au^n + B(u^n,u^n)]dt = bdt + \frac{1}{\sqrt{n}}dW(t) + \frac{1}{n}\int_X f(\zeta)\widetilde{N}_n(dt,d\zeta), \tag{2.14}$$

$$u^n(0) = u_0. \tag{2.15}$$

In [34], by a priori estimates, weak convergence method and monotonicity arguments, we studied 2D PEs with multiplicative Lévy noise and proved the existence and uniqueness of the solutions, using approaches similar to that in [34], we can easily show in this additive case that equation (2.14) has a unique solution in $D([0,1];H)\cap L^2([0,1];V)$.

THEOREM 2.1 (Well-posedness and a priori bounds). *For $i=1,2$, let the initial datum u_0 satisfy $\mathbb{E}|u_0|^{2i}<\infty$, $\mathbb{E}|\partial_z u_0|^{2i}<\infty$, and $b,\partial_z b\in L^4(\Omega;L^2(0,T;H))$, $f(x,z,t,\zeta,\omega),\partial_z f(x,z,t,\zeta,\omega)$ satisfy $\mathbb{H}^2_\nu([0,T]\times X;H)$, then there exists a unique weak solution u^n of the stochastic primitive problem (2.14) with initial condition $u^n(0)=u_0$. Furthermore, we have*

$$E\Big(\sup_{0\le t\le T}|u^n(t)|^{2i} + \Big(\int_0^T \|u^n(t)\|^2dt\Big)^i\Big) \le C\left(1+E|u_0|^{2i}\right), \tag{2.16}$$

and satisfy the additional regularity

$$\partial_z u^n(t) \in L^4(\Omega, L^\infty(0,T;H)\cap L^2(0,T;V)). \tag{2.17}$$

3. Exponential estimates

To establish the large deviation principle, we first prove some exponential estimates. Let u^n be the solution of the following stochastic PEs

$$u^n(t) = u_0 - \int_0^t Au^n(s)ds - \int_0^t B(u^n(s),u^n(s))ds + bt \\ + \frac{1}{\sqrt{n}}W(t) + \frac{1}{n}\int_0^t\int_X f(\zeta)\widetilde{N}_n(dt,d\zeta). \tag{3.1}$$

Let $X^n = nu^n$, then X^n is the solution of the following equation

$$\begin{aligned} X^n(t) = nu_0 - \int_0^t AX^n(s)ds - \frac{1}{n}\int_0^t B(X^n(s),X^n(s))ds + nbt \\ + \sqrt{n}W(t) + \int_0^t \int_X f(\zeta)\widetilde{N}_n(dt,d\zeta). \end{aligned} \tag{3.2}$$

Denote by $\{e_k\}_{k=1}^{\infty}$ an orthonormal basis of H that consists of eigenvectors of Q in V with $\{\lambda_k\}_{k=1}^{\infty}$ being the corresponding eigenvalues.
Applying Itô's formula [1], we get the following lemma as in [32,37].

Lemma 3.1. *For $g_1 \in C_b^2(H)$, moreover, $g_2 \in C_b^2(D(A))$, then $M_t^{g_i} = exp(g_i(X^n(t)) - g_i(nx) - \int_0^t h_i(X^n(s)ds))$, $i = 1,2$ is a $\mathcal{F}_t-$ local martingale and*

$$\begin{aligned} h_i(y) \ &= \ - < Ay + \frac{1}{n}B(y,y), g_i'(y) > + n(b, g_i'(y)) \\ &+ \frac{n}{2}\sum_{k=1}^{\infty}\lambda_k\Big([g_i'(y)\otimes g_i'(y) + g_i''(y)]e_k, e_k\Big) \\ &+ n\int_X \Big\{\exp[g_i(y+f(\zeta)) - g_i(y)] - 1 - (g_i'(y), f(\zeta))\Big\}\nu(d\zeta), \end{aligned} \tag{3.3}$$

where $C_b^2(H)$ (the definition of $C_b^2(D(A))$ is similar) is defined by

$$\begin{aligned} C_b^2(H) = \{ & g: H \to R | g \text{ is bounded continuously} \\ & \text{and twice Fréchet differentiable with derivatives} \\ & g': [0,T]\times H \to L(H,R) \cong H, \\ & g'': [0,T]\times H \to L(H)\}. \end{aligned}$$

In the rest of this section, for $\lambda > 0$, we set $g_1(y) := (1+\lambda|y|^2)^{\frac{1}{2}}$ and $g_2(y) := (1+\lambda|\partial_z y|^2)^{\frac{1}{2}}$. Then

$$g_1'(y) = \lambda(1+\lambda|y|^2)^{-\frac{1}{2}}y, \tag{3.4}$$
$$g_1''(y) = -\lambda^2(1+\lambda|y|^2)^{-\frac{3}{2}}y\otimes y + \lambda(1+\lambda|y|^2)^{-\frac{1}{2}}I, \tag{3.5}$$
$$g_2'(y) = \lambda(1+\lambda|\partial_z y|^2)^{-\frac{1}{2}}(-\partial_{zz}y), \tag{3.6}$$
$$g_2''(y) = -\lambda^2(1+\lambda|\partial_z y|^2)^{-\frac{3}{2}}(-\partial_{zz}y)\otimes(-\partial_{zz}y) + \lambda(1+\lambda|\partial_z y|^2)^{-\frac{1}{2}}(-\partial_{zz}I), \tag{3.7}$$

where I stands for the identity operator.

Due to equation (2.10), we have $< B(y,y), g_1'(y) > = \lambda(1+\lambda|y|^2)^{-\frac{1}{2}} < B(y,y), y > = 0$, then the following lemma is the same as Lemma 3.2 and Lemma 3.3 for 2D Navier–Stokes equations in [37].

Lemma 3.2.

$$\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\sup_{0\le t\le 1}|u^n| > r) = -\infty, \tag{3.8}$$

$$\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\Big((\int_0^1 \|u^n\|^2 dt)^{\frac{1}{2}} > r\Big) = -\infty. \tag{3.9}$$

Comparing to Navier–Stokes equation, due to the estimates of $B(u,u)$ in Lemma 2.1, we should give some exponential estimates of $\partial_z u^n$, by introducing $g_2(y) := (1+\lambda|\partial_z y|^2)^{\frac{1}{2}}$, we have following lemma.

