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A NOTE ON THE STABILITY OF IMPLICIT-EXPLICIT FLUX-SPLITTINGS FOR STIFF SYSTEMS OF HYPERBOLIC CONSERVATION LAWS*

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Abstract. We analyze the stability of implicit-explicit flux-splitting schemes for stiff systems of conservation laws. In particular, we study the modified equation of the corresponding linearized systems. We first prove that symmetric splittings are stable, uniformly in the singular parameter ε. Then, we study non-symmetric splittings. We prove that for the isentropic Euler equations, the Degond–Tang splitting [Degond & Tang, Comm. Comp. Phys., 10:1–31, 2011] and the Haack–Jin–Liu splitting [Haack, Jin Liu, Comm. Comp. Phys., 12:955–980, 2012], and for the shallow water equations the recent RS-IMEX splitting are *strictly stable* in the sense of Majda–Pego. For the full Euler equations, we find a small instability region for a flux splitting introduced by Klein [Klein, J. Comp. Phys., 121:213–237, 1995], if this splitting is combined with an IMEX scheme as in [Noelle, Bispen, Arun, Lukáčová, Munz, SIAM J. Sci. Comp., 36:B989–B1024, 2014].

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1. Introduction

The efficient and stable approximation of solutions to stiff differential equations is a longstanding challenge in numerical analysis. For systems of ordinary differential equations (ODEs), stiffness may be defined as the simultaneous occurrence of eigenvalues of different orders of magnitude. In the context of conservation laws, the key example is the low-Mach number flow in gas dynamics, where the speed of acoustic waves is much bigger than the advection speed. In the limit, the equations change the type from hyperbolic to hyperbolic-elliptic, or from compressible to incompressible flow. The numerical challenge is to establish a scheme which is efficient (*i.e.*, it has a time step independent of the Mach number ε), uniformly consistent and stable as $\varepsilon \to 0$. Moreover, the limit scheme should be a consistent and stable approximation of the incompressible Euler equations. In [19], Jin introduced the term *asymptotic preserving (AP)* for such schemes.

In the context of ODEs, implicit-explicit (or IMEX) schemes are a method of choice (see the classic textbook [16]). They split the system into a fast part and a slow part, and they treat the fast part implicitly and the slow part explicitly; see [1,4,5,30] for more details about the use of IMEX methods in constructing AP scheme for stiff systems. In the context of hyperbolic conservation laws, there are some classical approaches (see [7] for a concise review) such as the pre-conditioning method proposed by Chorin [6] and Turkel [37]. More recently, some pioneering papers such as Klein [21], Degond and Tang [8], and Haack, Jin, and Liu [15] split the flux function $F(U)$ into fast and slow fluxes, $\widetilde{F}(U)$ and $\widehat{F}(U)$, in such a way that the Jacobians $\widetilde{A}(U) := \widetilde{F}'(U)$ and $\widehat{A}(U) := \widehat{F}'(U)$ are hyperbolic. Then, they employ an implicit-explicit strategy to solve

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this split system (see also [3,7,28] and the references therein). In most of these works the uniform (asymptotic) consistency of the scheme has been shown, either by some analysis or by numerical experiments. However, there are only a few results regarding the uniform (asymptotic) stability, among them [9, 11, 39], using the von Neumann stability analysis, energy methods, or entropy stability. See also [12, 18, 23] for some results in the context of kinetic equations.

Recently in [28], the second author and collaborators studied an IMEX flux-splitting scheme for the full Euler equations, using a variant of Klein's auxiliary splitting [21]. Unexpectedly, the scheme required an ε-dependent time step for stability. Motivated by this, Schütz and Noelle [32] began a stability study of the modified equation of the linearized system in Fourier variables. Computations showed the instability of some Fourier modes for Klein's auxiliary splitting when used within the context of flux-splitting IMEX schemes, which is *not* the context of Klein's original algorithm, *cf.* Remark 5.2. In [32, Remark 9] the authors conjectured that the culprit is a resonance between the implicit and explicit parts of the algorithm, as expressed by the commutator $\widehat{A}\widetilde{A}-\widetilde{A}\widehat{A}$, which in general is $O(\varepsilon^{-1})$. They proved the stability of characteristic splittings in terms of the modified equations, for which the commutator vanishes. To go beyond characteristic splittings, the authors and collaborators developed a generalized version of linearly implicit methods, called *Reference Solution IMEX (RS-IMEX)* which de-singularizes the commutator; the reader can consult [20, 31, 38] for some results regarding formal/rigorous AP analysis and numerical experiments of the RS-IMEX scheme applied to the isentropic Euler equations, the shallow water equations, and the Van der Pol system.

In the present paper, we analyze the stability of the modified equation for a general class of IMEX flux splittings. We first point out a stability result for symmetric splittings and relate this stability result to the linear stability in the sense of Majda–Pego [25]. Then, given a general background state, we study Fourier symbols for linearized modified equations for non-symmetric flux splittings and apply this to several well-known IMEX schemes. Note that our analysis can be applied to a general background state and any frequency variable for which the modified equation is valid, while the previous work [32] evaluated the Fourier symbols numerically using fixed background states and frequencies.

The paper is organized as follows. In Section 2, and to review [32], we discuss the IMEX methodology applied to a linear hyperbolic system as well as its corresponding modified equation. In Sections 3 and 4, we assemble a number of stability results for symmetric and general non-symmetric splittings, respectively. Using the results of Section 4, we prove in Section 5 that the modified equations resulting from the well-known IMEX flux splittings in [8,15] for the one-dimensional isentropic Euler equations, as well as the RS-IMEX splitting for the one-dimensional shallow water equations (as discussed in [38]), are stable in the sense of Majda–Pego. We also study Klein's auxiliary splitting [21], and discover a small instability region for the example of two colliding pulses [21,28], for a moderate CFL number. This seems to give a hint at the numerical difficulties observed in [28].

2. Flux-splittings, IMEX schemes, and the modified equation

In this section, we review the linear framework introduced in [32]. We consider the linear system of hyperbolic conservation laws in the (one-dimensional) domain $\Omega\subset\mathbb{R}$

$$U_t+AU_x=0,\qquad U(\cdot,0)=U_0(x), \tag{2.1}$$

where $U:\Omega\times[0,+\infty)\to\mathbb{R}^m$ are conservative variables and $A\in\mathbb{R}^{m\times m}$ is a real diagonalizable constant matrix (depending only on the parameter ε) with eigenvalues of $\lambda_1 \geq \ldots \geq \lambda_m$. The flux vector is defined as $F(U) := AU$. We assume well-prepared initial data so that the time derivatives of the solution, $\partial_t^k u$, are bounded uniformly in ε for $k \in \mathbb{N}$, *cf.* [13,22]). Now, we decompose the matrix A into stiff and non-stiff parts in an admissible way, as defined below.

