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EXISTENCE OF WEAK SOLUTIONS TO A KINETIC FLOCKING MODEL WITH CUT-OFF INTERACTION FUNCTION*

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Abstract. We prove the existence of weak solutions to a kinetic flocking model with cut-off interaction function by using the weak convergence method. Under the natural assumption that the $\boldsymbol{v}$-support of the initial distribution function $f_0(\boldsymbol{x},\boldsymbol{v})$ is bounded, we show that the $\boldsymbol{v}$-support of the distribution function $f(t,\boldsymbol{x},\boldsymbol{v})$ is uniformly bounded in time. Employing this property, we remove the constraint in the paper of Karper, Mellet, and Trivisa (SIAM. J. Math. Anal., 45, 215–243, 2013) that the initial distribution function should have better integrability for large $|\boldsymbol{x}|$.

Keywords. Cucker–Smale model, kinetic flocking model, velocity averaging lemma, Schauder fixed point theorem.

AMS subject classifications. 35D30, 35Q83, 35Q92, 74H20.

1. Introduction

In this paper, we consider the existence of weak solutions for the following kinetic flocking model with cut-off interaction function:

$$\begin{cases} f_t+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f+\lambda\nabla_{\boldsymbol{v}}\cdot[(\boldsymbol{u}(t,\boldsymbol{x})-\boldsymbol{v})f]=0, \\ f|_{t=0}=f_0(\boldsymbol{x},\boldsymbol{v}), \end{cases} \tag{1.1}$$

where $f(t,x,v)$ is the distribution function and λ is a positive constant denoting the coupling strength. Define

$$\boldsymbol{j}_r(t,\boldsymbol{x})=\int_{|\boldsymbol{x}-\boldsymbol{y}|<r}\int_{\mathbb{R}^d} f(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}\,d\boldsymbol{w}\,d\boldsymbol{y},\quad \rho_r(t,\boldsymbol{x})=\int_{|\boldsymbol{x}-\boldsymbol{y}|<r}\int_{\mathbb{R}^d} f(t,\boldsymbol{y},\boldsymbol{w})\,d\boldsymbol{w}\,d\boldsymbol{y},$$

where $r>0$ denotes the neighborhood radius. Here $\boldsymbol{u}(t,x)$ is given by

$$\boldsymbol{u}(t,\boldsymbol{x})=\begin{cases} \dfrac{\boldsymbol{j}_r(t,\boldsymbol{x})}{\rho_r(t,\boldsymbol{x})}, & \rho_r(t,\boldsymbol{x})\neq 0, \\ 0, & \rho_r(t,\boldsymbol{x})=0. \end{cases} \tag{1.2}$$

This model is formally derived from the particle model by taking the mean-field limit. Now let us review some background related to it.

Collective behaviors are common phenomena in nature, such as flocking of birds, schooling of fish, and swarming of bacteria. These phenomena have drawn much attention from researchers in biology, physics, and mathematics. They try to understand the mechanisms that lead to these behaviors via modeling, numerical simulation, and mathematical analysis.

Vicsek et al. [30] put forward a simple discrete model composed of N autonomous agents moving in the plane with the same speed v. Their positions $(x_i,y_i)(1\le i\le N)$ and headings $\theta_i(1\le i\le N)$ are updated as follows:

$$\begin{cases} x_i(t+1)=x_i(t)+v\cos\theta_i(t), \\ y_i(t+1)=y_i(t)+v\sin\theta_i(t), \quad i=1,2,\cdots,N, \end{cases} \tag{1.3}$$

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$$\theta_i(t+1) = \arctan \frac{\sum_{j \in \mathcal{N}_i(t)} \sin\theta_j(t)}{\sum_{j \in \mathcal{N}_i(t)} \cos\theta_j(t)},$$

where $\mathcal{N}_i(t) = \left\{ j : \sqrt{(x_j(t)-x_i(t))^2 + (y_j(t)-y_i(t))^2} < r \right\}$ denotes the neighbors of agent i at the instant t.

