FAST COMMUNICATION

ON ROBUST WIDTH PROPERTY FOR LASSO AND DANTZIG SELECTOR[*]

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Abstract. Recently, Cahill and Mixon completely characterized sensing operators in many compressed sensing instances with a robust width property, which allows uniformly stable and robust reconstruction via certain convex optimization. However, their current theory does not cover the Lasso and Dantzig selector models, both of which are popular alternatives in statistics and optimization community. In this short note, we show that the robust width property can be perfectly applied to these two types of models as well. Our main results affirmatively answer the question left by Cahill and Mixon.

Keywords. robust width property, compressed sensing, Lasso model, Dantzig selector model.

AMS subject classifications. 65K05, 65F22, 90C25.

1. Introduction

One of the main assignments of compressed sensing is to understand when it is possible to recover structured solutions to underdetermined systems of linear equations [4]. During the past decade, there have developed many reconstruction guarantees provided the sensing operator satisfies certain sufficient conditions. Well-known examples include the restricted isometry property, the null space property, the coherence property, and the dual certificate. The interested reader is referred to [6, 12, 13] for detailed information. However, none of these sufficient conditions is proved necessary for uniformly stable and robust reconstruction. Recently, Cahill and Mixon in [2] introduced a new notion—robust width property—which completely characterizes sensing operators in many compressed sensing instances. By restricting to the following constrained convex optimization model,

$$\min \|x\|_\sharp, \ \ \text{subject to} \ \ \|\Phi x - y\|_2 \le \epsilon, \qquad (Q_\epsilon)$$

they proved that the robust width property is necessary and sufficient for uniformly stable and robust reconstruction via the model above. Here, $\|\cdot\|_\sharp$ is some norm used to promote certain structured solutions, sensing operator Φ and observed data y are given, and ϵ measures the error. Unfortunately, it remains unclear whether the robust width property can be used to characterize the Lasso/Basis Pursuit and Dantzig selector models [3, 9], both of which are popular alternatives in the statistics and optimization community. In this study, we answer this question in the affirmative. We note that while this work was under review, the authors of [10] proposed a weak range space property of the sensing operator, which turns out to be a necessary and sufficient condition for the standard ℓ_1-minimization method to be weakly stable. But their theory heavily relies on the classic Hoffman's lemma concerning the error bound of linear systems [7] and hence is limited to ℓ_1-minimization problems. On the other hand, the Lasso and Dantzig selector models have not been considered under their theoretical framework as

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well, that may be addressed in future by the method presented in this work. In the following, we recall some notation appearing in [2].

Let $x^\natural$ be some unknown member of a finite-dimensional Hilbert space $\mathcal{H}$, and let $\Phi : \mathcal{H} \to \mathbb{F}^M$ denote some known linear operator, called the sensing operator, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Subset $\mathcal{A} \subseteq \mathcal{H}$ is a particular subset that consists of some type of structured members, such as sparse vectors in compressed sensing and low-rank matrices in matrix completion [4]. $B_\sharp := \{x \in \mathcal{H} : \|x\|_\sharp \leq 1\}$ is the unit $\sharp$-ball.

2. Robust width

The robust width property was formally proposed in [2]. We write down the definition and its equivalent form as follows.

DEFINITION 2.1 (Cahill and Mixon [2]). *We say a linear operator $\Phi : \mathcal{H} \to \mathbb{F}^M$ satisfies the (ρ, α)-robust width property over $B_\sharp$ if*

$$\|x\|_2 \leq \rho \|x\|_\sharp$$

for every $x \in \mathcal{H}$ such that $\|\Phi x\|_2 < \alpha \|x\|_2$; or equivalently if

$$\|\Phi x\|_2 \geq \alpha \|x\|_2$$

for every $x \in \mathcal{H}$ such that $\|x\|_2 > \rho \|x\|_\sharp$.