Definition 2.1 (Admissible splitting [32]). *The splitting $A = \widetilde{A} + \widehat{A}$, with "stiff" $\widetilde{A}$ and "non-stiff" $\widehat{A}$, is called to be admissible provided that*

(i) both $\widetilde{A}$ and $\widehat{A}$ induce a hyperbolic system, i.e., they have real eigenvalues and a complete set of eigenvectors;

(ii) the eigenvalues of $\widehat{A}$ are bounded independently of ε, e.g., $\mathcal{O}(1)$, and at least one of the eigenvalues of $\widetilde{A}$ is $\mathcal{O}(\frac{1}{\varepsilon})$.

As in [32], we choose a Rusanov-type scheme for both implicit and explicit parts in the computational domain Ω_N, with the time step Δt and the spatial step $\Delta x := \frac{|\Omega|}{N}$, where N is the number of computational cells. Such a scheme can be written either in the *un-split form*

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(\widetilde{\mathcal{F}}_{j+1/2}^{n+1} - \widetilde{\mathcal{F}}_{j-1/2}^{n+1} + \widehat{\mathcal{F}}_{j+1/2}^{n} - \widehat{\mathcal{F}}_{j-1/2}^{n}\right),$$

or the *split form* with an explicit step (from the temporal step n to some intermediate step $n+1/2$) and an implicit step (from the intermediate step $n+1/2$ to the new temporal step $n+1$)

$$\begin{cases} U_j^{n+1/2} = U_j^n - \frac{\Delta t}{\Delta x}\left(\widehat{\mathcal{F}}_{j+1/2}^{n} - \widehat{\mathcal{F}}_{j-1/2}^{n}\right), \\ U_j^{n+1} = U_j^{n+1/2} - \frac{\Delta t}{\Delta x}\left(\widetilde{\mathcal{F}}_{j+1/2}^{n+1} - \widetilde{\mathcal{F}}_{j-1/2}^{n+1}\right), \end{cases}$$

where the numerical fluxes are defined as

$$\begin{aligned} \widetilde{\mathcal{F}}_{j+1/2}^{n+1} &:= \frac{1}{2}\widetilde{A}\left(U_{j+1}^{n+1} + U_j^{n+1}\right) - \frac{\widetilde{\alpha}}{2}\left(U_{j+1}^{n+1} - U_j^{n+1}\right), \\ \widehat{\mathcal{F}}_{j+1/2}^{n} &:= \frac{1}{2}\widehat{A}\left(U_{j+1}^{n} + U_j^{n}\right) - \frac{\widehat{\alpha}}{2}\left(U_{j+1}^{n} - U_j^{n}\right), \end{aligned}$$

with $\mathcal{O}(1)$ numerical diffusion coefficients $\widetilde{\alpha}$ and $\widehat{\alpha}$ for stiff and non-stiff parts, respectively. Then, the modified equation (see [32, eq. (10)]) reads

$$U_t + AU_x = \frac{\Delta t}{2}\left(\frac{\alpha \Delta x}{\Delta t}\mathbb{I}_m - \widehat{A}^2 + \widetilde{A}^2 + [\widetilde{A}, \widehat{A}]\right)U_{xx}, \tag{2.2}$$

where $\alpha := \widetilde{\alpha} + \widehat{\alpha}$ and $[\widetilde{A}, \widehat{A}] := \widetilde{A}\widehat{A} - \widehat{A}\widetilde{A}$ is the commutator of the stiff and non-stiff Jacobians.

Assuming that $U \in [L_2(\Omega)]^m$, one can apply the Fourier transform to equation (2.2), which yields

$$\widehat{U}_t + \left(-i\xi A - \frac{\xi^2 \Delta t}{2}\left(\frac{\alpha \Delta x}{\Delta t}\mathbb{I}_m - \widehat{A}^2 + \widetilde{A}^2 + [\widetilde{A}, \widehat{A}]\right)\right)\widehat{U} = 0, \tag{2.3}$$

with ξ denoting the frequency variable. This gives the following convenient stability result.

LEMMA 2.1 (Corollary 1, [32]). *The modified equation (2.2) is L_2-stable if the frequency matrix*

$$\mathcal{P}(\xi) := -iA\xi - \xi^2 D, \quad D := \frac{\Delta t}{2}\Big(\frac{\alpha \Delta x}{\Delta t}\mathbb{I}_m - \widehat{A}^2 + \widetilde{A}^2 + [\widetilde{A}, \widehat{A}]\Big) \tag{2.4}$$

only has eigenvalues with negative real parts.

DEFINITION 2.1. *If $\mathcal{P}(\xi)$ satisfies equation (2.4), we call $\mathcal{P}(\xi)$ a stable matrix, i.e., it only has eigenvalues with negative real parts. In this case, we say that the IMEX splitting satisfies condition ($\mathcal{A}$).*

REMARK 2.1.

(i) From now on, whenever we talk about stability, it means stability in the sense of condition ($\mathcal{A}$), unless explicitly stated otherwise.

(ii) The modified equation (2.2) is derived formally by truncating Taylor expansions in space and time. We conjecture that a rigorous justification will have to rely on a "low-frequency assumption" such as

$$\|\xi^k A^k\| = \mathcal{O}(1) \quad \text{for } k = 3, 4 \ldots, \tag{2.5}$$

together with a suitable CFL condition.

(iii) Recalling a famous result of Gel'fand [10], if one considers a convection-diffusion system of equations like (2.4), the well-posedness requires the viscosity matrix (in this case D) to be *parabolic*, *i.e.*, its eigenvalues should have positive real parts. So, in general, parabolicity is a necessary condition for stability. Since in [10] this necessity has been justified only for very high-frequency modes, it cannot be applied in the context of this paper, since the frequencies are small due to a low-frequency assumption. Hence, the parabolicity is not a necessary condition anymore.

Unfortunately, without any additional structural assumption, obtaining a general stability condition for $\mathcal{P}$ is very delicate. For example, in [32] the authors introduce a characteristic splitting, for which the Jacobians are simultaneously diagonalizable and hence the commutator $[\tilde{A}, \hat{A}]$ vanishes. This immediately provides stability of the modified equation; see also Remark 4.2(i) below for ℓ_2-stability. In Section 3, we study the eigenvalues of $\mathcal{P}$ assuming symmetry of the system and its splitting and relate it to the strict stability in the sense of Majda–Pego [25], which seems to be a promising framework for stability. Also, in Section 4, given a general background state, we study Fourier symbols for linearized modified equations of non-symmetric flux-splitting. The results will be employed in Section 5 to obtain linearized stability or instability for a number of recent splittings.