Through simulations, Vicsek et al. found that this system can synchronize, that is, all agents move in the same direction when the density is large and the noise is small. Following this, mathematicians have tried to give a rigorous theoretical analysis. They found that the connectivity of the neighbor graph is crucial in the proof (cf. [22, 25]). However, in general, the verification of connectivity is difficult. One way to avoid this difficulty is to modify the Vicsek et al. model from local interactions to global ones. In 2007, Cucker and Smale [9] proposed the following model:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \\ \dfrac{d\boldsymbol{v}_i}{dt} = \dfrac{\lambda}{N} \displaystyle\sum_{j=1}^{N} \psi(|\boldsymbol{x}_j - \boldsymbol{x}_i|)(\boldsymbol{v}_j - \boldsymbol{v}_i), \quad i = 1, 2, \cdots, N, \end{cases} \tag{1.4}$$

where $\psi(\cdot)$ is a positive non-increasing function denoting the interactions between agents. The Cucker–Smale model has attracted much interest from mathematicians since it was put forward. Nowadays, studies of the Cucker–Smale model from particle to kinetic and hydrodynamic descriptions have been launched (see [2–5, 18, 20] and the references therein).

However, the Cucker–Smale model will lead to this problem: if a small group is located far from a much larger group, then the dynamics of the small group is almost halted because of the normalization factor $\frac{1}{N}$ in equation (1.4). To remedy this deficiency, Motsch and Tadmor [27] proposed the following model:

$$\begin{cases} \dfrac{d\boldsymbol{x}_i}{dt} = \boldsymbol{v}_i, \\ \dfrac{d\boldsymbol{v}_i}{dt} = \dfrac{\lambda}{\sum_{j=1}^{N} \psi(|\boldsymbol{x}_j - \boldsymbol{x}_i|)} \displaystyle\sum_{j=1}^{N} \psi(|\boldsymbol{x}_j - \boldsymbol{x}_i|)(\boldsymbol{v}_j - \boldsymbol{v}_i), \quad i = 1, 2, \cdots, N. \end{cases} \tag{1.5}$$

They also analyzed the flocking of this non-symmetric model by introducing the active set.

However, when the number of agents is large, it is convenient to study the average interaction between the large number of agents. Following the strategy from statistical physics, they formally derived the following kinetic model:

$$f_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} f + \lambda \nabla_{\boldsymbol{v}} \cdot (L[f] f) = 0, \tag{1.6}$$

where $L[f]$ is given by

$$L[f](t, \boldsymbol{x}, \boldsymbol{v}) = \frac{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(|\boldsymbol{x} - \boldsymbol{y}|) f(t, \boldsymbol{y}, \boldsymbol{w})(\boldsymbol{w} - \boldsymbol{v}) d\boldsymbol{w} d\boldsymbol{y}}{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi(|\boldsymbol{x} - \boldsymbol{y}|) f(t, \boldsymbol{y}, \boldsymbol{w}) d\boldsymbol{w} d\boldsymbol{y}}.$$

In some biological systems, such as opinion formation systems, fish schools, and honeybee groups, each agent in a group mainly detects the information around itself. So, a more realistic requirement is that the interaction function ψ is rapidly decaying or cut-off at a finite distance. Recently, Karper, Mellet, and Trivisa [23] studied model (1.6) with the interaction function ψ being compactly supported. They obtained the existence of the weak solution. For simplicity, we use their notations in this paper and restate their result as follows.

THEOREM 1.1 (Karper–Mellet–Trivisa). *Assume that* $f_0 \geq 0$ *satisfies*

$$f_0 \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d}) \quad \text{and} \quad (|\boldsymbol{v}|^2 + |\boldsymbol{x}|^2) f_0 \in L^1(\mathbb{R}^{2d}).$$

Suppose that ψ *is a smooth non-negative function such that*

$$\psi(x) > 0 \quad \text{for } |x| \leq r, \qquad \psi(x) = 0 \quad \text{for } |x| \geq R.$$

Then there exists a weak solution to model (1.6) *in the sense of distributions.*