LEMMA 3.3.

$$\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\sup_{0\le t\le 1}|\partial_z u^n|>r) = -\infty. \tag{3.10}$$

Proof. In equality (3.3),

$$\begin{aligned}<-Ay, g_2'(y)> &= \lambda(1+\lambda|\partial_z y|^2)^{-\frac{1}{2}} <-Ay, -\partial_{zz}y> \\ &= -\lambda(1+\lambda|\partial_z y|^2)^{-\frac{1}{2}}\|\partial_z y\|^2 \\ &\le 0.\end{aligned}$$

Applying equation (2.11), we get

$$<-B(y,y), g_2'(y)> = \lambda(1+\lambda|\partial_z y|^2)^{-\frac{1}{2}} <-B(y,y), -\partial_{zz}y> = 0.$$

Let $G(y) = e^{g_2(y)}$, by Taylor's expansion, there exist θ between 0 and 1 such that

$$\begin{aligned}&\exp[g_2(y+f(\zeta)) - g_2(y)] - 1 - (g_2'(y), f(\zeta)) \\ =& e^{-g_2(y)}[G(y+f(\zeta)) - G(y) - G(y)(g_2'(y), f(\zeta))] \\ =& \frac{1}{2}e^{-g_2(y)}(G''(y+\theta f(\zeta)), f(\zeta)\otimes f(\zeta)).\end{aligned} \tag{3.11}$$

Note that

$$G''(y) = G(y)g_2'(y)\otimes g_2'(y) + G(y)g_2''(y).$$

Given $w(x)\in V$, by equations (3.6) and (3.7), we deduce that,

$$\begin{aligned}(g_2'(y), w(x)) &= \lambda(1+\lambda|\partial_z y|^2)^{-\frac{1}{2}}(\partial_z y, \partial_z w) \\ &\le \frac{\lambda|\partial_z y|}{\sqrt{1+\lambda|\partial_z y|^2}}|\partial_z w| \\ &\le \lambda^{\frac{1}{2}}|\partial_z w|.\end{aligned} \tag{3.12}$$

$$\begin{aligned}&(g_2'(y)\otimes g_2'(y) + g_2''(y), w(x)\otimes w(x)) \\ =& \lambda^2(1+\lambda|\partial_z y|^2)^{-1}(\partial_z y, \partial_z w)(\partial_z y, \partial_z w) \\ &-\lambda^2(1+\lambda|\partial_z y|^2)^{-\frac{3}{2}}(\partial_z y, \partial_z w)(\partial_z y, \partial_z w) + \lambda(1+\lambda|\partial_z y|^2)^{-\frac{1}{2}}(\partial_z w, \partial_z w) \\ \le& \frac{\lambda^2|\partial_z y|^2}{1+\lambda|\partial_z y|^2}|\partial_z w|^2 + \frac{\lambda^2|\partial_z y|^2}{\sqrt{(1+\lambda|\partial_z y|^2)^3}}|\partial_z w|^2 + \frac{\lambda}{\sqrt{1+\lambda|\partial_z y|^2}}|\partial_z w|^2 \\ \le& 3\lambda|\partial_z w|^2.\end{aligned} \tag{3.13}$$

Then by conclusions (3.11)-(3.13), we have

$$\begin{aligned}&|\exp[g_2(y+f(\zeta)) - g_2(y)] - 1 - (g_2'(y), f(\zeta))| \\ \le& \frac{3}{2}\lambda\exp\Big(g_2(y+\theta f(\zeta)) - g_2(y)\Big)|\partial_z f(\zeta)|^2\end{aligned}$$

$$\leq \frac{3}{2}\lambda \exp[(g_2'(y+\theta_1 f(\zeta)),\theta f(\zeta))]|\partial_z f(\zeta)|^2$$
$$\leq \frac{3}{2}\lambda \exp\left(\lambda^{\frac{1}{2}}|\partial_z f(\zeta)|\right)|\partial_z f(\zeta)|^2.$$

Thus, by assumption (2.13), we have

$$\begin{aligned}
h(y) = & -< Ay + \frac{1}{n}B(y,y), g_2'(y) > + n(b, g_2'(y)) \\
& + \frac{n}{2}\sum_{k=1}^{\infty}\lambda_k\left([g_2'(y)\otimes g_2'(y) + g_2''(y)]e_k, e_k\right) \\
& + n\int_X \left\{\exp[g_2(y+f(\zeta)) - g_2(y)] - 1 - (g_2'(y), f(\zeta))\right\}\nu(d\zeta) \\
\leq & \, n|\partial_z b|\lambda^{\frac{1}{2}} + cn\lambda TrQ + \frac{3}{2}n\int_X \lambda \exp\left(\lambda^{\frac{1}{2}}|\partial_z f(\zeta)|\right)|\partial_z f(\zeta)|^2\nu(d\zeta) \\
\leq & \, n|\partial_z b|\lambda^{\frac{1}{2}} + cn\lambda TrQ + \frac{3}{2}nM_\lambda.
\end{aligned} \tag{3.14}$$

Observe that

$$\begin{aligned}
& P(\sup_{0\leq t\leq 1}|\partial_z u^n| > r) = P(\sup_{0\leq t\leq 1}|\partial_z X^n| > nr) \\
= & P\Big(\sup_{0\leq t\leq 1} g_2(X^n) > (1+\lambda n^2 r^2)^{\frac{1}{2}}\Big) \\
= & P\Big(\sup_{0\leq t\leq 1}[g_2(X^n) - g_2(nu_0) - \int_0^1 h(X^n)ds + g_2(nu_0) + \int_0^1 h(X^n)ds] \\
& > (1+\lambda n^2 r^2)^{\frac{1}{2}}\Big) \\
\leq & P\Big(\sup_{0\leq t\leq 1}[g_2(X^n) - g_2(nu_0) - \int_0^1 h(X^n)ds] + g_2(nu_0) \\
& + \sup_{0\leq t\leq 1}\int_0^1 h(X^n)ds > (1+\lambda n^2 r^2)^{\frac{1}{2}}\Big) \\
\leq & P\Big(\sup_{0\leq t\leq 1}[g_2(X^n) - g_2(nu_0) - \int_0^1 h(X^n)ds] \\
& > -g_2(nu_0) - \sup_{0\leq t\leq 1}\int_0^1 h(X^n)ds + (1+\lambda n^2 r^2)^{\frac{1}{2}}\Big).
\end{aligned} \tag{3.15}$$

