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ERGODICITY AND DYNAMICS FOR THE STOCHASTIC 3D NAVIER–STOKES EQUATIONS WITH DAMPING*

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Abstract. The stochastic 3D Navier–Stokes equation with damping driven by a multiplicative noise is considered in this paper. The existence of invariant measures is proved for $3<\beta\le 5$ with any $\alpha>0$ and $\alpha\ge\frac{1}{2}$ as $\beta=3$. Using asymptotic strong Feller property, the uniqueness of invariant measures is obtained for the degenerate additive noise. The existence of a random attractor for the random dynamical systems generated by the solution of stochastic 3D Navier–Stokes equations with damping is proved for $\beta>3$ with any $\alpha>0$ and $\alpha\ge\frac{1}{2}$ as $\beta=3$.

Keywords. stochastic Navier–Stokes equations; existence; uniqueness; ergodicity; random attractor.

AMS subject classifications. 35Q30; 76D05; 37A25; 37L40; 35B41.

1. Introduction

We concern with a classes of stochastic Navier–Stokes equations with damping. The damping is from the resistance to the motion of the flow, it describes various physical situations such as drag or friction effects, and some dissipative mechanisms [21]. Importance of such problems for climate modeling and physical fluid dynamics are well known [19, 38].

In this paper, we consider the following stochastic three-dimensional Navier–Stokes equations with damping

$$\begin{cases} du-\nu\Delta u dt+(u\cdot\nabla)udt+\nabla pdt+\alpha|u|^{\beta-1}udt=\sigma(t,u)dW(t),\\ \nabla\cdot u=0,\\ u(t,x)|_{\partial D}=0,\\ u(x,0)=u_0(x). \end{cases} \tag{1.1}$$

Where $D\subset R^3$ is an bounded domain with smooth boundary ∂D, $u=(u_1,u_2,u_3)$ is the velocity, p is the pressure, $\nu>0$ is the kinematic viscosity, $\beta\ge 1$, $\alpha>0$ and $t\in[0,T]$, u_0 is the initial velocity, and $\sigma(t,u)dW(t)$, stands for the random forces, where W is a Wiener process. In the following, for simplicity, we set $\nu=1$.

The deterministic three-dimensional Navier–Stokes equation with damping has been extensively investigated. For instance, Cai and Jiu have studied the existence and regularity of solutions for three-dimensional Navier–Stokes equations with damping [6], they obtained the global weak solution for $\beta\ge 1$, the global strong solution for $\beta\ge\frac{7}{2}$ and that the strong solution was unique for any $\frac{7}{2}\le\beta\le 5$. Based on it, Song and Hou considered the global attractor in [35] and [36]. In [23], the authors considered the L^2 decay of weak solutions with $\beta\ge\frac{10}{3}$, the optimal upper bounds of the higher-order derivative of the strong solution for $\frac{7}{2}\le\beta<5$ and the asymptotic stability of the large solution to the system with $\beta\ge\frac{7}{2}$. In [40], the authors considered the regularity criterion

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of the three-dimensional Navier–Stokes equations with nonlinear damping. In [43], for $\alpha=1$, Zhou proved that the strong solution exists globally for $\beta\geq 3$ and strong-weak uniqueness for $\beta\geq 1$, and established two regularity criteria as $1\leq\beta\leq 3$. Oliveira has studied the existence of weak solutions for the generalized Navier–Stokes equations with damping [29]. By using Fourier splitting method, the L_2 decay of weak solutions for 3D Navier–Stokes equations with damping was proved for $\beta>2$ with any $\alpha>0$ in [25].

The existence of a unique strong solution to a stochastic tamed 3D Navier–Stokes equations in the whole space was proved in [32]. The existence and uniqueness of the solution for the 2D stochastic Navier–Stokes equations driven by jump noise were studied in [5]. A small time large deviation principle for the stochastic non-Newtonian fluids driven by multiplicative noise was proved in [26]. Using Galerkin's approximation and compactness method, Liu and Gao in [24] proved the existence of martingale solutions, existence and uniqueness of strong solution and small time large deviation principles for the stochastic 3D Navier–Stokes equations with damping for $\beta>3$ with any $\alpha>0$ and $\alpha\geq\frac{1}{2}$ as $\beta=3$.

The irreducibility and the strong Feller property imply the uniqueness of the invariant measure as explored in [12, 14, 15, 20]. Using a sort of ground state transformation, exponential ergodicity for stochastic Burgers and 2D Navier–Stokes equations was shown in [18]. Existence and uniqueness of invariant measure for stochastic 2D Navier–Stokes equations with Lévy noise were proved in [13]. Romito and Xu proved that any Markov solution to the 3D stochastic Navier–Stokes equations driven by a mildly degenerate noise is uniquely ergodic in [33]. The existence and uniqueness of invariant measure for a class of stochastic Boussinesq equations driven by Lévy processes were obtained in [42].

It is known that the asymptotic behaviour of a random dynamical system is confirmed by a random attractor. The existence of random attractors has been researched by many authors refer in [1, 8, 11, 39]. The asymptotic dynamics for stochastic reaction-diffusion equations with multiplicative noise defined on unbounded domains were proved in [41]. Using the Faedo–Galerkin method, the existence and uniqueness of a weak solution were proved. Meanwhile, random dynamics of the 3D stochastic Navier–Stokes–Voight equations were obtained in [17]. The existence of a random attractor for the random dynamic systems generated by the solution of stochastic 2D fractional Ginzburg–Landau equation with multiplicative was proved in [27]. The well-posedness and dynamics of the stochastic 2D incompressible fractional magneto-hydrodynamic equations driven by Gaussian multiplicative noise were obtained in [22].

To obtain ergodicity and dynamics for the stochastic 3D Navier–Stokes equations with damping driven by noise, the main difficulty lies in dealing with the nonlinear term $B(u,u)=P((u\cdot\nabla)u)$ and $g(u)=\alpha|u|^{\beta-1}u$. Using the nonlinear structure and delicated analysis, we get the estimate of $\int_0^t\int_D|u|^{\beta-1}|\nabla u|^2dxds$ for $\beta\geq 3$ and use this term to control the estimate of $\int_0^t\int_D|u|^2|\nabla u|^2dxds$. Using the Krylov–Bogoliubov method and by the delicated estimate for the nonlinear term, we obtain the existence of invariant measure to the problem (1.1) for $3<\beta\leq 5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{2}$ as $\beta=3$. For the existence of random attractor for the random dynamical systems generated by the solution of stochastic 3D Navier–Stokes equations with damping, we need make detailed estimate for $2z^{-1}(t)b(B(v,v),\Delta v)$; here $z(t)$ comes from the Ornstein–Uhlenbeck process.

This paper is organized as follows. In Section 2, we recall some fundamental concepts and some lemmas which are used in the sequel. In Section 3, we will prove the existence of invariant measure to the problem (1.1) for $3<\beta\leq 5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{2}$ as $\beta=3$. Using the notion of asymptotic strong Feller property, the uniqueness of invariant measures is proved for the degenerate additive noise. In Section 4, the existence of random attractor for the random dynamical systems generated by the solution of stochastic 3D Navier–Stokes equations with damping is proved for $\beta>3$ with any $\alpha>0$ and $\alpha\geq\frac{1}{2}$ as $\beta=3$.

2. Preliminaries

Let D be a bounded domain in R^3 with sufficiently smooth boundary and $C_0^\infty(D,R^3)$ be the set of all smooth functions from D to R^3 with compact support. Let $W(t)$ be a sequence of independent one-dimensional standard Brownian motions on some complete filtration probability space $(\Omega,\mathcal{F},P;(\mathcal{F}_t)_{t\geq 0})$. We define the usual function spaces

$$\begin{aligned}\mathcal{V}&=\{u\in(C_0^\infty(D,R^3))^3:\mathrm{div}u=0\},\\ H&=\text{the closure of }\mathcal{V}\text{ in }L^2(D),\\ V&=\text{the closure of }\mathcal{V}\text{ in }H_0^1(D).\end{aligned}$$

It is well known that H, V are separable Hilbert spaces and identify H and its dual H', we have $V\hookrightarrow H\hookrightarrow V'$ with dense and continuous injections, and $V\hookrightarrow H$ is compact. H and V endowed, respectively, with the inner products

$$\begin{aligned}(u,v)&=\int_D u\cdot v dx,\quad \forall u,v\in H,\\ ((u,v))&=\sum_{i=1}^3\int_D \nabla u_i\cdot\nabla v_i dx,\quad \forall u,v\in V,\end{aligned}$$

and norms $|\cdot|_2=(\cdot,\cdot)^{\frac{1}{2}}$, $||\cdot||^2=((\cdot,\cdot))$. Let $||\cdot||_s$ be the H^s Sobolev-norm for $s\in R$. Let $\mathbb{P}$ be the orthogonal projection of $L^2(D;R^3)$ to H. For $u,v\in L^2(D;R^3)$, $Au=-\mathbb{P}\Delta u$ is the Stokes operator defined by $\langle Au,v\rangle=((u,v))$. $B:V\times V\to V'$ is a bilinear operator defined by $\langle B(u,v),w\rangle=b(u,v,w)$, $B(u)=B(u,u)$, where

$$b(u,v,w)=\sum_{i=1}^3\int_D u_i\frac{\partial v_j}{\partial x_i}w_j dx,$$

and $\langle\cdot,\cdot\rangle$ is the duality product between V and V'. For simplicity, we set $g(u)=\alpha|u|^{\beta-1}u$ and $F(u)=-(A(u)+B(u)+g(u))$. Here, $\sigma(t,u)$ is a mappings from V into V or $H\to H$.

Assumption 2.1. *Assume that there exist nonnegative constants k_0,k_1,k_2,l_1,l_2 and l such that for all $t\in[0,T]$ and $u,v\in V$,*

(A1) $|\sigma(t,u)|_2^2\leq k_0+k_1|u|_2^2$.

(A2) $|\sigma(t,u)-\sigma(t,v)|_2^2\leq l_1|u-v|_2^2$.

(A3) $|\partial_u\sigma(t,u)|\leq k_2+l|u|$.

(A4) $||\sigma(t,u)-\sigma(t,v)||^2\leq l_2||u-v||^2$.

Remark 2.1. If $\sigma(t,u)=\rho u$ and $\rho\leq k_2$, (A3) is satisfied.

