

## ON THE 3D EULER EQUATIONS WITH CORIOLIS FORCE IN BORDERLINE BESOV SPACES\*

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**Abstract.** We consider the 3D Euler equations with Coriolis force (EC) in the whole space. We show long-time solvability in Besov spaces for high speed of rotation  $\Omega$  and arbitrary initial data. For that, we obtain  $\Omega$ -uniform estimates and a blow-up criterion of BKM type in our framework. Our initial data class is larger than previous ones considered for (EC) and covers borderline cases of the regularity. The uniqueness of solutions is also discussed.

**Keywords.** Euler equations; Coriolis force; Long-time solvability; Blow up; Besov-spaces.

**AMS subject classifications.** 35Q31; 76U05; 76B03; 35A07; 42B35.

### 1. Introduction

We consider the free incompressible Euler equations with Coriolis force

$$\begin{cases} \frac{\partial u}{\partial t} + \mathbb{P}\Omega e_3 \times u + \mathbb{P}(u \cdot \nabla)u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^3 \end{cases}, \quad (1.1)$$

where  $u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$  stands for the velocity field,  $\mathbb{P} = (\delta_{jk} + R_j R_k)_{1 \leq j, k \leq 3}$  is the Leray–Helmholtz projection and  $R_j$  denotes the  $j$ -th Riesz transform. The Coriolis parameter  $\Omega \in \mathbb{R}$  corresponds to twice the speed of rotation around the vertical unit vector  $e_3 = (0, 0, 1)$ . The initial velocity is denoted by  $u_0 = u_0(x) = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$  and satisfies the compatibility condition  $\nabla \cdot u_0 = 0$ . The reader is referred to the book [12] for more details about the physical model. Throughout the paper, we denote spaces of scalar and vector functions abusively in the same way; for example, we write  $u_0 \in H^s(\mathbb{R}^3)$  instead of  $u_0 \in (H^s(\mathbb{R}^3))^3$ .

The system (1.1) has been studied by several authors in the case  $\Omega = 0$  that corresponds to the classical Euler equations (E). In what follows we give a brief review of some of these results. In the framework of Sobolev spaces, Kato [20] showed that (E) has a unique local-in-time solution  $u \in C([0, T]; H^s(\mathbb{R}^3)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^3))$  for  $u_0 \in H^s(\mathbb{R}^3)$  with an integer  $s \geq 3$  where  $T = T(\|u_0\|_{H^s(\mathbb{R}^3)})$ . In [21], Kato and Ponce proved that if  $s > 2/p + 1$ ,  $1 < p < \infty$  and  $u_0 \in H_p^s(\mathbb{R}^2)$ , then there exists a unique 2D global solution  $u \in C([0, \infty); H_p^s(\mathbb{R}^2))$ . Later, in [22] they considered  $n \geq 2$  and proved that for  $s > n/p + 1$ ,  $1 < p < \infty$  and  $u_0 \in H_p^s(\mathbb{R}^n)$ , there exist  $T > 0$  and a unique solution  $u \in C([0, T]; H_p^s(\mathbb{R}^n)) \cap C^1([0, T]; H_p^{s-1}(\mathbb{R}^n))$ . Temam [29] extended the results of Kato [20] to  $H^m$  and  $W^{m,p}$  in bounded domains (see also Ebin–Marsden [15] and Bourguignon–Brezis [6]). For existence and uniqueness results in Holder  $C^{k,\gamma}$  and Triebel–Lizorkin  $F_{p,q}^s$  spaces, the reader is referred to [11] and [7, 8], respectively.

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In the context of Besov spaces, Chae [9] and Zhou [31] proved that (E) has a unique solution  $u \in C([0, T]; B_{p,1}^{n/p+1}(\mathbb{R}^n))$  for  $1 < p < \infty$  and  $n \geq 3$  (see also [30] for  $n = 2$ ). After, the borderline cases  $p = \infty$  [25] and  $p = 1$  [26] was considered by Pak and Park. Takada [28] showed existence-uniqueness in Besov type spaces based on weak- $L^p$  with  $1 < p < \infty$  and  $n \geq 3$ . The exponent  $s = \frac{n}{p} + 1$  is critical for (E) in  $H_p^s$  and  $B_{p,q}^s$ -spaces. In fact, Bourgain and Li [5] showed that (E) is ill-posed in  $H_p^s$  and  $B_{p,q}^{n/p+1}$  for  $1 \leq p < \infty$ ,  $1 < q \leq \infty$  and  $n = 2, 3$ . So, it is natural to consider  $q = 1$  when  $s = \frac{n}{p} + 1$ . The critical case is also of special interest because the regularity index  $s - 1 = \frac{n}{p}$  of the vorticity  $\nabla \times u$  corresponds to a critical case of Sobolev type embeddings. Motivated by the symbol  $>$ , the case  $s > \frac{n}{p} + 1$  has been named in the literature as supercritical.

For  $\Omega \neq 0$ , Dutrifoy [14] showed long-time existence of solutions for system (1.1) with lower bound on the existence-time  $T_\Omega \gtrsim \log \log |\Omega|$  provided that  $|\Omega|$  is large enough and  $u_0$  belongs to a certain Sobolev type class. Also, Dutrifoy [13] and Charve [10] obtained analogous results for quasigeostrophic systems. Recently, for  $s > s_0 = \frac{3}{2} + 1$  and  $u_0 \in H^s(\mathbb{R}^3)$ , Koh, Lee and Takada [23] proved that there exists a unique local in time solution  $u$  for system (1.1) in the class  $C([0, T]; H^s(\mathbb{R}^3)) \cap C^1([0, T]; H^{s-1}(\mathbb{R}^3))$ . Moreover, assuming that  $s > s_1 = \frac{5}{2} + 1$ , they showed that their solutions can be extended to long-time intervals  $[0, T_\Omega]$  provided that the speed of rotation is large enough. For the viscous case, we refer the reader to the works [1, 2, 12, 18] for global well-posedness in Sobolev spaces with  $|\Omega|$  large enough and to the papers [16, 19] (and their references) for results about global well-posedness with  $\Omega$ -uniform smallness condition on initial data in different types of critical spaces (e.g., in Fourier Besov spaces).

In view of the previous results for system (1.1) and (E), it is natural to wonder about the borderline cases  $s_0$  and  $s_1$ . In this paper we extend the results of [23] by treating these two cases in the framework of Besov spaces. To be more precise, we consider the critical regularity  $s_0$  and show local-in-time existence and uniqueness of solutions for initial data in the critical Besov space  $B_{2,1}^{s_0}$  with smallness condition on the existence-time uniformly in  $\Omega \in \mathbb{R}$ . After, for large Coriolis parameter  $|\Omega|$ , we obtain long-time solvability of system (1.1) in  $B_{2,1}^{s_1}$  in the borderline case  $s = s_1$ . It is worth to observe that  $H^s \subset B_{2,1}^{5/2}$  and  $H^s \subset B_{2,1}^{7/2}$  for  $s > \frac{3}{2} + 1$  and  $s > \frac{5}{2} + 1$ , respectively, and so our result provides a larger class for both local and long-time solvability of system (1.1).

Our main result reads as follows.

**THEOREM 1.1.**

- (i) Let  $u_0 \in B_{2,1}^{5/2}(\mathbb{R}^3)$  satisfy  $\nabla \cdot u_0 = 0$ . There exists  $T = T(\|u_0\|_{B_{2,1}^{5/2}}) > 0$  such that system (1.1) has a unique solution  $u \in C([0, T]; B_{2,1}^{5/2}(\mathbb{R}^3)) \cap C^1([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3))$ , for all  $\Omega \in \mathbb{R}$ .
- (ii) Let  $0 < T < \infty$  and  $u_0 \in B_{2,1}^{7/2}(\mathbb{R}^3)$  be such that  $\nabla \cdot u_0 = 0$ . There exists  $\Omega_0 = \Omega_0(T, \|u_0\|_{B_{2,1}^{7/2}}) > 0$  such that system (1.1) has a unique solution  $u \in C([0, T]; B_{2,1}^{7/2}(\mathbb{R}^3)) \cap C^1([0, T]; B_{2,1}^{5/2}(\mathbb{R}^3))$  provided that  $|\Omega| \geq \Omega_0$ .

Considering  $\Omega = 0$ , item (i) recovers the local existence result by Chae [9] and Zhou [31] for Euler equations in  $B_{p,1}^{n/p+1}(\mathbb{R}^n)$  in the case  $p = 2$  and  $n = 3$ . Assuming further regularity on the initial data, item (ii) shows that local solutions can be extended to arbitrary large time  $T > 0$  provided that  $|\Omega|$  is large enough and so it resembles results for the 2D Euler equations (see [9, 30]). In fact, we recall that existence of smooth solutions for the 3D Euler equations is an outstanding open problem. Long-time solvability type

results for system (1.1) with arbitrary data show a smoothing effect connected to the speed of rotation  $\Omega$  (see [12]).

Finally, we comment on some technical points in our results. The general strategy of this paper consists in three basic steps: approximation scheme; *a priori*  $\Omega$ -uniform estimates and passing to the limit for obtaining local-in time solutions; blow-up criterion and long-time solvability. This is the same one employed by [23] in  $H^s$ -spaces however here we need to carry out the necessary estimates in the borderline Besov spaces  $B_{2,1}^{s_0}$  and  $B_{2,1}^{s_1}$ . In order to pass the limit in the approximation scheme  $\{u^\delta\}_{\delta>0}$ , the authors of [23] relied on the Hilbert structure of  $H^s$ -spaces. Since our setting has not such property, we need to control  $u^\delta$  by means of estimates involving localization and  $B_{2,1}^s$ -norms (see, e.g., Lemma 3.3, Proposition 3.1 and proof of Theorem 1.1). In order to cover the endpoints  $s_0$  and  $s_1$  of the ranges in [23], we are inspired by previous results for the Euler equations (E) [9, 25, 31] and consider  $B_{2,1}^s$ -spaces (and the embedding  $B_{2,1}^{3/2} \hookrightarrow L^\infty$  in  $\mathbb{R}^3$ ) that allow us to have  $\nabla u \in L^\infty$  for  $s = s_0$  (which is not true in  $H^s$ ) and control globally in time  $U(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$  for large  $\Omega$  when  $s = s_1$ . The quantity  $U(t)$  is used to derive a blow-up criterion and obtain long-time solutions. Also, we show Lemma 4.1 that deals with the time-continuity of weak solutions for system (1.1) and is useful to prove time-regularity of solutions obtained as limit of the approximation scheme. For that matter, we extend [26, Lemma 2.1] (that considered solutions of (E) in  $B_{1,1}^{d+1}(\mathbb{R}^d)$ ) to the Euler Coriolis equations in  $B_{p,1}^{3/p+1}(\mathbb{R}^3)$  with  $1 \leq p < \infty$ .