In this paper, we also study (1.6), with the interaction function being the characteristic function of a ball, and under the condition that the $\boldsymbol{v}$-support of the initial distribution function $f_0(\boldsymbol{x},\boldsymbol{v})$ is bounded. This condition is natural in view of the derivation of the kinetic model (1.6). Since the particle agents initially have bounded speeds, it is reasonable to assume that $f_0(\boldsymbol{x},\boldsymbol{v})$ has bounded $\boldsymbol{v}$-support after taking the mean-field limit. Then, using our technical Lemma 2.1, we show that the $\boldsymbol{v}$-support of $f(t,\boldsymbol{x},\boldsymbol{v})$ is uniformly bounded in time. Employing this property, we remove the constraint $|\boldsymbol{x}|^2 f_0 \in L^1(\mathbb{R}^{2d})$ in Theorem 1.1 [23].

Denote

$$M(t) = \sup\{|\boldsymbol{v}| : (\boldsymbol{x},\boldsymbol{v}) \in \operatorname{supp} f(t,\cdot,\cdot)\} \quad \text{and} \quad M_0 > 0.$$

Define $B(M_0)$ as a ball centered at 0 of radius M_0 in $\mathbb{R}^d$. Next we give the definition of the weak solution and present our main theorem.

DEFINITION 1.1. *Let* $0 \leq f_0(\boldsymbol{x},\boldsymbol{v}) \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$, $\operatorname{supp}_{\boldsymbol{v}} f_0(\boldsymbol{x},\cdot) \subseteq B(M_0)$. *We say* $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,T], L^1(\mathbb{R}^{2d}) - W) \cap L^\infty([0,T] \times \mathbb{R}^{2d}), \forall T > 0$ *is a weak solution to equation* (1.1) *if*

$$\int_0^T \int_{\mathbb{R}^{2d}} f\psi_t \, d\boldsymbol{x} d\boldsymbol{v} dt + \int_0^T \int_{\mathbb{R}^{2d}} f\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \psi \, d\boldsymbol{x} d\boldsymbol{v} dt + \lambda \int_0^T \int_{\mathbb{R}^{2d}} f(\boldsymbol{u} - \boldsymbol{v}) \cdot \nabla_{\boldsymbol{v}} \psi \, d\boldsymbol{x} d\boldsymbol{v} dt$$
$$+ \int_{\mathbb{R}^{2d}} f_0 \psi(0) \, d\boldsymbol{x} d\boldsymbol{v} = 0 \quad \forall \psi(t,\boldsymbol{x},\boldsymbol{v}) \in \mathcal{D}([0,T) \times \mathbb{R}^{2d}),$$

and $f|_{t=0} = f_0(\boldsymbol{x},\boldsymbol{v})$ *in* $L^1(\mathbb{R}^{2d})$. *Here* $C([0,T], L^1(\mathbb{R}^{2d}) - W)$ *means that* f *is continuous in* $[0,T]$ *with respect to the weak topology in* $L^1(\mathbb{R}^{2d})$.

THEOREM 1.2. *Assume* $0 \leq f_0(\boldsymbol{x},\boldsymbol{v}) \in L^1(\mathbb{R}^{2d}) \cap L^\infty(\mathbb{R}^{2d})$ *and* $\operatorname{supp}_{\boldsymbol{v}} f_0(\boldsymbol{x},\cdot) \subseteq B(M_0)$. *Then equation* (1.1) *admits a weak solution* $f(t,\boldsymbol{x},\boldsymbol{v}) \in C([0,T], L^1(\mathbb{R}^{2d}) - W) \cap L^\infty([0,T] \times \mathbb{R}^{2d})$, $\forall T > 0$. *Moreover,* $f(t,\boldsymbol{x},\boldsymbol{v})$ *and* $M(t)$ *satisfy*

(1) $0 \leq f(t,\boldsymbol{x},\boldsymbol{v}) \leq \|f_0\|_{L^\infty(\mathbb{R}^{2d})} e^{\lambda dt}, \quad$ *a.e.* $(t,\boldsymbol{x},\boldsymbol{v}) \in [0,T] \times \mathbb{R}^{2d}$;

(2) $\|f(t)\|_{L^p(\mathbb{R}^{2d})} \leq e^{\frac{\lambda d(p-1)t}{p}} \|f_0\|_{L^p(\mathbb{R}^{2d})}, \ 1 \leq p < \infty, \quad \forall t \in [0,T]$;

(3) $M(t) \leq M_0, \ \forall t \in [0,T]$.