Lemma 2.1 (Burkholder–Davis–Gundy Inequality [7]). *For every $p\geq 1$, there exists a constant C_p such that for any real-valued square integrable martingale M with $M(0)=0$, and for any $T>0$,*

$$C_p^{-1}\mathbb{E}[M]_T^{p/2}\leq\mathbb{E}(\sup_{0\leq t\leq T}|M(t)|^p)\leq C_p\mathbb{E}[M]_T^{p/2},\tag{2.1}$$

where $[M]_T$ is called the quadratic variation of M.

Now we present some concepts related to a random dynamic system refer in [1–4, 9, 16, 34].

Let $(X,||\cdot||_X)$ be a separable Hilbert space, $(\Omega,\mathcal{F},P)$ be a probability space and $\{\theta_t:\Omega\mapsto\Omega,\ t\in R\}$ be a family of measure preserving transformations such that $(t,\omega)\mapsto\theta_t\omega$ is measurable, $\theta_0=Id_\Omega$, $\theta_{t+s}=\theta_t\theta_s$, for all $s,t\in R$. The space $(\Omega,\mathcal{F},P,(\theta_t)_{t\in R})$ is a (measurable) dynamical system. We define the mapping $S(t,s;\omega):X\to X$, $-\infty<s<t<\infty$, dependence on $\omega\in\Omega$.

We define the random omega limit set of a bounded set $B\subset X$ at time t as

$$\mathcal{A}(B,t,\omega)=\bigcap_{T\le t}\overline{\bigcup_{s\le T}S(t,s;\omega)B}.$$

According to [9, 10], we get the following theorem about the existence of random attractors.

THEOREM 2.1. *Let $S(t,s;\omega)$ be a stochastic dynamical system with the following properties:*

(1) $S(t,r;\omega)S(r,s;\omega)x=S(t,s;\omega)x$, for all $s\le r\le t$ and $x\in X$;

(2) $S(t,s;\omega)$ is continuous in X, for all $s\le t$;

(3) the mapping $\omega\mapsto S(t,s;\omega)x$ is measurable for all $s<t$ and $x\in X$;

(4) the mapping $s\mapsto S(t,s;\omega)x$ is right continuous for all $t,x\in X$ and $P-a.e.$ and $\omega\in\Omega$.

Suppose that there exists a measure preserving mapping $\theta_t,t\in R$ such that

$$S(t,s;\omega)x=S(t-s,0;\theta_s\omega)x$$

holds for $P-a.e.$ $\omega\in\Omega$. Then there exists a compact attracting set $K(\omega)$ at time 0. For $P-a.e.$ $\omega\in\Omega$, we set $\mathcal{A}(\omega)=\overline{\bigcup_{B\subset X}\mathcal{A}(B,\omega)}$ where the union is taken over all the bounded subsets of X, and $\mathcal{A}(B,\omega)$ is defined by

$$\mathcal{A}(B,\omega)=\bigcap_{T\le 0}\overline{\bigcup_{s\le T}S(0,s;\omega)B}.$$

Hence, $\mathcal{A}(\omega)$ is the random attractor.

3. Ergodicity

We now rewrite problem (1.1) as follows in the abstract form:

$$\begin{aligned} &du(t)=-(Au(t)+B(u(t))+g(u(t)))dt+\sigma(t,u)dW(t),\\ &u(x,0)=u_0,\qquad x\in D. \end{aligned}\tag{3.1}$$

THEOREM 3.1. *Suppose that Assumption 2.1 holds and $\beta>3$ with any $\alpha>0$ and $\alpha\ge\frac{1}{2}$ as $\beta=3$. Then for any $\mathcal{F}_0-$adapted V-valued function u_0 satisfying $\mathbb{E}||u_0||^{2p}<\infty$ for $p\ge1$, then there exists a unique strong solution $u(t)$ to the problem (3.1) with the initial condition $u(0)=u_0$, and $u(t)\in L^2(\Omega,L^2([0,T],H^2))\cap L^2(\Omega,L^\infty([0,T],V))$, it satisfies the following inequality*

$$\mathbb{E}(\sup_{0\le t\le T}||u(t)||^{2p}+\int_0^T||u(s)||^{2p-2}||\nabla u(s)||^2ds+\int_0^T||u(s)||^{2p-2}|\nabla|u|^{\frac{\beta+1}{2}}|_2^2ds$$

$$+\int_0^T\int_D ||u(s)||^{2p-2}|u|^{\beta-1}|\nabla u|^2 dxds) \leq C(\mathbb{E}||u_0||^{2p}+1).$$

Proof. By using monotonicity method, we will prove in Appendix. ∎

REMARK 3.1. Using the damping term, we get the estimate of $\int_0^t\int_D |u|^{\beta-1}|\nabla u|^2 dxds$ for $\beta\geq 3$ and use this term to control the estimate of $\int_0^t\int_D |u|^2|\nabla u|^2 dxds$. We overcome the convective term $(u\cdot\nabla)u$ in the space V and prove the existence and uniqueness of strong solution for the stochastic 3D Navier–Stokes equations with damping. In this paper [24], using the existence of martingale solutions plus pathwise uniqueness implies the existence of a unique strong solution. However, we get the existence and uniqueness of strong solution for the stochastic 3D Navier–Stokes equations with damping by using monotonicity method.

Let $C_b(V)$ be the set of all bounded and locally uniformly continuous functions on V and $\mathcal{M}(V)$ denote the probability measure space on V. Under the norm

$$||\varphi||_\infty = \sup_{u\in V}|\varphi(u)|,$$

then $C_b(V)$ is a Banach space. Let $\mathcal{P}_t$ be a Markov semigroup in the space $C_b(V)$ defined as

$$\mathcal{P}_t\varphi(u_0) = \mathbb{E}\varphi(u(t,\cdot;0,u_0)), \quad t\geq 0,\ u_0\in V,\ \varphi\in C_b(V).$$

$\mathcal{P}_t^*$ define on the probability measure space $\mathcal{M}(V)$ and is defined by

$$\int_V \varphi d(\mathcal{P}_t^*\mu) = \int_V \mathcal{P}_t\varphi d\mu, \quad \forall\varphi\in C_b(V), \quad \forall\mu\in\mathcal{M}(V).$$

If $\mathcal{P}_t^*\mu=\mu$ for any $t\geq 0$, then a measure $\mu\in\mathcal{M}(V)$ is called invariant.

3.1. Existence of invariant measures. For fixed initial value $u_0\in V$, we denote the unique solution in Theorem 3.1 by $u(t,u_0)$. Then $\{u(t,u_0): u_0\in V, t\geq 0\}$ forms a strong Markov process with state space V. For two initial data $u_{01},u_{02}\in V$, we denote $u_i=u(t,u_{0i})$ the solutions starting from u_{0i}, $i=1,2$. Let us define the stopping time

$$\tau_N = \inf\{t\geq 0: ||u(t,u_{01})||\vee||u(t,u_{02})||\geq N\}.$$

LEMMA 3.1. *Suppose that Assumption 2.1 holds and $3<\beta\leq 5$ with any $\alpha>0$ and $\alpha\geq\frac{1}{2}$ as $\beta=3$, then there exists a constant C_N such that*

$$\mathbb{E}||u(t\wedge\tau_N,u_{01})-u(t\wedge\tau_N,u_{02})||^2 \leq C_N||u_{01}-u_{02}||^2. \tag{3.2}$$

Proof. Let $\omega(t)=u(t,u_{01})-u(t,u_{02})$. Applying the Itô formula to $||\omega(t\wedge\tau_N)||^2$, we deduce

$$\begin{aligned}||\omega(t\wedge\tau_N)||^2 = &||\omega(0)||^2 - 2\int_0^{t\wedge\tau_N}((Au_1-Au_2,\omega(s)))ds\\ &-2\int_0^{t\wedge\tau_N}((B(u_1(s))-B(u_2(s)),\omega(s)))ds\\ &-2\int_0^{t\wedge\tau_N}((g(u_1(s))-g(u_2(s)),\omega(s)))ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t\wedge\tau_N}||\sigma(s,u_1(s))-\sigma(s,u_2(s))||^2ds\\
&+2\int_0^{t\wedge\tau_N}((\omega(s),\sigma(s,u_1(s))-\sigma(s,u_2(s))))dW(s)\\
=&||\omega(0)||^2+I_1+I_2+I_3+I_4+I_5. \qquad (3.3)
\end{aligned}
$$

By estimating the first term, we get

$$
\begin{aligned}
I_1=&-2\int_0^{t\wedge\tau_N}((Au_1-Au_2,\omega(s)))ds\\
=&-2\int_0^{t\wedge\tau_N}||\nabla\omega(s)||^2ds. \qquad (3.4)
\end{aligned}
$$

For I_2, we have the following inequalities

$$
\begin{aligned}
I_2=&-2\int_0^{t\wedge\tau_N}((B(u_1,\omega)+B(\omega,u_2),\omega(s)))ds\\
\le&\int_0^{t\wedge\tau_N}(\varepsilon||\nabla\omega||^2+C|u_1\cdot\nabla\omega|_2^2+C|\omega\cdot\nabla u_2|_2^2)ds\\
\le&\int_0^{t\wedge\tau_N}(\varepsilon||\nabla\omega||^2+C|u_1|_\infty^2|\nabla\omega|_2^2+C|\omega|_\infty^2|\nabla u_2|_2^2)ds\\
\le&\int_0^{t\wedge\tau_N}(\varepsilon||\nabla\omega||^2+C||u_1||||\nabla u_1||||\omega||^2+C||\omega||||\nabla\omega||||u_2||^2)ds\\
\le&\int_0^{t\wedge\tau_N}(2\varepsilon||\nabla\omega||^2+C||u_1||||\nabla u_1||||\omega||^2+C||\omega||^2||u_2||^4)ds. \qquad (3.5)
\end{aligned}
$$

For I_3, we deduce

$$
\begin{aligned}
I_3=&-2\int_0^{t\wedge\tau_N}((g(u_1(s))-g(u_2(s)),\omega(s)))ds\\
\le&\int_0^{t\wedge\tau_N}||\nabla\omega||^2ds+C\int_0^{t\wedge\tau_N}\int_D[|\omega|(|u_1|^{\beta-1}+|u_2|^{\beta-1})]^2dxds\\
\le&\int_0^{t\wedge\tau_N}||\nabla\omega||^2ds+C\int_0^{t\wedge\tau_N}|\omega|_6^2\left||u_1|^{\beta-1}+|u_2|^{\beta-1}\right|_3^2ds\\
\le&\int_0^{t\wedge\tau_N}||\nabla\omega||^2ds+C\int_0^{t\wedge\tau_N}||\omega||^2(|u_1|_{3(\beta-1)}^{2(\beta-1)}+|u_2|_{3(\beta-1)}^{2(\beta-1)})ds. \qquad (3.6)
\end{aligned}
$$