The plan of this paper is as follows. The next section is devoted to some preliminaries about product and commutator estimates in Besov spaces and projection operators linked to the Coriolis term. In Section 3, we deal with the approximation scheme  $\{u^\delta\}_{\delta>0}$  and show local existence on  $[0, T]$  with  $T > 0$  independent of  $\delta$  and  $\Omega$ . The proof of Theorem 1.1 is given in Section 4 through three subsections: item (i) in Subsection 4.1, blow-up criterion in Subsection 4.2, and item (ii) in Subsection 4.3.

## 2. Function spaces and projection operators

This section is devoted to some preliminaries about Besov spaces. We refer the reader to [4] for more details on these spaces and their properties. Also, we recall two projection operators that will be useful for our purposes.

Let  $\mathcal{S}(\mathbb{R}^3)$  and  $\mathcal{S}'(\mathbb{R}^3)$  stand for the Schwartz class and the space of tempered distributions, respectively. Let  $\widehat{f}$  denote the Fourier transform of  $f \in \mathcal{S}'$ . Consider a nonnegative radial function  $\phi_0 \in \mathcal{S}(\mathbb{R}^3)$  satisfying  $0 \leq \widehat{\phi}_0(\xi) \leq 1$  for all  $\xi \in \mathbb{R}^3$ ,  $\text{supp } \widehat{\phi}_0 \subset \{\xi \in \mathbb{R}^3 : \frac{1}{2} \leq |\xi| \leq 2\}$  and

$$\sum_{j \in \mathbb{Z}} \widehat{\phi}_j(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^3 \setminus \{0\},$$

where  $\phi_j(x) := 2^{3j} \phi_0(2^j x)$ . For  $k \in \mathbb{Z}$ , we define the function  $S_k \in \mathcal{S}$  as

$$\widehat{S}_k(\xi) = 1 - \sum_{j \geq k+1} \widehat{\phi}_j(\xi)$$

and denote  $\psi = S_0$ . For  $f \in \mathcal{S}'(\mathbb{R}^3)$ , the Littlewood–Paley operator  $\Delta_j$  is defined by  $\Delta_j f := \phi_j * f$ .

Let  $s \in \mathbb{R}$  and  $1 \leq p, q \leq \infty$  and let  $\mathcal{P}$  denote the set of polynomials with 3 variables. The homogeneous Besov space  $\dot{B}_{p,q}^s(\mathbb{R}^3)$  is the set of all  $f \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}$  such that

$$\|f\|_{\dot{B}_{p,q}^s} := \|\{2^{sj} \|\Delta_j f\|_{L^p}\}_{j \in \mathbb{Z}}\|_{l^q(\mathbb{Z})} < \infty.$$

The inhomogeneous version of  $\dot{B}_{p,q}^s$ , denoted by  $B_{p,q}^s(\mathbb{R}^3)$ , is defined as the set of all  $f \in \mathcal{S}'(\mathbb{R}^3)$  such that

$$\|f\|_{B_{p,q}^s} := \|\{2^{sj} \|\Delta_j f\|_{L^p}\}_{j \in \mathbb{N}}\|_{l^q(\mathbb{N})} + \|\psi * f\|_{L^p}.$$

The pairs  $(\dot{B}_{p,q}^s, \|\cdot\|_{\dot{B}_{p,q}^s})$  and  $(B_{p,q}^s, \|\cdot\|_{B_{p,q}^s})$  are Banach spaces. For  $s > 0$ , we have the equivalence

$$\|f\|_{B_{p,q}^s} \sim \|f\|_{\dot{B}_{p,q}^s} + \|f\|_{L^p}. \quad (2.1)$$

LEMMA 2.1 (Bernstein inequality). *Assume that  $f \in L^p$ ,  $1 \leq p \leq \infty$ , and  $\text{supp } \widehat{f} \subset \{\xi \in \mathbb{R}^3 : 2^{j-2} \leq |\xi| < 2^j\}$ . Then there exists a constant  $C = C(k) > 0$  such that*

$$C^{-1}2^{jk} \|f\|_{L^p} \leq \|D^k f\|_{L^p} \leq C2^{jk} \|f\|_{L^p}.$$

REMARK 2.1. As a consequence of the above lemma, we have the following equivalence

$$\|D^k f\|_{\dot{B}_{p,q}^s} \sim \|f\|_{\dot{B}_{p,q}^{s+k}}. \quad (2.2)$$

We also recall the estimate (see, e.g., [28])

$$\|f\|_{L^\infty} \leq C \|f\|_{B_{p,q}^s}, \quad (2.3)$$

where  $s > n/p$  with  $1 \leq p, q \leq \infty$ , or  $s = n/p$  with  $1 \leq p \leq \infty$  and  $q = 1$ . Thus, for  $s > n/p + 1$  with  $1 \leq p, q \leq \infty$  or  $s = n/p + 1$  with  $1 \leq p \leq \infty$  and  $q = 1$ , we have the estimates

$$\|\nabla f\|_{L^\infty} \leq \|\nabla f\|_{B_{p,q}^{s-1}} \leq \|f\|_{B_{p,q}^s}. \quad (2.4)$$

The following lemma contains product estimates in the framework of Besov spaces (see [9]).

LEMMA 2.2. *Let  $s > 0$ ,  $1 \leq p, q \leq \infty$ ,  $1 \leq p_1, p_2 \leq \infty$  and  $1 \leq r_1, r_2 \leq \infty$  satisfy  $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r_1} + \frac{1}{r_2}$ . Then there exists a universal constant  $C > 0$  such that*

$$\begin{aligned} \|fg\|_{\dot{B}_{p,q}^s} &\leq C(\|f\|_{\dot{B}_{p_1,q}^s} \|g\|_{L^{p_2}} + \|g\|_{\dot{B}_{r_1,q}^s} \|f\|_{L^{r_2}}), \\ \|fg\|_{B_{p,q}^s} &\leq C(\|f\|_{B_{p_1,q}^s} \|g\|_{L^{p_2}} + \|g\|_{B_{r_1,q}^s} \|f\|_{L^{r_2}}). \end{aligned}$$

In the next two lemmas we recall estimates in  $\dot{B}_{p,q}^s$  and  $B_{p,q}^s$  for the commutator (see [9, 28])

$$[v \cdot \nabla, \Delta_j]u = v \cdot \nabla(\Delta_j u) - \Delta_j(v \cdot \nabla u).$$

LEMMA 2.3. *Let  $1 < p < \infty$  and  $1 \leq q \leq \infty$ .*

(i) *Let  $s > 0$ ,  $v \in \dot{B}_{p,q}^s(\mathbb{R}^n)$  with  $\nabla v \in L^\infty(\mathbb{R}^n)$  and  $\nabla \cdot v = 0$ , and  $\theta \in \dot{B}_{p,q}^s(\mathbb{R}^n)$  with  $\nabla \theta \in L^\infty(\mathbb{R}^n)$ . Then, there exists a universal constant  $C > 0$  such that*

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \|[v \cdot \nabla, \Delta_j] \theta\|_{L^p}^q \right)^{1/q} \leq C \left( \|\nabla v\|_{L^\infty} \|\theta\|_{\dot{B}_{p,q}^s} + \|\nabla \theta\|_{L^\infty} \|v\|_{\dot{B}_{p,q}^s} \right).$$

(ii) Let  $s > -1$ ,  $v \in \dot{B}_{p,q}^{s+1}(\mathbb{R}^n)$  with  $\nabla v \in L^\infty(\mathbb{R}^n)$  and  $\nabla \cdot v = 0$ , and  $\theta \in \dot{B}_{p,q}^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ . Then, there exists a universal constant  $C > 0$  such that

$$\left( \sum_{j \in \mathbb{Z}} 2^{sjq} \|[v \cdot \nabla, \Delta_j] \theta\|_{L^p}^q \right)^{1/q} \leq C \left( \|\nabla v\|_{L^\infty} \|\theta\|_{\dot{B}_{p,q}^s} + \|\theta\|_{L^\infty} \|v\|_{\dot{B}_{p,q}^{s+1}} \right).$$

LEMMA 2.4. Let  $1 < p < \infty$  and let  $s > 3/p + 1$  with  $1 \leq q \leq \infty$  or  $s = 3/p + 1$  with  $q = 1$ . Then, there exists a constant  $C > 0$  such that

$$\left( \sum_{j \in \mathbb{Z}} 2^{jq} \|(S_{j-2} u \cdot \nabla) \Delta_j u - \Delta_j (u \cdot \nabla) u\|_{L^p}^q \right)^{1/q} \leq C \|\nabla u\|_{L^\infty} \|u\|_{B_{p,q}^s},$$

for all  $u \in B_{p,q}^s(\mathbb{R}^3)$  with  $\nabla \cdot u = 0$ .

In order to handle the Coriolis term, we will need the following projection operators  $P_\pm : L^2(\mathbb{R}^3)^3 \rightarrow L^2(\mathbb{R}^3)^3$  given by

$$P_\pm v := \frac{1}{2} \left( \mathbb{P}v \pm i \frac{D}{|D|} \times v \right),$$

where  $\frac{D}{|D|} \times$  is defined by means of the Fourier transform as  $(\frac{D}{|D|} \times v)^\wedge(\xi) := \frac{\xi}{|\xi|} \times \widehat{v}(\xi)$ .

The next lemma contains basic properties of  $P_\pm$  and can be found in [14, 23].