Here, we would like to point out the definition above (or say, its intrinsic idea) is not completely new. In fact, when restricted to the case of ℓ_1-minimization, it reduces to the ℓ_1-constrained minimal singular value property, which was originally defined in [8].

DEFINITION 2.2. *For any $k \in \{1, 2, \cdots, N\}$ and matrix $\Phi \in \mathbb{R}^{M \times N}$, define the ℓ_1-constrained minimal singular value of Φ by*

$$r_k(\Phi) = \min_{x \neq 0, x \in S_k} \frac{\|\Phi x\|_2}{\|x\|_2},$$

where $S_k = \{x \in \mathbb{R}^N : \|x\|_1 \leq \sqrt{k} \|x\|_2\}$. If $r_k(\Phi) > 0$, then we say Φ satisfies the ℓ_1-constrained minimal singular value property with $r_k(\Phi)$.

The geometrical aspect of the ℓ_1-constrained minimal singular value property was exploited in [11].

3. Main results

We first introduce the definition of a compressed sensing space.

DEFINITION 3.1 (Cahill and Mixon [2]). *A compressed sensing space $(\mathcal{H}, \mathcal{A}, \|\cdot\|_\sharp)$ with bound L consists of a finite-dimensional Hilbert space $\mathcal{H}$, a subset $\mathcal{A} \subseteq \mathcal{H}$, and a norm $\|\cdot\|_\sharp$ on $\mathcal{H}$ with following properties:*

(i) $0 \in \mathcal{A}$.

(ii) For every $a \in \mathcal{A}$ and $v \in \mathcal{H}$, there exists a decomposition $v = z_1 + z_2$ such that

$$\|a + z_1\|_\sharp = \|a\|_\sharp + \|z_1\|_\sharp, \qquad \|z_2\|_\sharp \leq L \|v\|_2.$$

The subdifferential $\partial f(x)$ of a convex function f at x is the set-valued operator [1] given by

$$\partial f(x) := \{u \in \mathcal{H} : f(y) \geq f(x) + \langle u, y - x \rangle, \forall y \in \mathcal{H}\}.$$

The following lemma will be useful to establish our main results.

LEMMA 3.1. *Let $\|\cdot\|_\diamond$ be the dual norm of $\|\cdot\|_\sharp$ on $\mathcal{H}$. If $u \in \partial\|x\|_\sharp$, then $\|u\|_\diamond \le 1$. If $x \ne 0$ and $u \in \partial\|x\|_\sharp$, then $\|u\|_\diamond = 1$.*

Proof. From the convexity of $\|\cdot\|_\sharp$ and the subdifferential definition, for any $u \in \partial\|x\|_\sharp$ and $v \in \mathcal{H}$ it holds

$$\|v\|_\sharp \ge \|x\|_\sharp + \langle u, v-x \rangle.$$

Set $v = 0$ and $v = 2x$ to get $\langle u, x \rangle \ge \|x\|_\sharp$ and $\langle u, x \rangle \le \|x\|_\sharp$ respectively. This implies $\langle u, x \rangle = \|x\|_\sharp$ and hence $\langle u, v \rangle \le \|v\|_\sharp$. Similarly, by taking $-v \in \mathcal{H}$, we can get $-\langle u, v \rangle \le \|v\|_\sharp$. Thus, $|\langle u, v \rangle| \le \|v\|_\sharp$. Therefore,

$$\|u\|_\diamond = \sup_{\|v\|_\sharp \le 1} |\langle u, v \rangle| \le \sup_{\|v\|_\sharp \le 1} \|v\|_\sharp \le 1.$$

In the case of $x \ne 0$, by the Cauchy–Schwartz inequality, we get that $\|x\|_\sharp = \langle u, x \rangle \le \|x\|_\sharp \|u\|_\diamond$ and hence $\|u\|_\diamond \ge 1$. So we must have $\|u\|_\diamond = 1$. $\square$

Now, we state the characterization of uniformly stable and robust reconstruction via the Lasso/Basis Pursuit type model by utilizing the (ρ, α)-robust width property.