Taking expectations of equality (3.3) and combining itmes (3.4)-(3.6) and (3.3), we deduce

$$
\begin{aligned}
\mathbb{E}||\omega(t\wedge\tau_N)||^2+2\mathbb{E}\int_0^{t\wedge\tau_N}||\nabla\omega(s)||^2ds\le&\mathbb{E}\int_0^{t\wedge\tau_N}(C||u_1||||\nabla u_1||||\omega||^2\\
&+C||\omega||^2||u_2||^4)ds+(1+2\varepsilon)\mathbb{E}\int_0^{t\wedge\tau_N}||\nabla\omega||^2ds\\
&+C\mathbb{E}\int_0^{t\wedge\tau_N}||\omega||^2(|u_1|_{3(\beta-1)}^{2(\beta-1)}+|u_2|_{3(\beta-1)}^{2(\beta-1)})ds\\
&+C\mathbb{E}\int_0^{t\wedge\tau_N}||\omega||^2ds+||\omega(0)||^2. \qquad (3.7)
\end{aligned}
$$

For $|\nabla|u_i|^{\frac{\beta+1}{2}}|^2$, because $H^1 \hookrightarrow L^6$ and by Theorem 3.1, we have

$$\int_0^{t\wedge\tau_N} |u_i|_{3(\beta+1)}^{\beta+1} ds \le c\int_0^{t\wedge\tau_N} |\nabla|u_i|^{\frac{\beta+1}{2}}|_2^2 ds \le C, \quad i=1,2. \tag{3.8}$$

Since

$$0 < \frac{2(\beta-3)}{\beta-1} \le 1, \quad \text{for } 3<\beta\le 5,$$

hence, we deduce

$$\begin{aligned}\int_0^{t\wedge\tau_N} |u_1|_{3(\beta-1)}^{2(\beta-1)} ds \le & C\int_0^{t\wedge\tau_N} |u_1|_{3(\beta+1)}^{\frac{2(\beta-3)(\beta+1)}{\beta-1}} ||u_1||^{\frac{8}{\beta-1}} ds \\ \le & C \sup_{0\le s\le t\wedge\tau_N} ||u_1||^{\frac{8}{\beta-1}} (\int_0^{t\wedge\tau_N} |u_1|_{3(\beta+1)}^{\beta+1} ds)^{\frac{2(\beta-3)}{\beta-1}} (t\wedge\tau_N)^{\frac{5-\beta}{\beta-1}}. \end{aligned}\tag{3.9}$$

Similarly,

$$\begin{aligned}\int_0^{t\wedge\tau_N} |u_2|_{3(\beta-1)}^{2(\beta-1)} ds \le & C\int_0^{t\wedge\tau_N} |u_2|_{3(\beta+1)}^{\frac{2(\beta-3)(\beta+1)}{\beta-1}} ||u_2||^{\frac{8}{\beta-1}} ds \\ \le & C \sup_{0\le s\le t\wedge\tau_N} ||u_2||^{\frac{8}{\beta-1}} (\int_0^{t\wedge\tau_N} |u_2|_{3(\beta+1)}^{\beta+1} ds)^{\frac{2(\beta-3)}{\beta-1}} (t\wedge\tau_N)^{\frac{5-\beta}{\beta-1}}. \end{aligned}\tag{3.10}$$

Choosing sufficiently small ε and applying the Gronwall lemma, we get

$$\mathbb{E}||\omega(t\wedge\tau_N)||^2 \le C_N ||u_{01}-u_{02}||^2. \tag{3.11}$$

For $\alpha \ge \frac{1}{2}$ as $\beta=3$, we can get the above estimate easily. This completes the proof of Lemma 3.1. □

THEOREM 3.2. *Suppose that Assumption 2.1 holds and $3<\beta\le 5$ with any $\alpha>0$ and $\alpha\ge\frac{1}{2}$ as $\beta=3$, for any $t>0$, $\mathcal{P}_t$ maps $C_b(V)$ into $C_b(V)$. Then $\mathcal{P}_t$ is a Feller semigroup on $C_b(V)$.*

Proof. We will prove that $\mathcal{P}_t$ is locally uniformly continuous. Our main task is to show that for any $\varepsilon>0$, $t>0$ and $m\in N$, there exists $\delta>0$ such that

$$\sup_{||u_{01}-u_{02}||\le\delta} |\mathcal{P}_t\varphi(u_{01}) - \mathcal{P}_t\varphi(u_{02})| < \varepsilon, \tag{3.12}$$

where $u_{01}, u_{02} \in B_m$ and $B_m = \{u\in V : ||u|| \le m\}$. Let us also define the stopping time

$$\tau_N = \inf\{t\ge 0 : ||u(t,u_{01})|| \vee ||u(t,u_{02})|| \ge N\}.$$

First, by the Theorem 3.1, we deduce that

$$\begin{aligned}\mathbb{E}|\varphi(u(t,u_{0i})) - \varphi(u(t\wedge\tau_N, u_{0i}))| \le & 2|\varphi|_\infty P\{\tau_N < t\} \\ \le & 2|\varphi|_\infty \sup_{u_{0i}\in B_m} \mathbb{E}[\sup_{s\in[0,t]} ||u(s,u_{0i})||^2]/N^2 \\ \le & 2C|\varphi|_\infty/N^2, \quad i=1,2.\end{aligned}$$

Choosing $N>m$ sufficiently large such that we have the following inequality

$$\mathbb{E}|\varphi(u(t,u_{0i})) - \varphi(u(t\wedge\tau_N, u_{0i}))| \le \frac{\varepsilon}{4}, \quad i=1,2. \tag{3.13}$$

Next, for any $u_1, u_2 \in B_m$ and fixed N with $||u_1 - u_2|| \leq \delta_N$, there exists $\delta_N > 0$ such that

$$|\varphi(u_1) - \varphi(u_2)| \leq \frac{\varepsilon}{4}.$$

Using Lemma 3.1 and Chebyshev inequality and choosing $||u_{01} - u_{02}||^2 \leq \frac{\varepsilon \delta_N^2}{8C|\varphi|_\infty}$, we deduce

$$\begin{aligned}
&\mathbb{E}|\varphi(u(t\wedge\tau_N, u_{01})) - \varphi(u(t\wedge\tau_N, u_{02}))| \\
&= \int_{\Omega_1} |\varphi(u(t\wedge\tau_N, u_{01})) - \varphi(u(t\wedge\tau_N, u_{02}))| P(d\omega) \\
&\quad + \int_{\Omega_2} |\varphi(u(t\wedge\tau_N, u_{01})) - \varphi(u(t\wedge\tau_N, u_{02}))| P(d\omega) \\
&\leq \frac{\varepsilon}{4} + 2|\varphi|_\infty P\{||u(t\wedge\tau_N, u_{01}) - u(t\wedge\tau_N, u_{02})|| > \delta_N\} \\
&\leq \frac{\varepsilon}{4} + 2|\varphi|_\infty \frac{\mathbb{E}||u(t\wedge\tau_N, u_{01}) - u(t\wedge\tau_N, u_{02})||^2}{\delta_N^2} \leq \frac{\varepsilon}{2},
\end{aligned} \tag{3.14}$$

where $\Omega_1 = \{\omega : ||u(t\wedge\tau_N, u_{01}) - u(t\wedge\tau_N, u_{02})|| \geq \delta_N\}$ and $\Omega_2 = \{\omega : ||u(t\wedge\tau_N, u_{01}) - u(t\wedge\tau_N, u_{02})|| < \delta_N\}$. By inequalities (3.13) and (3.14), inequality (3.12) is proved, this completes the proof of Theorem 3.2. ∎

THEOREM 3.3. *Suppose that Assumption 2.1 holds and $3 < \beta \leq 5$ with any $\alpha > 0$ and $\alpha \geq \frac{1}{2}$ as $\beta = 3$, then there exists an invariant measure μ_* associated to the semigroup $\mathcal{P}_t$ such that for any $t \geq 0$ and $\varphi \in C_b(V)$,*

$$\int_V \mathcal{P}_t \varphi(u) \mu_*(du) = \int_V \varphi(u) \mu_*(du).$$

Proof. By the classical Krylov–Bogoliubov method [12], we need to show that for any $\varepsilon > 0$, there exists $M > 0$ such that for any $T > 1$,

$$\frac{1}{T} \int_0^T P(||u(s)||_{H^2}^2 > M) ds < \varepsilon. \tag{3.15}$$

Let $u_0 = 0$. Applying Itô's formula to the process $|u(t)|_2^2$, we have for any $t \geq 0$

$$\mathbb{E}|u(t)|_2^2 + \int_0^t \mathbb{E}||u(s)||^2 ds + 2\int_0^t \mathbb{E}|u(s)|_{\beta+1}^{\beta+1} ds \leq Ct.$$

Applying Itô formula to $||u(t)||^2$, we have

$$\begin{aligned}
&\mathbb{E}(||u(t)||^2 + \int_0^t \int_D |u|^{\beta-1} |\nabla u|^2 dx ds + \int_0^t |\nabla |u|^{\frac{\beta+1}{2}}|_2^2 ds) \\
&\leq -\frac{1}{4} \int_0^t \mathbb{E}||\nabla u(s)||^2 ds + C\int_0^t \mathbb{E}||u(s)||^2 ds + Ct \\
&\leq -\frac{1}{4} \int_0^t \mathbb{E}||\nabla u(s)||^2 ds + Ct.
\end{aligned}$$

Then we deduce

$$\frac{1}{t} \int_0^t \mathbb{E}||u(s)||_{H^2}^2 ds < C.$$

Hence, inequality (3.15) is proved, this completes the proof of Theorem 3.3. ∎

3.2. Ergodicity. In this subsection, we prove the uniqueness of invariant measures for the degenerate additive noise.

For any $m\in\mathbb{N}$, let $\Omega=C_0(R_+;R^m)$ be the space of all continuous functions with initial values 0. P the standard Wiener measure on $\mathcal{F}=\mathcal{B}(C_0(R_+;R^m))$. The coordinate process

$$W(t,\omega)=\omega(t),\quad \omega\in\Omega,$$

is a standard Wiener process on $(\Omega,\mathcal{F},P)$.