LEMMA 2.5. The projections  $P_\pm$  satisfy  $P_\pm \mathbb{P} = P_\pm$ . Moreover, if  $\nabla \cdot v = 0$  we have that  $v = P_+ v + P_- v$ ,  $\mathbb{P}(e_3 \times v) = -i \frac{D_3}{|D|} (P_+ v - P_- v)$ ,  $P_\pm P_\pm = P_\pm$ , and  $P_+ P_- = 0$ .

### 3. Approximation scheme

Let  $u_0$  be the initial velocity in system (1.1). For  $0 < \delta < 1$ , we consider the approximate parabolic problem

$$\begin{cases} \frac{\partial u^\delta}{\partial t} - \delta \Delta u^\delta + \mathbb{P} \Omega e_3 \times u^\delta + \mathbb{P}(u^\delta \cdot \nabla) u^\delta = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ \nabla \cdot u^\delta = 0 & \text{in } \mathbb{R}^3 \times (0, \infty), \\ u^\delta(x, 0) = u_0(x) & \text{in } \mathbb{R}^3. \end{cases} \quad (3.1)$$

We are going to show that the above problem has a solution for each  $\delta > 0$  in a suitable class involving Besov spaces. For that matter, first we recall some estimates for the heat semigroup  $\{e^{t\Delta}\}_{t \geq 0}$  in  $B_{p,q}^s$  (see, e.g., [24]).

LEMMA 3.1. Let  $s_0 \leq s_1$  and  $1 \leq p, q \leq \infty$ . Then there exists a constant  $C > 0$  (independent of  $p, q$  and  $t > 0$ ) such that

$$\|e^{t\Delta} f\|_{B_{p,q}^{s_1}} \leq C(1 + t^{-\frac{1}{2}(s_1 - s_0)}) \|f\|_{B_{p,q}^{s_0}},$$

for all  $f \in B_{p,q}^{s_0}(\mathbb{R}^3)$ .

We start by showing estimates for the bilinear term of the mild formulation for system (3.1).

LEMMA 3.2. Let  $0 < T < \infty$ ,  $0 < \delta < 1$  and  $1 < p < \infty$ .

(i) There exists  $C > 0$  such that

$$\begin{aligned} & \sup_{0 < t < T} \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P}(u(\tau) \cdot \nabla) v(\tau) \, d\tau \right\|_{B_{p,1}^{3/p}} \\ & \leq C \sup_{0 < t < T} \|u(t)\|_{B_{p,1}^{3/p}} \|v\|_{L^1(0,T;B_{p,1}^{3/p+1})}, \end{aligned} \quad (3.2)$$

for all  $u \in C([0,T];B_{p,1}^{3/p}(\mathbb{R}^3))$  and  $v \in L^1(0,T;B_{p,1}^{3/p+1}(\mathbb{R}^3))$ .

(ii) Let  $k=1,2$ . There exists  $C > 0$  such that

$$\begin{aligned} & \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P}(u(\tau) \cdot \nabla) v(\tau) \, d\tau \right\|_{L^1(0,T;B_{p,1}^{3/p+k})} \\ & \leq C \left( T + T^{\frac{1}{2}} \delta^{-\frac{1}{2}} \right) \sup_{0 < t < T} \|u(t)\|_{B_{p,1}^{3/p+k-1}} \|v\|_{L^1(0,T;B_{p,1}^{3/p+k})}, \end{aligned} \quad (3.3)$$

for all  $u \in C([0,T];B_{p,1}^{3/p}(\mathbb{R}^3))$  with  $\nabla \cdot u = 0$  and  $v \in L^1(0,T;B_{p,1}^{3/p+k}(\mathbb{R}^3))$ .

*Proof.* For  $1 < p < \infty$ , we have that  $\|e^{\delta t \Delta}\|_{B_{p,1}^{3/p} \rightarrow B_{p,1}^{3/p}} \leq 1$  and  $\mathbb{P}$  is bounded in  $B_{p,1}^{3/p}$ . So, we can estimate

$$\left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P}(u(\tau) \cdot \nabla) v(\tau) \, d\tau \right\|_{B_{p,1}^{3/p}} \leq C \int_0^t \|(u(\tau) \cdot \nabla) v(\tau)\|_{B_{p,1}^{3/p}} \, d\tau.$$

From Lemmas 2.2 and 2.1, it follows that

$$\|(u(\tau) \cdot \nabla) v(\tau)\|_{B_{p,1}^{3/p}} \leq C \|u(\tau)\|_{B_{p,1}^{3/p}} \|v(\tau)\|_{B_{p,1}^{3/p+1}}$$

and then

$$\begin{aligned} & \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P}(u(\tau) \cdot \nabla) v(\tau) \, d\tau \right\|_{B_{p,1}^{3/p}} \leq C \int_0^t \|u(\tau)\|_{B_{p,1}^{3/p}} \|v(\tau)\|_{B_{p,1}^{3/p+1}} \, d\tau \\ & \leq C \sup_{0 < t < T} \|u(t)\|_{B_{p,1}^{3/p}} \|v\|_{L^1(0,T;B_{p,1}^{3/p+1})}, \end{aligned}$$

for all  $0 < t < T$ , which gives estimate (3.2).

Next, we turn to item (ii). By Minkowski inequality and Lemmas 3.1, 2.2 and 2.1, we have that

$$\begin{aligned} & \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P}(u(\tau) \cdot \nabla) v(\tau) \, d\tau \right\|_{B_{p,1}^{3/p+k}} \\ & \leq C \int_0^t \left\{ 1 + \delta^{-\frac{1}{2}} (t-\tau)^{-\frac{1}{2}} \right\} \left( \|u(\tau)\|_{B_{p,1}^{3/p}} \|v(\tau)\|_{B_{p,1}^{3/p+k}} + \|u(\tau)\|_{B_{p,1}^{3/p+k-1}} \|v(\tau)\|_{B_{p,1}^{3/p+1}} \right) \, d\tau \\ & \leq C \int_0^t \left\{ 1 + \delta^{-\frac{1}{2}} (t-\tau)^{-\frac{1}{2}} \right\} \|u(\tau)\|_{B_{p,1}^{3/p+k-1}} \|v(\tau)\|_{B_{p,1}^{3/p+k}} \, d\tau \\ & \leq C \sup_{0 < t < T} \|u(t)\|_{B_{p,1}^{3/p+k-1}} \left\{ \|v\|_{L^1(0,T;B_{p,1}^{3/p+k})} + \delta^{-\frac{1}{2}} \int_0^t (t-\tau)^{-\frac{1}{2}} \|v(\tau)\|_{B_{p,1}^{3/p+k}} \, d\tau \right\}, \end{aligned} \quad (3.4)$$

for all  $0 < t < T$  and  $k=1,2$ . We can now compute the norm  $\|\cdot\|_{L^1(0,T)}$  in estimate (3.4) to obtain

$$\left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P}(u(\tau) \cdot \nabla) v(\tau) \, d\tau \right\|_{L^1(0,T;B_{p,1}^{3/p+k})}$$

$$\begin{aligned}
&\leq C \sup_{0 < t < T} \|u(t)\|_{B_{p,1}^{3/p+k-1}} \left\{ T \|v\|_{L^1(0,T;B_{p,1}^{3/p+k})} \right. \\
&\quad \left. + \delta^{-\frac{1}{2}} \int_0^T \|v(\tau)\|_{B_{p,1}^{3/p+k}} \int_\tau^T (t-\tau)^{-\frac{1}{2}} dt d\tau \right\} \\
&\leq C \left( T + T^{\frac{1}{2}} \delta^{-\frac{1}{2}} \right) \sup_{0 < t < T} \|u(t)\|_{B_{p,1}^{3/p+k-1}} \|v\|_{L^1(0,T;B_{p,1}^{3/p+k})},
\end{aligned}$$

which yields the desired estimate.  $\square$

Before proceeding, we recall that  $AC([0,T];X)$  denotes the set of all  $X$ -valued absolutely continuous functions on  $[0,T]$ . The next lemma ensures the existence of strong solution for system (3.1). The proof follows essentially the same steps of [23, Lemma 3.1.] but using estimates in Besov spaces instead of Sobolev spaces.

**LEMMA 3.3.** *Let  $1 < p < \infty$ ,  $\delta \in (0,1)$  and  $\Omega \in \mathbb{R}$ . Assume that  $u_0 \in B_{p,1}^{3/p+1}(\mathbb{R}^3)$  and  $\nabla \cdot u_0 = 0$ . Then there exists a positive time  $T_{\delta,\Omega} = T(\delta, |\Omega|, \|u_0\|_{B_{p,1}^{3/p+1}})$  such that system (3.1) has a unique strong solution  $u^\delta$  satisfying*

$$u^\delta \in C([0, T_{\delta,\Omega}]; B_{p,1}^{3/p+1}(\mathbb{R}^3)) \cap AC([0, T_{\delta,\Omega}]; B_{p,1}^{3/p}(\mathbb{R}^3)) \cap L^1(0, T_{\delta,\Omega}; B_{p,1}^{3/p+2}(\mathbb{R}^3)). \quad (3.5)$$

*Proof.* Firstly, we consider the mild formulation for system (3.1)

$$u^\delta(t) = e^{\delta t \Delta} u_0 - \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \Omega e_3 \times u^\delta(\tau) d\tau - \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} (u^\delta(\tau) \cdot \nabla) u^\delta(\tau) d\tau \quad (3.6)$$

and show the existence of a local in time solution. Lemma 3.1 yields the estimate

$$\|e^{\delta t \Delta} f\|_{L^1(0,T;B_{p,1}^{3/p+2})} \leq C(T + T^{\frac{1}{2}} \delta^{-\frac{1}{2}}) \|f\|_{B_{p,1}^{3/p+1}},$$

for all  $f \in B_{p,1}^{3/p+1}$ . Thus, for all  $0 < T < \infty$  we have

$$\sup_{0 \leq t \leq T} \|e^{\delta t \Delta} u_0\|_{B_{p,1}^{3/p+1}} + L_{\delta,T}^{-1} \|e^{\delta t \Delta} u_0\|_{L^1(0,T;B_{p,1}^{3/p+2})} \leq C_0 \|u_0\|_{B_{p,1}^{3/p+1}}, \quad (3.7)$$

where  $C_0 > 0$  is a constant and  $L_{\delta,T} = (T + T^{\frac{1}{2}} \delta^{-\frac{1}{2}})$ .