By inequality (3.14) and Doob's inequality,

$$\begin{aligned}
& P\Big(\sup_{0\leq t\leq 1}[g_2(X^n) - g_2(nu_0) - \int_0^1 h(X^n)ds] \\
& \quad > -g_2(nu_0) - \sup_{0\leq t\leq 1}\int_0^1 h(X^n)ds + (1+\lambda n^2 r^2)^{\frac{1}{2}}\Big) \\
\leq & P\Big(\sup_{0\leq t\leq 1}[g_2(X^n) - g_2(nu_0) - \int_0^1 h(X^n)ds] \\
& \quad > -g_2(nu_0) - n|\partial_z b|\lambda^{\frac{1}{2}} - cn\lambda TrQ - \frac{3}{2}nM_\lambda + (1+\lambda n^2 r^2)^{\frac{1}{2}}\Big)
\end{aligned}$$

$$\leq \Big(\sup_{0\leq t\leq 1} E[\exp(g_2(X^n)-g_2(nu_0)-\int_0^1 h(X^n)ds)]\Big)$$
$$\times \exp[g_2(nu_0)+n|\partial_z b|\lambda^{\frac{1}{2}}+cn\lambda TrQ+\frac{3}{2}nM_\lambda-(1+\lambda n^2r^2)^{\frac{1}{2}}]$$
$$\leq \exp[g_2(nu_0)+n|\partial_z b|\lambda^{\frac{1}{2}}+cn\lambda TrQ+\frac{3}{2}nM_\lambda-(1+\lambda n^2r^2)^{\frac{1}{2}}], \tag{3.16}$$

where in the last step, we used the fact that

$$\sup_{0\leq t\leq 1} E[\exp(g_2(X^n)-g_2(nu_0)-\int_0^1 h(X^n)ds)]\leq 1,$$

because $\exp(g_2(X^n)-g_2(nu_0)-\int_0^1 h(X^n)ds)$ is a non-negative local martingale with the initial value 1. By inequalities (3.15) and (3.16), we have

$$\frac{1}{n}\log P(\sup_{0\leq t\leq 1}|\partial_z u^n|>r)\leq \frac{(1+\lambda n^2|\partial_z u_0|^2)^{\frac{1}{2}}}{n}+|\partial_z b|\lambda^{\frac{1}{2}}$$
$$+c\lambda TrQ+\frac{3}{2}M_\lambda-\frac{(1+\lambda n^2 r^2)^{\frac{1}{2}}}{n},$$

then taking $n\to\infty$, and $r\to\infty$, we obtain the limit (3.10). □

By the projection operator P_m

$$P_m x:=\sum_{i=1}^m (x,e_i)e_i, x\in H,$$

and let $Z^{n,m}$, Z^n be the solution of the following linear equations respectively,

$$Z^{n,m}=-\int_0^t AZ^{n,m}ds+\frac{1}{n}\int_0^t\int_X P_m f(\zeta)\widetilde{N}_n(ds,d\zeta), \tag{3.17}$$

and

$$Z^n=-\int_0^t AZ^n ds+\frac{1}{n}\int_0^t\int_X f(\zeta)\widetilde{N}_n(ds,d\zeta). \tag{3.18}$$

The existence and uniqueness of $Z^{n,m}$, Z^n can refer to [32].

Put $\widetilde{Z}^{n,m}:=n(Z^{n,m}-Z^n)$, then $\widetilde{Z}^{n,m}$ is the solution of the equation

$$\widetilde{Z}^{n,m}=-\int_0^t A\widetilde{Z}^{n,m}ds+\int_0^t\int_X (P_m f(\zeta)-f(\zeta))\widetilde{N}_n(ds,d\zeta). \tag{3.19}$$

Similar to obtain Lemma 3.1, for $i=1,2$, one has

$$\exp\Big(g_i(\widetilde{Z}^{n,m}(t))-g_i(\widetilde{Z}^{n,m}(0))-\int_0^t \widetilde{h}(\widetilde{Z}^{n,m})ds\Big) \tag{3.20}$$

is a $\mathcal{F}_t-$ local martingale, where

$$\widetilde{h}(y)=-<Ay,g_i'(y)>+n\int_X\Big\{\exp[g_i(y+P_m f(\zeta)-f(\zeta))-g(y)]$$

$$-1-(g_i'(y),P_m f(\zeta)-f(\zeta))\Big\}\nu(d\zeta).$$

Note that expression (3.20) is similar to equality (3.3), even easier, as result, for $i=1$, we have the following lemma.

LEMMA 3.4. *for any* $\delta>0$,

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\sup_{0\le t\le 1}|Z^{n,m}-Z^n|>\delta)=-\infty$$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\int_0^t\|Z^{n,m}-Z^n\|^2ds>\delta)=-\infty.$$

For $i=2$, using equations (3.6) and (3.7) and following the same steps of the proof of the limit (3.10), by λ is arbitrary, taking $\lambda\to\infty$, we easily obtain Lemma 3.5.

LEMMA 3.5. *For any given* $\delta>0$, *we have*

$$\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\sup_{0\le t\le 1}|\partial_z Z^{n,m}|>r)=-\infty, \tag{3.21}$$

$$\lim_{r\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\sup_{0\le t\le 1}|\partial_z Z^{n}|>r)=-\infty, \tag{3.22}$$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P(\sup_{0\le t\le 1}|\partial_z Z^{n,m}-\partial_z Z^n|>\delta)=-\infty. \tag{3.23}$$

4. Large deviation principle

For $l\in H$, define

$$F(l):=\int_X[\exp(f(\zeta),l)-1-(f(\zeta),l)]\nu(d\zeta)+(Ql,l)+(b,l).$$

Set, for $z\in H$,

$$F^*(z)=\sup_{l\in H}[(z,l)-F(l)],$$

let

$$L_t^n:=bt+\frac{1}{\sqrt{n}}W(t)+\frac{1}{n}\int_0^t\int_X f(\zeta)\widetilde{N}_n(ds,d\zeta).$$

Then by [7], the laws of $\{L_t^n,n\ge 1\}$ satisfy a large deviation principle on $D([0,1];H)$ with rate function I_0, which is defined on $D([0,1];H)$ as follows: if $g\in D([0,1];H)$ and $g'\in L^1([0,1];H)$, $I_0(g)=\int_0^1 F^*(g'(s))ds$; otherwise $I_0(g)=\infty$.

