COMMUN. MATH. SCI.
Vol. 15, No. 8, pp. 2373–2386

FAST COMMUNICATION

HELFRICH'S ENERGY AND CONSTRAINED MINIMISATION*

STEPHAN WOJTOWYTSCH†

Abstract. For every non-negative integer, we construct a smooth surface of the genus given by the integer and embedded into the unit ball such that the embedded manifold has surface area exactly twice as large as the unit sphere and Willmore energy only slightly larger than that of two spheres. From this we deduce that a minimising sequence for Willmore's energy in the class of surfaces with some prescribed genus and area 8π embedded in the unit ball converges to a doubly covered sphere. We obtain the same result for certain Canham–Helfrich energies without genus constraint and show that Canham–Helfrich energies in another parameter regime are not bounded from below in the class of smooth surfaces with prescribed area which are embedded into a fixed bounded domain.

Furthermore, we prove that the class of connected surfaces embedded in a bounded domain with uniformly bounded Willmore energy and area is compact under varifold convergence.

Keywords. Helfrich energy, Willmore energy, constrained minimisation, topological type, varifold.

AMS subject classifications. 49Q10, 49Q20, 53C80.

1. Introduction and main results

Let M be a closed C^2-surface embedded in $\mathbb{R}^3$. Then the Canham–Helfrich energy of M is defined by

$$\mathcal{E}(M) = \int_M \chi_H (H - H_0)^2 + \chi_K K \,\mathrm{d}\mathcal{H}^2$$

where H and K are the mean and Gaussian curvatures of M respectively and χ_H, χ_K and H_0 are material parameters. This bending energy is commonly used in the modelling of thin elastic structures, such as lipid bilayers in biology [3,7,11,15] or thin sheets [5]. The application we have in mind is to inner mitochondrial membranes composed of lipid bilayers which are connected, almost inextensible (area constraint), and confined to a small container compared to their area by the outer mitochondrial membrane (confinement constraint). Under the microscope, complex handlebar structures are observed, which leads us to the investigation of the topology of energy minimisers. The most commonly studied version of Helfrich's energy corresponds to $\chi_H \equiv \frac{1}{4}$ and $\chi_K, H_0 \equiv 0$, and is known as Willmore's energy:

$$\mathcal{W}(M) = \frac{1}{4}\int_M |H|^2 \,\mathrm{d}\mathcal{H}^2.$$

A well-known (non-compact) example of Große–Brauckmann shows that if $H_0 \neq 0$, $\mathcal{E}$ need not be lower semi-continuous under varifold convergence. Namely, in [6], a sequence of surfaces M_k is constructed which converges to a multiplicity-two plane in a measure sense and satisfies $H_{M_k} \equiv 1$ for all $k \in \mathbb{N}$. If $H_0 = 0$, in contrast to the case in which $H_0 \neq 0$, the energy does not depend on the orientation of M.

*Received: February 25, 2017; accepted (in revised form): July 25, 2017. Communicated by John Lowengrub.

†Department of Mathematical Sciences, Durham University, Durham DH1 1PT, United Kingdom, (s.j.wojtowytsch@durham.ac.uk).

The parameter χ_H must be positive to obtain an energy bound from below. Due to the Gauss–Bonnet theorem, if χ_K is constant then the second term in the Helfrich functional is of topological nature as

$$\int_M K\,\mathrm{d}\mathcal{H}^2 = 4\pi(1-g)$$

where $g\in\mathbb{N}_0$ is the genus of the surface M. Thus, when Helfrich's energy is supposed to be minimised in a certain genus class, the second term is usually dropped. We investigate the influence of prescribed topological genus on the minimisation problem for Willmore's energy and of the parameter χ_K on Helfrich's energy without prescribed genus.

Let $g\in\mathbb{N}_0$, $S>0$, and $\Omega\subset\mathbb{R}^3$ open. Denote by $\mathcal{M}_{g,S,\Omega}$ the space of closed connected orientable genus g surfaces which are C^2-embedded in Ω with surface area S, and by $\mathcal{M}_{S,\Omega}$ the union of all $\mathcal{M}_{g,S,\Omega}$ over $g\in\mathbb{N}_0$. Our main result is the following:

THEOREM 1.1. *Let $m\in\mathbb{N}$, $m\geq 2$, $g\in\mathbb{N}_0$, and $\varepsilon>0$. Then there exists $M\in\mathcal{M}_{g,4\pi m,B_1(0)}$ such that*

$$\mathcal{W}(M) < 4\pi m + \varepsilon.$$

For the proof, we show that we can connect two concentric spheres with almost equal radii by an arbitrary number of catenoids. This does not change the area or Willmore's energy very much since catenoids are minimal surfaces, but does change the topology to arbitrary genus. A further perturbation with small Willmore energy adds a sufficient amount of area. The argument is similar to that in [14], where two spheres were connected by one catenoid. Our construction is more analytic than geometric and allows for any finite number of catenoids, whereas the construction of [14] requires (almost) a whole hemisphere per catenoid. This has important implications for curvature energies.

COROLLARY 1.1. *Let $g\in\mathbb{N}_0$ be arbitrary, $m\in\mathbb{N}, m\geq 2$, and consider $\Omega=B_1(0)$ and $S=4m\pi$. Then every sequence $M_k\in\mathcal{M}_{g,S,\Omega}$ such that*

$$\mathcal{W}(M_k)\to\inf\{\mathcal{W}(M)\,|\,M\in\mathcal{M}_{g,S,\Omega}\}$$

converges to an m-fold covered unit sphere as varifolds. The infimum energy is $4m\pi$ for any g.

Convergence holds in the sense of varifolds (see below) and in particular as Radon measures on $\mathbb{R}^3$. This result differs from the unconstrained case [2] or minimisation among C^2-boundaries with prescribed isoperimetric ratio [9]. In both cases, there exists a smooth embedded (i.e. multiplicity-1) surface of genus g which minimises Willmore's energy among all surfaces of genus g (and which, in the second case, bound a domain with certain isoperimetric ratio).