Consider the map

$$\mathcal{B}(u^\delta)(t) = e^{\delta t \Delta} u_0 - \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \Omega e_3 \times u^\delta(\tau) d\tau - \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} (u^\delta(\tau) \cdot \nabla) u^\delta(\tau) d\tau$$

and the complete metric space

$$\begin{aligned}
Z_T := &\left\{ u \in C([0,T]; B_{p,1}^{3/p+1}(\mathbb{R}^3)) \cap L^1(0,T; B_{p,1}^{3/p+2}(\mathbb{R}^3)); \right. \\
&\quad \left. \nabla \cdot u = 0 \text{ and } \|u\|_{Z_T} \leq 2C_0 \|u_0\|_{B_{p,1}^{3/p+1}} \right\}
\end{aligned}$$

whose norm is given by

$$\|u\|_{Z_T} := \sup_{0 \leq t \leq T} \|u(t)\|_{B_{p,1}^{3/p+1}} + L_{\delta,T}^{-1} \|u\|_{L^1(0,T;B_{p,1}^{3/p+2})}.$$

We claim that the map  $\mathcal{B}$  is a contraction map on  $Z_T$  for small  $T > 0$ .

In fact, using that  $\|e^{\delta t \Delta}\|_{B_{p,1}^{3/p+1} \rightarrow B_{p,1}^{3/p+1}} \leq 1$  and  $\mathbb{P}$  is bounded in  $B_{p,1}^{3/p+1}$  for  $1 < p < \infty$ , we have that

$$\begin{aligned} & \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \Omega e_3 \times u(\tau) \, d\tau \right\|_{B_{p,1}^{3/p+1}} \\ & \leq C|\Omega| \int_0^t \|e_3 \times u(\tau)\|_{B_{p,1}^{3/p+1}} \, d\tau \leq C|\Omega|T \sup_{0 \leq t \leq T} \|u(t)\|_{B_{p,1}^{3/p+1}}. \end{aligned}$$

Taking the supremum over  $t \in [0, T]$ , we get a constant  $C > 0$  such that

$$\sup_{0 \leq t \leq T} \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \Omega e_3 \times u(\tau) \, d\tau \right\|_{B_{p,1}^{3/p+1}} \leq C|\Omega|T \sup_{0 \leq t \leq T} \|u(t)\|_{B_{p,1}^{3/p+1}}, \quad (3.8)$$

for all  $u \in C([0, T]; B_{p,1}^{3/p+1}(\mathbb{R}^3))$ . Similarly,

$$\begin{aligned} & \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \Omega e_3 \times u(\tau) \, d\tau \right\|_{B_{p,1}^{3/p+2}} \leq C|\Omega| \int_0^t \|e_3 \times u(\tau)\|_{B_{p,1}^{3/p+2}} \, d\tau \\ & \leq C|\Omega| \|u\|_{L^1(0, T; B_{p,1}^{3/p+2})}, \end{aligned} \quad (3.9)$$

for all  $t \in [0, T]$ . An integration of estimate (3.9) over  $[0, T]$  yields the estimate

$$\left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \Omega e_3 \times u(\tau) \, d\tau \right\|_{L^1(0, T; B_{p,1}^{3/p+2})} \leq C|\Omega|T \|u\|_{L^1(0, T; B_{p,1}^{3/p+2})}, \quad (3.10)$$

for all  $u \in L^1(0, T; B_{p,1}^{3/p+2}(\mathbb{R}^3))$ .

Next we can apply estimates (3.8) and (3.10) and Lemma 3.2 in order to estimate

$$\begin{aligned} \|\mathcal{B}(u^\delta) - \mathcal{B}(v^\delta)\|_{Z_T} &= \left\| \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \Omega e_3 \times (u^\delta(\tau) - v^\delta(\tau)) \, d\tau \right. \\ & \quad + \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} \{ (u^\delta(\tau) - v^\delta(\tau)) \cdot \nabla \} u^\delta(\tau) \, d\tau \\ & \quad \left. + \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P} (v^\delta(\tau) \cdot \nabla) (u^\delta(\tau) - v^\delta(\tau)) \, d\tau \right\|_{Z_T} \\ & \leq C_1 |\Omega| T \|u^\delta - v^\delta\|_{Z_T} + C_2 L_{\delta, T} (\|u^\delta\|_{Z_T} + \|v^\delta\|_{Z_T}) \|u^\delta - v^\delta\|_{Z_T} \\ & \leq \left\{ C_1 |\Omega| T + 4C_0 C_2 L_{\delta, T} \|u_0\|_{B_{p,1}^{3/p+1}} \right\} \|u^\delta - v^\delta\|_{Z_T}, \end{aligned} \quad (3.11)$$

for all  $u^\delta, v^\delta \in Z_T$ . Moreover, using estimates (3.7) and (3.11) with  $v^\delta = 0$ , we obtain

$$\begin{aligned} \|\mathcal{B}(u^\delta)\|_{Z_T} &\leq \|e^{\delta t \Delta} u_0\|_{Z_T} + \|\mathcal{B}(u^\delta) - \mathcal{B}(0)\|_{Z_T} \\ &\leq C_0 \|u_0\|_{B_{p,1}^{3/p+1}} + \left\{ C_1 |\Omega| T + 4C_0 C_2 L_{\delta, T} \|u_0\|_{B_{p,1}^{3/p+1}} \right\} \|u^\delta\|_{Z_T} \\ &\leq C_0 \|u_0\|_{B_{p,1}^{3/p+1}} \left\{ 1 + 2C_1 |\Omega| T + 8C_0 C_2 L_{\delta, T} \|u_0\|_{B_{p,1}^{3/p+1}} \right\}, \end{aligned} \quad (3.12)$$



for all  $u^\delta \in Z_T$ . Next we choose  $T = T_{\delta, \Omega} = T(\delta, |\Omega|, \|u_0\|_{B_{p,1}^{3/p+1}}) > 0$  such that

$$2C_1|\Omega|T_{\delta, \Omega} + 8C_0C_2\|u_0\|_{B_{p,1}^{3/p+1}} \left( T_{\delta, \Omega} + T_{\delta, \Omega}^{\frac{1}{2}} \delta^{-\frac{1}{2}} \right) < 1. \quad (3.13)$$

Inserting inequality (3.13) into estimates (3.12) and (3.11), we get that  $\mathcal{B}(Z_{T_{\delta, \Omega}}) \subset Z_{T_{\delta, \Omega}}$  and

$$\|\mathcal{B}(u^\delta) - \mathcal{B}(v^\delta)\|_{Z_{T_{\delta, \Omega}}} \leq \frac{1}{2} \|u^\delta - v^\delta\|_{Z_{T_{\delta, \Omega}}}, \text{ for all } u^\delta, v^\delta \in Z_{T_{\delta, \Omega}},$$

which gives the claim. By the Banach Fixed Point Theorem, there exists a unique solution  $u^\delta \in Z_{T_{\delta, \Omega}}$  for equation (3.6).

We claim that  $u^\delta \in Z_{T_{\delta, \Omega}}$  is a strong solution for system (3.1) in the class (3.5). By the above estimates and using that  $u^\delta \in C([0, T]; B_{p,1}^{3/p+1}(\mathbb{R}^3)) \cap L^1(0, T; B_{p,1}^{3/p+2}(\mathbb{R}^3))$ , it is not difficult to see that

$$\mathbb{P}\Omega e_3 \times u^\delta + \mathbb{P}(u^\delta \cdot \nabla) u^\delta \in L^1(0, T_{\delta, \Omega}; B_{p,1}^{3/p+1}(\mathbb{R}^3))$$

and  $\delta \Delta v^\delta \in L^1(0, T_{\delta, \Omega}; B_{p,1}^{3/p}(\mathbb{R}^3))$  where

$$v^\delta(t) := - \int_0^t e^{\delta(t-\tau)\Delta} \mathbb{P}\{\Omega e_3 \times u^\delta(\tau) + (u^\delta(\tau) \cdot \nabla) u^\delta(\tau)\} d\tau.$$

Thus,  $\partial_t v^\delta \in L^1(0, T_{\delta, \Omega}; B_{p,1}^{3/p}(\mathbb{R}^3))$  and then  $v^\delta \in AC([0, T_{\delta, \Omega}]; B_{p,1}^{3/p}(\mathbb{R}^3))$ . Moreover,  $e^{\delta t \Delta} u_0 \in AC([0, T_{\delta, \Omega}]; B_{p,1}^{3/p}(\mathbb{R}^3))$ . By standard arguments (see Kato [20] and Pazy [27]), we obtain the desired claim. For more details see [23]. The uniqueness follows from the fact that  $u^\delta$  is the unique solution for equation (3.6) in the class  $Z_{T_{\delta, \Omega}}$ .  $\square$

In what follows, we prove that there exists  $T > 0$  independent of  $\delta \in (0, 1)$  and  $\Omega \in \mathbb{R}$  such that the solution  $u^\delta$  exists on  $[0, T]$ . For that, we need some *a priori* uniform estimates for  $u^\delta$  in the space  $B_{2,1}^{5/2}$ .

**PROPOSITION 3.1.** *Assume that  $u_0 \in B_{2,1}^{5/2}(\mathbb{R}^3)$  and  $\nabla \cdot u_0 = 0$ . There exists  $T = T(\|u_0\|_{B_{2,1}^{5/2}}) > 0$  such that system (3.1) has a unique strong solution*

$$u^\delta \in C([0, T]; B_{2,1}^{5/2}(\mathbb{R}^3) \cap AC([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3))),$$

for all  $0 < \delta < 1$  and  $\Omega \in \mathbb{R}$ . Furthermore,  $\{u^\delta\}_{\delta \in (0,1)}$  is bounded in  $C([0, T]; B_{2,1}^{5/2}(\mathbb{R}^3))$ .