Consider the following stochastic 3D Navier–Stokes equations with damping:

$$du(t)=-(Au(t)+Bu(t)+g(u(t)))dt+d\mathbf{w}(t), \tag{3.16}$$

here, $u(0)=u_0\in V$ and $\mathbf{w}(t)=QW(t)$ is the noise, and the linear map $Q:R^m\to V$ is defined by

$$Qe_i=q_ie_i,\quad q_i>0, i=1,2,\cdots,m,$$

where, e_i is the canonical basis of R^m and e_i is orthonormal basis of V satisfy

$$\mathbb{P}\Delta e_i=-\lambda_i e_i,$$

where $0<\lambda_1\le\lambda_2\le\cdots\le\lambda_m\to\infty$.

Let

$$\zeta_0=\sum_{i=1}^m\frac{q_i^2}{\lambda_i},\quad \zeta_1=\sum_{i=1}^m q_i^2.$$

Moreover, the quadratic variation of $\mathbf{w}(t)$ in H and V are defined by

$$[\mathbf{w}]_H(t)=\zeta_0 t,\quad [\mathbf{w}]_V(t)=\zeta_1 t.$$

THEOREM 3.4. *Let $\{\mathcal{P}_t\}_{t\ge0}$ be the transition semigroup associated with equation (3.16), then for any large sufficient m_*, there exists a unique invariant probability measure associated with $\{\mathcal{P}_t\}_{t\ge0}$ for $3<\beta\le5$ with any $\alpha>0$ and $\alpha\ge\frac{1}{2}$ as $\beta=3$.*

The proof of Theorem 3.4 is divided into two parts. In the first part, we show the asymptotic strong Feller property of $\mathcal{P}_t$. In the second part, we prove that the origin 0 belongs to the support of each invariant measure. By [20], these two parts will imply Theorem 3.4.

3.2.1. Asymptotically strong Feller property. For $0\le s<t$, let $J_{s,t}v_0$ be the solution of the linearized equation

$$\partial_t J_{s,t}v_0=\Delta J_{s,t}v_0+\mathcal{K}(u(t,\omega;u_0),J_{s,t}v_0), J_{s,s}v_0=v_0, \tag{3.17}$$

here, $\mathcal{K}$ is linear with respect to the second component and defined by

$$\mathcal{K}(u,v)=-\mathbb{P}((v\cdot\nabla)u+(u\cdot\nabla)v)-\mathbb{P}(\alpha|u|^{\beta-1}v+\alpha(\beta-1)|u|^{\beta-3}\langle u,v\rangle_{R^3}u).$$

Inspired by [30, 32], we get that for each ω

$$(\mathcal{J}_{0,t}v_0)(\omega)=\lim_{\epsilon\to0}\frac{u(t,\omega;u_0+\epsilon v_0)-u(t,\omega;u_0)}{\epsilon}. \tag{3.18}$$

Let $\mathcal{H}$ be the Cameron–Martin space and for any $v\in\mathcal{H}$, the Malliavin derivative is given by

$$D^v u(t,\omega;u_0)=\lim_{\epsilon\to 0}\frac{u(t,\omega+\epsilon v;u_0)-u(t,\omega;u_0)}{\epsilon},\quad \mathrm{P-a.s.} \tag{3.19}$$

Observe that v can be random and possibly nonadapted to the filtration generated by W. Let $\mathcal{A}_t v=D^v u(t,\omega;u_0)$, then

$$\partial_t\mathcal{A}_t v=\Delta\mathcal{A}_t v+\mathcal{K}(u(t,\omega;u_0),\mathcal{A}_t v)+Q\dot{v}(t),\quad \mathcal{A}_0 v=0, \tag{3.20}$$

where, $\dot{v}(t)$ is the derivative of $v(t)$ with respect to t.
Next, $\mathcal{A}_t$ is defined by

$$\mathcal{A}_t v=\int_0^t \mathcal{J}_{s,t}Q\dot{v}(s)ds.$$

For $v_0\in V$ and $v\in\mathcal{H}$, let

$$v(t)=\mathcal{J}_{0,t}v_0-\mathcal{A}_t v.$$

We have

$$\partial_t v(t)=\Delta v(t)+\mathcal{K}(u(t),v(t))-Q\dot{v}(t),\quad v(0)=v_0.$$

Let $\mathbb{H}_l^1$ be the following finite-dimensional "low-frequency" subsequence of V,

$$\mathbb{H}_l^1=\mathrm{span}\{e_1,e_2,\cdots,e_m\},$$

and $\mathbb{H}_h^1$ be the "high-frequency" subspace of V. We get the following direct sum decomposition:

$$\mathbb{H}=\mathbb{H}_l^1\oplus\mathbb{H}_h^l,$$

such that for any $u\in V$

$$u=u_l+u_h,\quad u_l\in\mathbb{H}_l^1,\quad u_h\in\mathbb{H}_h^1.$$

For any $v\in H$, we give

$$\Pi_l v=\sum_{i=1}^m\langle(-\Delta)e_i,v\rangle e_i\in\mathbb{H}_l^1,$$

and

$$\Pi_h v=v-\Pi_l v.$$

LEMMA 3.2. *For any $u,v\in H^2$, let*

$$\mathcal{N}(u)=||\nabla u||^2+|u|_{3(\beta-1)}^{2(\beta-1)}. \tag{3.21}$$

We deduce

$$\langle v_h,\mathcal{K}(u,v)\rangle_V\le\frac{1}{2}||\nabla v_h||^2+C\mathcal{N}(u)(||v_h||^2+||v_l||^2), \tag{3.22}$$

and

$$||\Pi_l\mathcal{K}(u,v)||^2 \le C_m|v|_2(1+|u|_{2(\beta-1)}^{\beta-1}), \tag{3.23}$$

here, C_m only depends on m.

Proof. For the first inequality (3.22), we deduce

$$\langle v_h, \mathcal{K}(u,v)\rangle_V = I_1 + I_2 + I_3, \tag{3.24}$$

where,

$$\begin{aligned}
I_1 &= -\langle v_h, \mathbb{P}((v\cdot\nabla)u)\rangle_V,\\
I_2 &= -\langle v_h, \mathbb{P}((u\cdot\nabla)v)\rangle_V,\\
I_3 &= -\langle v_h, \mathbb{P}(\alpha|u|^{\beta-1}v + \alpha(\beta-1)|u|^{\beta-3}\langle u,v\rangle_{R^3}u)\rangle_V.
\end{aligned}$$

For the term I_1, applying the Young's inequality, we deduce

$$\begin{aligned}
I_1 &\le \frac{1}{8}||\nabla v_h||^2 + 2||v|\cdot|\nabla u||_2^2\\
&\le \frac{1}{8}||\nabla v_h||^2 + 2|v|_6^2\cdot|\nabla u|_3^2\\
&\le \frac{1}{8}||\nabla v_h||^2 + C||v_h||^2\cdot||\nabla u||^2 + C||v_l||^2\cdot||\nabla u||^2.
\end{aligned} \tag{3.25}$$

For the term I_2, we get

$$\begin{aligned}
I_2 &\le \frac{1}{8}||\nabla v_h||^2 + 2||u|\cdot|\nabla v||_2^2\\
&\le \frac{1}{8}||\nabla v_h||^2 + 2|u|_\infty^2\cdot|\nabla v|_2^2\\
&\le \frac{1}{8}||\nabla v_h||^2 + C||v_h||^2\cdot||\nabla u||^2 + C||v_l||^2\cdot||\nabla u||^2.
\end{aligned} \tag{3.26}$$

For the term I_3, we get

$$\begin{aligned}
I_3 &\le \frac{1}{4}||\nabla v_h||^2 + C||u|^{\beta-1}\cdot|v||_2^2\\
&\le \frac{1}{4}||\nabla v_h||^2 + C|v|_6^2|u|_{3(\beta-1)}^{2(\beta-1)}\\
&\le \frac{1}{4}||\nabla v_h||^2 + C||v_h||^2|u|_{3(\beta-1)}^{2(\beta-1)} + C||v_l||^2|u|_{3(\beta-1)}^{2(\beta-1)}.
\end{aligned} \tag{3.27}$$

By the above calculations, this completes the proof of inequality (3.22).

For the second inequality (3.23), we have

$$||\Pi_l\mathcal{K}(u,v)||^2 = \sum_{i=1}^{m}\langle e_i, \mathcal{K}(u,v)\rangle_V^2 = \sum_{i=1}^{m}\sum_{j=1}^{3} J_{ij}, \tag{3.28}$$

here,

$$\begin{aligned}
J_{i1} &= -\langle e_i, \mathbb{P}((v\cdot\nabla)u)\rangle_V,\\
J_{i2} &= -\langle e_i, \mathbb{P}((u\cdot\nabla)v)\rangle_V,
\end{aligned}$$

$$J_{i3}=-\langle e_i,\mathbb{P}(\alpha|u|^{\beta-1}v+\alpha(\beta-1)|u|^{\beta-3}\langle u,v\rangle_{R^3}u)\rangle_V.$$

For the term J_{i1}, we deduce

$$\begin{aligned}J_{i1}&=-\langle e_i,\mathbb{P}((v\cdot\nabla)u)\rangle_V\\&\le|\nabla\Delta e_i|_\infty||v|u|_{L^1}\\&\le C_{e_i}|v|_2|u|_2.\end{aligned}\tag{3.29}$$

Similarly, we get

$$J_{i2}\le C_{e_i}|v|_2|u|_2,\tag{3.30}$$
$$J_{i3}\le C_{e_i}|v|_2|u|^{\beta-1}_{2(\beta-1)}.\tag{3.31}$$

We get by adding them together

$$||\Pi_l\mathcal{K}(u,v)||^2\le C_m|v|_2(1+|u|^{\beta-1}_{2(\beta-1)}),$$

where, C_m only depends on m. This completes the proof of inequality (3.23). □

LEMMA 3.3. *For any $\eta>0$, then there exist positive constants C_1, $C_2>0$ such that for any $t>0$ and $u_0\in V$,*

$$\mathbb{E}\exp\{\eta\int_0^t\mathcal{N}(u(s;u_0))ds\}\le\exp\{C_1||u_0||^2+C_2t\},\tag{3.32}$$

here, $\mathcal{N}(u)$ is given by definition (3.21).