COROLLARY 1.2. *Denote by $\mathcal{E}$ Helfrich's energy with constant parameters $\chi_K<0<\chi_H$ and $H_0=0$. Let $m\in\mathbb{N}, m\geq 2$, and specify $\Omega=B_1(0)$, $S=4m\pi$. Then the following hold true:*

(1) Every sequence $M_k\in\mathcal{M}_{S,\Omega}$ such that

$$\mathcal{E}(M_k)\to\inf\{\mathcal{E}(M)\,|\,M\in\mathcal{M}_{S,\Omega}\}$$

converges to a higher multiplicity unit sphere $\mu=m\cdot\mathcal{H}^2|_{S^3}$ as varifolds and we have $\mathcal{E}(\mu)<\liminf_{k\to\infty}\mathcal{E}(M_k)$;

(2) *The energy infimum is* $4\pi(4\chi_H m - |\chi_K|)$*;*

(3) *If* $M\in\mathcal{M}_{S',B_1(0)}$ *for some* $S'>0$ *and*

$$\mathcal{E}(M)\le 4\chi_H S'$$

holds, then M *is a topological sphere; and*

(4) *If* $\chi_K<-4\chi_H$*, then for any open* $\Omega\Subset\mathbb{R}^3$*,* $S>0$*,* $C>0$ *the functional* $\mathcal{E}$ *is bounded from below in the class of smooth manifolds* $\mathcal{M}_{S,\Omega}$ *and in the closure of*

$$\mathcal{B}_{S,\Omega,C}:=\{M\in\mathcal{M}_{S,\Omega}\,|\,\mathcal{E}(M)<C\}$$

under varifold convergence (see below), but not in the union of the closures

$$\bigcup_{k=1}^{\infty}\overline{\mathcal{B}_{S,\Omega,k}}.$$

The theorem has some implications for the use of Helfrich's energy in the modelling of lipid bilayers. The multiple covering of a single sphere is unphysical since a biological membrane separating two domains is usually the location of chemical exchange. The higher multiplicity does not increase effective surface area. On the contrary, it would make the transport of any exchanged species more difficult. It is expected that in regions of self-contact, any simple geometric model would cease to be valid and a more precise description of the bilayer structure of biological membranes has to be taken into account. Obviously, the situation of the corollary is highly idealised, but it is probable that similar phenomena could be observed under more generic conditions.

We also suggest that it might be more appropriate to consider the lower semi-continuous envelope with respect to varifold convergence of Helfrich's energy in the class of C^2-boundaries than its direct extension to curvature varifolds. The lower semi-continuous envelope captures the minimal genus of approximating sequences while the direct extension may have lower energy when small handles 'collapse away'.

The case $\chi_K>0$ is entirely unphysical. Here we can consider non-constant material parameters. Assume that there are measurable functions χ_H,χ_K, and H_0 associated to each surface $M\in\mathcal{M}_{S,\Omega}$.

Corollary 1.3. *Let* $\Omega\subset\mathbb{R}^3$ *open and* $r>0$ *such that* $\overline{B_r(x)}\subset\Omega$ *for some* $x\in\mathbb{R}^3, r>0$. *Let* $\mathcal{E}$ *be Helfrich's energy with parameters* χ_H,χ_K*, and* H_0 *satisfying the bounds*

$$||\chi_H||_{L^\infty(M)}\le C,\qquad ||H_0||_{L^2(M)}\le C,\qquad \delta\le\chi_K\le C$$

for some $C,\delta>0$ *independent of* $M\in\mathcal{M}_{S,\Omega}$*. Assume that* $\mu=4\pi r^2\cdot\delta_x$ *is a point measure or* $\mu=\mathcal{H}^2|_{\partial B_r(x)}$*. Then there exists a sequence* $M_k\in\mathcal{M}_{4\pi r^2,\Omega}$ *such that* $\mathcal{H}^2|_{M_k}\overset{*}{\rightharpoonup}\mu$ *as Radon measures and* $\mathcal{E}(M_k)\to-\infty$*. In the second case, even varifold convergence holds.*

Corollaries 1.1, 1.2, and 1.3 easily follow from Theorem 1.1 and the reverse estimate given in Lemma 1.1. We remark that unlike genus, connectedness is stable in the minimisation problem. While we will only need the case $n=3$, the proof goes through in arbitrary co-dimension.

Theorem 1.2. *Let* $K,M>0$ *and* $\Omega\Subset\mathbb{R}^n$ *open. The class of integral* 2*-varifolds* μ *in* $\mathbb{R}^n$ *satisfying the following two conditions,*

(1) $\mathrm{spt}(\mu)\subset\overline{\Omega}$, $\quad\mu(\overline{\Omega})\le M$, $\quad\mathcal{W}(\mu)\le K$; *and*

(2) $\mathrm{spt}(\mu)$ *is connected,*

is (sequentially) compact under the convergence of varifolds.

REMARK 1.1. *The closure (with respect to varifold convergence) of the subclass given by connected manifolds C^2-embedded into Ω with surface area bounded by M and Willmore energy bounded by C is by definition closed, and hence also compact.*

This has a direct implication for minimising Willmore's energy in a suitable topological class.

COROLLARY 1.4. *Let $\Omega \Subset \mathbb{R}^n$ be open and $S > 0$. Then there exists an integral 2-varifold V with mass measure μ such that*

(1) $\operatorname{spt}(\mu) \subset \overline{\Omega}$ *is connected,* $\mu(\overline{\Omega}) = S$*; and*

(2) V minimises $\mathcal{W}$ among the integral 2-varifolds satisfying (1).

There also exists a minimiser in the smaller class of varifolds arising as the limit of connected embedded C^2-surfaces with uniformly bounded Willmore energy and surface area S.

Corollary 1.4 follows directly from Theorem 1.2, the definition of varifold convergence, and the lower-semicontinuity of Willmore's energy. The method of proof could easily be extended to show statements like Theorem 1.1 and Corollary 1.1 also hold for the generalised Willmore functional

$$\mathcal{W}_p(M) = 2^{-p} \int_M |H|^p \, \mathrm{d}\mathcal{H}^2$$

for $1 < p < \infty$. For application perspectives, we will focus on the case $p = 2$.

1.1. Varifolds. Varifolds are a class of generalised surfaces defined as Radon measures in which integral energy functionals of curvatures naturally have minimisers. We give only a very brief introduction. For more information, see the classic sources [1, 16] or the recent review paper [13]. We are interested in the sub-class of *integral k-varifolds* which can be described as

$$\mu = \sum_{i=1}^{\infty} \theta_i \cdot \mathcal{H}^k|_{M_i}$$

where $M_i \subset \mathbb{R}^n$ are k-dimensional C^1-manifolds with $1 \leq k < n$, $\mathcal{H}^k|_{M_i}$ is the restriction of the k-dimensional Hausdorff measure in $\mathbb{R}^n$ to M_i, and $\theta_i : M_i \to \mathbb{N}_0$ is an $\mathcal{H}^k$-measurable function. We set $\theta_i = 0$ outside M_i, write $M = \bigcup_i M_i$, and write $\mu = \theta \cdot \mathcal{H}^k|_M$ with $\theta = \sum_i \theta_i$. An integral k-varifold is a Radon measure μ of the form above (in particular, it is locally finite).