3. Stability of symmetric splittings

In this section, we assume that A, $\widehat{A}$ and $\widetilde{A}$ are symmetric, and point out the stability of such a splitting in Corollary 3.1. Then, by introducing the notion of strict stability in the sense of Majda–Pego [25], we generalize condition ($\mathcal{A}$) (for "*linear*" systems) to "*linearized*" systems. For symmetric splittings, this notion gives a more general stability result for symmetrizable systems (see Theorem 3.2). Non-symmetric splittings are treated in Section 4.

Note that for any symmetric matrix A, a symmetric splitting is always possible, *e.g.*, if one chooses $\widehat{A} = \operatorname{diag}(A|_{\varepsilon=1})$. For any symmetric splitting, the commutator is a skew-Hermitian matrix. Therefore,

$$\mathcal{P}(\xi) = -\Bigg[\underbrace{iA\xi + \xi^2\frac{\Delta t}{2}[\widetilde{A},\widehat{A}]}_{=:\mathscr{A}} + \underbrace{\xi^2\frac{\Delta t}{2}\Big(\frac{\alpha\Delta x}{\Delta t}\mathbb{I}_m - \widehat{A}^2 + \widetilde{A}^2\Big)}_{=:\mathscr{H}}\Bigg], \tag{3.1}$$

where $\mathscr{A}$ and $\mathscr{H}$ are skew-Hermitian and Hermitian matrices, respectively. One may conjecture that the eigenvalues of $\mathscr{H}$ would be positive. The following lemma verifies this conjecture.

LEMMA 3.1. *The Hermitian matrix $\mathscr{H}$ is positive-definite under a non-restrictive CFL condition, independently of ε.*

Proof. One way to conclude the lemma is to use the *eigenvalue stability inequality* (see [34, eq. 1.64]), which states that for two Hermitian matrices L and M of size m, the following holds

$$|\lambda_k(L+M) - \lambda_k(L)| \leq \|M\|_{\mathrm{op}}, \qquad k=1,\dots,m, \tag{3.2}$$

where the *operator norm* is defined as $\|M\|_{\mathrm{op}} := \max\big(|\lambda_1(M)|,|\lambda_m(M)|\big)$. So, if one puts $L=\widetilde{A}^2$ and $M=-\widehat{A}^2$ in inequality (3.2), it yields

$$-\underline{\lambda} < \lambda_k(\widetilde{A}^2) - \|\widehat{A}\|^2_{\mathrm{op}} \leq \lambda_k(-\widehat{A}^2+\widetilde{A}^2) \leq \lambda_k(\widetilde{A}^2) + \|\widehat{A}\|^2_{\mathrm{op}},$$

with $\underline{\lambda} \geq 0$. Due to the order of magnitude of eigenvalues, $\underline{\lambda}$ can be chosen to be positive and $\mathcal{O}(1)$, namely $\underline{\lambda} > \|\widehat{A}\|^2_{\mathrm{op}}$ which implies the time step restriction $\Delta t < \alpha\Delta x/\|\widehat{A}\|^2_{\mathrm{op}}$. This CFL condition shifts the eigenvalues to the right (by $\frac{\alpha\Delta x}{\Delta t}\mathbb{I}_m$), so that the eigenvalues of $\mathscr{H}$ are positive.

Another way to conclude the same result is to use the sub-additivity of the numerical range: to show that the numerical range of $\widetilde{A}^2$ is positive, and to put the numerical range of $\frac{\alpha\Delta x}{\Delta t}\mathbb{I}_m - \widehat{A}^2$ in the right half-plane under some CFL condition. ∎

Given these properties of $\mathscr{A}$ and $\mathscr{H}$, there is a sum of a Hermitian and a skew-Hermitian matrix in equation (3.1), and one can use the Bendixon's theorem in [17] (see [2] for the original work which is limited to real matrices), which shows that given a Hermitian matrix with stable eigenvalues in the left half-plane and a skew-Hermitian matrix, the sum will have stable eigenvalues, *i.e.*, the eigenvalues have negative real parts. To recall, we restate the theorem from [17].

THEOREM 3.1 (Theorem II, [17]). *Consider the matrix $M \in \mathbb{K}^{m\times m}$ with $\mathbb{K}=\mathbb{C}$ or $\mathbb{R}$, where $\lambda_k\big(\mathcal{H}(M)\big) = p_k \in \mathbb{R}$ for $k=1,\dots,m$ and $\mathcal{H}$ stands for the Hermitian part. Then, the following holds*

$$\min_k p_k \leq \Re\big[\lambda_k(M)\big] \leq \max_k p_k.$$

From Lemma 3.1 and Theorem 3.1, one can conclude immediately the following corollary.

COROLLARY 3.1. *Under a non-restrictive CFL condition, an admissible symmetric splittings is stable, i.e., it satisfies condition ($\mathcal{A}$).*

REMARK 3.1. One could also use an energy estimate to show that for the hyperbolic-parabolic system (2.2) with a symmetric matrix A, the positive-definiteness of the viscosity matrix D is necessary and sufficient for L_2-stability.

In order to generalize Lemma 2.1 for systems linearized around an arbitrary state U_0, we introduce the notion of *strict stability* in the sense of Majda–Pego [25] below.

Definition 3.1. *For the non-linear system $\partial_t U + \partial_x F(U)_x = \partial_x(D(U)U_x)$, the viscosity matrix D is strictly stable at U_0 if and only if there exists a $\delta > 0$ such that the eigenvalues $\lambda_k(\xi)$ of the matrix $\mathcal{P}(\xi) := -f'(U_0)i\xi - \xi^2 D(U_0)$ satisfy the following algebraic condition*

$$\Re\big[\lambda_k(\xi)\big] \leq -\delta|\xi|^2, \quad \textit{for all } \xi \in \mathbb{R}.$$

This definition also provides some non-linear stability results; see [25]. Note that Definition 3.1 refers to a given state (arbitrary, but fixed) U_0, around which the system is linearized. To keep the notation simpler and when there is no confusion, we suppress the dependence on U_0. Using this framework, one can also find the generalization of the stability of symmetric splittings at U_0, as in [25, 27]:

Theorem 3.2. *Consider the frequency matrix (2.4) and let $\mathcal{M}(U_0)$ be a real symmetric positive-definite matrix, symmetrizing $A(U_0)$ from the left, i.e., $(\mathcal{M}A)|_{U_0}$ is symmetric. Then if $(\mathcal{M}D)|_{U_0}$ is positive-definite, the frequency matrix (2.4), and so the modified equation (2.2), is strictly stable at U_0, i.e., there exists a $\delta > 0$ such that $\Re\big[\lambda_k(\mathcal{P}(\xi))\big] \leq -\delta|\xi|^2$.*

It is clear that for symmetric splittings, the identity matrix can play the role of $\mathcal{M}$ and Theorem 3.2 is reduced to the arguments we have presented above, leading to Corollary 3.1.