For $g\in D([0,1];V)$, define $\phi_t(g)\in D([0,1];H)\cap L^2([0,1];V)$ as the solution to the following equation:

$$\phi_t(g)=u_0-\int_0^t A\phi_s(g)ds-\int_0^t B(\phi_s(g),\phi_s(g))ds+g(t). \tag{4.1}$$

As we obtain Theorem 2.1, by a priori estimates, weak convergence method and monotonicity arguments, for $g\in D([0,1];V)$, we easily obtain the existence and uniqueness of solution $\phi_t(g)\in D([0,1];H)\cap L^2([0,1];V)$. A priori estimates can also be found in following Lemma 4.1.

For $h\in D([0,1];H)$,

$$I(h):=\inf\{I_0(g):h=\phi(g),g\in D([0,1];H)\}$$

with the convention $\inf\{\emptyset\}=\infty$.

In this section, we establish a large deviation principle of PEs, it is easy to see that $\phi(L_t^n)$ is the solution of the equation (2.14). As we known, by contraction principle (Theorem 4.2.1 in [10]), LDP is preserved under continuous mappings, however, the contraction principle cannot be applied directly because the mapping ϕ defined in equation (4.1) is continuous only on $D([0,1];V)$ (see Lemma 4.1), not necessarily on $D([0,1];H)$. Instead, by exponential approximations plus continuous contractions, we obtain the LDP, the corresponding result is Theorem 4.2.23 in [10] which can be applied thanks to Lemma 4.1, Lemma 4.2 and Lemma 4.3 of the paper. We state the Theorem as follows.

THEOREM 4.1. *Let $\{\mu_\varepsilon\}$ be a family of probability measures that satisfies the LDP with a good rate function I on a Hausdorff topological space $\mathcal{X}$, and for $m=1,2,\ldots$, let $f_m:\mathcal{X}\to\mathcal{Y}$ be continuous functions, with $(\mathcal{Y},d)$ a metric space. Assume there exists a measurable map $f:\mathcal{X}\to\mathcal{Y}$ such that for every $\alpha<\infty$,*

$$\limsup_{m\to\infty}\sup_{x:I(x)\le\alpha} d(f_m(x),f(x))=0.$$

Then any family of probability measures $\{\widetilde{\mu}_\varepsilon\}$ for which $\{\mu_\varepsilon\circ f_m^{-1}\}$ are exponentially good approximations satisfies the LDP in $\mathcal{Y}$ with the good rate function $I'(y)=\inf\{I(x):y=f(x)\}$.

Thus, equation (2.14) is projected onto subspaces of V, yielding exponential approximations to the original sequence of solutions and the large deviation principle (Theorem 4.2) has been obtained from Lemma 4.1, Lemma 4.2 and Lemma 4.3. The derivation of the large deviation principle in the paper follows closely a work by Tiange Xu and Tusheng Zhang (reference [37]), where the LDP for a class of two-dimensional stochastic Navier–Stokes equations with Lévy noise is established. It is well known that the dynamics of PEs are of the similar form as Navier–Stokes equations , but with a different nonlinear operator B. Compared with Navier–Stokes equations, almost without exception, main job of all previous papers related to PEs is to deal with the nonlinear term B. In this paper, to obtain large deviation principle, we need to prove a number of exponential estimates for the energy of the solutions as well as the exponential convergence of the approximating solutions in $D([0,1];H)\cap L^2([0,1];V)$. Since the highly nonlinear term B, we can't get there estimates directly as in [37], by introducing $g_2(y)$, we do some more complex estimates with $\partial_z u$, on this basis, we solved this problem and these estimates about $B(u^n)$ are our main original contribution of this paper.

Then we state the main result of this paper.

THEOREM 4.2. *Let μ_n be the law of the solution u^n of the equations (2.14) and (2.15), then $\{\mu_n,n\ge 1\}$ satisfies a large deviation principle on $D([0,1];H)$ endowed with the uniform topology with the rate function $I(\cdot)$, i.e.,*

(i) For any closed subset $F\subset D([0,1];H)$,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(F)\le-\inf_{h\in F}I(h).$$

(ii) For any open set $G \subset D([0,1];H)$,

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(G) \le -\inf_{h\in G} I(h).$$

LEMMA 4.1. *The mapping ϕ defined in equation (4.1) is continuous form $D([0,1];V)$ into $D([0,1];H)\cap L^2([0,1];V)$ in the topology of uniform convergence.*

Proof. Let $v_t(g)=\phi_t(g)-g(t)$, then $v_t(g)$ satisfies the following equation

$$v_t(g)=u_0-\int_0^t Av_s(g)ds-\int_0^t Ag(s)ds-\int_0^t B(v_s(g)+g(s),v_s(g)+g(s))ds. \quad (4.2)$$

We have

$$v(\cdot): D([0,1];V)\to D([0,1];H)\cap L^2([0,1];V)$$

is continuous, that is, take $\{g_n\}_{n=1}^\infty$, $g\in D([0,1];V)$ such that $\lim_{n\to\infty}\sup_{0\le t\le 1}\|g_n(t)-g(t)\|=0$, then

$$\lim_{n\to\infty}\Big(\sup_{0\le t\le 1}|v_t(g)-v_t(g_n)|^2+\int_0^t \|v_s(g)-v_s(g_n)\|^2 ds\Big)=0. \quad (4.3)$$

To this end, we need some energy estimates for $v_t(g)$. In view of Lemma 2.1 for nonlinear term $B(u,u)$, we have

$$\begin{aligned}
&|v_t(g)|^2+2\int_0^t \|v_s(g)\|^2 ds\\
&=|u_0|^2-2\int_0^t <Ag(s),v_s(g)>ds-2\int_0^t \mathfrak{b}(v_s(g)+g(s),v_s(g)+g(s),v_s(g))ds\\
&\le |u_0|^2+\frac{1}{4}\int_0^t \|v_s(g)\|^2ds+C\int_0^t \|g(s)\|^2ds\\
&\qquad +2\int_0^t |\mathfrak{b}(v_s(g),g(s),v_s(g))|ds+2\int_0^t |\mathfrak{b}(g(s),g(s),v_s(g))|ds\\
&\le |u_0|^2+\frac{1}{4}\int_0^t \|v_s(g)\|^2ds+C\int_0^t \|g(s)\|^2ds+C\int_0^t |v_s(g)|\|v_s(g)\|\|g(s)\|ds\\
&\qquad +C\int_0^t |v_s(g)|^{\frac{1}{2}}\|v_s(g)\|^{\frac{3}{2}}\|g(s)\|ds+C\int_0^t \|g(s)\|^2\|v_s(g)\|ds\\
&\le |u_0|^2+\frac{1}{4}\int_0^t \|v_s(g)\|^2ds+C\int_0^t \|g(s)\|^2ds\\
&\quad +\frac{1}{4}\int_0^t \|v_s(g)\|^2ds+C\int_0^t |v_s(g)|^2\|g(s)\|^2ds\\
&\qquad +\frac{1}{4}\int_0^t \|v_s(g)\|^2ds+C\int_0^t |v_s(g)|^2\|g(s)\|^4ds\\
&\qquad +\frac{1}{4}\int_0^t \|v_s(g)\|^2ds+C\int_0^t \|g(s)\|^4ds.
\end{aligned} \quad (4.4)$$