Proof. Applying Itô's formula to $|u(t)|_2^2$, we get

$$|u(t)|_2^2=|u_0|_2^2-2\int_0^t||u(s)||^2ds-2\int_0^t|u(s)|^{\beta+1}_{\beta+1}ds+2\int_0^t\langle u(s),d\mathbf{w}(s)\rangle+\zeta_0t.\tag{3.33}$$

Applying Hölder's inequality, then there exists a constant $C>0$ such that

$$|u(t)|_2^2-C\le|u(t)|^{\beta+1}_{\beta+1}.$$

We have

$$|u(t)|_2^2\le|u_0|_2^2+2\int_0^t(C-||u(s)||^2-|u(s)|_2^2)ds+2\int_0^t\langle u(s),d\mathbf{w}(s)\rangle+\zeta_0t.\tag{3.34}$$

Applying ([32], Lemma 6.2), we have that for any $t,\eta>0$,

$$\mathbb{E}\exp\{\eta\int_0^t||u(s)||^2ds\}\le\exp\{\eta|u_0|_2^2+C_\eta t\}.\tag{3.35}$$

Next, applying the Itô's formula to $||u(t)||^2$, we get

$$\begin{aligned}||u(t)||^2\le&||u_0||^2-\int_0^t||\nabla u(s)||^2ds-\int_0^t\Big||\nabla u||u|^{\frac{\beta-1}{2}}\Big|_2^2ds-\frac{4(\beta-1)}{(\beta+1)^2}\int_0^t|||u|^{\frac{\beta+1}{2}}||^2ds\\&+C\int_0^t||u(s)||^2ds+2\int_0^t\langle u(s),d\mathbf{w}(s)\rangle_V+\int_0^t\zeta_1ds.\end{aligned}\tag{3.36}$$

Applying Poincare's inequality, then there exists a constant $c>0$ such that

$$|u(s)|_{3(\beta-1)}^{2(\beta-1)}-c\leq\frac{4(\beta-1)}{(\beta+1)^2}\||u|^{\frac{\beta+1}{2}}\|^2.$$

Then, we have

$$\begin{aligned}||u(t)||^2\leq&||u_0||^2+C\int_0^t(-||\nabla u(s)||^2-|u(s)|_{3(\beta-1)}^{2(\beta-1)}+||u(s)||^2)ds\\&+2\int_0^t\langle u(s),d\mathbf{w}(s)\rangle_V+\int_0^t(c+\zeta_1)ds.\end{aligned}\tag{3.37}$$

Applying inequality (3.37) and exponential martingales, then we obtain inequality (3.32). ∎

Inspired by ([20], Proposition 3.12), we show the following proposition which imply the asymptotically strong Feller property of $(\mathcal{P}_t)_{t\geq 0}$.

PROPOSITION 3.1. *Let $(\mathcal{P}_t)_{t\geq 0}$ be the semigroup associate with the 3D SNS with damping (3.16). Then there exist constant $m_*\in N$ and some constants $C_0,C_1,\gamma>0$ such that for any $t>0$, $u_0\in V$, and any Fréchet differentiable function φ on V with $|\varphi|_\infty$, $\|\varphi\|_\infty<\infty$,*

$$||\nabla\mathcal{P}_t\varphi(u_0)||\leq C_0\exp\{C_1||u_0||^2\}\cdot(|\varphi|_\infty+e^{-\gamma t}\|\varphi\|_\infty).$$

Proof. For any $v_0\in V$ with $||v_0||=1$, denote

$$v_l(t)=\begin{cases}v_{0l}(1-\frac{t}{2||v_{0l}||}), & t\in[0,2||v_{0l}||],\\ 0, & t\in(2\|v_{0l}\|,\infty).\end{cases}\tag{3.38}$$

Suppose that $v_h(t)$ satisfies the following linear evolution equation:

$$\partial_t v_h(t)=\Delta\Pi_h v_h(t)+\Pi_h\mathcal{K}(u(t),v_h(t)+v_l(t)),\quad v_h(0)=v_{0h}.$$

Let

$$v(t)=v_h(t)+v_l(t),$$

and

$$\dot{v}(t)=Q^{-1}(\frac{v_l\cdot 1_{\{t<2||v_l||\}}}{2||v_l||}+\Delta v_l(t)+\Pi_l\mathcal{K}(u(t),v(t))).$$

Applying the chain rule and integration by parts formula in the Malliavin calculus [28], we deduce

$$\begin{aligned}\langle\nabla\mathcal{P}_t\varphi(u_0),v_0\rangle_V=&\mathbb{E}\langle(\nabla\varphi)(u(t;u_0)),\mathcal{J}_{0,t}v_0\rangle_V\\=&\mathbb{E}\langle(\nabla\varphi)(u(t;u_0)),\mathcal{A}_t v(t)\rangle_V+\mathbb{E}\langle(\nabla\varphi)(u(t;u_0)),v(t)\rangle_V\\=&\mathbb{E}(D^\nu(\varphi(u(t;u_0))))+\mathbb{E}\langle(\nabla\varphi)(u(t;u_0)),v(t)\rangle_V\\=&\mathbb{E}(\varphi(u(t;u_0))\cdot\int_0^t\dot{v}(s)dW(s))+\mathbb{E}\langle(\nabla\varphi)(u(t;u_0)),v(t)\rangle_V\\\leq&|\varphi|_\infty\int_0^t\mathbb{E}|\dot{v}(s)|^2ds+||\varphi||_\infty E||v(t)||.\end{aligned}\tag{3.39}$$

Applying the chain rule and Lemmas 3.2 and 3.3, we get

$$\begin{aligned}\frac{d}{dt}||v_h(t)||^2 &= -2||\nabla_h v_h(t)||^2 + 2\langle v_h(t), \Pi_h \mathcal{K}(u(t), v(t))\rangle_V \\ &\leq -||\nabla_h v_h(t)||^2 + C\cdot\mathcal{N}(u(t))\cdot(||v_h(t)||^2 + ||v_l(t)||^2) \\ &\leq (-\lambda_m + C\cdot\mathcal{N}(u(t)))\cdot||v_h(t)||^2 + C\cdot\mathcal{N}(u(t))\cdot||v_l(t)||^2.\end{aligned}$$

Notice that

$$v_l(t) = 0 \quad \text{for } t \geq 2,$$

applying Gronwall's inequality, we have

$$\begin{aligned}||v_h(t)||^2 \leq & ||v_h(0)||^2 \exp\{-\lambda_m t + C\int_0^t \mathcal{N}(u(s))ds\} \\ & + \exp\{-\lambda_m(t-2) + C\int_0^t \mathcal{N}(u(s))ds\}\int_0^2 ||v_l(s)||^2 ds.\end{aligned}$$

By virtue of Lemma 3.3, since $\lambda_m \to \infty$ as $m \to \infty$ and $||v_l(t)|| \leq 1$ for $0 \leq t \leq 2$, then there exist positive constants γ and m_* such that

$$E||v_h(t)||^2 \leq C_{\lambda_1}\cdot \mathrm{e}^{C||u_0||^2 - \gamma t}.$$

Therefore, for all $t \geq 2$

$$E||v(t)|| \leq C_{\lambda_1}\cdot \mathrm{e}^{C||u_0||^2 - \gamma t}. \tag{3.40}$$

By Lemma 3.2, we get

$$\begin{aligned}E|\dot{v}(t)|^2 &\leq C_m(1 + \mathbb{E}(|v(t)|_2(1 + |u(t)|^{\beta-1}_{2(\beta-1)}))) \\ &\leq C_m(1 + (\mathbb{E}|v(t)|_2^2)^{\frac{1}{2}}(1 + \mathbb{E}|u(t)|^{2(\beta-1)}_{2(\beta-1)})^{\frac{1}{2}}) \\ &\leq C_m(1 + (\mathbb{E}|v(t)|_2^2)^{\frac{1}{2}}(1 + \mathbb{E}|u(t)|^{2(\beta-1)}_{2(\beta-1)})).\end{aligned}$$

Since $\int_0^t \mathbb{E}|u(s)|^{2(\beta-1)}_{2(\beta-1)} ds < C$ and applying inequality (3.40), we get

$$\begin{aligned}\int_0^\infty E|\dot{v}(t)|^2 dt &\leq C_{m,\lambda_1,\gamma}\cdot \mathrm{e}^{C||u_0||^2}\cdot(1 + \int_0^\infty \mathrm{e}^{-\gamma t}\mathbb{E}|u(t)|^{2(\beta-1)}_{2(\beta-1)} dt) \\ &\leq C_{m,\lambda_1,\gamma}\cdot \mathrm{e}^{C||u_0||^2}.\end{aligned}$$

This completes the proof of Proposition 3.1. □

3.2.2. A support property of invariant measures. Inspired by [12,20,30,32], we prove the following proposition.

PROPOSITION 3.2. *0 belongs to the support of any invariant measure of* $\{\mathcal{P}_t\}_{t\geq 0}$.

We need the following lemma result.