THEOREM 3.1. *For any CS space $(\mathcal{H}, \mathcal{A}, \|\cdot\|_\sharp)$ with bound L and any linear operator $\Phi : \mathcal{H} \to \mathbb{F}^M$, the following are equivalent up to constants:*

(a) Φ satisfies the (ρ, α)-robust width property over $B_\sharp$.

(b) For every $x^\natural \in \mathcal{H}, \kappa \in (0,1), \lambda > 0$ and $\omega \in \mathbb{F}^M$ satisfying $\|\Phi^T \omega\|_\diamond \le \kappa\lambda$, any solution x^* to the following unconstrained convex optimization model, called the Lasso/Basis Pursuit type model in this paper:

$$\min \frac{1}{2}\|\Phi x - (\Phi x^\natural + \omega)\|_2^2 + \lambda\|x\|_\sharp \qquad (P_\lambda)$$

satisfies $\|x^* - x^\natural\|_2 \le C_0\|x^\natural - a\|_\sharp + C_1 \cdot \lambda$ for every $a \in \mathcal{A}$.

In particular, (a) implies (b) with

$$C_0 = \left(\frac{1-\kappa}{2\rho} - L\right)^{-1}, \qquad C_1 = \frac{1+\kappa}{\alpha^2 \rho}$$

provided $\rho < \frac{1-\kappa}{2L}$. Also, (b) implies (a) with

$$\rho = 2C_0, \qquad \alpha = \frac{\kappa}{2\tau C_1},$$

where $\tau = \sup_{\|x\|_\sharp \le 1} \|\Phi x\|_2$.

Proof. Let $z = x^* - x^\natural$. We divide the proof of $(a) \Rightarrow (b)$ into four steps for clarity. They are partially inspired by [5] and [2].

Step 1: Prove the first relationship:

$$\|x^*\|_\sharp - \kappa\|z\|_\sharp \le \|x^\natural\|_\sharp. \qquad (3.1)$$

Since x^* is a minimizer to (P_λ), we have

$$\frac{1}{2}\|\Phi x^* - (\Phi x^\natural + w)\|_2^2 + \lambda\|x^*\|_\sharp \le \frac{1}{2}\|\Phi x^\natural - (\Phi x^\natural + w)\|_2^2 + \lambda\|x^\natural\|_\sharp.$$

Hence,

$$\frac{1}{2}\|(\Phi x^* - \Phi x^\natural) - w\|_2^2 + \lambda\|x^*\|_\sharp \leq \frac{1}{2}\|w\|_2^2 + \lambda\|x^\natural\|_\sharp.$$

Rearrange terms to give

$$\lambda\|x^*\|_\sharp \leq -\frac{1}{2}\|\Phi(x^* - x^\natural)\|_2^2 + \langle \Phi(x^* - x^\natural), w\rangle + \lambda\|x^\natural\|_\sharp \leq \langle x^* - x^\natural, \Phi^T w\rangle + \lambda\|x^\natural\|_\sharp.$$

By the Cauchy–Schwartz inequality and the condition $\|\Phi^T w\|_\diamond \leq \kappa\lambda$, we obtain

$$\langle x^* - x^\natural, \Phi^T w\rangle \leq \|x^* - x^\natural\|_\sharp \|\Phi^T w\|_\diamond \leq \kappa\lambda\|x^* - x^\natural\|_\sharp.$$

Thus, we get

$$\lambda\|x^*\|_\sharp \leq \kappa\lambda\|x^* - x^\natural\|_\sharp + \lambda\|x^\natural\|_\sharp,$$

from which the first relationship follows.