Applying Gronwall's inequality, we have

$$\sup_{0\le s\le t}|v_s(g)|^2\le\Big(|u_0|^2+Ct\sup_{0\le s\le t}(\|g(s)\|^2+\|g(s)\|^4)\Big)\exp\Big(Ct\sup_{0\le s\le t}(\|g(s)\|^2+\|g(s)\|^4)\Big)$$

$$= M_g(t). \tag{4.5}$$

Furthermore,

$$\begin{aligned}
&\int_0^t \|v_s(g)\|^2 ds \\
&\le |u_0|^2 + Ct\Big(\sup_{0\le s\le t}\|g(s)\|^2 + \sup_{0\le s\le t}\|g(s)\|^4 + \sup_{0\le s\le t}(\|g(s)\|^2+\|g(s)\|^4)M_g(t)\Big) \\
&= C_g(t).
\end{aligned} \tag{4.6}$$

Since $\lim_{n\to\infty}\sup_{0\le t\le 1}\|g_n(t)-g(t)\|=0$, it is easy to see that, inequalities (4.5) and (4.6) still hold for g_n. Then we have

$$\begin{aligned}
&|v_t(g_n)-v_t(g)|^2 + 2\int_0^t \|v_s(g_n)-v_s(g)\|^2 ds \\
&= -2\int_0^t < Ag_n(s) - Ag(s), v_s(g_n)-v_s(g) > ds \\
&\quad -2\int_0^t < B(v_s(g_n)+g_n(s)) - B(v_s(g)+g(s)), v_s(g_n)-v_s(g) > ds \\
&\le \frac{1}{2}\int_0^t \|v_s(g_n)-v_s(g)\|^2 ds + C\int_0^t \|g_n(s)-g(s)\|^2 ds \\
&\quad +2\int_0^t |< B(v_s(g_n)+g_n(s)) - B(v_s(g)+g(s)), v_s(g_n)-v(g(s)) >| ds,
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
&|< B(v_s(g_n)+g_n(s)) - B(v_s(g)+g(s)), v_s(g_n)-v_s(g) >| \\
&\le \Big|\mathfrak{b}\Big(v_s(g_n)-v_s(g), v_s(g_n)+g_n(s), v_s(g_n)-v_s(g)\Big)\Big| \\
&\quad + \Big|\mathfrak{b}\Big(g_n(s)-g(s), v_s(g_n)+g_n(s), v_s(g_n)-v_s(g)\Big)\Big| \\
&\quad + \Big|\mathfrak{b}\Big(v_s(g_n)+g(s), g_n(s)-g(s), v_s(g_n)-v_s(g)\Big)\Big| \\
&= I_1 + I_2 + I_2.
\end{aligned} \tag{4.8}$$

By Lemma 2.1,

$$\begin{aligned}
I_1 &\le C|v_s(g_n)-v_s(g)|\|v_s(g_n)-v_s(g)\|\|v_s(g_n)+g_n(s)\| \\
&\quad + C|v_s(g_n)-v_s(g)|^{\frac{1}{2}}\|v_s(g_n)-v_s(g)\|^{\frac{3}{2}}|\partial_z v_s(g_n)+\partial_z g_n(s)| \\
&\le \frac{1}{8}\|v_s(g_n)-v_s(g)\|^2 \\
&\quad + C\Big(\|v_s(g_n)\|^2 + \|g_n(s)\|^2 + |\partial_z v_s(g_n)|^4 + |\partial_z g_n(s)|^4\Big)|v_s(g_n)-v_s(g)|^2; \\
I_2 &\le C\|g_n(s)-g(s)\|\|v_s(g_n)+g_n(s)\|\|v_s(g_n)-v_s(g)\| \qquad (4.9) \\
&\le \frac{1}{16}\|v_s(g_n)-v_s(g)\|^2 + C\Big(\|v_s(g_n)\|^2+\|g_n(s)\|^2\Big)\|g_n(s)-g(s)\|^2; \\
I_3 &\le C\|v_s(g)+g(s)\|\|g_n(s)-g(s)\|\|v_s(g_n)-v_s(g)\| \qquad (4.10) \\
&\le \frac{1}{16}\|v_s(g_n)-v_s(g)\|^2 + C\Big(\|v_s(g)\|^2+\|g(s)\|^2\Big)\|g_n(s)-g(s)\|^2. \qquad (4.11)
\end{aligned}$$

Combining estimates (4.9)-(4.10), one obtains

$$
\begin{aligned}
&|v_t(g_n)-v_t(g)|^2+\int_0^t \|v_s(g_n)-v_s(g)\|^2 ds \\
\le{}& C\int_0^t \|g_n(s)-g(s)\|^2 ds + C\int_0^t \Big(\|v_s(g_n)\|^2+\|g_n(s)\|^2\Big)\|g_n(s)-g(s)\|^2 ds \\
&+C\int_0^t \Big(\|v_s(g)\|^2+\|g(s)\|^2\Big)\|g_n(s)-g(s)\|^2 ds \\
&+C\int_0^t \Big(\|v_s(g_n)\|^2+\|g_n(s)\|^2+|\partial_z v_s(g_n)|^4+\|g_n(s)\|^4\Big)|v_s(g_n)-v_s(g)|^2 ds. \qquad (4.12)
\end{aligned}
$$