LEMMA 3.4. *For any* r_1, $r_2 > 0$, *then there exists* $T > 0$ *such that*

$$\inf_{||u_0|| \leq r_1} P\{\omega : ||u(T,\omega;u_0)|| \leq r_2\} > 0.$$

Proof. Let

$$v(t)=u(t)-\mathbf{w}(t),$$

then

$$\begin{aligned}v'(t)=&-A(v(t)+\mathbf{w}(t))-((v(t)+\mathbf{w}(t))\cdot\nabla)(v(t)+\mathbf{w}(t))\\&-\alpha|v(t)+\mathbf{w}(t)|^{\beta-1}(v(t)+\mathbf{w}(t)).\end{aligned}\tag{3.41}$$

Let $T>0$ and $\epsilon\in(0,1)$, to be determined below. Suppose that

$$\sup_{t\in[0,T]}||\mathbf{w}(t)||_{H^6}<\epsilon.\tag{3.42}$$

Taking the inner product of equation (3.41) with $v(t)$ in L^2, we get

$$\frac{d}{dt}|v(t)|_2^2=J_1+J_2+J_3+J_4,\tag{3.43}$$

where,

$$\begin{aligned}J_1&=-2||v(t)||^2+2\langle\Delta\mathbf{w}(t),v(t)\rangle,\\J_2&=-2\langle v(t),((v(t)+\mathbf{w}(t))\cdot\nabla)(v(t)+\mathbf{w}(t))\rangle,\\J_3&=-2\langle v(t)+\mathbf{w}(t),\alpha|v(t)+\mathbf{w}(t)|^{\beta-1}(v(t)+\mathbf{w}(t))\rangle,\\J_4&=2\langle\mathbf{w}(t),\alpha|v(t)+\mathbf{w}(t)|^{\beta-1}(v(t)+\mathbf{w}(t))\rangle.\end{aligned}$$

For the term J_1, by the above inequality (3.42), we get

$$J_1\le-2||v(t)||^2+C\epsilon|v(t)|_2.\tag{3.44}$$

For the term J_2, applying the Sobolev inequality, we get

$$\begin{aligned}J_2=&-2\langle v(t),((v(t)+\mathbf{w}(t))\cdot\nabla)(v(t)+\mathbf{w}(t))\rangle,\\\le&C||\mathbf{w}(t)||_\infty|v(t)+\mathbf{w}(t)|_2^2\\\le&C\epsilon|v(t)|_2^2+C\epsilon.\end{aligned}\tag{3.45}$$

For the term J_3, we get

$$J_3\le-|v(t)|_{\beta+1}^{\beta+1}+C\epsilon.\tag{3.46}$$

For the term J_4, we have

$$\begin{aligned}J_4\le&2|\mathbf{w}(t)|_\infty|v(t)+\mathbf{w}(t)|_\beta^\beta\\\le&C\epsilon|v(t)|_\beta^\beta+C\epsilon\\\le&C\epsilon|v(t)|_{\beta+1}^{\beta+1}+C\epsilon|v(t)|_2^2+C\epsilon.\end{aligned}\tag{3.47}$$

We have by adding them together

$$\frac{d}{dt}|v(t)|_2^2\le-2\lambda_1|v(t)|_2^2+C\epsilon|v(t)|_2^2+(C\epsilon-1)|v(t)|_{\beta+1}^{\beta+1}+C\epsilon.\tag{3.48}$$

Hence, by Lemma 6.1 in [32], we deduce that for any $\delta, h > 0$, then there exist $T_0 > 0$ large enough and ϵ sufficiently small such that

$$\sup_{t\in[0,T_0]} |v(t)|_2 \leq 2r_1, \tag{3.49}$$

and

$$\sup_{t\in[T_0,T_0+h]} |v(t)|_2 \leq \delta. \tag{3.50}$$

Taking the inner product of equation (3.41) with $-\Delta v(t)$ in L^2, we get

$$\frac{d}{dt}||v(t)||^2 \leq I_1 + I_2 + I_3 + I_4, \tag{3.51}$$

where

$$\begin{aligned}
&I_1 = 2\langle F(v(t)+\mathbf{w}(t)), v(t)+\mathbf{w}(t)\rangle_V,\\
&I_2 = 2|\langle \Delta^2 \mathbf{w}(t), v(t)+\mathbf{w}(t)\rangle|,\\
&I_3 = 2|\langle \Delta \mathbf{w}(t), ((v(t)+\mathbf{w}(t))\cdot\nabla)(v(t)+\mathbf{w}(t))\rangle|,\\
&I_4 = 2|\langle \Delta \mathbf{w}(t), \alpha|v(t)+\mathbf{w}(t)|^{\beta-1}(v(t)+\mathbf{w}(t))\rangle|.
\end{aligned}$$

For the term I_1, we get

$$\begin{aligned}
I_1 \leq & -||\nabla(v(t)+\mathbf{w}(t))||^2 - \left||v(t)+\mathbf{w}(t)|^{\frac{\beta-1}{2}}\nabla(v(t)+\mathbf{w}(t))\right|_2^2 + C||v(t)+\mathbf{w}(t)||^2\\
& - \frac{4(\beta-1)}{(\beta+1)^2}\left|\nabla|v(t)+\mathbf{w}(t)|^{\frac{\beta+1}{2}}\right|_2^2\\
\leq & -\frac{1}{2}||\nabla v(t)||^2 + C||v(t)||^2 - c|v(t)|_{\beta+1}^{\beta+1} + C\epsilon\\
\leq & -\frac{1}{4}||\nabla v(t)||^2 + C|v(t)|_2^2 - c|v(t)|_{\beta+1}^{\beta+1} + C\epsilon.
\end{aligned} \tag{3.52}$$

For the term I_2, by inequality (3.42) we get

$$I_2 \leq C\epsilon|v(t)|_2^2 + C\epsilon. \tag{3.53}$$

For the term I_3, by inequality (3.42) we get

$$\begin{aligned}
I_3 \leq & 2|\nabla\Delta\mathbf{w}|_\infty |v(t)+\mathbf{w}(t)|_2^2\\
\leq & C\epsilon|v(t)|_2^2 + C\epsilon.
\end{aligned} \tag{3.54}$$

For the term I_4, we get

$$\begin{aligned}
I_4 \leq & C\epsilon|v(t)|_\beta^\beta + C\epsilon\\
\leq & C\epsilon|v(t)|_{\beta+1}^{\beta+1} + C\epsilon|v(t)|_2^2 + C\epsilon.
\end{aligned} \tag{3.55}$$

We have by adding them together

$$\frac{d}{dt}||v(t)||^2 \leq -C_0||v(t)||^2 + C|v(t)|_2^2 + C\epsilon|v(t)|_2^2 + (C\epsilon - c)|v(t)|_{\beta+1}^{\beta+1} + C\epsilon. \tag{3.56}$$

Applying Gronwall's inequality, for any $0<t_1<t_2$, we get

$$
\begin{aligned}
||v(t_2)||^2 \le & e^{-C_0(t_2-t_1)}||v(t_1)||^2 + \frac{1}{C_0}(C \sup_{t\in[t_1,t_2]} |v(t)|_2^2 \\
& + C\epsilon \sup_{t\in[t_1,t_2]} |v(t)|_2^2 + (C\epsilon - c) \sup_{t\in[t_1,t_2]} |v(t)|_{\beta+1}^{\beta+1} + C\epsilon).
\end{aligned} \tag{3.57}
$$

Let $t_1=0$ and $t_1=T_0$, and by inequality (3.49), we have

$$
\begin{aligned}
||v(T_0)||^2 \le & r_1^2 + \frac{1}{C_0}(C \sup_{t\in[0,T_0]} |v(t)|_2^2 \\
& + C\epsilon \sup_{t\in[0,T_0]} |v(t)|_2^2 + (C\epsilon - c) \sup_{t\in[0,T_0]} |v(t)|_{\beta+1}^{\beta+1} + C\epsilon) \\
\le & C(r_1^2+1).
\end{aligned} \tag{3.58}
$$

Next, let $t_1=T_0$ and $t_2=T_0+h$, we have

$$
\begin{aligned}
||v(T_0+h)||^2 \le & e^{-C_0 h} C(r_1^2+1) + \frac{1}{C_0}(C \sup_{t\in[T_0,T_0+h]} |v(t)|_2^2 \\
& + C\epsilon \sup_{t\in[T_0,T_0+h]} |v(t)|_2^2 + (C\epsilon - c) \sup_{t\in[T_0,T_0+h]} |v(t)|_{\beta+1}^{\beta+1} + C\epsilon),
\end{aligned} \tag{3.59}
$$

which together with inequality (3.50) implies that there exist a T sufficiently large and ϵ sufficiently small such that

$$
||v(T)|| \le \frac{r_2}{2}. \tag{3.60}
$$

Hence, then there exist T large enough and $\epsilon\in(0,1)$ small enough such that for any $||u_0||\le r_1$

$$
||u(T,\omega;u_0)|| \le r_2. \tag{3.61}
$$

Let

$$
\Omega_\epsilon = \{\omega : \sup_{t\in[0,T]} ||\mathbf{w}(t,\omega)||_{H^6} < \epsilon\}, \tag{3.62}
$$

then

$$
\Omega_\epsilon \subset \cap_{||u_0||\le r_1} \{\omega : ||u(T,\omega;u_0)|| \le r_2\}. \tag{3.63}
$$

Since Ω_ϵ is an open set and $P(\Omega_\epsilon)>0$, this completes the proof of Lemma 3.4. ∎

Proof. (**Proof of Proposition 3.2.**) Suppose that $B_r=\{u_0\in V : ||u_0||\le r\}$ denotes the ball in V for $r>0$. For any invariant measure μ, then there exists a constant $r_1>0$ such that

$$
\mu(B_{r_1}) \ge \frac{1}{2}.
$$

By the above lemma, we get for any $r_2>0$,

$$
\mu(B_{r_2}) = \mathcal{P}_t^* \mu(B_{r_2}) = \int_V \mathcal{P}_t(x, B_{r_2}) \mu(dx)
$$

$$
\begin{aligned}
&= \int_V \mathcal{P}_t 1_{B_{r_2}}(x)\mu(dx) \geq \int_{B_{r_1}} \mathcal{P}_t 1_{B_{r_2}}(x)\mu(dx) \\
&\geq \mu(B_{r_1}) \inf_{x\in B_{r_1}} \mathcal{P}_t 1_{B_{r_2}}(x) > 0,
\end{aligned}
$$

which implies that 0 belongs to the support of μ. This completes the proof of Proposition 3.2. □

4. Random attractor

In this section, let $\sigma(t,u)=\rho u$, we consider the following stochastic 3D Navier–Stokes equations with damping driven by a multiplicative white noise.

$$
\begin{cases}
du-\nu\Delta u dt+(u\cdot\nabla)udt+\nabla pdt+\alpha|u|^{\beta-1}udt=\rho udW(t),\\
\nabla\cdot u=0,\\
u(t,x)|_{\partial D}=0,\\
u(x,0)=u_0(x).
\end{cases}
\tag{4.1}
$$

Where $W(t)$ is two-sided real-valued Wiener process on a probability space $(\Omega,\mathcal{F},P)$. We consider the probability space $(\Omega,\mathcal{F},P)$, where

$$\Omega=\{\omega\in C(R,R):\omega(0)=0\}.$$

$\mathcal{F}$ is the Borel $\sigma-$algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega,\mathcal{F})$.