Step 2: Prove the second relationship:

$$\|z\|_\sharp \leq \frac{2}{1-\kappa}\|x^\natural - a\|_\sharp + \frac{2L}{1-\kappa}\|z\|_2, \quad \forall\, a \in \mathcal{A}. \tag{3.2}$$

Pick $a \in \mathcal{A}$, and decompose $z = x^* - x^\natural = z_1 + z_2$ according to the property (ii) in Definition 3.1 so that $\|a + z_1\|_\sharp = \|a\|_\sharp + \|z_1\|_\sharp$ and $\|z_2\|_\sharp \leq L\|z\|_2$. In light of relationship (3.1), we derive that

$$\begin{aligned}
\|a\|_\sharp + \|x^\natural - a\|_\sharp &\geq \|x^\natural\|_\sharp \\
&\geq \|x^*\|_\sharp - \kappa\|z\|_\sharp \\
&= \|x^\natural + (x^* - x^\natural)\|_\sharp - \kappa\|x^* - x^\natural\|_\sharp \\
&= \|a + (x^\natural - a) + z_1 + z_2\|_\sharp - \kappa\|z_1 + z_2\|_\sharp \\
&\geq \|a + z_1\|_\sharp - \|x^\natural - a\|_\sharp - (1+\kappa)\|z_2\|_\sharp - \kappa\|z_1\|_\sharp \\
&= \|a\|_\sharp + \|z_1\|_\sharp - \|x^\natural - a\|_\sharp - (1+\kappa)\|z_2\|_\sharp - \kappa\|z_1\|_\sharp \\
&= \|a\|_\sharp + (1-\kappa)\|z_1\|_\sharp - \|x^\natural - a\|_\sharp - (1+\kappa)\|z_2\|_\sharp.
\end{aligned}$$

Rearrange terms to give

$$\|z_1\|_\sharp \leq \frac{2}{1-\kappa}\|x^\natural - a\|_\sharp + \frac{1+\kappa}{1-\kappa}\|z_2\|_\sharp$$

which implies

$$\|z\|_\sharp \leq \|z_1\|_\sharp + \|z_2\|_\sharp \leq \frac{2}{1-\kappa}\|x^\natural - a\|_\sharp + \frac{2}{1-\kappa}\|z_2\|_\sharp.$$

Thus, the second relationship follows by invoking $\|z_2\|_\sharp \leq L\|z\|_2$.

Step 3: Derive the upper bound:

$$\|\Phi z\|_2^2 \leq (1+\kappa)\lambda\|z\|_\sharp. \tag{3.3}$$

The optimality condition of (P_λ) reads

$$\Phi^T(\Phi x^\natural + w - \Phi x^*) \in \lambda \cdot \partial\|x^*\|_\sharp.$$

By using Lemma 3.1, we get $\|\Phi^T(\Phi x^\natural + w - \Phi x^*)\|_\diamond \leq \lambda$. Thus,

$$\begin{aligned}
\|\Phi^T \Phi z\|_\diamond &= \|\Phi^T \Phi(x^* - x^\natural)\|_\diamond \\
&\leq \|\Phi^T(\Phi x^* - \Phi x^\natural - w)\|_\diamond + \|\Phi^T w\|_\diamond \\
&\leq \lambda + \kappa\lambda = (1 + \kappa)\lambda.
\end{aligned}$$

Therefore,

$$\|\Phi z\|_2^2 = \langle z, \Phi^T \Phi z\rangle \leq \|z\|_\sharp \cdot \|\Phi^T \Phi z\|_\diamond \leq (1 + \kappa)\lambda\|z\|_\sharp,$$

where the first inequality follows from the Cauchy–Schwartz inequality.