Then we estimate $|\partial_z v_t(g_n)|$, taking inner product with $-\partial_{zz}\phi_t(g_n)$ in equation (4.1) for g instead of g_n, and using equation (2.11), one obtains

$$
\begin{aligned}
|\partial_z \phi_t(g_n)|^2 &= |\partial_z u_0|^2 - 2\int_0^t \|\partial_z \phi_s(g_n)\|^2 ds + 2\int_0^t (\partial_z g_n'(s), \partial_z \phi_s(g_n)) ds \\
&\le |\partial_z u_0|^2 + \int_0^t |\partial_z g_n'(s)|^2 ds + \int_0^t |\partial_z \phi_s(g_n)|^2 ds. \qquad (4.13)
\end{aligned}
$$

By Gronwall's inequality, we obtain, there exist $N_g(t)$, such that, $\sup_{0\le s\le t}|\partial_z\phi_s(g_n)|^2 \le N_g(t)$. since $\partial_z v_t(g_n)=\phi_t(g_n)-g_n(t)$ and $g_n\in D([0,1];V)$, we have

$$
\sup_{0\le s\le t}|\partial_z v_s(g_n)|^2 \le N_g(t)+\sup_{0\le s\le t}\|g_n(s)\|^2. \qquad (4.14)
$$

For inequality (4.12), applying Gronwall's inequality again, and inequalities (4.6) and (4.14), moreover, letting $n\to\infty$, we arrive at the limit (4.3). ∎

Now, let $u^{n,m}$ be the solution of the equation

$$
\begin{aligned}
u^{n,m}(t) = u_0 - \int_0^t Au^{u,m} ds - \int_0^t B(u^{n,m}) ds + b^m t \\
+\frac{1}{\sqrt{n}} W^m(t) + \frac{1}{n}\int_0^t\int_X f^m(\zeta)\widetilde{N}_n(dx,d\zeta), \qquad (4.15)
\end{aligned}
$$

where $b^m = P_m b$, $W^m(t) = P_m W(t)$, $f^m = P_m f(\zeta)$, let $\overline{u}^{n,m} = u^{n,m} - Z^{n,m}$, $\overline{u}^n = u^n - Z^n$, then $\overline{u}^{n,m}$ and $\overline{u}^n$ satisfy

$$
\overline{u}^{n,m}(t) = u_0 - \int_0^t A\overline{u}^{n,m} ds - \int_0^t B(\overline{u}^{n,m}+Z^{n,m}) ds + b^m t + \frac{1}{\sqrt{n}} W^m(t),
$$

and

$$
\overline{u}^{n}(t) = u_0 - \int_0^t A\overline{u}^{n} ds - \int_0^t B(\overline{u}^{n}+Z^{n}) ds + bt + \frac{1}{\sqrt{n}} W(t).
$$

LEMMA 4.2. *For any $\delta>0$,*

$$
\lim_{m\to\infty}\limsup_{n\to\infty}\frac{1}{n}\log P\big(\sup_{0\le t\le 1}|u^{n,m}(t)-u^n(t)|>\delta\big) = -\infty. \qquad (4.16)
$$

Proof. Our primary work on the estimate of $B(u,u)$. By Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5, for $\delta_0>0$, we can define a series of stopping times, such as

$$\tau_{\delta_0}^{n,m}=\inf\{t\geq 0,|Z^{n,m}-Z^n|>\delta_0,|\partial_z Z^{n,m}-\partial_z Z^n|>\delta_0,\int_0^t\|Z^{n,m}-Z^n\|^2ds>\delta_0\}.$$

and

$$\begin{aligned}\tau_M^{n,m}=\inf\Big\{&t\geq 0,|u^n|>M,|u^{n,m}|>M,|\partial_z u^n|>M,|\partial_z u^{n,m}|>M,\\&|Z^n|>M,|Z^{n,m}|>M,|\partial_z Z^n|>M,|\partial_z Z^{n,m}|>M,\\&\int_0^t\|u^n\|^2ds>M,\int_0^t\|u^{n,m}\|^2ds>M,\\&\int_0^t\|Z^n\|^2ds>M,\int_0^t\|Z^{n,m}\|^2ds>M\Big\}.\end{aligned}$$

Applying Itô's formula to $|\overline{u}^{n,m}_{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}|^2-|\overline{u}^{n}_{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}|^2$, we have

$$\begin{aligned}&\sup_{0\leq s\leq t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}|\overline{u}^{n,m}-\overline{u}^n|^2+2\int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\|\overline{u}^{n,m}-\overline{u}^n\|^2ds\\&\leq 2\int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\Big|<B(\overline{u}^{n,m}+Z^{n,m})-B(\overline{u}^n+Z^n),\overline{u}^{n,m}-\overline{u}^n>\Big|ds\\&\quad+2\int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\Big|(\overline{u}^{n,m}-\overline{u}^n,b^m-b)\Big|ds+\frac{2}{n}\int_0^{t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\sum_{i=m+1}^{\infty}\lambda_i ds\\&\quad+\frac{2}{\sqrt{n}}\sup_{0\leq s\leq t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\Big|\int_0^s(\overline{u}^{n,m}-\overline{u}^n,dW^m-dW)\Big|.\end{aligned}\tag{4.17}$$

Then we mainly estimate the following nonlinear term,

$$\begin{aligned}&\int_0^t\Big|<B(\overline{u}^{n,m}+Z^{n,m})-B(\overline{u}^n+Z^n),\overline{u}^{n,m}-\overline{u}^n>\Big|ds\\&=\int_0^t\mathfrak{b}(\overline{u}^{n,m},\overline{u}^{n,m},\overline{u}^{n,m}-\overline{u}^n)-\mathfrak{b}(\overline{u}^n,\overline{u}^n,\overline{u}^{n,m}-\overline{u}^n)ds\\&\quad+\int_0^t\mathfrak{b}(\overline{u}^{n,m},Z^{n,m},\overline{u}^{n,m}-\overline{u}^n)-\mathfrak{b}(\overline{u}^n,Z^n,\overline{u}^{n,m}-\overline{u}^n)ds\\&\quad+\int_0^t\mathfrak{b}(Z^{n,m},\overline{u}^{n,m},\overline{u}^{n,m}-\overline{u}^n)-\mathfrak{b}(Z^n,\overline{u}^n,\overline{u}^{n,m}-\overline{u}^n)ds\\&\quad+\int_0^t\mathfrak{b}(Z^{n,m},Z^{n,m},\overline{u}^{n,m}-\overline{u}^n)-\mathfrak{b}(Z^n,Z^n,\overline{u}^{n,m}-\overline{u}^n)ds\\&=J_1+J_2+J_3+J_4.\end{aligned}\tag{4.18}$$