*Proof.* Applying the Littlewood–Paley operator  $\Delta_j$  to the first equation in system (3.1), taking the  $L^2$ -norm product with  $\Delta_j u^\delta(t)$ , and using  $\nabla \cdot \Delta_j u^\delta = 0$  and the skew-symmetric of  $e_3 \times$ , we have that

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j u^\delta(t)\|_{L^2}^2 + \delta \langle -\Delta \Delta_j u^\delta(t), \Delta_j u^\delta(t) \rangle_{L^2} = - \langle \Delta_j (u^\delta(t) \cdot \nabla) u^\delta(t), \Delta_j u^\delta(t) \rangle_{L^2}. \quad (3.14)$$

Notice that the second term in the right-hand side of equation (3.14) is non-negative. So, using that

$$\langle (u^\delta(t) \cdot \nabla) \Delta_j u^\delta(t), \Delta_j u^\delta(t) \rangle_{L^2} = 0$$

and recalling the definition of the commutator  $[u^\delta(t) \cdot \nabla, \Delta_j]$ , we get

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j u^\delta(t)\|_{L^2}^2 \leq \langle [u^\delta(t) \cdot \nabla, \Delta_j] u^\delta(t), \Delta_j u^\delta(t) \rangle_{L^2}.$$

By the Cauchy–Schwarz inequality, it follows that

$$\frac{d}{dt} \|\Delta_j u^\delta(t)\|_{L^2} \leq \|[u^\delta(t) \cdot \nabla, \Delta_j] u^\delta(t)\|_{L^2}.$$

Multiplying by  $2^{5/2j}$ , applying the  $l^1(\mathbb{Z})$ -norm and Lemma 2.3, we can estimate

$$\begin{aligned} \frac{d}{dt} \|u^\delta(t)\|_{\dot{B}_{2,1}^{5/2}} &= \sum_{j \in \mathbb{Z}} 2^{5/2j} \frac{d}{dt} \|\Delta_j u^\delta(t)\|_{L^2} \\ &\leq \sum_{j \in \mathbb{Z}} 2^{5/2j} \|[u^\delta(t) \cdot \nabla, \Delta_j] u^\delta(t)\|_{L^2} \\ &\leq C \|\nabla u^\delta(t)\|_{L^\infty} \|u^\delta(t)\|_{\dot{B}_{2,1}^{5/2}}. \end{aligned}$$

By Remark 2.1, it follows that

$$\frac{d}{dt} \|u^\delta(t)\|_{\dot{B}_{2,1}^{5/2}} \leq C \|u^\delta(t)\|_{\dot{B}_{2,1}^{5/2}}^2. \quad (3.15)$$

On the other hand, taking the  $L^2$ -norm product with  $u^\delta(t)$  in system (3.1), we arrive at

$$\frac{1}{2} \frac{d}{dt} \|u^\delta(t)\|_{L^2}^2 + \langle -\delta \Delta u^\delta(t), u^\delta(t) \rangle_{L^2} = 0.$$

Above, we have used the skew-symmetric of  $e_3 \times$  and  $\langle (u^\delta(t) \cdot \nabla) u^\delta(t), u^\delta(t) \rangle_{L^2} = 0$  because  $\nabla \cdot u^\delta = 0$ . Then,

$$\frac{d}{dt} \|u^\delta(t)\|_{L^2} \leq 0. \quad (3.16)$$

Denote by  $\|\cdot\|_{B_{2,1}^s}^*$  the equivalent norm  $\|\cdot\|_{L^2} + \|\cdot\|_{\dot{B}_{2,1}^s}$  in  $B_{2,1}^s$  (see the equivalence (2.1)). By inequalities (3.15) and (3.16), we have that

$$\begin{aligned} \frac{d}{dt} \|u^\delta(t)\|_{B_{2,1}^{5/2}}^* &= \frac{d}{dt} \left( \|u^\delta(t)\|_{\dot{B}_{2,1}^{5/2}} + \|u^\delta(t)\|_{L^2} \right) \\ &\leq \frac{d}{dt} \|u^\delta(t)\|_{\dot{B}_{2,1}^{5/2}} \\ &\leq C \|u^\delta(t)\|_{\dot{B}_{2,1}^{5/2}}^2 \\ &\leq C (\|u^\delta(t)\|_{B_{2,1}^{5/2}}^*)^2. \end{aligned} \quad (3.17)$$

Using inequality (3.17) and that  $K_1 \|\cdot\|_{B_{2,1}^s} \leq \|\cdot\|_{B_{2,1}^s}^* \leq K_2 \|\cdot\|_{B_{2,1}^s}$  for some  $K_1, K_2 > 0$ , it follows that

$$\|u^\delta(t)\|_{B_{2,1}^{5/2}} \leq \frac{1}{K_1} \|u^\delta(t)\|_{B_{2,1}^{5/2}}^* \leq \frac{1}{K_1} \frac{\|u_0\|_{B_{2,1}^{5/2}}^*}{1 - C \|u_0\|_{B_{2,1}^{5/2}}^* t} \leq \frac{1}{K_1} \frac{K_2 \|u_0\|_{B_{2,1}^{5/2}}}{1 - CK_2 \|u_0\|_{B_{2,1}^{5/2}} t},$$

for  $0 \leq t < (CK_2 \|u_0\|_{B_{2,1}^{5/2}})^{-1}$ . Taking  $T = T(\|u_0\|_{B_{2,1}^{5/2}}) = (2CK_2 \|u_0\|_{B_{2,1}^{5/2}})^{-1}$  and  $L = 2K_2/K_1$ , we obtain

$$\|u^\delta(t)\|_{B_{2,1}^{5/2}} \leq L \|u_0\|_{B_{2,1}^{5/2}}, \quad \text{for all } t \in [0, T]. \quad (3.18)$$

Notice that  $T > 0$  is independent of  $\delta \in (0, 1)$  and  $\Omega \in \mathbb{R}$ . If  $T_{\delta, \Omega} < T$ , by inequalities (3.13) and (3.18) we can take  $T'_{\delta, \Omega} = T'_{\delta, \Omega}(\|u_0\|_{B_{2,1}^{5/2}}) > 0$  small enough and solve system (3.1) on  $[T_{\delta, \Omega}, T_{\delta, \Omega} + T'_{\delta, \Omega}]$  with the initial value  $u^\delta(T_{\delta, \Omega}) \in B_{2,1}^{5/2}(\mathbb{R}^3)$ . It follows that the solution  $u^\delta$  can be extended to the interval  $[0, T_{\delta, \Omega} + T'_{\delta, \Omega}]$ . Invoking again the same procedure, we can extend  $u^\delta$  (if necessary) to  $[0, T_{\delta, \Omega} + 2T'_{\delta, \Omega}]$ ,  $[0, T_{\delta, \Omega} + 3T'_{\delta, \Omega}]$  and so on, and obtain a solution  $u^\delta$  for system (3.1) on  $[0, T]$  satisfying inequality (3.18).  $\square$

#### 4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 through three subsections.

##### 4.1. Proof of item (i).

*Proof.*

**Existence part.** For  $0 < \delta_1 < \delta_2 < 1$ , we can write

$$\begin{cases} \partial_t(u^{\delta_1} - u^{\delta_2}) - \delta_1 \Delta(u^{\delta_1} - u^{\delta_2}) + (\delta_2 - \delta_1) \Delta u^{\delta_2} = -\mathbb{P}\Omega e_3 \times (u^{\delta_1} - u^{\delta_2}) \\ -\mathbb{P}\{(u^{\delta_1} - u^{\delta_2}) \cdot \nabla\} u^{\delta_1} - \mathbb{P}(u^{\delta_2} \cdot \nabla)(u^{\delta_1} - u^{\delta_2}), \\ \nabla \cdot u^{\delta_1} = \nabla \cdot u^{\delta_2} = 0, \\ (u^{\delta_1} - u^{\delta_2})(0, x) = 0. \end{cases} \quad (4.1)$$

We will show that there exists a limit  $u \in C([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3))$  such that

$$u^\delta(t) \rightarrow u(t) \text{ in } B_{2,1}^{3/2} \text{ uniformly for } t \in [0, T]. \quad (4.2)$$

We start by obtaining estimates in  $B_{2,1}^{3/2}$  for the difference  $u^{\delta_1} - u^{\delta_2}$  uniformly in  $[0, T]$ . Computing the  $L^2$ -inner product of system (4.1) with  $u^{\delta_1} - u^{\delta_2}$ , and afterwards using the skew-symmetry of  $(e_3 \times \cdot)$ ,  $\nabla \cdot (u^{\delta_1} - u^{\delta_2}) = 0$ , Hölder's inequality, and Remark 2.1, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2}^2 \\ & \leq (\delta_2 - \delta_1) \|-\Delta u^{\delta_2}(t)\|_{L^2} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2} + \|\nabla u^{\delta_1}(t)\|_{L^\infty} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2}^2 \\ & \leq C\delta_2 \|-\Delta u^{\delta_2}(t)\|_{L^2} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2} + \|u^{\delta_1}(t)\|_{B_{2,1}^{5/2}} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2}^2. \end{aligned}$$

Integrating over  $(0, t)$  and using inequality (3.18), we arrive at the estimate

$$\begin{aligned} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2} & \leq C\delta_2 \int_0^t \|-\Delta u^{\delta_2}(\tau)\|_{L^2} d\tau + C \int_0^t \|u^{\delta_1}(\tau)\|_{B_{2,1}^{5/2}} \|(u^{\delta_1} - u^{\delta_2})(\tau)\|_{L^2} d\tau \\ & \leq C\delta_2 T \|u^{\delta_2}\|_{L^\infty(0, T; B_{2,1}^{5/2})} + C \int_0^t \|u^{\delta_1}(\tau)\|_{B_{2,1}^{5/2}} \|(u^{\delta_1} - u^{\delta_2})(\tau)\|_{L^2} d\tau \\ & \leq C\delta_2 T \|u_0\|_{B_{2,1}^{5/2}} + C \|u_0\|_{B_{2,1}^{5/2}} \int_0^t \|(u^{\delta_1} - u^{\delta_2})(\tau)\|_{L^2} d\tau. \end{aligned} \quad (4.3)$$

By Gronwall's inequality and estimate (4.3), we have that there exists  $C > 0$  such that

$$\|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2} \leq C\delta_2 T \|u_0\|_{B_{2,1}^{5/2}} \exp\{C\|u_0\|_{B_{2,1}^{5/2}} t\}$$

and, consequently, as  $\delta_2 \rightarrow 0^+$  we have

$$\sup_{0 < t < T} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{L^2} \leq C\delta_2 T \|u_0\|_{B_{2,1}^{5/2}} \exp\{C\|u_0\|_{B_{2,1}^{5/2}} T\} \rightarrow 0. \tag{4.4}$$