The function θ is called the density of μ and can be computed $\mathcal{H}^k$-almost everywhere by

$$\theta(x) = \lim_{r \searrow 0} \frac{\mu(B_r(x))}{\omega_k r^k}$$

where ω_k is the volume of the unit ball in k dimensions. Furthermore, for $\mathcal{H}^k$-almost every $x \in \bigcup_{i=1}^{\infty} M_i$ there exists a unique unoriented k-plane $T_x\mu$ such that

$$\lim_{r \searrow 0} r^{-k} \int_{\mathbb{R}^n} f\left(\frac{y-x}{r}\right) \mathrm{d}\mu_y = \theta(x) \int_{T_x\mu} f \, \mathrm{d}\mathcal{H}^k \qquad \forall\ f \in C_c(\mathbb{R}^n)$$

and the map

$$\tau:\bigcup_{i=1}^{\infty}M_i\to G(n,k),\qquad x\mapsto T_x\mu$$

into the Grassmannian of unoriented k-planes in $\mathbb{R}^n$ is measurable. In fact, $T_x\mu=T_xM_i$ for μ-almost every $x\in M_i$. Using this first order information, the measure μ induces a unique Radon measure V on the product of $\mathbb{R}^n\times G(n,k)$ by

$$V(f)=\int_{\mathbb{R}^n}f(x,T_x\mu)\,\mathrm{d}\mu\qquad\forall\ f\in C_c(\mathbb{R}^n\times G(n,k)).$$

Often, V is referred to as the integral varifold and μ as its mass measure. We will also refer to μ as an integral varifold, using the one-to-one correspondence between μ and V. We will say that integral varifolds μ_k converge as Radon measures if the measures μ_k converge (weakly as Radon measures) and that they converge as varifolds if the associated measures V_k converge (weakly as Radon measures). The second mode of convergence is strictly stronger.

An integral varifold μ has a generalised mean curvature H if there exists $H\in L^1_{loc}(\mu,\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n}\mathrm{div}_{T_x\mu}X\,\mathrm{d}\mu=-\int_{\mathbb{R}^n}\langle H,X\rangle\,\mathrm{d}\mu\qquad\forall\ X\in C^1_c(\mathbb{R}^n,\mathbb{R}^n).\tag{1.1}$$

A deep result of Menne's [12] states that if μ has a generalised mean curvature (or more generally, locally bounded first variation), the manifolds M_i in the decomposition can even be chosen to be C^2-smooth and the mean curvature of μ agrees with the mean curvature of M_i μ-almost everywhere on M_i.

Willmore's energy is defined for integral varifolds μ (or for the corresponding V) through the integral over the squared weak mean curvature

$$\mathcal{W}(V)=\mathcal{W}(\mu)=\frac{1}{4}\int|H|^2\,\mathrm{d}\mu$$

if $H\in L^2(\mu)$. A special instance of Allard's compactness theorem [1] states that the set of integral k-varifolds satisfying uniform bounds

$$\mu(\mathbb{R}^n)+\mathcal{W}(\mu)\le C$$

and $\mathrm{spt}(\mu)\subset K$ for some compact $K\subset\mathbb{R}^n$ is (sequentially) compact under varifold convergence. Recall that the support of a measure is

$$\mathrm{spt}(\mu)=\{x\in\mathbb{R}^n\,|\,\mu(B_r(x))>0\quad\forall\ r>0\}.$$

We also extend Helfrich's energy to a suitable class of varifolds, specifically Hutchinson's curvature varifolds [8]. We will only calculate the Gauss curvature of varifolds that are constant integer density multiples of smooth surfaces and our results only depend on the fact that M and an integer multiple of M have the same Gauss curvature. This assumption is sensible, since the total curvature cannot distinguish between the immersed manifolds $N_x:=M\cup(M+x)$ for different $x\in\mathbb{R}^3$ which are the union of M and a translate of itself. The multiplicity-two case can be thought of as $x=0$.

Let us give the more technical definition. In [8], Hutchinson introduced a second fundamental form A for varifolds. Unlike the weak mean curvature H, the second fundamental form A has to be considered on the space $\mathbb{R}^n \times G(n,k)$ and the induced measure V has to be considered. We set

$$\mathcal{E}(V) = \chi_H \int_{\mathbb{R}^n} |H|^2 \,\mathrm{d}\mu + \chi_K \int_{\mathbb{R}^n \times G(n,k)} K \,\mathrm{d}V$$

where $2K = |H\circ\pi|^2 - |A|^2$ whenever $A \in L^2(V)$ and $\pi : \mathbb{R}^n \times G(n,k) \to \mathbb{R}^n$ is the projection on the first component. Here $|A|$ is the Frobenius norm of the second fundamental form. This definition gives the usual Gaussian curvature on C^2-manifolds or integer multiples thereof, since the density cancels out in the equation defining A. More generally, if V is a *curvature varifold* (i.e., a varifold with weak second fundamental form) then the integral varifold μ has locally finite first variation and is therefore C^2-rectifiable by [12]. The second fundamental form of V at $(x, T_x\mu)$ agrees with the second fundamental form of M_i at x for $\mathcal{H}^k$-almost every $x \in M_i$ where μ has a weak tangent space.

Recall the following result from [14] about varifolds supported in the unit ball.

Lemma 1.1 (Müller and Röger, Theorem 1). *Let μ be an integral 2-varifold with square integrable mean curvature H such that $\mathrm{spt}(\mu) \subset \overline{B_1(0)}$. Then we have*

$$\mathcal{W}(\mu) \geq \mu(B)$$

and equality holds if and only if $\mu = k\cdot\mathcal{H}^2|_{S^2}$ for an integer $k \in \mathbb{N}$.

Finally, recall the following localised Li–Yau inequality originally due to L. Simon. A proof formulated for manifolds can be found as Lemma 1 of [17], but the same argument applies to integral varifolds.