Step 4: Finish the proof. Assume $\|z\|_2 > C_0 \cdot \|x^\natural - a\|_\sharp$, since otherwise we are done. In light of relationship (3.2), we obtain

$$\|z\|_\sharp < \left[\frac{2}{C_0(1 - \kappa)} + \frac{2L}{1 - \kappa}\right]\|z\|_2 = \rho^{-1}\|z\|_2,$$

i.e., $\|z\|_2 > \rho\|z\|_\sharp$. By the (ρ, α)-robust width property of Φ, we have $\|\Phi z\|_2 \geq \alpha\|z\|_2$. Utilizing the upper bound of $\|\Phi z\|_2^2$ obtained in Step 3, we derive that

$$\alpha^2\|z\|_2^2 \leq \|\Phi z\|_2^2 \leq (1 + \kappa)\lambda\|z\|_\sharp < \frac{(1 + \kappa)\lambda}{\rho}\|z\|_2.$$

Thus,

$$\|z\|_2 \leq \frac{(1 + \kappa)\lambda}{\alpha^2 \rho} = C_1 \cdot \lambda \leq C_0\|x^\natural - a\|_\sharp + C_1 \cdot \lambda.$$

This completes the proof of $(a) \Rightarrow (b)$.

Now, we turn to the proof of $(b) \Rightarrow (a)$. Pick $x^\natural$ such that $\|\Phi x^\natural\|_2 < \alpha\|x^\natural\|_2$. By the expression of $\tau = \sup_{\|x\|_\sharp \leq 1}\|\Phi x\|_2$ and using the Cauchy–Schwartz inequality, we derive that

$$\begin{aligned}
\tau \cdot \alpha\|x^\natural\|_2 > \tau \cdot \|\Phi x^\natural\|_2 &= \sup_{\|x\|_\sharp \leq 1}\|\Phi x\|_2 \cdot \|\Phi x^\natural\|_2 \\
&\geq \sup_{\|x\|_\sharp \leq 1}\langle \Phi x, \Phi x^\natural\rangle = \sup_{\|x\|_\sharp \leq 1}\langle x, \Phi^T \Phi x^\natural\rangle \\
&= \|\Phi^T \Phi x^\natural\|_\diamond.
\end{aligned}$$

Let $\lambda = \kappa^{-1}\tau\alpha\|x^\natural\|_2$ and $\omega = -\Phi x^\natural$. Then, we have

$$\kappa\lambda = \tau \cdot \alpha\|x^\natural\|_2 \geq \|\Phi^T \Phi x^\natural\|_\diamond = \|\Phi^T w\|_\diamond,$$

which implies that the choice of λ and ω satisfies the constrained condition $\|\Phi^T w\|_\diamond \leq \kappa\lambda$. Thereby, we can take $\omega = -\Phi x^\natural$ and hence conclude that $x^* = 0$ is a minimizer of (P_λ). Thus,

$$\|x^\natural\|_2 = \|x^* - x^\natural\|_2 \leq C_0\|x^\natural\|_\sharp + C_1\lambda = C_0\|x^\natural\|_\sharp + C_1\kappa^{-1}\tau\alpha\|x^\natural\|_2.$$

Take $\alpha = \frac{\kappa}{2\tau C_1}$ and $\rho = 2C_0$ and rearrange terms to give

$$\|x^\natural\|_2 \leq \frac{C_0}{1 - C_1\kappa^{-1}\tau\alpha}\|x^\natural\|_\sharp = \rho\|x^\natural\|_\sharp.$$

So the (ρ,α)-robust width property of Φ holds.   □

REMARK 3.1.   In the paper [2], to obtain a corresponding result for (Q_ϵ), it suffices for $\|\cdot\|_\sharp$ to satisfy:

(i)$\|x\|_\sharp \geq \|0\|_\sharp$ for every $x \in \mathcal{H}$, and

(ii) $\|x+y\|_\sharp \leq \|x\|_\sharp + \|y\|_\sharp$ for every $x,y \in \mathcal{H}$.