By Lemma 2.1, we have

$$\begin{aligned}|J_1|&\leq\int_0^t|\mathfrak{b}(\overline{u}^{n,m}-\overline{u}^n,\overline{u}^n,\overline{u}^{n,m}-\overline{u}^n)|ds\\&\leq C\int_0^t|\overline{u}^{n,m}-\overline{u}^n|\|\overline{u}^n\|\|\overline{u}^{n,m}-\overline{u}^n\|ds\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t |\overline{u}^{n,m}-\overline{u}^n|^{\frac{1}{2}}\|\overline{u}^{n,m}-\overline{u}^n\|^{\frac{3}{2}}|\partial_z\overline{u}^n|ds\\
&\leq \frac{1}{4}\int_0^t \|\overline{u}^{n,m}-\overline{u}^n\|^2ds+C\int_0^t |\overline{u}^{n,m}-\overline{u}^n|^2(\|\overline{u}^n\|^2+|\partial_z\overline{u}^n|^4)ds,
\end{aligned}
\tag{4.19}
$$

$$
\begin{aligned}
|J_2|\leq &\int_0^t |\mathfrak{b}(\overline{u}^{n,m}-\overline{u}^n,Z^{n,m},\overline{u}^{n,m}-\overline{u}^n)|ds+\int_0^t |\mathfrak{b}(\overline{u}^n,Z^{n,m}-Z^n,\overline{u}^{n,m}-\overline{u}^n)|ds\\
\leq &C\int_0^t |\overline{u}^{n,m}-\overline{u}^n|\|\overline{u}^{n,m}-\overline{u}^n\|\|Z^{n,m}\|ds\\
&+C\int_0^t |\overline{u}^{n,m}-\overline{u}^n|^{\frac{1}{2}}\|\overline{u}^{n,m}-\overline{u}^n\|^{\frac{3}{2}}|\partial_z Z^{n,m}|ds\\
&+C\int_0^t |\overline{u}^n|^{\frac{1}{2}}\|\overline{u}^n\|^{\frac{1}{2}}|Z^{n,m}-Z^n|^{\frac{1}{2}}\|Z^{n,m}-Z^n\|^{\frac{1}{2}}\|\overline{u}^{n,m}-\overline{u}^n\|ds\\
&+C\int_0^t \|\overline{u}^n\||\partial_z\overline{u}^{n,m}-\partial_z\overline{u}^n|\|Z^{n,m}-Z^n\|ds\\
\leq &\frac{1}{4}\int_0^t \|\overline{u}^{n,m}-\overline{u}^n\|^2ds+C\int_0^t |\overline{u}^{n,m}-\overline{u}^n|^2(\|Z^{n,m}\|^2+|\partial_z Z^{n,m}|^4)ds\\
&+C\sup_{0\leq s\leq t}\Big(|Z^{n,m}(s-)-Z^n(s-)||\overline{u}^n(s-)|\Big)\cdot\Big(\int_0^t \|Z^{n,m}-Z^n\|^2ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|\overline{u}^n\|^2ds\Big)^{\frac{1}{2}}\\
&+C\sup_{0\leq s\leq t}(|\partial_z\overline{u}^{n,m}(s-)|+|\partial_z\overline{u}^n(s-)|)\cdot\Big(\int_0^t \|Z^{n,m}-Z^n\|^2ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|\overline{u}^n\|^2ds\Big)^{\frac{1}{2}},
\end{aligned}
\tag{4.20}
$$

$$
\begin{aligned}
|J_3|\leq &\int_0^t |\mathfrak{b}(Z^{n,m}-Z^n,\overline{u}^n,\overline{u}^{n,m}-\overline{u}^n)|ds\\
\leq &C\int_0^t |Z^{n,m}-Z^n|^{\frac{1}{2}}\|Z^{n,m}-Z^n\|^{\frac{1}{2}}|\overline{u}^n|^{\frac{1}{2}}\|\overline{u}^n\|^{\frac{1}{2}}\|\overline{u}^{n,m}-\overline{u}^n\|ds\\
&+C\int_0^t \|Z^{n,m}-Z^n\||\partial_z\overline{u}^{n,m}-\partial_z\overline{u}^n|\|\overline{u}^n\|ds\\
\leq &\frac{1}{4}\int_0^t \|\overline{u}^{n,m}-\overline{u}^n\|^2ds\\
&+C\sup_{0\leq s\leq t}\Big(|Z^{n,m}(s-)-Z^n(s-)||\overline{u}^n(s-)|\Big)\cdot\Big(\int_0^t \|Z^{n,m}-Z^n\|^2ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|\overline{u}^n\|^2ds\Big)^{\frac{1}{2}}\\
&+C\sup_{0\leq s\leq t}(|\partial_z\overline{u}^{n,m}(s-)|+|\partial_z\overline{u}^n(s-)|)\cdot\Big(\int_0^t \|Z^{n,m}-Z^n\|^2ds\Big)^{\frac{1}{2}}\Big(\int_0^t \|\overline{u}^n\|^2ds\Big)^{\frac{1}{2}},
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
|J_4|\leq &\int_0^t |\mathfrak{b}(Z^{n,m}-Z^n,Z^{n,m},\overline{u}^{n,m}-\overline{u}^n)|ds+\int_0^t |\mathfrak{b}(Z^n,Z^{n,m}-Z^n,\overline{u}^{n,m}-\overline{u}^n)|ds\\
\leq &C\int_0^t |Z^{n,m}-Z^n|^{\frac{1}{2}}\|Z^{n,m}-Z^n\|^{\frac{1}{2}}|Z^{n,m}|^{\frac{1}{2}}\|Z^{n,m}\|^{\frac{1}{2}}\|\overline{u}^{n,m}-\overline{u}^n\|ds
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_0^t \|Z^{n,m}-Z^n\||\partial_z\overline{u}^{n,m}-\partial_z\overline{u}^n|\|Z^{n,m}\|ds\\
&+C\int_0^t |Z^{n,m}-Z^n|^{\frac12}\|Z^{n,m}-Z^n\|^{\frac12}|Z^n|^{\frac12}\|Z^n\|^{\frac12}\|\overline{u}^{n,m}-\overline{u}^n\|ds\\
&+C\int_0^t \|Z^n\||\partial_z\overline{u}^{n,m}-\partial_z\overline{u}^n|\|Z^{n,m}-Z^n\|ds\\
\leq&\frac14\int_0^t\|\overline{u}^{n,m}-\overline{u}^n\|^2ds+C\sup_{0\leq s\leq t}\Big(|Z^{n,m}(s-)-Z^n(s-)||Z^{n,m}(s-)|\Big)\\
&\cdot(\int_0^t\|Z^{n,m}-Z^n\|^2ds)^{\frac12}(\int_0^t\|Z^{n,m}\|^2ds)^{\frac12}\\
&+C\sup_{0\leq s\leq t}\Big(|Z^{n,m}(s-)-Z^n(s-)||Z^n(s-)|\Big)\cdot(\int_0^t\|Z^{n,m}-Z^n\|^2ds)^{\frac12}(\int_0^t\|Z^n\|^2ds)^{\frac12}\\
&+C\sup_{0\leq s\leq t}(|\partial_z\overline{u}^{n,m}(s-)|+|\partial_z\overline{u}^n(s-)|)\cdot(\int_0^t\|Z^{n,m}-Z^n\|^2ds)^{\frac12}(\int_0^t\|Z^{n,m}\|^2ds)^{\frac12}\\
&+C\sup_{0\leq s\leq t}(|\partial_z\overline{u}^{n,m}(s-)|+|\partial_z\overline{u}^n(s-)|)\cdot(\int_0^t\|Z^{n,m}-Z^n\|^2ds)^{\frac12}(\int_0^t\|Z^n\|^2ds)^{\frac12}
\end{aligned}
\tag{4.22}
$$