Taking $W(t,\omega)=\omega(t)$, the time shift is defined by

$$\theta_t\omega(s)=\omega(t+s)-\omega(t), \quad \text{for } s,t\in R.$$

The process

$$z(t)=e^{-\rho W(t)}$$

satisfies the following stochastic differential equation:

$$dz(t)=\frac{1}{2}\rho^2 zdt-\rho zdW(t).$$

Let $v(t)=u(t)z(t)$, we get the following random differential equation

$$\frac{dv(t)}{dt}=-Av(t)-z^{-1}(t)B(v,v)+\frac{1}{2}\rho^2 v-z^{-(\beta-1)}\alpha|v|^{\beta-1}v. \tag{4.2}$$

For each $\omega\in\Omega$, if $s\in\mathbb{R}$, there exists a unique solution to equation (4.2) defined on $[s,\infty)$ such that

$$v(s,\omega)=v_s(\omega), \qquad \mathrm{P}-\text{a.s.}$$

LEMMA 4.1. *Suppose that $\beta>3$ with any $\alpha>0$ and $\alpha\geq\frac{1}{2}$ as $\beta=3$, and $u_s\in L^2$ with $|u(s)|_2\leq R$, then there exists a random radius $r_0(\omega)$ such that, for any given $R>0$, there exists $s_1(\omega)\leq -1$ such that for all $s\leq s_1(\omega)$, for $\mathbb{P}-a.s.$ $\omega\in\Omega$, we have the following inequality*

$$|v(t)|_2^2+e^{-t}\int_s^t e^{\tau}|v(\tau)|_2^2 d\tau\leq r_0^2(\omega), \quad \forall t\in[-1,0], \tag{4.3}$$

here, r_0 is to be determined.

Proof. Multiplying the equation (4.2) in L^2 by v, we deduce that

$$\frac{1}{2}\frac{d}{dt}|v|_2^2+||v||^2+\alpha z^{-(\beta-1)}|v|_{\beta+1}^{\beta+1}=-z^{-1}(t)b(B(v,v),v)+\frac{1}{2}\rho^2|v|_2^2. \tag{4.4}$$

Using Young's inequality, then there exists a constant C such that

$$(\rho^2+2)|v|_2^2\leq \alpha z^{-(\beta-1)}|v|_{\beta+1}^{\beta+1}+Cz^2.$$

Hence, we have

$$\frac{d}{dt}|v|_2^2+2||v||^2+az^{-(\beta-1)}|v|_{\beta+1}^{\beta+1}+2|v|_2^2\leq Cz^2. \tag{4.5}$$

For any $t>s$, we have

$$\begin{aligned}|v(t)|_2^2+e^{-t}\int_s^t e^{\tau}|v(\tau)|_2^2d\tau \leq& e^{-t}(e^s|v(s)|_2^2+C\int_s^t e^{\tau}z^2(\tau)d\tau)\\ =&e^{-t}(e^sz^2|u(s)|_2^2+C\int_s^t e^{\tau}z^2(\tau)d\tau).\end{aligned} \tag{4.6}$$

Since

$$\lim_{s\to-\infty}\frac{W(s)}{s}=0,\quad \mathrm{P-a.s},$$

let $\varepsilon=\frac{1}{4\rho}$, then exists a function $s_1'(\omega)\leq -1$ such that

$$\left|\frac{W(s)}{s}\right|<\varepsilon,\quad \text{as } s<s_1'(\omega).$$

Hence, we have

$$\begin{aligned}e^sz^2(s)=&e^{s(1-2\rho W(s)/s)}\\ <&e^{\frac{1}{2}s}.\end{aligned}$$

As $s\to-\infty$, it is easy to get that

$$e^sz^2(s)\to 0,\quad \mathrm{P-a.s.}$$

For the last term on the right of inequality (4.6), we have

$$\begin{aligned}C\int_s^t e^{\tau}z^2(\tau)d\tau \leq& C\int_{-\infty}^0 e^{\tau}e^{-2\rho W(\tau)}d\tau\\ \leq& C(\int_{s_1'(\omega)}^0 e^{\tau-2\rho W(\tau)}d\tau+\int_{-\infty}^{s_1'(\omega)}e^{\frac{\tau}{2}}d\tau).\end{aligned}$$

For $u_s\in L^2$ with $|u(s)|_2\leq R$, then there exists a time $s_1(\omega)\leq s_1'(\omega)$ such that

$$e^sz^2|u(s)|_2^2\leq e^sz^2R^2\leq 1,\quad \mathrm{P-a.s.},\ \text{for } \forall s\leq s_1(\omega). \tag{4.7}$$

For $t\in[-1,0]$, we get

$$|v(t)|_2^2+e^{-t}\int_s^t e^{\tau}|v(\tau)|_2^2 d\tau \le e^2(1+C(\int_{s_1'(\omega)}^0 e^{\tau-2\rho W(\tau)}d\tau+\int_{-\infty}^{s_1'(\omega)} e^{\frac{\tau}{2}}d\tau))$$
$$\triangleq r_0^2(\omega). \tag{4.8}$$

This completes the proof of Lemma 4.1. □

LEMMA 4.2. *Suppose that $\beta>3$ with any $\alpha>0$ and $\alpha\ge\frac{1}{2}$ as $\beta=3$, and $u_s\in V$ with $||u(s)||\le R$, then there exists a random radius $r_1(\omega)$ such that, for any given $R>0$, there exists $s_2(\omega)\le -1$ such that for all $s\le s_2(\omega)$, for $\mathbb{P}-a.s.$ $\omega\in\Omega$, we have the following inequality*

$$||v(t)||^2+e^{-t}\int_s^t e^{\tau}||\nabla v(\tau)||^2 d\tau \le r_1^2(\omega), \quad \forall t\in[-1,0], \tag{4.9}$$

here, $r_1^2=e^2+Cr_0^2(\omega)$.

Proof. Multiplying equation (4.2) in L^2 by $-\Delta v$, we deduce that

$$\frac{d}{dt}||v||^2+2||\nabla v||^2-2\alpha z^{-(\beta-1)}(|v|^{\beta-1}v,\Delta v)=2z^{-1}(t)b(B(v,v),\Delta v)+\rho^2||v||^2. \tag{4.10}$$

Moreover,

$$\frac{d}{dt}||v||^2+2||\nabla v||^2+2\alpha z^{-(\beta-1)}\int_D |v|^{\beta-1}|\nabla v|^2 ds+\frac{\alpha(\beta-1)}{2}z^{-(\beta-1)}\int_D |v|^{\beta-3}|\nabla|v|^2|^2 ds$$
$$=2z^{-1}(t)b(B(v,v),\Delta v)+\rho^2||v||^2. \tag{4.11}$$

For the first term on the right-hand side, since

$$0<\frac{2}{\beta-1}<1, \quad \text{for } \beta>3, \tag{4.12}$$

using Young's inequality, we deduce

$$\begin{aligned} 2z^{-1}(t)b(B(v,v),\Delta v) &\le \frac{1}{2}|\Delta v|_2^2+Cz^{-2}(t)|v\cdot\nabla v|_2^2 \\ &\le \frac{1}{2}||\nabla v||^2+C\int_D z^{-2}(t)|v|^2|\nabla v|^{\frac{4}{\beta-1}}|\nabla v|^{2-\frac{4}{\beta-1}}ds \\ &\le \frac{1}{2}||\nabla v||^2 \\ &\quad +C[\int_D (z^{-2}|v|^2|\nabla v|^{\frac{4}{\beta-1}})^{\frac{\beta-1}{2}}ds]^{\frac{2}{\beta-1}}[\int_D (|\nabla v|^{2-\frac{4}{\beta-1}})^{\frac{\beta-1}{\beta-3}}ds]^{\frac{\beta-3}{\beta-1}} \\ &\le \frac{1}{2}||\nabla v||^2+\varepsilon\int_D z^{-(\beta-1)}|\nabla v|^2|v|^{\beta-1}ds+C(\varepsilon)||v||^2. \end{aligned} \tag{4.13}$$

Using the Gagliardo–Nirenberg inequality, we have

$$(C(\varepsilon)+\rho^2+1)||v||^2\le\frac{1}{2}||\nabla v||^2+C|v|_2^2. \tag{4.14}$$

Hence

$$\frac{d}{dt}||v||^2+||\nabla v||^2+(2\alpha-\varepsilon)z^{-(\beta-1)}\int_D |v|^{\beta-1}|\nabla v|^2 ds$$

$$+\frac{\alpha(\beta-1)}{2}z^{-(\beta-1)}\int_D |v|^{\beta-3}|\nabla|v|^2|^2ds+||v||^2\le C|v|_2^2. \tag{4.15}$$

Choosing sufficiently small ε and multiplying equation (4.15) by e^t and integrating over (s,t), we get

$$\begin{aligned}||v(t)||^2+e^{-t}\int_s^t e^{\tau}||\nabla v||^2d\tau \le & e^{-t}(e^s||v(s)||^2+C\int_s^t e^{\tau}|v|_2^2d\tau)\\ =& e^{-t}(e^s z^2(s)||u(s)||^2+C\int_s^t e^{\tau}|v|_2^2d\tau)\\ \le & e^{-t}e^s z^2(s)||u(s)||^2+Cr_0^2(\omega).\end{aligned}$$

For $u_s\in V$ with $||u(s)||\le R$, then there exists a time $s_2(\omega)\le s_2'(\omega)$ such that

$$e^s z^2||u(s)||^2\le e^s z^2R^2\le 1, \quad \mathrm{P-a.s.}, \text{ for } \forall s\le s_2(\omega). \tag{4.16}$$

For $t\in[-1,0]$, we get

$$\begin{aligned}||v(t)||^2+e^{-t}\int_s^t e^{\tau}||\nabla v||^2d\tau \le & e^2+Cr_0^2(\omega)\\ \triangleq & r_1^2(\omega).\end{aligned}$$

For $\alpha\ge\frac{1}{2}$ as $\beta=3$, we can get the above estimate easily. This completes the proof of Lemma 4.2. ∎

Let $K(\omega)$ be the ball in V with radius $r_0(\omega)+r_1(\omega)$. Lemmas 4.1-4.2 prove that for any B in V, then there exists an $s_2(\omega)$ such that for any $s\le s_2$,

$$S(0,s;\omega)B\subset K(\omega) \quad \mathrm{P-a.e.} \quad \omega\in\Omega.$$

This implies that $K(\omega)$ is an attracting set at time 0. Hence, by Theorem 2.1, we get the following result refer in [9].

THEOREM 4.1. *Assume the conditions of Lemmas 4.1-4.2 hold. Then the random dynamical system generated by the solutions of the stochastic 3D Navier–Stokes equations with damping has a compact random attractors in V.*

Appendix A.