Let  $0 < \theta < 1$  and  $s_1, s_2, s_3 \geq 0$  be such  $s_3 = (1 - \theta)s_1 + \theta s_2$ . By Gagliardo–Nirenberg type inequality in Besov spaces (see [17]), we can estimate

$$\|(u^{\delta_1} - u^{\delta_2})(t)\|_{B_{2,1}^{s_3}} \leq C \|(u^{\delta_1} - u^{\delta_2})(t)\|_{B_{2,1}^{s_2}}^{1-\theta} \|(u^{\delta_1} - u^{\delta_2})(t)\|_{B_{2,1}^{s_1}}^\theta. \tag{4.5}$$

Considering  $s_1 = 0$ ,  $s_2 = 5/2$  and  $s_3 = \theta s_2$  in inequality (4.5), and using inequality (3.18),  $B_{2,2}^0 = L^2$  and property (4.4), we obtain

$$\|u^{\delta_1} - u^{\delta_2}\|_{L^\infty(0,T;B_{2,1}^{s_3})} \leq C \|u_0\|_{B_{2,1}^{5/2}}^\theta \|u^{\delta_1} - u^{\delta_2}\|_{L^\infty(0,T;L^2)}^{1-\theta} \rightarrow 0, \text{ as } \delta_2 \rightarrow 0^+,$$

for each fixed  $\theta \in (0, 1)$ . Hence, by completeness and uniqueness of the limit in the distributional sense,  $u^\delta \rightarrow u$  in  $L^\infty(0, T; B_{2,1}^{\tilde{s}})$  for all  $0 < \tilde{s} < 5/2$ . In particular, taking  $\tilde{s} = 3/2$  and recalling that  $u^\delta \in C([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3))$ , we obtain the convergence (4.2).

Also, in view of inequality (3.18), it follows that  $(u^\delta)_{\delta \in (0,1)}$  is bounded in  $L^\infty(0, T; B_{2,1}^{5/2}(\mathbb{R}^3))$ . Then, we can extract a subsequence  $(u^{\delta^{(j)}})_{j=1}^\infty$  that converges to  $u$  weakly- $\star$  in  $L^\infty(0, T; B_{2,1}^{5/2}(\mathbb{R}^3))$ . Thus we have that

$$u \in L^\infty(0, T; B_{2,1}^{5/2}(\mathbb{R}^3)) \cap C([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3)) \tag{4.6}$$

and

$$\|u\|_{L^\infty(0,T;B_{2,1}^{5/2})} \leq \liminf_{j \rightarrow \infty} \|u^{\delta^{(j)}}\|_{L^\infty(0,T;B_{2,1}^{5/2})} \leq L \|u_0\|_{B_{2,1}^{5/2}}. \tag{4.7}$$

Next we claim that  $u$  is a solution for system (1.1). For the nonlinear term, by using integration by parts, Lemma 2.2, Remark 2.1, and inequalities (3.18) and (4.7), we can estimate

$$\begin{aligned} & \int_0^t \|\mathbb{P}\nabla \cdot [u^\delta(\tau) \otimes u^\delta(\tau) - u(\tau) \otimes u(\tau)]\|_{B_{2,1}^{3/2}} d\tau \\ &= \int_0^t \|\mathbb{P}\nabla \cdot [(u^\delta(\tau) - u(\tau)) \otimes u^\delta(\tau) + u(\tau) \otimes (u^\delta(\tau) - u(\tau))]\|_{B_{2,1}^{3/2}} d\tau \\ &\leq C \int_0^t \left\{ \|u^\delta(\tau)\|_{B_{2,1}^{5/2}} + \|u(\tau)\|_{B_{2,1}^{5/2}} \right\} \|u^\delta(\tau) - u(\tau)\|_{B_{2,1}^{3/2}} d\tau \\ &\leq CT \|u_0\|_{B_{2,1}^{5/2}} \sup_{0 < t < T} \|u^\delta(t) - u(t)\|_{B_{2,1}^{3/2}} \rightarrow 0, \text{ as } \delta \rightarrow 0^+, \end{aligned}$$

which implies

$$\int_0^t \mathbb{P}(u^\delta(\tau) \cdot \nabla) u^\delta(\tau) d\tau \rightarrow \int_0^t \mathbb{P}(u(\tau) \cdot \nabla) u(\tau) d\tau \text{ in } L^\infty((0, T); B_{2,1}^{3/2}), \text{ as } \delta \rightarrow 0^+. \tag{4.8}$$

Also, we have that

$$\begin{aligned} \delta \int_0^t \|\Delta u^\delta(\tau)\|_{B_{2,1}^{1/2}} d\tau &\leq \delta \int_0^t \|u^\delta(\tau)\|_{B_{2,1}^{5/2}} d\tau \\ &\leq \delta T \|u^\delta\|_{L^\infty(0,T;B_{2,1}^{5/2})} \leq C\delta T \|u_0\|_{B_{2,1}^{5/2}} \rightarrow 0 \end{aligned}$$

and

$$\int_0^t \|\mathbb{P}\Omega e_3 \times (u^\delta(\tau) - u(\tau))\|_{B_{2,1}^{3/2}} d\tau \leq CT|\Omega| \sup_{0 < t < T} \|u^\delta(t) - u(t)\|_{B_{2,1}^{3/2}} \rightarrow 0, \text{ as } \delta \rightarrow 0^+.$$

Then

$$\begin{aligned} \delta \int_0^t -\Delta u^\delta(\tau) d\tau &\rightarrow 0 \text{ in } L^\infty((0,T);B_{2,1}^{1/2}) \\ \int_0^t \mathbb{P}\Omega e_3 \times u^\delta(\tau) d\tau &\rightarrow \int_0^t \mathbb{P}\Omega e_3 \times u(\tau) d\tau \text{ in } L^\infty((0,T);B_{2,1}^{3/2}), \text{ as } \delta \rightarrow 0^+. \end{aligned} \quad (4.9)$$

Therefore, since  $u^\delta$  satisfies system (3.1), we obtain from the limits (4.8) and (4.9) and the continuous inclusion  $B_{2,1}^{3/2} \subset B_{2,1}^{1/2}$  that

$$u(t) - u_0 = \int_0^t \{\mathbb{P}\Omega e_3 \times u(\tau) + \mathbb{P}(u(\tau) \cdot \nabla)u(\tau)\} d\tau \text{ in } B_{2,1}^{1/2}(\mathbb{R}^3). \quad (4.10)$$

In view of the above estimates and property (4.6), we can see that both sides of equation (4.10) belong to  $C([0,T];B_{2,1}^{3/2}(\mathbb{R}^3))$ . Thus, equality (4.10) holds in  $B_{2,1}^{3/2}(\mathbb{R}^3)$  and  $u \in AC([0,T];B_{2,1}^{3/2}(\mathbb{R}^3)) \cap L^\infty(0,T;B_{2,1}^{5/2}(\mathbb{R}^3))$  is a solution for system (1.1), as claimed.

The next lemma deals with the time-continuity of solutions for system (1.1). In particular, for  $p=2$  it implies the time-continuity of the solution  $u$  in  $B_{2,1}^{5/2}(\mathbb{R}^3)$  obtained as limit of the approximation scheme. Notice that in fact it holds for  $1 \leq p < \infty$ .

**LEMMA 4.1.** *Let  $0 < T < \infty$  and  $1 \leq p < \infty$ . If  $u$  is a solution for system (1.1) in  $L^\infty(0,T;B_{p,1}^{3/p+1}(\mathbb{R}^3))$  with initial velocity  $u_0 \in B_{p,1}^{3/p+1}(\mathbb{R}^3)$  satisfying  $\nabla \cdot u_0 = 0$ , then  $u \in C([0,T];B_{p,1}^{3/p+1}(\mathbb{R}^3))$ .*

*Proof.* Firstly, by Lemma 2.2, we have that  $\partial_t u \in L^\infty(0,T;B_{p,1}^{3/p}(\mathbb{R}^3))$ . Thus

$$u \in W^{1,\infty}([0,T];B_{p,1}^{3/p}(\mathbb{R}^3)) \subset C([0,T];B_{p,1}^{3/p}(\mathbb{R}^3)).$$

For every  $k \in \mathbb{N}$ , we denote  $w_k := S_k u$ . We are going to prove that the sequence  $\{w_k\}_{k \in \mathbb{N}}$  converges to  $u$  in  $L^\infty(0,T;B_{p,1}^{3/p}(\mathbb{R}^3))$ . Applying the Littlewood–Paley operator in system (1.1), for each  $j \in \mathbb{N}$  we obtain

$$\partial_t \Delta_j u + (S_j u \cdot \nabla) \Delta_j u = (S_j u \cdot \nabla) \Delta_j u - \Delta_j (u \cdot \nabla) u - \Delta_j \nabla p - \Omega e_3 \times \Delta_j u.$$

Since  $\Delta_j u$  is absolutely continuous on  $[0,T]$  with values in  $L^p(\mathbb{R}^3)$  and  $\nabla \cdot S_{j-2} u = 0$ , we can estimate

$$\begin{aligned} \|\Delta_j u(t)\|_{L^p} &\leq \|\Delta_j u_0\|_{L^p} + \int_0^t \|\Delta_j \nabla p\|_{L^p} d\tau \\ &\quad + \int_0^t \|(S_j u \cdot \nabla) \Delta_j u - \Delta_j (u \cdot \nabla) u\|_{L^p} d\tau + \int_0^t \|\Omega e_3 \times \Delta_j u\|_{L^p} d\tau. \end{aligned}$$