Putting estimates (4.17), (4.19)-(4.22) together, applying Gronwall's inequality and the martingale inequality in [6] to $\frac{2}{\sqrt{n}}\sup_{0\leq s\leq t\wedge\tau_M^{n,m}\wedge\tau_{\delta_0}^{n,m}}\left|\int_0^s(\overline{u}^{n,m}-\overline{u}^n,dW^m-dW)\right|$, we can prove the limit (4.16). ∎

Define $\phi_t^m(g)$ be the solution of the following equation

$$\phi_t^m(g)=u_0-\int_0^t A\phi_s^m(g)ds-\int_0^t B(\phi_s^m(g))ds+P_mg(t). \tag{4.23}$$

The well-posedness of $\phi_t^m(g)\in D([0,1];H)\cap L^2([0,1];V)$ is the same as equation (4.1).

LEMMA 4.3. *For any $r>0$,*

$$\lim_{m\to\infty}\sup_{\{g:I_0(g)\leq r\}}\sup_{0\leq t\leq 1}|\phi_t^m(g)-\phi_t(g)|=0. \tag{4.24}$$

Proof. Taking inner product with $\phi_t^m(g)$ in equation (4.23), with $(B(\phi_t^m(g)),\phi_t^m(g))=0$, and

$$\sup_{\{g:I_0(g)\leq r\}}\int_0^1|g'(s)|ds\leq C,$$

by Gronwall's inequality, we can obtain

$$\sup_{\{g:I_0(g)\leq r\}}\sup_{0\leq t\leq 1}|\phi_t^m(g)|^2\leq C, \tag{4.25}$$

$$\sup_{\{g:I_0(g)\leq r\}}\int_0^1\|\phi_s^m(g)\|^2ds\leq C. \tag{4.26}$$

Note that the bounds (4.25), (4.26) also hold for $\phi_t(g)$. Moreover, taking inner product with $\partial_{zz}\phi_t^m(g)$ in equation (4.23), with $(B(\phi_t^m(g)),\partial_{zz}\phi_t^m(g))=0$, we can also obtain

$$\sup_{\{g:I_0(g)\leq r\}}\sup_{0\leq t\leq 1}|\partial_z\phi_t^m(g)|^2\leq C. \tag{4.27}$$

Then we have

$$\begin{aligned}\phi_t^m(g)-\phi_t(g)=&-\int_0^t A(\phi_s^m(g)-\phi_s(g)ds\\&-\int_0^t (B(\phi_s^m(g))-B(\phi_s(g))ds+P_mg(t)-g(t).\end{aligned}\tag{4.28}$$

Taking inner product, we deduce

$$\begin{aligned}&\sup_{0\le t\le 1}|\phi_t^m(g)-\phi_t(g)|^2+2\int_0^1\|\phi_s^m(g)-\phi_s(g)\|^2ds\\ \le&\int_0^1\Big|(B(\phi_s^m(g))-B(\phi_s(g),\phi_s^m(g)-\phi_s(g))\Big|ds\\&+\frac{1}{2}\sup_{0\le t\le 1}|\phi_t^m(g)-\phi_t(g)|^2+C(\int_0^1|P_mg'-g'|ds)^2.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}&\int_0^1\Big|(B(\phi_s^m(g))-B(\phi_s(g),\phi_s^m(g)-\phi_s(g))\Big|ds\\ \le&\int_0^1|\mathfrak{b}(\phi_s^m(g)-\phi_s(g),\phi_s(g),\phi_s^m(g)-\phi_s(g)|ds\\ \le&\int_0^1|\phi_s^m(g)-\phi_s(g)|\|\phi_s^m(g)-\phi_s(g)\|\|\phi_s(g)\|ds\\&+\int_0^1|\phi_s^m(g)-\phi_s(g)|^{\frac12}\|\phi_s^m(g)-\phi_s(g)\|^{\frac32}|\partial_z\phi_s(g)|ds\\ \le&\int_0^1\|\phi_s^m(g)-\phi_s(g)\|^2ds\\&+C\int_0^1|\phi_s^m(g)-\phi_s(g)|^2\Big(\|\phi_s(g)\|^2+|\partial_z\phi_s(g)|^4\Big)ds.\end{aligned}\tag{4.29}$$

Then, we obtain

$$\begin{aligned}&\sup_{0\le t\le 1}|\phi_t^m(g)-\phi_t(g)|^2+2\int_0^1\|\phi_s^m(g)-\phi_s(g)\|^2ds\\ \le& C(\int_0^1|P_mg'(s)-g'(s)|ds)^2+C\int_0^1|\phi_s^m(g)-\phi_s(g)|^2\Big(\|\phi_s(g)\|^2+|\partial_z\phi_s(g)|^4\Big)ds.\end{aligned}$$

By Gronwall's inequality, the bounds (4.25), (4.26), (4.27) and letting $m\to\infty$, we obtain the limit (4.24). ∎

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