4. Stability of non-symmetric splittings

In this section, we study the stability of $\mathcal{P}$ without the symmetry assumption so that the commutator contributes to the real parts of the eigenvalues of $\mathcal{P}$, and hence to the stability.

Let us denote the spectrum of $\mathcal{P}$ as $\lambda(\mathcal{P})$. Then, by the theorem of spectral inclusion [14, Theorem 1.2-1], this spectrum (and in particular its convex hull) is contained in the closure of the numerical range of $\mathcal{P}$. In other words, $\mathrm{Conv}(\lambda(\mathcal{P})) \subseteq W(\mathcal{P})$, where the numerical range $W(\mathcal{P})$ is defined as $W(\mathcal{P}) := \{\langle v, \mathcal{P}v\rangle, v \in \mathbb{C}^m, \|v\|_{\ell_2} = 1\}$. In fact, the real part of the numerical range of $\mathcal{P}$ is bounded by the spectrum of its Hermitian part, *i.e.*,

$$\Re[W(\mathcal{P})] = -\mathrm{Conv}\left(\lambda\big(\xi\mathcal{H}(iA) + \xi^2\mathcal{H}(D)\big)\right).$$

Due to the eigenvalue stability inequality, one can find the upper-bound of the numerical range,

$$\Re[W(\mathcal{P})] \leq -\Big(\xi^2\lambda_m\big(\mathcal{H}(D)\big) - \xi\|\mathcal{H}(iA)\|_{\mathrm{op}}\Big), \tag{4.1}$$

where $\lambda_m(\mathcal{H}(D))$ denotes the smallest eigenvalue. So, for the stability of $\mathcal{P}$, it is sufficient to set the numerical range to be in the left half-plane. Thus, as a subsidiary result, for symmetric systems $\mathcal{H}(iA)$ vanishes and one can conclude immediately from upper bound (4.1) that the positive-definiteness of D implies strict stability with $\delta = \lambda_m(\mathcal{H}(D))$, the smallest eigenvalues of $\mathcal{H}(D)$. For a non-symmetric A, although the positive-definiteness of D does not necessarily imply condition $(\mathcal{A})$, we suggest the stability by a modified version of positivity in Theorem 4.1 below; this is the same result as [25, Thm. 2.1].

THEOREM 4.1. *For a strictly hyperbolic system with $A\in\mathbb{R}^{m\times m}$, and with eigenvector matrix V, the positivity of $V^{-1}DV$ is sufficient for the stability in terms of condition $(\mathcal{A})$.*

Proof. By construction, A is hyperbolic and can be diagonalized as $A=V\Lambda V^{-1}$, where V is the matrix of eigenvectors. Substituting this into the definition of $\mathcal{P}$ yields

$$\mathcal{P}(\xi)=-i\xi A-\xi^2 D=V\Big(-i\xi\Lambda-\xi^2\widetilde{D}\Big)V^{-1},$$

where $\widetilde{D}:=V^{-1}DV$. Since similarity transformations do not change the spectrum, we instead study the eigenvalues of $\widetilde{\mathcal{P}}(\xi)$ defined as $\widetilde{\mathcal{P}}(\xi):=-i\xi\Lambda-\xi^2\widetilde{D}$.

One can decompose $\widetilde{D}$ as the sum of Hermitian and skew-Hermitian matrices, *i.e.*, $\mathcal{H}(\widetilde{D})+\mathcal{A}(\widetilde{D}_\nu)$. From positivity $-\xi^2\mathcal{H}(\widetilde{D})$ is stable and by Theorem 3.1, the addition of skew-Hermitian matrices $-\xi^2\mathcal{A}(\widetilde{D}_\nu)$ and $-i\xi\Lambda$ cannot destabilize a stable Hermitian matrix. So $\mathcal{P}$ is stable. ∎

As the term $\frac{\alpha\Delta x}{\Delta t}\mathbb{I}_m$ in $\widetilde{D}$ only shifts the eigenvalues, we only need to study $\widetilde{D'}:=V^{-1}(\widetilde{A}-\widehat{A})(\widetilde{A}+\widehat{A})V$ (instead of $\widetilde{D}$) as claimed by the following lemma. Note that $\widehat{A}^2+\widetilde{A}^2+[\widetilde{A},\widehat{A}]=(\widetilde{A}-\widehat{A})(\widetilde{A}+\widehat{A})$.

LEMMA 4.1. *Consider the linear system (2.1). Let V be the eigenvector matrix of A, and suppose that the splitting $A=\widehat{A}+\widetilde{A}$ is admissible in the sense of Definition 2.1. If there exists a lower-bound $\underline{\lambda}_{\mathcal{H}(\widetilde{D'})}$ for the eigenvalues of the Hermitian part of $\widetilde{D'}:=V^{-1}(\widetilde{A}-\widehat{A})(\widetilde{A}+\widehat{A})V$, such that $\underline{\lambda}_{\mathcal{H}(\widetilde{D'})}=\mathcal{O}(1)$, the splitting is strictly stable in the sense of Majda–Pego.*

REMARK 4.1. It is important to emphasizes that, compared to the full frequency matrix $\mathcal{P}(\xi)$ used in Lemma 2.1, Lemma 4.1 is a convenient simplification since $\mathcal{H}(\widetilde{D'})$ does not depend on ξ.

One can go one step further and use the structure of the viscosity matrix, which is known for equation (2.2). Because of the hyperbolicity assumption, A and $\widehat{A}$ are real diagonalizable, *i.e.*, $A=V\Lambda V^{-1}$ and $\widetilde{A}=U\widetilde{\Lambda}U^{-1}$, where V and U are matrices of eigenvectors and $U=VJ$, where J stands for the change of basis matrix. Substituting these into the definition of $\mathcal{P}$ gives $\widetilde{\mathcal{P}}$ as

$$\widetilde{\mathcal{P}}(\xi)=-i\xi\Lambda-\xi^2\underbrace{\frac{\Delta t}{2}\Big[\frac{\alpha\Delta x}{\Delta t}\mathbb{I}_m-\Lambda^2+2J\widetilde{\Lambda}J^{-1}\Lambda\Big]}_{=\widetilde{D}}.$$

This form of $\widetilde{D}$ reveals more of its structure. As one can see, the role of J is crucial for the positivity of $\widetilde{D}$. For example, for the admissible characteristic splitting, $J=\mathbb{I}_m$ and the positivity (and stability) is clear since the components of $\widetilde{D}$ are diagonal. So, it is plausible to claim that the splittings whose eigenspaces are close to each other are more likely to be stable. This indeed matches the results of [24, Sect. D.7]: if the eigenspaces of the split matrices coincide, the power-boundedness of each step is enough for the stability of the whole scheme.