DEFINITION A.1. *A $\mathcal{F}_t-$adapted V-valued process $u(t)$ is said to be a strong solution of problem (3.1) if the following conditions are satisfied*

(1) $u(t)\in L^2(\Omega,L^2([0,T],H^2))\cap L^2(\Omega,L^{\infty}([0,T],V))$;

(2) For every $t\in[0,T]$ and $\mathcal{F}_0-$adapted V-valued function u_0, the following equality holds P-a.s.

$$u(t)=u_0-\int_0^t Au(s)ds-\int_0^t B(u(s))ds-\int_0^t g(u(s))ds+\int_0^t \sigma(s,u(s))dW(s).$$

Proof. **(Proof of Theorem 3.1.)** We first consider the Galerkin approximation of (3.1). The following lemma provides the existence and uniqueness of approximate solutions and uniform estimate. This is the main preliminary step in the proof of Theorem 3.1. ∎

LEMMA A.1. *Under the same assumptions as in Theorem 3.1, if* $p \geq 1$*, there exists a positive constant* C *such that*

$$\sup_{n\geq 1}\mathbb{E}\Big(\sup_{0\leq t\leq T}||u_n(t)||^{2p}+\int_0^T||u_n(s)||^{2p-2}||\nabla u_n(s)||^2ds+\int_0^T||u_n(s)||^{2p-2}|\nabla|u_n|^{\frac{\beta+1}{2}}|_2^2ds$$
$$+\int_0^T\int_D||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2dxds\Big)\leq C(\mathbb{E}||u_0||^{2p}+1).$$

Proof. Applying Itô formula to $||u_n(t)||^{2p}$, we get

$$\begin{aligned}||u_n(t)||^{2p}\leq&||u_0||^{2p}-2p\int_0^t||u_n(s)||^{2p-2}||\nabla u_n(s)||^2ds\\&-2p\alpha\int_0^t\int_D||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2dxds\\&-\frac{8p\alpha(\beta-1)}{(\beta+1)^2}\int_0^t||u_n(s)||^{2p-2}|\nabla|u_n|^{\frac{\beta+1}{2}}|_2^2ds\\&+2p\int_0^t||u_n(s)||^{2p-2}|\langle\Delta u_n,B(u_n)\rangle|ds\\&+2p\int_0^t||u_n(s)||^{2p-2}(\nabla u_n(s),\nabla\sigma_n(s,u_n(s))dW(s))\\&+p(2p-1)\int_0^t||u_n(s)||^{2p-2}||\sigma_n(s,u_n(s))||^2ds.\end{aligned}\tag{A.1}$$

Taking the supremum over the interval $[0,t]$ on the inequality (A.1), we get

$$\begin{aligned}&\sup_{s\in[0,t]}||u_n(t)||^{2p}+\sup_{s\in[0,t]}2p\int_0^t||u_n(s)||^{2p-2}||\nabla u_n(s)||^2ds\\&\quad+\sup_{s\in[0,t]}2p\alpha\int_0^t\int_D||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2dxds\\&\quad+\sup_{s\in[0,t]}\frac{8p\alpha(\beta-1)}{(\beta+1)^2}\int_0^t||u_n(s)||^{2p-2}|\nabla|u_n|^{\frac{\beta+1}{2}}|_2^2ds\\&\leq||u_0||^{2p}+2p\sup_{s\in[0,t]}\int_0^t||u_n(s)||^{2p-2}(\nabla u_n(s),\nabla\sigma_n(s,u_n(s))dW(s))\\&\quad+p(2p-1)\sup_{s\in[0,t]}\int_0^t||u_n(s)||^{2p-2}||\sigma_n(s,u_n(s))||^2ds\\&\quad+2p\sup_{s\in[0,t]}\int_0^t||u_n(s)||^{2p-2}|\langle\Delta u_n,B(u_n)\rangle|ds\\&=||u_0||^{2p}+I_1+I_2+I_3.\end{aligned}\tag{A.2}$$

Applying the Burkholder–Davis–Gundy inequality and Young's inequality, we have

$$\begin{aligned}\mathbb{E}I_1(t)\leq&C\mathbb{E}\Big(\int_0^t||u_n(s)||^{4(p-1)}||u_n(s)||^2||\sigma_n(s,u_n(s))||^2ds\Big)^{\frac12}\\\leq&C\mathbb{E}\Big(\int_0^t\int_D||u_n(s)||^{2p}||u_n(s)||^{2p-2}(1+|u_n|^2)|\nabla u_n|^2dxds\Big)^{\frac12}\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}||u_n(s)||^{2p} + C\mathbb{E}\int_0^t\int_D ||u_n(s)||^{2p-2}(1+|u_n|^2)|\nabla u_n|^2 dxds \\
&\leq \frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}||u_n(s)||^{2p} + C\mathbb{E}\int_0^t ||u_n(s)||^{2p} ds \\
&\quad + \mathbb{E}\int_0^t ||u_n(s)||^{2p-2}(\varepsilon\int_D |u_n|^{\beta-1}|\nabla u_n|^2 dx + C_\varepsilon ||u_n(s)||^2) ds \\
&\leq \frac{1}{2}\mathbb{E}\sup_{0\leq s\leq t}||u_n(s)||^{2p} + \varepsilon\mathbb{E}\int_0^t\int_D ||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2 dxds \\
&\quad + C_\varepsilon\mathbb{E}\int_0^t ||u_n(s)||^{2p} ds. \qquad (A.3)
\end{aligned}
$$

By the Assumption 2.1, we have

$$
\begin{aligned}
\mathbb{E}I_2 &\leq C\mathbb{E}\int_0^t\int_D ||u_n(s)||^{2p-2}(1+|u_n|^2)|\nabla u_n|^2 dxds \\
&\leq \varepsilon\mathbb{E}\int_0^t\int_D ||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2 dxds \\
&\quad + C_\varepsilon\mathbb{E}\int_0^t ||u_n(s)||^{2p} ds. \qquad (A.4)
\end{aligned}
$$

Since

$$
0 < \frac{2}{\beta-1} < 1, \qquad \text{for } \beta > 3, \qquad (A.5)
$$

using Young's inequality and (A.5) to estimate $I_3(t)$, we deduce

$$
\begin{aligned}
&\mathbb{E}I_3(t) \leq 2p\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}(\frac{1}{2}|\Delta u_n|_2^2 + \frac{1}{2}|u_n\cdot\nabla u_n|_2^2) ds \\
&\leq p\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}||\nabla u_n||^2 ds \\
&\quad + p\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}\int_D |u_n|^2|\nabla u_n|^{\frac{4}{\beta-1}}|\nabla u_n|^{2-\frac{4}{\beta-1}} dxds \\
&\leq p\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}||\nabla u_n||^2 ds \\
&\quad + p\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}[\int_D (|u_n|^2|\nabla u_n|^{\frac{4}{\beta-1}})^{\frac{\beta-1}{2}} dx]^{\frac{2}{\beta-1}}[\int_D (|\nabla u_n|^{2-\frac{4}{\beta-1}})^{\frac{\beta-1}{\beta-3}} dx]^{\frac{\beta-3}{\beta-1}} ds \\
&\leq p\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}||\nabla u_n||^2 ds + C_\varepsilon\mathbb{E}\int_0^t ||u_n(s)||^{2p} ds \\
&\quad + \varepsilon\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}\int_D |\nabla u_n|^2|u_n|^{\beta-1} dxds. \qquad (A.6)
\end{aligned}
$$

Putting (A.3)-(A.6) into (A.2), we have

$$
\mathbb{E}\sup_{s\in[0,t]}||u_n(s)||^{2p} + 2p\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}||\nabla u_n(s)||^2 ds
$$

$$
\begin{aligned}
&+2p\alpha\mathbb{E}\int_0^t\int_D ||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2dxds\\
&+\frac{8p\alpha(\beta-1)}{(\beta+1)^2}\mathbb{E}\int_0^t ||u_n(s)||^{2p-2}|\nabla|u_n|^{\frac{\beta+1}{2}}|_2^2ds\\
\le&\mathbb{E}||u_0||^{2p}+\frac{1}{2}\mathbb{E}\sup_{0\le s\le t}||u_n(s)||^{2p}\\
&+3\varepsilon\mathbb{E}\int_0^t\int_D ||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2dxds\\
&+C_\varepsilon\mathbb{E}\int_0^t ||u_n(s)||^{2p}ds+p\int_0^t ||u_n(s)||^{2p-2}||\nabla u_n||^2ds. \qquad \text{(A.7)}
\end{aligned}
$$

Choosing sufficiently small ε and applying Gronwall lemma, we have

$$
\begin{aligned}
\mathbb{E}(\sup_{0\le t\le T}||u_n(t)||^{2p}&+\int_0^T ||u_n(s)||^{2p-2}||\nabla u_n(s)||^2ds+\int_0^T ||u_n(s)||^{2p-2}|\nabla|u_n|^{\frac{\beta+1}{2}}|_2^2ds\\
&+\int_0^T\int_D ||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2dxds)\le C(\mathbb{E}||u_0||^{2p}+1).
\end{aligned}
$$

Since the constant C is independent of n, we have

$$
\begin{aligned}
\sup_{n\ge1}\mathbb{E}(\sup_{0\le t\le T}||u_n(t)||^{2p}&+\int_0^T ||u_n(s)||^{2p-2}||\nabla u_n(s)||^2ds+\int_0^T ||u_n(s)||^{2p-2}|\nabla|u_n|^{\frac{\beta+1}{2}}|_2^2ds\\
&+\int_0^T\int_D ||u_n(s)||^{2p-2}|u_n|^{\beta-1}|\nabla u_n|^2dxds)\le C(\mathbb{E}||u_0||^{2p}+1). \qquad \text{(A.8)}
\end{aligned}
$$

For $\alpha\ge\frac{1}{2}$ as $\beta=3$, we can get the above estimate easily. This completes the proof of Lemma A.1. □

By using monotonicity method, we get the existence of the strong solution for problem (3.1) for $\beta>3$ with any $\alpha>0$ and $\alpha\ge\frac{1}{2}$ as $\beta=3$. The uniqueness of the strong solution for problem (3.1) also is obtained by using ordinary method. The remain section of Theorem 3.1 is proved by using similar method in [31,37]. This completes the proof of Theorem 3.1.

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