Lemma 1.2. *Let μ be an integral 2-varifold with $H \in L^2(\mu)$ and $r > 0$. Then*

$$\Theta^2(x) := \limsup_{s\to 0} \frac{\mu(B_s(x))}{\pi s^2} \leq \frac{\mu(B_r(x))}{\pi r^2} + \frac{1}{4\pi}\int_{B_r(x)} |H|^2 \,\mathrm{d}\mu \tag{1.2}$$

Since we only consider surfaces in this article, all varifolds in the following will be integral 2-varifolds.

2. Proofs

2.1. Approximation of spheres. In this section, we will prove Theorem 1.1. First we flatten the unit sphere slightly to have a flat segment on which we can easily glue two surfaces together. We denote by $D_r = B_r(0)$ the disc of radius r around the origin in $\mathbb{R}^2$.

Lemma 2.1 ("flattening a sphere"). *Let $\varepsilon > 0$. Then there exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ there exists a convex closed C^∞-sphere $M_\varepsilon \subset B_1(0)$ in $\mathbb{R}^3$ such that*

$$M_\varepsilon \cap [D_{2\delta} \times (0,\infty)] = \{x_3 = 1 - 3\delta\} \cap [D_{2\delta} \times (0,\infty)],$$
$$M_\varepsilon \setminus [D_{4\delta} \times (0,\infty)] = S^2 \setminus [D_{4\delta} \times (0,\infty)],$$

and

$$\mathcal{W}(M_\varepsilon) < 4\pi + \varepsilon.$$

Proof. Take $f \in C^\infty(-1,1)$, $f(t) = \sqrt{1-t^2}$, and

$$f_\delta : B_1(0) \to \mathbb{R}, \qquad f_\delta(x) = f\circ r_\delta(|x|) = \sqrt{1 - r_\delta^2(|x|)}$$

where $r_\delta \in C^\infty[0,1]$ satisfies

$$r_\delta(t) = \begin{cases} 3\delta & t \le 2\delta \\ t & t \ge 4\delta \end{cases}, \qquad 0 \le r_\delta' \le 1, \qquad 0 \le r_\delta'' \le \frac{4}{\delta}.$$

Then

$$\partial_i f_\delta(x) = (f' \circ r_\delta)\, r_\delta' \,\frac{x_i}{|x|} \tag{2.1}$$

and

$$\partial_{ij}^2 f_\delta(x) = (f'' \circ r_\delta)\,(r_\delta')^2 \,\frac{x_i x_j}{|x|^2} + (f' \circ r_\delta)\, r_\delta'' \,\frac{x_i x_j}{|x|^2} + (f' \circ r_\delta)\, r_\delta' \left[\frac{\delta_{ij}}{|x|} - \frac{x_i x_j}{|x|^3}\right]. \tag{2.2}$$

It is easy to see that $D^2 f_\delta$ is negative semi-definite since all three terms in the sum in equation (2.2) are negative semi-definite, so f_δ is concave. Thus

$$M^\delta := \{x \in S^2 \,|\, x_3 \le 0\} \cup \{(x, f(x)) \,|\, x \in D_1\}$$

is a convex sphere. The topological type can also be found through the Gaussian curvature integral, which coincides for f and f_δ since their boundary values agree by the Gauss–Bonnet theorem.

When we denote $f_0(x) = \sqrt{1 - |x|^2}$, we observe that $|f_\delta - f_0| \le 3\delta$,

$$|\partial_i f_\delta - \partial_i f_0| = \left[(f' \circ r_\delta) r_\delta' - (f' \circ r_0)\right] \frac{x_i}{|x|},$$

and

$$|\partial_{ij}^2 f_\delta - \partial_{ij}^2 f_0| = \left[(f'' \circ r_\delta)\,(r_\delta')^2 - (f'' \circ r_0|)\right] \frac{x_i x_j}{|x|^2} + (f' \circ r_\delta)\, r_\delta'' \,\frac{x_i x_j}{|x|^2}$$
$$- \left[(f' \circ r_\delta) r_\delta' - (f' \circ r_0)\right] \left[\frac{x_i x_j}{|x|^3} - \frac{\delta_{ij}}{|x|}\right].$$

The first term is small since $x_i/|x|$ is bounded and $f'(0) = 0$, so we can choose δ small enough to make f_δ and f_0 close in $C^1(\overline{B_1(0)})$. Curvature prevents us from making them C^2-close, but they are clearly $W^{2,2}$-close since

$$\left|\left|\left[(f'' \circ r_\delta)\,(r_\delta')^2 - (f'' \circ r_0)\right] \frac{x_i x_j}{|x|^2}\right|\right|_{L^2} \le 2\,||f''||_{L^\infty(-4\delta,4\delta)} \sqrt{\pi\,(4\delta)^2},$$

$$\left|\left|(f' \circ r_\delta)\, r_\delta'' \,\frac{x_i x_j}{|x|^2}\right|\right|_{L^2} \le ||f'||_{L^\infty(-4\delta,4\delta)} \left(\int_{D_{4\delta}} (4/\delta)^2 \,\mathrm{d}x\right)^{1/2},$$

and

$$\left|\left|\left[(f' \circ r_\delta) r_\delta' - (f' \circ r_0)\right] \left[\frac{x_i x_j}{|x|^3} - \frac{\delta_{ij}}{|x|}\right]\right|\right|_{L^2} \le 2\left(\int_{D_{4\delta}} \left(\frac{2}{|x|}\right)^2 2\left[||f''||_{L^\infty(-4\delta,4\delta)} |x|\right]^2 \mathrm{d}x\right)^{\frac{1}{2}}$$

all become small linearly with δ. Since mean curvature H_f, volume element dA_f, and Willmore integrand w_f of the graph

$$\Gamma_f = \{(x, f(x)) \,|\, x \in B_1(0) \subset \mathbb{R}^2\}$$

of f are given by

$$H_f = \frac{(1+f_y^2)\,f_{xx} - 2\,f_x f_y\,f_{xy} + (1+f_x^2)\,f_{yy}}{(1+f_x^2+f_y^2)^{3/2}},$$
$$\mathrm{dA}_f = \sqrt{1+f_x^2+f_y^2}, \qquad \text{and}$$
$$w_f = H_f^2\,\mathrm{dA}_f,$$

we see that $||w_f - w_g||_{L^1}$ is small if $||f-g||_{C^1}$ and $||f-g||_{W^{2,2}}$ are both small for some $g \in W^{2,2}(D_1)$. So we can choose δ small enough to make the additional Willmore energy of the new surface as small as we need for $g = f_\delta$. □

Next we create the handles by which we will connect spheres.