In contrast, Theorem 3.1 not only requires (i) and (ii) above, but also utilizes the convexity of $\|\cdot\|_\sharp$ and its dual norm. In other words, the additional requirement of convexity excludes the cases of nonconvex $\|\cdot\|_\sharp$. For example, the case of

$$\|x\|_\sharp = \|x\|_p^p := \sum_{i=1}^N |x_i|^p, \quad 0<p<1$$

is not covered by Theorem 3.1.

With very similar arguments, we can show the following result which characterizes the uniformly stable and robust reconstruction via the Dantzig type model by utilizing the (ρ,α)-robust width property.

THEOREM 3.2.   For any CS space $(\mathcal{H},\mathcal{A},\|\cdot\|_\sharp)$ with bound L and any linear operator $\Phi:\mathcal{H}\to\mathbb{F}^M$, the following are equivalent up to constants:

(a) Φ satisfies the (ρ,α)-robust width property over $B_\sharp$.

(b) For every $x^\natural \in \mathcal{H}, \lambda>0$ and $\omega\in\mathbb{F}^M$ satisfying $\|\Phi^T\omega\|_\diamond \leq \lambda$, any solution x^* to the following optimization model, called the Dantzig type model,

$$\min\|x\|_\sharp, \quad subject\ to \quad \|\Phi^T(\Phi x-(\Phi x^\natural+\omega))\|_\diamond \leq \lambda \qquad (R_\lambda)$$

satisfies $\|x^*-x^\natural\|_2 \leq C_0\|x^\natural-a\|_\sharp + C_1\cdot\lambda$ for every $a\in\mathcal{A}$.

In particular, (a) implies (b) with

$$C_0 = \left(\frac{1}{2\rho}-L\right)^{-1}, \quad C_1 = \frac{2}{\alpha^2\rho}$$

provided $\rho<\frac{1}{2L}$. Also, (b) implies (a) with

$$\rho=2C_0, \quad \alpha=\frac{\kappa}{2\tau C_1},$$

where $\tau=\sup_{\|x\|_\sharp\leq1}\|\Phi x\|_2$.

Proof.   The proof below follows from the pattern used for that of Theorem 3.1. Let $z=x^*-x^\natural$.

Step 1: Since x^* is a minimizer of (R_λ), it holds that $\|x^*\|_\sharp \leq \|x^\natural\|_\sharp$. Now, repeat the argument for Step 2 in the proof of Theorem 3.1 to give

$$\|z\|_\sharp \leq 2\|x^\natural-a\|_\sharp + 2L\|z\|_2.$$

Step 2: Prove the upper bound:

$$\|\Phi z\|_2^2 \leq 2\lambda\|z\|_\sharp.$$

This can be done by using the following inequalities:

$$\|\Phi^T\Phi z\|_\diamond \leq \|\Phi^T(\Phi x^*-(\Phi x^\natural+w))\|_\diamond + \|\Phi^T w\|_\diamond \leq 2\lambda,$$

and

$$\|\Phi z\|_2^2 = \langle z, \Phi^T \Phi z\rangle \leq \|z\|_\sharp \cdot \|\Phi^T \Phi z\|_\diamond.$$

The remaining proof of $(a) \Rightarrow (b)$ follows by repeating the argument for Step 4 in the proof of Theorem 3.1.

The proof of $(b) \Rightarrow (a)$ is as follows. Pick $x^\natural$ such that $\|\Phi x^\natural\|_2 < \alpha \|x^\natural\|_2$. Let $\lambda = \tau \alpha \|x^\natural\|_2$ and $\omega = -\Phi x^\natural$. We have proved in the proof of Theorem 3.1 that such a choice of λ and ω satisfies the constrained condition of $\|\Phi^T w\|_\diamond \leq \lambda$, and hence $x^* = 0$ is the unique minimizer of (R_λ). The proof of $(b) \Rightarrow (a)$ follows by repeating the corresponding part in the proof of Theorem 3.1. □

Note that the convexity of $\|\cdot\|_\sharp$ is not involved in the proof of Theorem 3.2.

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