It follows that

$$\begin{aligned}
\|u(t) - w_k(t)\|_{B_{p,1}^{3/p+1}} &\leq C \sum_{j \geq k} 2^{j(3/p+1)} \|\Delta_j u(t)\|_{L^p} \\
&\leq C \left( \sum_{j \geq k} 2^{j(3/p+1)} \|\Delta_j u_0\|_{L^p} + \int_0^t \sum_{j \geq k} 2^{j(3/p+1)} \|\Delta_j \nabla p\|_{L^p} d\tau \right. \\
&\quad + \int_0^t \sum_{j \geq k} 2^{j(3/p+1)} \|(S_j u \cdot \nabla) \Delta_j u - \Delta_j (u \cdot \nabla) u\|_{L^p} d\tau \\
&\quad \left. + |\Omega| \int_0^t \sum_{j \geq k} 2^{j(3/p+1)} \|\Delta_j u\|_{L^p} d\tau \right).
\end{aligned}$$

The first term in the right-hand side converges to zero as  $k \rightarrow \infty$  because  $u_0 \in B_{p,1}^{3/p+1}(\mathbb{R}^3)$ . By Lemma 2.2, Lemma 2.4 and the fact that  $u(t) \in B_{p,1}^{3/p+1}(\mathbb{R}^3)$ , we have that the second and third terms in the right-hand side also converge to zero as  $k \rightarrow \infty$ . Therefore, the sequence  $\{w_k\}_{k \in \mathbb{N}}$  converges to  $u$  in  $L^\infty(0, T; B_{p,1}^{3/p+1}(\mathbb{R}^3))$ . Moreover, we get

$$\begin{aligned}
\|w_k(s) - w_k(t)\|_{B_{p,1}^{3/p+1}} &= \|S_k(u(s) - u(t))\|_{B_{p,1}^{3/p+1}} \\
&\leq C \sum_{j=-1}^{k+1} 2^{j(3/p+1)} \|\Delta_j(u(s) - u(t))\|_{L^p} \\
&\leq C 2^{k+1} \|u(s) - u(t)\|_{B_{p,1}^{3/p}}.
\end{aligned} \tag{4.11}$$

Estimate (4.11) and the fact that  $u \in C([0, T]; B_{p,1}^{3/p}(\mathbb{R}^3))$  imply that each  $w_k \in C([0, T]; B_{p,1}^{3/p+1}(\mathbb{R}^3))$ . Therefore, the limit  $u$  also belongs to  $C([0, T]; B_{p,1}^{3/p+1}(\mathbb{R}^3))$ .  $\square$

Now, taking  $p=2$  in Lemma 4.1, since  $u \in L^\infty(0, T; B_{2,1}^{5/2}(\mathbb{R}^3))$  and  $u_0 \in B_{2,1}^{5/2}(\mathbb{R}^3)$  we have that  $u \in C([0, T]; B_{2,1}^{5/2}(\mathbb{R}^3))$ , and then  $u$  satisfies

$$\partial_t u = -\mathbb{P}\Omega e_3 \times u - \mathbb{P}(u \cdot \nabla) u \in C([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3)). \tag{4.12}$$

This shows that  $u \in C^1([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3))$ , and therefore  $u$  is a strong solution for system (1.1) in the class

$$C([0, T]; B_{2,1}^{5/2}(\mathbb{R}^3)) \cap C^1([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3)). \tag{4.13}$$

**Uniqueness part.** Let  $u$  and  $v$  be strong solutions for system (1.1) in the class (4.13) with the same initial data  $u_0(x)$ . Subtracting the corresponding equations satisfied by  $u$  and  $v$ , we get

$$\begin{cases} \partial_t(u-v) + \mathbb{P}\Omega e_3 \times (u-v) + \mathbb{P}\{(u-v) \cdot \nabla\}u + \mathbb{P}(v \cdot \nabla)(u-v) = 0, \\ \nabla \cdot u = \nabla \cdot v = 0, \\ (u-v)(0, x) = 0. \end{cases} \tag{4.14}$$

Computing the  $L^2$ -inner product of system (4.14) with  $u - v$ , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(u-v)(t)\|_{L^2}^2 &= -\langle (u-v)(t), \mathbb{P}\{(u-v)(t) \cdot \nabla\}u(t)\rangle_{L^2} \\ &\leq \|\nabla u(t)\|_{L^\infty} \|(u-v)(t)\|_{L^2}^2 \\ &\leq C \|u(t)\|_{B_{2,1}^{5/2}} \|(u-v)(t)\|_{L^2}^2, \end{aligned}$$

and then

$$\begin{aligned} \|(u-v)(t)\|_{L^2} &\leq C \int_0^t \|u(\tau)\|_{B_{2,1}^{5/2}} \|(u-v)(\tau)\|_{L^2} d\tau \\ &\leq C \|u\|_{L^\infty(0,T;B_{2,1}^{5/2})} \int_0^t \|(u-v)(\tau)\|_{L^2} d\tau. \end{aligned} \quad (4.15)$$

Since  $\|u\|_{L^\infty(0,T;B_{2,1}^{5/2})} < \infty$ , we can use Gronwall inequality to obtain  $\|u(t) - v(t)\|_{L^2} = 0$  for all  $t \in [0, T]$ , and then  $u \equiv v$ .  $\square$

**4.2. Blow-up criterion.** In this part, we prove a blow-up criterion of BKM type (see [3]). We will use it to prove item (ii) of Theorem 1.1.

PROPOSITION 4.1. *Let  $u_0 \in B_{2,1}^{5/2}(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$ . Assume that*

$$u \in C([0, T]; B_{2,1}^{5/2}(\mathbb{R}^3)) \cap C^1([0, T]; B_{2,1}^{3/2}(\mathbb{R}^3)) \quad (4.16)$$

*is a solution for system (1.1). For some  $T' > T$ ,  $u$  can be extended to  $[0, T']$  with  $u \in C([0, T']; B_{2,1}^{5/2}(\mathbb{R}^3)) \cap C^1([0, T']; B_{2,1}^{3/2}(\mathbb{R}^3))$  provided that  $\int_0^{T'} \|\nabla u(t)\|_{L^\infty} dt < \infty$ .*

*Proof.* Item (i) of Theorem 1.1 assures that the existence-time  $T > 0$  depends only on the initial data norm  $\|u_0\|_{B_{2,1}^{5/2}}$ . Computing the  $L^2$ -inner product of system (1.1) with  $u$ , using the symmetry of  $e_3 \times u$  and  $\nabla \cdot u = 0$ , one can deduce

$$\|u(t)\|_{L^2} = \|u_0\|_{L^2} \quad \text{for all } t \in [0, T]. \quad (4.17)$$

Moreover, we can apply the operator  $\Delta_j$  in system (1.1), multiply the result by  $\Delta_j u$  and after use  $\langle (u \cdot \nabla) \Delta_j u, \Delta_j u \rangle_{L^2} = 0$  to get the identity

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j u(t)\|_{L^2}^2 = -\langle \Delta_j(u(t) \cdot \nabla)u(t), \Delta_j u(t) \rangle_{L^2} = \langle [u(t) \cdot \nabla, \Delta_j]u(t), \Delta_j u(t) \rangle_{L^2}. \quad (4.18)$$

Using the Schwartz inequality and integrating identity (4.18) over  $(0, t)$ , we obtain

$$\|\Delta_j u(t)\|_{L^2} \leq \|\Delta_j u_0\|_{L^2} + \int_0^t \|[u(\tau) \cdot \nabla, \Delta_j]u(\tau)\|_{L^2} d\tau. \quad (4.19)$$

Now we multiply estimate (4.19) by  $2^{(5/2)j}$  and afterwards take the  $l^1(\mathbb{Z})$ -norm to deduce

$$\|u(t)\|_{\dot{B}_{2,1}^{5/2}} \leq \|u_0\|_{\dot{B}_{2,1}^{5/2}} + \int_0^t \sum_{j \in \mathbb{Z}} 2^{(5/2)j} \|[u(\tau) \cdot \nabla, \Delta_j]u(\tau)\|_{L^2} d\tau.$$

By Lemma 2.3 (i), there exists  $C > 0$  such that

$$\|u(t)\|_{\dot{B}_{2,1}^{5/2}} \leq \|u_0\|_{\dot{B}_{2,1}^{5/2}} + C \int_0^t \|\nabla u(\tau)\|_{L^\infty} \|u(\tau)\|_{B_{2,1}^{5/2}} d\tau. \quad (4.20)$$

Putting together property (4.17) and estimate (4.20), we have that

$$\|u(t)\|_{B_{2,1}^{5/2}} \leq C_3 \|u_0\|_{B_{2,1}^{5/2}} + C_4 \int_0^t \|\nabla u(\tau)\|_{L^\infty} \|u(\tau)\|_{B_{2,1}^{5/2}} d\tau,$$

where  $C_3$  and  $C_4$  are positive constants. By Gronwall inequality, we get

$$\|u(t)\|_{B_{2,1}^{5/2}} \leq C_3 \|u_0\|_{B_{2,1}^{5/2}} \exp \left\{ C_4 \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \right\}, \text{ for all } t \in [0, T]. \quad (4.21)$$

Therefore, by standard arguments, if  $\int_0^T \|\nabla u(t)\|_{L^\infty} dt < \infty$  then  $u$  can be continued to  $[0, T']$  and so to  $[0, T')$  for some  $T' > T$  (by item (i) of Theorem 1.1).  $\square$

The contrapositive assertion of Proposition 4.1 gives the following remark.

REMARK 4.1. Let  $u_0 \in B_{2,1}^{5/2}(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$ . Assume that  $u$  is a solution for system (1.1) in the class (4.16). If  $T = T^* < \infty$  is the maximal existence-time, then

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} dt = \infty.$$

**4.3. Proof of item (ii).**

*Proof.* Let  $u_0 \in B_{2,1}^{7/2}(\mathbb{R}^3)$  with  $\nabla \cdot u_0 = 0$ . Proceeding similarly to the case  $u_0 \in B_{2,1}^{5/2}(\mathbb{R}^3)$  (see the class (4.13)), we can obtain a maximal existence-time  $T_* > 0$  such that the solution  $u \in C([0, T_*]; B_{2,1}^{7/2}(\mathbb{R}^3)) \cap C^1([0, T_*]; B_{2,1}^{5/2}(\mathbb{R}^3))$ .