Lemma 2.2 ("flattening a catenoid"). *Let $R \gg 1$. Then there exists a connected orientable C^∞-manifold $\Sigma \subset \mathbb{R}^3$ such that*

$$\Sigma \setminus Z_R \quad = \quad \big(\{x_3 = R+1/2\} \cup \{x_3 = -(R+1/2)\}\big) \setminus \overline{Z_R}$$

where Z_R is the cylinder $Z_R = D_{\cosh(R+1)} \times (-R+1/2, R+1/2)$, and furthermore

$$\mathcal{W}(\Sigma) = O(e^{-2R}), \qquad \int_\Sigma K\,\mathrm{d}\mathcal{H}^2 = -4\pi$$

where K denotes the Gaussian curvature of Σ.

Proof. Define the surface of revolution

$$\Sigma = \left\{ \begin{pmatrix} f(t)\cos\phi \\ f(t)\sin\phi \\ g(t) \end{pmatrix} \,\middle|\, t,\phi \in \mathbb{R} \right\}.$$

If $f(t) = \cosh(t)$ and $g(t) = t$, Σ is the usual catenoid. We consider $f = \cosh$ and an even C^∞-function g satisfying

$$g(t) = \begin{cases} t & |t| \le R \\ R+1/2 & |t| \ge R+1 \end{cases},$$
$$0 < g'(t) \le 1 \text{ for } |t| < R+1, \quad \text{and} -4 \le g''(t) \le 0 \text{ for } t \ge 0.$$

Then clearly Σ is connected as the continuous image of a connected set and given as the union of two planes outside the cylinder Z_R. The volume element dA and the mean curvature of Σ are

$$\mathrm{dA} = f\sqrt{(g')^2 + (f')^2},$$

and

$$\begin{aligned} H &= \frac{f f'' g' - f f' g'' - g'(f')^2 - (g')^3}{f\,[(f')^2+(g')^2]^{3/2}} \\ &= \frac{g'(f f'' - (f')^2 - 1) + g'(1-(g')^2) - f f' g''}{f\,[(f')^2+(g')^2]^{3/2}} \\ &= \frac{g'(1-(g')^2) - f f' g''}{f\,[(f')^2+(g')^2]^{3/2}} \end{aligned}$$

respectively since $ff''-(f')^2-1=0$ for $f=\cosh$. Thus,

$$\begin{aligned}\mathcal{W}(\Sigma)&=2\pi\int_0^\infty \frac{\left[g'\left(1-(g')^2\right)-ff'g''\right]^2}{f\left[(f')^2+(g')^2\right]^{5/2}}\,\mathrm{d}t\\ &\leq 4\pi\int_R^{R+1}\frac{(g')^2(1-(g')^2)^2}{f\left[(f')^2+(g')^2\right]^{5/2}}+\frac{f(f')^2(g'')^2}{\left[(f')^2+(g')^2\right]^{5/2}}\,\mathrm{d}t\\ &\leq 4\pi\int_R^{R+1}\frac{1}{f(f')^5}\,\mathrm{d}t+4\pi\int_R^{R+1}\frac{f(g'')^2}{|f'|^3}\,\mathrm{d}t\\ &=O(e^{-2R}).\end{aligned}$$

It remains to show that the total Gaussian curvature is -4π. When we orient Σ by the choice of the normal vector

$$\nu=\frac{1}{f\sqrt{(f')^2+(g')^2}}\begin{pmatrix}-fg'\cos\phi\\ -fg'\sin\phi\\ ff'\end{pmatrix},$$

we see that every unit vector $\nu=(\sin\theta)e_\phi+(\cos\theta)e_z\neq(0,0,\pm1)$ is the normal ν_x at the unique point $x\in\Sigma$ determined by the ϕ-coordinate and t given by

$$\tan\theta=-\frac{g'(t)}{f'(t)}.$$

This is uniquely solvable except for $\tan\theta=0$ by construction of g. Since $f''\geq0$ and $f'g''\leq0$, we know that

$$K=\frac{-(g')^2f''+f'g'g''}{f\left[(f')^2+(g')^2\right]^2}\leq0$$

is the determinant of the Gauss map $G:\Sigma\to S^2$, $G(x)=\nu_x$, so

$$4\pi=\mathcal{H}^2(S^2)=\mathcal{H}^2(G(\Sigma))=\int_\Sigma|K|\,\mathrm{d}\mathcal{H}^2=-\int_\Sigma K\,\mathrm{d}\mathcal{H}^2.$$

□

Now we are ready to prove this section's main statement.

Proof. **(Proof of Theorem 1.1.)** We first give the proof for $m=2$. Let $\beta>0$ be chosen later depending on $\varepsilon,\delta>0$. Take M_β constructed as in Lemma 2.1, and $\delta>0$ such that M_β coincides with the plane $\{x_3=1-3\delta\}$ inside the cylinder $D_{2\delta}\times(0,1)$. We may specify δ to be taken sufficiently small later. Take $0<\rho<\delta/2$ such that there are $g+1$ points $x_1,\dots,x_{g+1}$ in $D_{\delta/2}$ such that the discs $D_\rho(x_i)$ are pairwise disjoint.

Choose $R>0$ and Σ as in Lemma 2.2, such that $\mathcal{W}(\Sigma)=O(e^{-2R})<\beta$. Then choose $\eta>0$ such that $\eta\cosh(R+1)<\rho$ and $\eta R<\delta^3$. Finally, define

$$r=\frac{1-3\delta-(2R+1)\eta}{1-3\delta}<1,\qquad \tilde M=M_\beta\cup r\cdot M_\beta.$$