REMARK 4.2.
(i) The IMEX scheme based on the characteristic splitting is uniformly stable not only in the sense that its modified equation is stable (as discussed in [32]), but also in ℓ_2-norm. This is because for such a splitting, one can decouple the system into q scalar

equations $\partial_t w_k + \lambda_k \partial_x w_k = 0$ for $k = 1, \ldots, m$; then, by the von Neumann stability analysis, both explicit and implicit steps can be shown to be ℓ_2-stable, respectively, under an appropriate (and ε-uniform) CFL condition, and unconditionally (see [35, Sect. 3.3.4]).

(ii) In the light of [15, Lemma 3.1], the stability of each step (in some norm) is clearly enough for the stability of the whole scheme; however, it is far from being necessary in most cases, and almost often not practical to be fulfilled. For instance, notice that the example in [32, Sect. 7] does not have stable steps. One could confirm numerically that for both stable and unstable settings (with $\varepsilon_1 = 10^{-1}$ and $\varepsilon_2 = 10^{-2}$ respectively) the implicit operator $\widetilde{\mathcal{S}}$ is power-bounded while the explicit operator $\widehat{\mathcal{S}}$ is not. Nonetheless, their multiplication $\widetilde{\mathcal{S}}\widehat{\mathcal{S}}$ makes one case stable and the other unstable. For further details about the stability of the difference equations, the reader can consult [24, Appendix D] and [36, Chap. 4].

(iii) It seems plausible to conjecture that if the commutator is $\mathcal{O}(1)$, the viscosity matrix is parabolic under a suitable and non-restrictive choice of CFL condition, using the *continuity of eigenvalues* [29, Appendix K] [26, Chap. 1]. But on one hand, the constant of that continuity grows as $\varepsilon \to 0$, and on the other, it is not even clear if the parabolicity is a relevant condition to be used for low-frequency modes, as mentioned earlier in Remark 2.1.

5. Applications

In this section, we show that Lemma 4.1 provides the linearized stability at any given state U_0 of several splittings used in practice, namely the splitting of Haack–Jin–Liu [15] (abbreviated as HJL hereafter), Degond–Tang [8] (DT hereafter), and the RS-IMEX splitting (in the context of [38]). We also discuss the numerical instability which has been reported in [28] for Klein's auxiliary splitting of the Euler equations [21]. Recall that our analysis is based on the modified equation (2.2), hence on the implicit-explicit Euler time integration accompanied with Rusanov-type numerical fluxes.

5.1. Haack–Jin–Liu splitting. For the isentropic Euler equations written as $\partial_t U + \partial_x F(U) = 0$, the system and the flux Jacobian are

$$U = \begin{bmatrix} \varrho \\ \varrho u \end{bmatrix}, \qquad F = \begin{bmatrix} \varrho u \\ \varrho u^2 + \frac{1}{\varepsilon^2} p(\varrho) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 \\ -u^2 + \frac{1}{\varepsilon^2} p'(\varrho) & 2u \end{bmatrix}, \tag{5.1}$$

where ϱ, u, and $p(\varrho) = \kappa \varrho^\gamma$ are the density, velocity, and pressure. The HJL splitting [15] decomposes the Jacobian of the flux function as

$$\widehat{A} = \begin{bmatrix} 0 & \beta \\ -u^2 + \dfrac{p'(\varrho) - a(t)}{\varepsilon^2} & 2u \end{bmatrix}, \qquad \widehat{\lambda} = u \pm \sqrt{(1-\beta)u^2 + \frac{\beta\big(p'(\varrho) - a(t)\big)}{\varepsilon^2}},$$

$$\widetilde{A} = \begin{bmatrix} 0 & 1-\beta \\ \frac{1}{\varepsilon^2} a(t) & 0 \end{bmatrix}, \qquad \widetilde{\lambda} = \pm \frac{\sqrt{a(t)(1-\beta)}}{\varepsilon},$$

where $\beta \in [0,1]$ is a parameter to be chosen (note that it is called α in [15]) and $a(t) := \min_x p'$. With these settings, the splitting is admissible in the sense of Definition 2.1; see [15] for further details.

Assume that the system has been linearized around an arbitrary state $U_0 = (\varrho_0, u_0)^T$. Then, in light of Lemma 4.1, we have to study the positivity of $\widetilde{D'}$. With the aid of

Maple$^{\text{TM}}$, one can get

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2\lambda^{1,2}_{\mathcal{H}(\widetilde{D}')}\right)=\lim_{\varepsilon\to 0}\left[\varepsilon^2(\beta-2)u_0^2+\Big(a_0-\beta p_0'\pm\big((\beta-1)p_0'+a_0\big)\Big)\right]$$
$$=\lim_{\varepsilon\to 0}\left[\varepsilon^2(\beta-2)u_0^2+\Big(a_0\pm(-p_0'+a_0)+\beta(-p_0'\pm p_0')\Big)\right].$$

Owing to the formal analysis for $\varepsilon\ll 1$, the asymptotic expansion gives $p_0'-a_0=\mathcal{O}(\varepsilon^2)$, so

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2\lambda^{1,2}_{\mathcal{H}(\widetilde{D}')}\right)=a_0,(1-2\beta)a_0.$$

Since $a_0>0$, both eigenvalues are nonnegative in the limit and one can find the lower-bound $\underline{\lambda}_{\mathcal{H}(\widetilde{D'})}=\mathcal{O}(1)$, provided we set $\beta\le 1/2$. So, for $\beta\le 1/2$, due to Lemma 4.1, the scheme is strictly stable in the sense of Majda–Pego under a non-restrictive CFL condition. Note that for the numerical experiments presented in [15], β is chosen in this stable region and is often of $\mathcal{O}(\varepsilon^2)$.

5.2. Degond–Tang splitting. In [8] and for the isentropic Euler equations with the pressure function $p(\varrho)=\kappa\varrho^\gamma$ (like the case of HJL splitting), the following splitting has been proposed for A in equation (5.1):

$$\widehat{A}=\begin{bmatrix}0 & 0\\ -u^2+\theta p'(\varrho) & 2u\end{bmatrix},\qquad \widehat{\lambda}=0,2u,$$
$$\widetilde{A}=\begin{bmatrix}0 & 1\\ \dfrac{(1-\theta\varepsilon^2)}{\varepsilon^2}p'(\varrho) & 0\end{bmatrix},\qquad \widetilde{\lambda}=\pm\frac{\sqrt{(1-\theta\varepsilon^2)p'(\varrho)}}{\varepsilon},$$

where θ is an ad-hoc parameter to be chosen between 0 and $1/\varepsilon^2$. Note that it is discussed in [7,8,33] that taking $\theta=\mathcal{O}(1)$ leads to the AP property; so, we assume θ to be $\mathcal{O}(1)$. Then, one can clearly confirm that this splitting is admissible in the sense of Definition 2.1.