Next, we apply the projection operators  $P_\pm$  in system (1.1) in order to get

$$\partial_t P_\pm u \mp i\Omega \frac{D_3}{|D|} P_\pm u + P_\pm (u \cdot \nabla) u = 0 \text{ with } P_\pm u(0, x) = P_\pm u_0.$$

Denoting  $A_\pm := \pm i\Omega \frac{D_3}{|D|}$  and using Duhamel principle, we have that

$$P_\pm u(t) = e^{\pm i\Omega t \frac{D_3}{|D|}} P_\pm u_0 - \int_0^t e^{\pm i\Omega(t-\tau) \frac{D_3}{|D|}} P_\pm (u(\tau) \cdot \nabla) u(\tau) d\tau. \quad (4.22)$$

Before proceeding, we recall the Strichartz estimates of [23] which states that if  $2 \leq r, \theta \leq \infty$  with  $(r, \theta) \neq (2, \infty)$  and  $\frac{1}{r} + \frac{1}{\theta} \leq \frac{1}{2}$  then

$$\|e^{\pm i\Omega t \frac{D_3}{|D|}} f\|_{L^r(0, \infty; L^\theta)} \leq C \|f\|_{L^2}. \quad (4.23)$$

Let  $2 < r < \infty$ . A scaling argument in estimate (4.23) leads us to

$$\|\Delta_j e^{\pm i\Omega t \frac{D_3}{|D|}} f\|_{L^r(0, \infty; L^\infty)} \leq C 2^{\frac{3}{2}j} |\Omega|^{-\frac{1}{r}} \|\Delta_j f\|_{L^2}, \quad (4.24)$$

for all  $j \in \mathbb{Z}$  and  $\Omega \in \mathbb{R} \setminus \{0\}$ , where  $C = C(r)$  is a constant.

In what follows, we derive an estimate in  $B_{\infty,1}^1$  for the solution  $u$ . Using  $u = P_+ u + P_- u$  (see Lemma 2.5), we only need to show the estimate for  $P_+ u$  and  $P_- u$ .



First notice that

$$\begin{aligned}
\|e^{\pm i\Omega t \frac{D_3}{|D|}} P_{\pm} u_0\|_{L^r(0,\infty; \dot{B}_{\infty,1}^1)} &= \left\| \sum_{j \in \mathbb{Z}} 2^j \|\Delta_j e^{\pm i\Omega t \frac{D_3}{|D|}} P_{\pm} u_0\|_{L^\infty} \right\|_{L_t^r(0,\infty)} \\
&\leq \sum_{j \in \mathbb{Z}} 2^j \|\Delta_j e^{\pm i\Omega t \frac{D_3}{|D|}} P_{\pm} u_0\|_{L_t^r(0,\infty; L^\infty)} \\
&\leq C|\Omega|^{-\frac{1}{r}} \sum_{j \in \mathbb{Z}} 2^j (2^j)^{3/2} \|\Delta_j P_{\pm} u_0\|_{L^2} \\
&= C|\Omega|^{-\frac{1}{r}} \|P_{\pm} u_0\|_{\dot{B}_{2,1}^{5/2}}.
\end{aligned}$$

Moreover, by estimate (4.23), we have that

$$\|e^{\pm i\Omega t \frac{D_3}{|D|}} P_{\pm} u_0\|_{L^r(0,\infty; L^\infty)} \leq C|\Omega|^{-\frac{1}{r}} \|P_{\pm} u_0\|_{L^2}.$$

For  $2 < r < \infty$  and  $\Omega \in \mathbb{R} \setminus \{0\}$ , the last two estimates yield

$$\|e^{\pm i\Omega t \frac{D_3}{|D|}} P_{\pm} u_0\|_{L^r(0,\infty; B_{\infty,1}^1)} \leq C|\Omega|^{-\frac{1}{r}} \|P_{\pm} u_0\|_{B_{2,1}^{5/2}}. \quad (4.25)$$

For the nonlinear term, using similar arguments we obtain

$$\begin{aligned}
&\left\| \int_0^t e^{\pm i\Omega(t-\tau) \frac{D_3}{|D|}} P_{\pm}(u(\tau) \cdot \nabla) u(\tau) \, d\tau \right\|_{L^r(0,T; L^\infty)} \\
&\leq C|\Omega|^{-\frac{1}{r}} \int_0^T \|P_{\pm}(u(\tau) \cdot \nabla) u(\tau)\|_{L^2} \, d\tau, \\
&\left\| \int_0^t e^{\pm i\Omega(t-\tau) \frac{D_3}{|D|}} P_{\pm}(u(\tau) \cdot \nabla) u(\tau) \, d\tau \right\|_{L^r(0,T; \dot{B}_{\infty,1}^1)} \\
&\leq C|\Omega|^{-\frac{1}{r}} \int_0^T \|P_{\pm}(u(\tau) \cdot \nabla) u(\tau)\|_{\dot{B}_{2,1}^{5/2}} \, d\tau.
\end{aligned}$$

Therefore

$$\left\| \int_0^t e^{\pm i\Omega(t-\tau) \frac{D_3}{|D|}} (u(\tau) \cdot \nabla) u(\tau) \, d\tau \right\|_{L^r(0,T; B_{\infty,1}^1)} \leq C|\Omega|^{-\frac{1}{r}} \int_0^T \|(u(\tau) \cdot \nabla) u(\tau)\|_{B_{2,1}^{5/2}} \, d\tau. \quad (4.26)$$

Estimates (4.25) and (4.26) imply that

$$\|u\|_{L^r(0,T; B_{\infty,1}^1)} \leq C|\Omega|^{-\frac{1}{r}} \left( \|u_0\|_{B_{2,1}^{5/2}} + \int_0^T \|(u(\tau) \cdot \nabla) u(\tau)\|_{B_{2,1}^{5/2}} \, d\tau \right), \quad (4.27)$$

for all  $0 < T < T_*$ . Next, we define

$$U(t) := \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau, \quad \text{for } 0 \leq t \leq T_*.$$

Using the embedding  $B_{\infty,1}^1(\mathbb{R}^3) \hookrightarrow W^{1,\infty}(\mathbb{R}^3)$  and estimates (4.21) and (4.27), we obtain

$$\begin{aligned}
U(t) &\leq \int_0^t \|u(\tau)\|_{B_{\infty,1}^1} d\tau \\
&\leq Ct^{1-\frac{1}{r}} \|u\|_{L^r(0,t;B_{\infty,1}^1)} \\
&\leq Ct^{1-\frac{1}{r}} |\Omega|^{-\frac{1}{r}} \left( \|u_0\|_{B_{2,1}^{\frac{5}{2}}} + \int_0^t \|(u(\tau) \cdot \nabla)u(\tau)\|_{B_{2,1}^{\frac{5}{2}}} d\tau \right) \\
&\leq Ct^{1-\frac{1}{r}} |\Omega|^{-\frac{1}{r}} \left( \|u_0\|_{B_{2,1}^{\frac{7}{2}}} + \int_0^t \|u(\tau)\|_{B_{2,1}^{\frac{7}{2}}}^2 d\tau \right) \\
&\leq Ct^{1-\frac{1}{r}} |\Omega|^{-\frac{1}{r}} \left( \|u_0\|_{B_{2,1}^{\frac{7}{2}}} + \|u_0\|_{B_{2,1}^{\frac{7}{2}}}^2 \int_0^t \exp(CU(\tau)) d\tau \right).
\end{aligned}$$

Then, there exist positive constants  $C_5$  and  $C_6$  (independent of  $\Omega$ ) such that

$$U(t) \leq C_5 t^{1-\frac{1}{r}} |\Omega|^{-\frac{1}{r}} \|u_0\|_{B_{2,1}^{\frac{7}{2}}} \left( 1 + \|u_0\|_{B_{2,1}^{\frac{7}{2}}} t \exp(C_6 U(t)) \right), \quad \forall t \in (0, T_*). \quad (4.28)$$

For  $0 < T < \infty$ , we consider

$$H_T = \{t \in [0, T] \cap [0, T_*] \mid U(t) \leq C_5 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{\frac{7}{2}}}\}, \quad \widehat{T}_* = \sup H_T.$$

We will show that  $\widehat{T}_* = \min\{T, T_*\}$ . For that, suppose that  $\widehat{T}_* < \min\{T, T_*\}$  by contradiction. Then there exists  $\widehat{T}$  such that  $\widehat{T}_* < \widehat{T} < \min\{T, T_*\}$ . In view of  $u \in C([0, \widehat{T}]; B_{2,1}^{7/2}(\mathbb{R}^3))$ , we have that  $U(t)$  is uniformly continuous on  $[0, \widehat{T}]$  and

$$U(\widehat{T}_*) \leq C_5 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}}. \quad (4.29)$$

Taking a sufficiently large  $\Omega \in \mathbb{R} \setminus \{0\}$  in such a way that

$$|\Omega|^{\frac{1}{r}} \geq 2 \left( 1 + \|u_0\|_{B_{2,1}^{7/2}} T \exp(C_5 C_6 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}}) \right), \quad (4.30)$$

and using estimates (4.28) and (4.29) and inequality (4.30), it follows that

$$\begin{aligned}
U(\widehat{T}_*) &\leq C_5 (\widehat{T}_*)^{1-\frac{1}{r}} |\Omega|^{-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}} \left( 1 + \|u_0\|_{B_{2,q}^{7/2}} \widehat{T}_* \exp(C_6 U(\widehat{T}_*)) \right) \\
&\leq C_5 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}} |\Omega|^{-\frac{1}{r}} \left( 1 + \|u_0\|_{B_{2,1}^{7/2}} T \exp(C_5 C_6 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}}) \right) \\
&\leq \frac{1}{2} C_5 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}}.
\end{aligned}$$

Thus, there exists  $L$  such that  $\widehat{T}_* < L < \widehat{T}$  and  $U(L) \leq C_5 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}}$ , contradicting the definition of  $\widehat{T}_*$ . Therefore, if inequality (4.30) holds true we have that  $\widehat{T}_* = \min\{T, T_*\}$ . If  $T_* < T$ , it follows that  $T_* = \widehat{T}_* = \sup H_T$  and then

$$U(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau \leq C_5 T^{1-\frac{1}{r}} \|u_0\|_{B_{2,1}^{7/2}} < \infty,$$

for all  $0 \leq t < T_*$ , and so  $U(T_*) < \infty$ . In view of the blow-up criterion (see Remark 4.1), we are done.  $\square$

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