Since M_β is convex, this is a smooth embedded manifold. By construction, inside the cylinders

$$Z_i:=D_\rho(x_i)\times\{0<z<1\},\qquad i=1,\dots,g+1$$

$\tilde{M}$ is given by the union of the planes $\{z=1-3\delta\}$ and $\{z=1-3\delta-(2R+1)\eta\}$, which have separation $(2R+1)\eta$. Since the Z_i are disjoint, we can replace $\tilde{M}$ inside each cylinder by

$$x_i+\frac{1+r}{2}e_z+\eta\cdot\left(\Sigma\cap Z_R\right).$$

We call the resulting manifold M. It is clear that M is a connected surface. Since both the total curvature integral and Willmore's energy are invariant under spacial rescaling, and since M is flat on the remaining segments, we have

$$\begin{aligned}\mathcal{W}(M)=2\mathcal{W}(M_\beta)+(g+1)\mathcal{W}(\Sigma) &\qquad <8\pi+2\varepsilon+(g+1)\beta, \quad \text{and}\\ \int_M K\,\mathrm{d}\mathcal{H}^2=2\int_{M_\beta}K\,\mathrm{d}\mathcal{H}^2+(g+1)\int_\Sigma K\,\mathrm{d}\mathcal{H}^2 &\qquad =2\cdot 4\pi+(g+1)\cdot(-4\pi)=4\pi(1-g),\end{aligned}$$

so that M is a closed smoothly embedded orientable genus g surface with small Willmore energy. Unfortunately, $M\not\subset B_1(0)$ and it is not clear after our modifications whether $\mathcal{H}^2(M)=8\pi$. If $\mathcal{H}^2(M)>8\pi$, we only need to choose $\beta=\varepsilon/(g+3)$ and set

$$M_g=\sqrt{\frac{8\pi}{\mathcal{H}^2(M)}}\cdot M. \tag{2.3}$$

The more complicated case is $\mathcal{H}^2(M)\le 8\pi$. Then at least

$$\mathcal{H}^2(M)\ge\left(1+r^2\right)\mathcal{H}^2\left(S^2\setminus[D_{4\delta}\times(0,1)]\right)\ge 8\pi-C\delta^2$$

since $\eta R<\delta^3$, and thus $r\ge 1-\delta^2$. Now consider only the inner sphere, which is still spherical around its south pole. Take a function $h\in C_c^\infty(D_r)$ on a small disc such that $h\ge 0$ and $h\not\equiv 0$. Then we may replace a neighbourhood of the south pole of the inner sphere by

$$\tilde{\Sigma}^t=\left\{\left(x,-\sqrt{r^2-|x|^2}+t\,h\left(\frac{x}{\alpha\sqrt{t}}\right)\right)\,\middle|\,x\in B_r(0)\right\}.$$

The resulting surface is denoted by M^t. Again, this does not change the topological type, but it changes the area and the Willmore functional by

$$\mathcal{H}^2(M^t)\ge\mathcal{H}^2(M)+ct^2,\qquad \mathcal{W}(M_t)\le\mathcal{W}(M)+Ct$$

as is computed in the proof of Proposition 2 in [14], at least for suitable spherically symmetric h. Thus we can take $t=O(\delta)$ such that $\mathcal{H}^2(M^t)>8\pi$ and define M_g again by equation (2.3), this time choosing both β and δ small enough depending on $\varepsilon>0$.

Now consider the case $m\in\mathbb{N}$, $m\ge 3$. We proceed inductively by first linking two spheres of similar radii $r_1<r_2<1$ with $g+1$ catenoids at the north pole as above. Then we link the outer sphere with one catenoid to a sphere of radius $r_3\in(r_2,1)$ at the south pole. The third sphere is connected with one catenoid at the north pole to a sphere of radius $r_4\in(r_3,1)$ and so on until we have linked m spheres with $m+g-1$ catenoids. The genus can be computed by the total integral of Gauss curvature. The resulting surface is orientable since it is embedded in $\mathbb{R}^3$. Finally, we modify the innermost sphere around its south pole as before to obtain the correct surface area. The smallness of the energy can be established as before when r_1 is close enough to 1. ∎

REMARK 2.1. If we fix a genus g, then we can even find a C^2-smooth map $f: (0,\varepsilon)\times M_\varepsilon \to \mathbb{R}^3$ which maps (t,M_ε) to M_t constructed above. In particular, $f(t,M_\varepsilon)$ is a C^∞-smooth manifold for all $t\in(0,\varepsilon)$. Clearly, the images converge as varifolds to an m-fold covered sphere as $t\searrow 0$. We can continue the evolution past the m-fold covered sphere in various ways. This describes a singularity in a geometric flow which may occur with decreasing Willmore energy in finite time. It is unclear whether such singularities would appear in the gradient flow of the Willmore functional.

2.2. Proofs of the corollaries. Let us use Theorem 1.1 to illustrate phenomena occurring when we minimise curvature energies under area constraint in the unit ball.

Proof. **(Proof of Corollary 1.1.)** By Theorem 1.1, there exists a sequence $N_k\in\mathcal{M}:=\mathcal{M}_{g,4m\pi,B_1(0)}$ such that $\mathcal{W}(N_k)<4m\pi+1/k$. So

$$4m\pi\le\inf\left\{\mathcal{W}(M)\,\middle|\,M\in\mathcal{M}\right\}\le 4m\pi$$

with the first estimate coming from Lemma 1.1. Now let M_k be a minimising sequence in $\mathcal{M}$. Take a subsequence of M_k. Due to Allard's compactness theorem [1], there exists an integral varifold μ with square integrable mean curvature H such that a further subsequence converges to μ as varifolds and

$$\mathcal{W}(\mu)\le\limsup_{k\to\infty}\mathcal{W}(M_k)=\inf\left\{\mathcal{W}(M)\,\middle|\,M\in\mathcal{M}\right\}\le 4m\pi.$$

The convergence of varifolds implies the convergence of their mass measures as Radon measures, so $\mu(\overline{B_1(0)})=4m\pi$ and $\mu(\mathbb{R}^n\setminus\overline{B_1(0)})=0$, whence $\mu=m\cdot\mathcal{H}^2|_{S^2}$ by Lemma 1.1. Since every subsequence has a further subsequence which converges to the same limit and varifold convergence is topological (as a convergence of Radon measures), we see that the whole sequence converges. □

REMARK 2.2. The radial symmetry of the sphere simplifies the calculations in Lemma 2.1, but in fact any C^2-surface can be locally flattened around a point when written as a graph over its tangent space. This would allow us to generalise the construction below to more generic containers Ω and areas S. We conjecture that any minimising sequence $M_k\in\mathcal{M}_{g,S,\Omega}$ for $\mathcal{W}$ converges to M for all $g\ge g_0\in\mathbb{N}_0$ if the following assumptions are met:

(1) There exists a *unique* minimiser μ of $\mathcal{W}$ in $\overline{\mathcal{M}_{S,\Omega}}$ such that
(2) $\mu\in\overline{\mathcal{M}_{g_0,S,\Omega}}$ and
(3) there exists at least one point $x_0\in\operatorname{spt}(\mu)$ with $\Theta^2(\mu,x_0)\ge 2$.
(4) Furthermore, the tangent space $T_{x_0}\mu$ exists and there is $r>0$ such that $\operatorname{spt}(\mu)\cap B_r(x_0)$ can be written as the union of finitely many $W^{2,2}\cap C^1$-graphs over $T_{x_0}\mu$ which only touch tangentially.