As for the HJL splitting and with $U_0=(\varrho_0,u_0)^T$, we study the positivity of $\widetilde{D}'$. With the aid of Maple$^{\text{TM}}$, one gets

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2\lambda^{1,2}_{\mathcal{H}(\widetilde{D}')}\right)=\lim_{\varepsilon\to 0}\left[-\varepsilon^2\left(\theta+2u_0^2\right)p_0'+p_0'\pm\mathcal{O}(\varepsilon^2)\right]=p_0'>0.$$

Thus, both eigenvalues are positive in the limit, and due to Lemma 4.1, the scheme is strictly stable in the sense of Majda–Pego under a non-restrictive CFL condition. Note that this stability does not depend on the choice of θ.

5.3. RS-IMEX splitting. Here, we consider the RS-IMEX splitting for the shallow water equations with a flat bottom topography:

$$U=\begin{bmatrix}z\\ m\end{bmatrix},\quad F=\begin{bmatrix}m\\ \dfrac{m^2}{h}+\dfrac{z^2\varepsilon^2-2zb}{2}\end{bmatrix},\quad A=\begin{bmatrix}0 & 1\\ -\dfrac{m^2}{(z\varepsilon^2-b)^2}+(z\varepsilon^2-b) & \dfrac{2m}{z\varepsilon^2-b}\end{bmatrix},\tag{5.2}$$

with the same notation as [3,38]: z denotes a scaled perturbation of the height from a constant reference, $h+b=\varepsilon^2 z$, $m:=(\varepsilon^2 z-b)u$ is the momentum and b is the bottom function which is negative and constant. For equations (5.2), the RS-IMEX splitting with the lake-at-rest reference solution gives the following flux splitting (see [38]):

$$\widehat{A} = \begin{bmatrix} 0 & 0 \\ z\varepsilon^2 - \dfrac{m^2\varepsilon^2}{(z\varepsilon^2-b)^2} & \dfrac{2m}{z\varepsilon^2-b} \end{bmatrix}, \qquad \widehat{\lambda} = 0, \frac{2m}{z\varepsilon^2-b},$$

$$\widetilde{A} = \begin{bmatrix} 0 & 1/\varepsilon^2 \\ -b & 0 \end{bmatrix}, \qquad \widetilde{\lambda} = \pm\frac{\sqrt{-b}}{\varepsilon^2},$$

So, it can be concluded that this splitting is admissible in the sense of Definition 2.1.

As for the HJL splitting and with $U_0=(z_0,m_0)^T$, one can obtain that

$$\lim_{\varepsilon\to 0}\left(\varepsilon^2\lambda^{1,2}_{\mathcal{H}(\widetilde{D}')}\right) = \lim_{\varepsilon\to 0}\frac{-b^5+\mathcal{O}(\varepsilon^2)}{(\varepsilon^2 z_0-b)^4} = -b>0, \tag{5.3}$$

since $b<0$. Hence, using Lemma 4.1 and similar to the HJL splitting example, the splitting is strictly stable in the sense of Majda–Pego under a non-restrictive CFL condition. Note that the leading orders $\varepsilon^2\lambda^{1,2}_{\mathcal{H}(\widetilde{D'})}$ have been the same for the HJL (with $\beta=\mathcal{O}(\varepsilon^2)$), DT, and RS-IMEX splittings.

Remark 5.1. It would be interesting to extend the stability result to equations with a varying bottom topography. Nonetheless, it is not clear how to linearize the Jacobian matrices $\widetilde{A}$ and $\widehat{A}$ (by freezing b), and also simultaneously, the source term (by freezing b_x). Thus, it is more difficult to understand the linearization error, and hence the validity of the stability analysis.

Example 5.1. *In addition to the previous analysis of the modified equation in the low-Mach and the low-Froude number limit ($\varepsilon\ll 1$), we now study $\lambda^{1,2}_{\mathcal{H}(\widetilde{D'})}$ for all $\varepsilon\in(0,1]$. To have the "same" settings for all these three splittings, we consider the pressure law $p(\varrho)=\varrho^2/2$ for the HJL and DT splittings so that they coincide with the pressure function of the shallow water equations. We also choose $(\varrho_0,u_0)=(1,1)$ for the HJL and DT splittings, and $(z_0,b,u_0)=(0,-1,1)$ for the RS-IMEX splitting. With these settings, all the systems are the same and can be compared to each other.*

We also set the ad-hoc parameters of HJL and DT splittings as the typical values, $\beta=\varepsilon^2$ and $\theta=1$. Figure 5.1 shows that $\lambda^{1,2}_{\mathcal{H}(\widetilde{D'})}$ are bounded from below. Indeed, $\lambda^{1}_{\mathcal{H}(\widetilde{D'})}$ is always positive, while $\lambda^{2}_{\mathcal{H}(\widetilde{D'})}$ is positive in the left of the kink—around $\varepsilon\in(0.4,0.6)$—and negative in the right, but uniformly bounded. Thus, owing to Lemma 4.1, all these splittings are asymptotically stable. Note that the plots of the RS-IMEX, HJL, and DT splittings are hardly distinguishable for small ε.

5.4. Klein's auxiliary splitting. In his influential paper [21], Klein introduced two flux-splittings for the full Euler equations:

$$U=\begin{bmatrix}\varrho\\ \varrho u\\ \varrho E\end{bmatrix}, \quad F=\begin{bmatrix}\varrho u\\ \varrho u^2+\frac{1}{\varepsilon^2}p\\ (\varrho E+p)u\end{bmatrix}, \quad A=\begin{bmatrix}0 & 1 & 0\\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \frac{\gamma-1}{\varepsilon^2}\\ -Hu+\frac{(\gamma-1)\varepsilon^2}{2}u^3 & H-\varepsilon^2(\gamma-1)u^2 & \gamma u\end{bmatrix}, \tag{5.4}$$

where the total energy ϱE satisfies the dimensionless equation of state $\varrho E=\frac{p}{\gamma-1}+\frac{\varepsilon^2}{2}\varrho|u|^2$, and $H:=E+\frac{p}{\varrho}$ stands for total enthalpy.