Proof. **(Proof of Corollary 1.2.) Ad (3).** Since $\int_M|H|^2\,\mathrm{d}\mathcal{H}^2>4\mathcal{H}^2(M)$ by Lemma 1.1 for manifolds in $B_1(0)$, a manifold M satisfying

$$4\chi_H\mathcal{H}^2(M)\ge\mathcal{E}(M)=\chi_H\int_M|H|^2\,\mathrm{d}\mathcal{H}^2+4\pi\chi_K\int_M K\,\mathrm{d}\mathcal{H}^2>4\chi_H\mathcal{H}^2(M)+4\pi\chi_K(1-g)$$

has genus $g=0$. **Ad (1).** As before,

$$\inf\left\{\mathcal{E}(M)\,|\,M\in\mathcal{M}_{4\pi m,B_1(0)}\right\}=16\pi m\chi_H-4\pi|\chi_K|$$

is realised by smooth spheres converging to a multiplicity-m sphere. **Ad (2).** As noted before, a smooth multiplicity-m sphere $\mu := m\cdot\mathcal{H}^2|_{S^2}$ has total Gaussian curvature $\int K\,\mathrm{d}V = 4\pi m$. Thus

$$\mathcal{E}(V) = 4\pi m(4\chi_H - |\chi_K|) < 16\pi m\chi_H - 4\pi|\chi_K| = \lim_{k\to\infty}\mathcal{E}(M_k).$$

Ad (4). If $\chi_K < -4\chi_H$, then multiplicity-m spheres illustrate that $\mathcal{E}$ is not bounded below on the varifold closure of smooth surfaces, since $m\cdot\mathcal{H}^2|_{S^2}$ can be approximated with finite energy $\mathcal{E}$.

Assume that M_k is a sequence of smooth surfaces with energy $\mathcal{E}$ bounded by C and that M_k converges to a varifold μ. This implies that their genera and Willmore energies are bounded by

$$g \le \frac{C}{4\pi|\chi_K|} + 1, \qquad \mathcal{W}(M_k) \le \mathcal{E}(M_k) + 4\pi|\chi_K|.$$

Therefore,

$$\int_{M_k} |A|^2\,\mathrm{d}\mathcal{H}^2 = \int_{M_k} |H|^2 - 2K\,\mathrm{d}\mathcal{H}^2 \le 4[\mathcal{E}(M_k) + 4\pi|\chi_K|] + 8\pi\left[\frac{C}{4\pi|\chi_K|} + 1\right].$$

This is uniformly bounded in k, so μ is a curvature varifold [8]. Clearly,

$$\mathcal{E}(V) \ge \chi_H\mathcal{W}(\mu) - 2|\chi_K|\mathcal{W}(\mu) \ge -|\chi_K|\int |A|^2\,\mathrm{d}V$$

is a uniform bound from below in $\overline{\mathcal{B}_{S,\Omega,C}}$. Here we used the inequality $2K \le |H|^2$ for the first estimate. ∎

The 'change of topological type' is mathematically meaningful. As the catenoid collapses away, two spheres remain in the limit. The Gaussian integral does not see that these spheres happen to coincide.

Proof. **(Proof of Corollary 1.3.)** To approximate a multiplicity-one sphere by manifolds M_k, insert a sphere of radius $1/k$ into a sphere of radius $\approx r$ and connect the two by g_k catenoids, $g_k \to \infty$. Willmore's energy is close to 8π, so the total energy is

$$\begin{aligned}
\mathcal{E}(M_k) &= \int_{M_k} \chi_H^k (H - H_0^k)^2\,\mathrm{d}\mathcal{H}^2 + \int_{M_k} \chi_K^k K\,\mathrm{d}\mathcal{H}^2\\
&\le C\int_{M_k} 2\left(|H|^2 + |H_0^k|^2\right)\mathrm{d}\mathcal{H}^2 + \delta\int_{M_k\cap\{K<0\}} K\,\mathrm{d}\mathcal{H}^2 + C\int_{M_k\cap\{K>0\}} K\,\mathrm{d}\mathcal{H}^2\\
&\le 2C\left(8\pi + 1 + C^2\right) + \delta\int_{M_k} K\,\mathrm{d}\mathcal{H}^2 + \frac{C}{4}\int_{M_k\cap\{K>0\}} H^2\,\mathrm{d}\mathcal{H}^2\\
&\le 2C\left(8\pi + 1 + C^2\right) + 4\pi\delta(1 - g_k) + C^3/4
\end{aligned}$$

since $K = \lambda_1\lambda_2 \le (\lambda_1 + \lambda_2)^2/4 = H^2/4$ if λ_1 and λ_2 have the same sign. Clearly, this goes to $-\infty$ as $g_k \to \infty$. To approximate a Dirac measure, we approximate a multiplicity-m sphere of radius $r_m = r/\sqrt{m}$ with genus g-manifolds $\tilde{M}_m$, $g \gg m$. ∎

2.3. Connectedness. We conclude this article with a more positive result.

Proof. **(Proof of Theorem 1.2.)** Since varifold convergence is weak* convergence and the pre-dual of varifolds is the space of continuous functions on a compact manifold

$\overline{\Omega}\times G(3,2)$ (in particular, it is separable), the topology is locally metrisable. Therefore, compactness and sequential compactness coincide on the bounded set we consider.

Take a sequence of integral 2-varifolds μ_k. By Allard's compactness theorem [1], there is a subsequence converging to an integral 2-varifold μ. Take a subsequence of μ_k for which the supports $\mathrm{spt}(\mu_k)$ converge to a compact set $K\subset\overline{B_R(0)}$ in the Hausdorff distance. We will show that points which lie in $K\setminus\mathrm{spt}(\mu)$ are atoms of mass at least 4π of the limit of measures associated to Willmore's energy. Since this is a finite measure, there can only be finitely many such points. If $\mathrm{spt}(\mu_k)$ is connected for all k, then the limit K is also connected and there cannot be any isolated points, so $K\subset\mathrm{spt}(\mu)$. The reverse inclusion always holds, so $\mathrm{spt}(\mu)=K$ is connected.

Assume that $x\in K\setminus\mathrm{spt}(\mu)$. Denote the weak* limit $\alpha=\lim_{k\to\infty}|H_k|^2\cdot\mu_k$ (for a subsequence along which it exists). We can take a sequence $x_k\in\mathrm{spt}(\mu_k)$, $x_k\to x$ with

$$\theta_k(x_k)=\Theta^2_{\mu_k}(x_k)=\limsup_{r\to 0}\frac{\mu_k(B_r(x_k))}{\pi r^2}\geq 1$$

since the set of points where $\theta_k(x)=\lim_{r\to 0}\frac{\mu_k(B_r(x))}{\pi r^2}$ (Lebesgue points of θ_k) is dense in $\mathrm{spt}(\mu_k)$. Take $\rho>0$ with $\mu(\overline{B_\rho(x)})=0$. Due to Lemma 1.2 applied to μ_k, we get

$$\begin{aligned}1&\leq\liminf_{k\to\infty}\left(\frac{\mu_k(B_{\rho/2}(x_k))}{\pi(\rho/2)^2}+\frac{1}{4\pi}\int_{B_{\rho/2}(x_k)}|H_k|^2\,\mathrm{d}\mu_k\right)\\&\leq\liminf_{k\to\infty}\frac{1}{4\pi}\int_{B_\rho(x)}|H_k|^2\,\mathrm{d}\mu_k\\&\leq\alpha\left(\overline{B_\rho(x)}\right)\end{aligned}$$

since the first term vanishes in the limit and $B_{\rho/2}(x_k)\subset B_\rho(x)$ for all large $k\in\mathbb{N}$. Taking $\rho\to 0$ shows that $\alpha(\{x\})\geq 4\pi$ and concludes the proof. Compare to Lemma 3.5 in [4] for a phase field version of this argument.

A diagonal sequence shows that the subclass of varifolds which arise as the weak* limits of embedded connected C^2-manifolds with suitable bounds is closed. Hence, it is compact as well. □

Remark 2.3. We would like to thank the referee for pointing out that a similar, but slightly different, proof of this result had already been given as the proof of Proposition A.2 in [10]. We believe that our proof holds independent interest since it easily yields Hausdorff convergence $\mathrm{spt}(\mu_k)\to\mathrm{spt}(\mu)$ if the stronger energy bound

$$\limsup_{k\to\infty}\int|H_k|^p\,\mathrm{d}\mu_k<\infty$$

for some $p>2$ is assumed, since in that case

$$\alpha(B_\rho(x))\leq\limsup_{k\to\infty}\int_{B_{2\rho}(x)}|H_k|^2\,\mathrm{d}\mu_k\leq\limsup_{k\to\infty}\left(\int_{B_{2\rho}(x)}|H_k|^2\,\mathrm{d}\mu_k\right)^{1-\frac{2}{p}}\mu_k(B_{2\rho}(x))^{\frac{2}{p}}=0$$

for $x\notin\mathrm{spt}(\mu)$ and some small $\rho>0$. Similarly, Hausdorff convergence can be established directly if $\limsup_{k\to\infty}\mathcal{W}(\mu_k)<\mathcal{W}(\mu)+4\pi$, whereas Proposition A.2 in [10] only yields an implicit bound. On the other hand, a lower bound on the quotients $\mu(B_r(x))/r^2$ for $x\in\mathrm{spt}(\mu)$ used in the proof of Proposition A.2 in [10] and established in Proposition A.1 in [10] may be of independent interest for different applications.

Acknowledgements. I would like to thank P.W. Dondl for his encouragement and for many stimulating discussions. I would also like to thank Durham University for financial support through a Durham Doctoral Studentship. I am grateful to the anonymous referee for the constructive feedback.

REFERENCES

[1] W.K. Allard, *On the first variation of a varifold*, Ann. of Math., 95(3):417–491, 1972.

[2] M. Bauer and E. Kuwert, *Existence of minimizing Willmore surfaces of prescribed genus*, Int. Math. Res. Not., 2003(10):553–576, 2003.

[3] P.B. Canham, *The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell*, Journal of Theoretical Biology, 26(1):61–81, 1970.

[4] P.W. Dondl and S. Wojtowytsch, *Uniform regularity and convergence of phase-fields for Willmore's energy*, Calc. Var. PDE, 56(4), 2017.

[5] G. Friesecke, R.D. James, and S. Müller, *A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity*, Commun. Pure Appl. Math., 55(11):1461–1506, 2002.

[6] K. Große-Brauckmann, *New surfaces of constant mean curvature*, Mathematische Zeitschrift, 214(1):527–565, 1993.

[7] W. Helfrich, *Elastic properties of lipid bilayers – theory and possible experiments*, Zeitschrift für Naturforschung C – a Journal of Biosciences, 28(11):693–703, 1973.

[8] J.E. Hutchinson, *Second fundamental form for varifolds and the existence of surfaces minimizing curvature*, Indiana University Math. J., 35(1):45–71, 1986.

[9] L.G.A. Keller, A. Mondino, and T. Rivière, *Embedded surfaces of arbitrary genus minimizing the Willmore energy under isoperimetric constraint*, Arch. Ration. Mech. Anal., 212(2):645–682, 2014.

[10] T. Lamm and R.M. Schätzle, *Optimal rigidity estimates for nearly umbilical surfaces in arbitrary codimension*, Geom. Funct. Anal., 24(6):2029–2062, 2014.

[11] L. Lussardi, M.A. Peletier, and M. Röger, *Variational analysis of a mesoscale model for bilayer membranes*, Journal of Fixed Point Theory and Applications, 15(1):217–240, 2014.

[12] U. Menne, *Second order rectifiability of integral varifolds of locally bounded first variation*, J. Geom. Anal., 23(2):709–763, 2013.

[13] U. Menne, *The concept of varifold*, arXiv:1705.05253 [math.DG], 05 2017.

[14] S. Müller and M. Röger, *Confined structures of least bending energy*, J. Diff. Geom., 97(1):109–139, 2014.

[15] M.A. Peletier and M. Röger, *Partial localization, lipid bilayers, and the elastica functional*, Archive for Rational Mechanics and Analysis, 193(3):475–537, 2009.

[16] L. Simon, *Lectures on Geometric Measure Theory*, Australian National University Centre for Mathematical Analysis, Canberra, 3, 1983.

[17] P. Topping, *Mean curvature flow and geometric inequalities*, J. Reine Angew. Math., 503:47–61, 1998.