The main splitting introduces two sub-systems, called system (I) and (II), given by [21, eqs. (3.1)–(3.2)]. In the second splitting, the system (I) is replaced by the so-called *auxiliary* system (I*), which is given by [21, eq. (3.8)]. In this section, we analyze

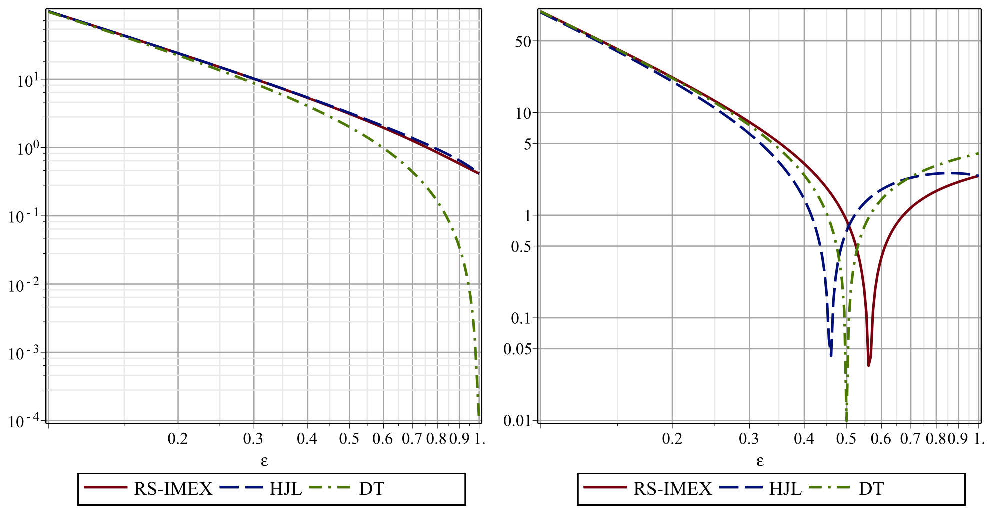

FIG. 5.1. $\left|\lambda^1_{\mathcal{H}(\widetilde{D'})}\right|$ *(left) and* $\left|\lambda^2_{\mathcal{H}(\widetilde{D'})}\right|$ *(right) for RS-IMEX, HJL, and DT splittings w.r.t.* ε.

the stability of a flux-splitting IMEX scheme, which uses Klein's auxiliary splitting as a building block (*cf.* [28]).

Here, the background state for the linearization is $U_0 = (\varrho_0, \varrho_0 u_0, \varrho_0 E_0)^T$. Following the derivation in [28], the auxiliary splitting (for $1 < \gamma \leq \frac{5}{3}$) is given by

$$\widetilde{A} = (1-\varepsilon^2)\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2}(\gamma-1)u^2 & -(\gamma-1)u & \dfrac{\gamma-1}{\varepsilon^2} \\ -u\dfrac{p-p_{\text{inf}}}{\varrho} + \dfrac{\gamma-1}{2}\varepsilon^2 u^3 & \dfrac{p-p_{\text{inf}}}{\varrho} - \varepsilon^2(\gamma-1)u^2 & (\gamma-1)u \end{bmatrix},$$

$$\widehat{A} = \begin{bmatrix} 0 & 1 & 0 \\ \left(\dfrac{(\gamma-1)\varepsilon^2}{2} - 1\right)u^2 & \left(2-(\gamma-1)\varepsilon^2\right)u & \gamma-1 \\ \widehat{A}_{31} & \widehat{A}_{32} & \widehat{A}_{33} \end{bmatrix},$$

$$\widehat{A}_{31} := -u\left[\left(1+\varepsilon^2(\gamma-1)\right)E - 2\varepsilon^4(\gamma-1)u^2 + (1-\varepsilon^2)\frac{p_{\text{inf}}}{\varrho}\right] + \frac{(\gamma-1)\varepsilon^4}{2}u^3,$$

$$\widehat{A}_{32} := E + \varepsilon^2(\gamma-1)\left(E - \frac{\varepsilon^2}{2}u^2\right) + (1-\varepsilon^2)\frac{p_{\text{inf}}}{\varrho} - (\gamma-1)\varepsilon^4 u^2,$$

$$\widehat{A}_{33} := \left(1+\varepsilon^2(\gamma-1)\right)u.$$

The choice of the parameter $p_{\text{inf}} := \min_x p(x,t)$ guarantees the hyperbolicity of split systems, whose eigenvalues are

$$\widehat{\lambda} = u, u \pm c^*, \qquad c^* := \sqrt{\frac{p+(\gamma-1)\Pi}{\varrho}}, \quad \Pi := (1-\varepsilon^2)p_{\text{inf}} + \varepsilon^2 p,$$

$$\widetilde{\lambda} = 0, \pm\frac{(1-\varepsilon^2)}{\varepsilon}\sqrt{\frac{(\gamma-1)(p-p_{\text{inf}})}{\varrho}}.$$

So, the splitting is admissible in the sense of Definition 2.1 (see [28]).

Our attempts to compute the eigenvalues of $\mathcal{H}(\widetilde{D'})$ for this full Euler case with Maple$^{\text{TM}}$ failed. Thus, we study the full frequency matrix $\mathcal{P}$ for the example of two colliding pulses [21], where we had previously found an ε-dependent time step [28]. Based on Remark 2.1, we should characterize the frequency ξ such that the condition (2.5) is fulfilled for the flux Jacobian defined in equation (5.4). It is straightforward to check that A^k has an entry (and so A^k has norm) of $\mathcal{O}(\varepsilon^{-2\lfloor (k+1)/2 \rfloor})$. Thus, assuming $\xi \sim \varepsilon^{\sigma_k}$ for each k, the condition $\|\xi^k A^k\| = \mathcal{O}(1)$ gives σ_k as $\sigma_k = \frac{\lfloor \frac{k+1}{2} \rfloor}{k/2}$, which has been computed in Table 5.1. Because $\sigma := \min_{k>2} \sigma_k = \frac{4}{3}$ and we aim to cut terms for $k>2$, the condition $\xi \sim \varepsilon^{4/3}$ justifies truncation of the modified equation.

k	1	2	3	4	5	even k	odd k	∞
σ_k	2	1	$\frac{4}{3}$	1	$\frac{6}{5}$	1	$\frac{k+1}{k}$	1

TABLE 5.1. *The values of* σ_k.

The domain is chosen to be $[-L, L]$ with $L := 2/\varepsilon^{4/3}$, for $\gamma = 1.4$ with the following initial data:

$$\varrho(x,0) = \varrho^{(0)} + \frac{\varepsilon}{2}\varrho^{(1)}\left(1 - \cos\left(\frac{2\pi x}{L}\right)\right), \quad \varrho^{(0)} = 0.955, \quad \varrho^{(1)} = 2, \tag{5.5a}$$

$$p(x,0) = p^{(0)} + \frac{\varepsilon}{2}p^{(1)}\left(1 - \cos\left(\frac{2\pi x}{L}\right)\right), \quad p^{(0)} = 1, \quad p^{(1)} = 2\gamma, \tag{5.5b}$$

$$u(x,0) = \frac{1}{2}u^{(0)}\operatorname{sign}(x)\left(1 - \cos\left(\frac{2\pi x}{L}\right)\right), \quad u^{(0)} = 2\sqrt{\gamma}. \tag{5.5c}$$

Note that in [21, 28], $L := 2/\varepsilon$ has been considered for which the initial data do not match the low frequency assumption (2.5).

In order to apply the Majda–Pego stability framework, we linearize around

$$\begin{aligned} \varrho_0 &= \varrho^{(0)} + \frac{\varepsilon}{2}\varrho^{(1)} = 0.955 + \varepsilon, \\ p_0 &= p^{(0)} + \frac{\varepsilon}{2}p^{(1)} = 1 + \varepsilon\gamma, \\ u_0 &= \sqrt{\gamma}, \end{aligned}$$

and $p_{\inf} = 1$. Note that we have replaced $1 - \cos\left(\frac{2\pi x}{L}\right)$ by its mean value of 1. The numerical diffusion and the grid parameters are chosen as in [28]:

$$\alpha = \sqrt{\frac{\gamma p_0}{\varrho_0}} + \max_x(u(x,0)), \quad \Delta x = 0.05, \quad \Delta t = \frac{\text{CFL}}{\max_x(u(x,0))}\Delta x.$$

We compute the real parts of the eigenvalues of the frequency matrix $\mathcal{P}$ of the modified equation numerically. Figure 5.2 displays $\Re(\lambda^1_{\mathcal{P}})$ for different CFL numbers and frequency variable $\xi = \varepsilon^{4/3}\pi$. Each sub-figure displays a specific region of ε. The figures reveal a small instability region near $\varepsilon \in (0.02, 0.06)$ and for CFL $= 0.45$. This seems to correspond closely to some of the numerical experiments in [28], where the CFL number needed to be reduced when changing ε from 0.1 to 0.05.

Note, however, that the lack of uniform stability in [28] is much stronger than the one seen in Figure 5.2, since in [28] the CFL number needed to decrease linearly with the Mach number, while in Figure 5.2, $\Re(\lambda^1_{\mathcal{P}}) \leq 0$ uniformly in ε, for fixed CFL$=$0.02. This discrepancy may possibly be due to a fundamental difference between the Fourier analysis in the present paper and the real computation in [28] as, based on Lemma 2.1, Figure 5.2 studies a single Fourier mode while, due to the sign(x) function in equation (5.5c), the initial data for the velocity contain a superposition of all Fourier modes, which may trigger instabilities not explained by the present analysis.

REMARK 5.2. It is important to point out some differences between the algorithms in [28] and [21]. Klein develops his approach using the more complex setting of multiple space variables and multiple pressures. Algorithmically, he "combines explicit predictor steps for long wave linear acoustics or global compression with a single implicit scalar Poisson-type corrector scheme" [21, p.3]. Thus, our stability analysis has no direct implication for the scheme proposed in [21]. Rather, it should be seen as a comment to [28].

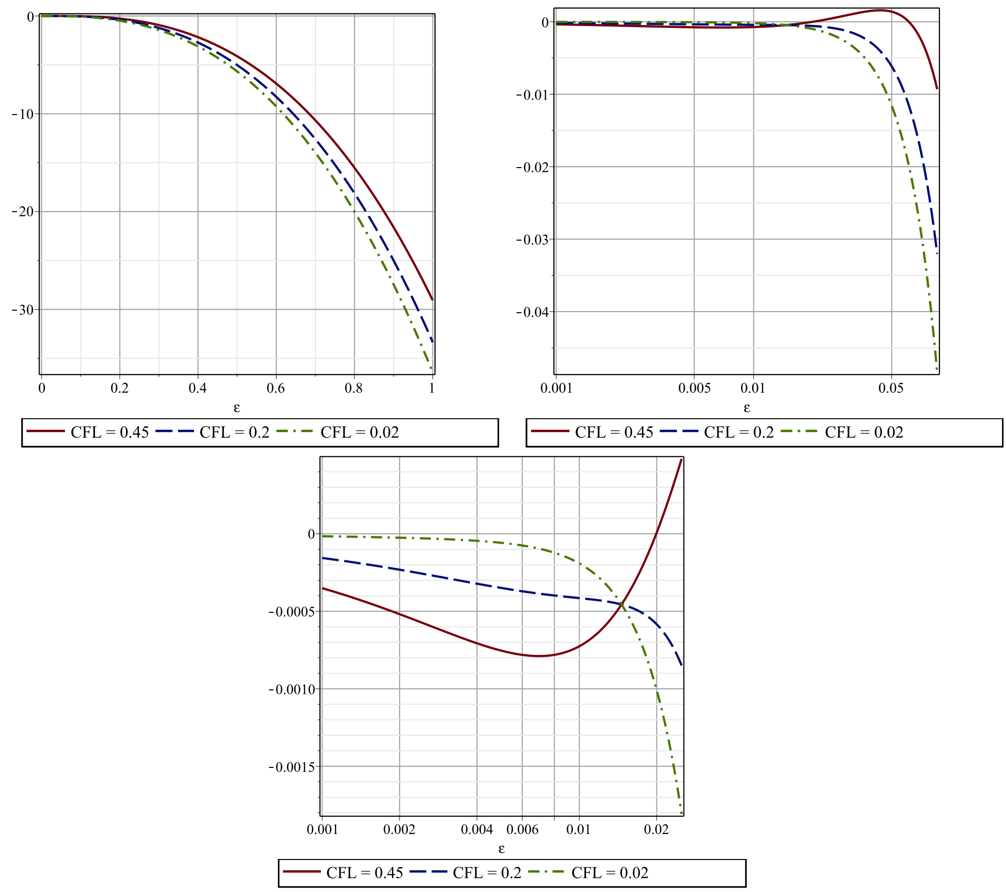

FIG. 5.2. $\Re(\lambda^1_{\mathcal{P}})$ *for Klein's auxiliary splitting w.r.t.* ε*, in different regions of* ε *and for* $\xi = \varepsilon^{4/3}\pi$.

6. Concluding remarks

In this paper, we reviewed the strict stability framework of Majda and Pego and employed it for the stability of modified equations. We also showed that for symmetric splittings the viscosity matrix is positive, which gives strict stability. Furthermore, we discussed a general class of splittings and showed that positivity of the viscosity matrix, after being transformed by the matrix of eigenvectors of the (linearized) flux Jacobian, is sufficient for strict stability, which matches the results of [25]. This criterion has been used to show the stability (of the modified equation) of several flux-splitting IMEX schemes for stiff systems of hyperbolic conservation laws: the Haack–Jin–Liu splitting [15] and the Degond–Tang splitting [8] for the isentropic Euler equations, and the recent RS-IMEX splitting for the shallow water equations [38] with a flat bottom topography. For the full Euler equation, we discovered a small region of instability for Klein's so-called auxiliary splitting [21] for the two colliding pulses example. This seems to confirm computational results in [28].

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