REMARK 1.1. The same results hold if we replace the interaction function with a non-negative bounded function with compact support. A little modification of steps 2 and

3 of the proof of Proposition 3.1 below gives this conclusion, since the the interaction function and its support can be controlled by a constant and a ball, respectively.

The rest of the paper is divided into four parts. In Section 2, we prove the well-posedness of the weak solution to the linearized equation. In Section 3, based on the results concerning the linear equation, we show that there exists a weak solution to the approximate equation by using the Schauder fixed point theorem. Section 4 is devoted to the proof of our theorem. In the last section, we summarize our paper and make a brief comment on it.

Notation. Throughout the paper, a superscript i of a vector denotes its i-th component, while a subscript denotes its order. We denote by C a general positive constant that may depend on λ, r, M_0 and $\|f_0\|_{L^\infty(\mathbb{R}^{2d})}$. Note that C may take different values in different expressions.

2. Preliminary

In this section, we linearize equation (1.1) by substituting the nonlinear term $\boldsymbol{u}$ with a given function E and study the well-posedness of the weak solution to the following linear equation:

$$\begin{cases} f_t+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f+\lambda\nabla_{\boldsymbol{v}}\cdot[(E(t,\boldsymbol{x})-\boldsymbol{v})f]=0 \quad \text{in } [0,T]\times\mathbb{R}^{2d}, \\ f|_{t=0}=f_0(\boldsymbol{x},\boldsymbol{v}), \end{cases} \tag{2.1}$$

with $E(t,\boldsymbol{x})\in[C([0,T]\times\mathbb{R}^{2d})]^d$ $(\forall T>0)$ satisfying

$$|E(t,\boldsymbol{x}_2)-E(t,\boldsymbol{x}_1)|\le K|\boldsymbol{x}_2-\boldsymbol{x}_1|, \quad \forall t\in[0,T], \tag{2.2}$$

where K is a positive constant. We denote by $(X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))$ the characteristic issuing from $(\boldsymbol{x}_0,\boldsymbol{v}_0)$. Then it satisfies

$$\begin{cases} \dfrac{dX}{dt}=V, \\ \dfrac{dV}{dt}=\lambda(E(t,X)-V), \end{cases} \tag{2.3}$$

$$X|_{t=0}=\boldsymbol{x}_0, \qquad V|_{t=0}=\boldsymbol{v}_0.$$

By virtue of the standard theory of ODEs, we know that

$$(X(t;\cdot,\cdot),V(t;\cdot,\cdot)):\ \mathbb{R}^{2d}\longrightarrow\mathbb{R}^{2d}$$

is a Lipschitz homomorphism. Thus, if the initial data are smooth, we can construct the unique classical solution by the method of characteristics. Since $C_0^\infty(\mathbb{R}^{2d})$ is dense in $L^1(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ with respect to the topology in $L^1(\mathbb{R}^{2d})$, a simple approximation yields Proposition 2.1. The results in this section are rather routine. For the reader's convenience, we present a simple proof.

PROPOSITION 2.1. *Assume $0\le f_0(\boldsymbol{x},\boldsymbol{v})\in L^1(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ and $E(t,\boldsymbol{x})\in[C([0,T]\times\mathbb{R}^{2d})]^d$ $(\forall T>0)$ satisfies equation (2.2). Then equation (2.1) admits a unique weak solution $f(t,\boldsymbol{x},\boldsymbol{v})\in C([0,T],L^1(\mathbb{R}^{2d}))\cap L^\infty([0,T]\times\mathbb{R}^{2d})$. Moreover, $f(t,\boldsymbol{x},\boldsymbol{v})$ satisfies*

(1) $0\le f(t,\boldsymbol{x},\boldsymbol{v})\le\|f_0\|_{L^\infty(\mathbb{R}^{2d})}e^{\lambda dt}, \quad a.e.\ (t,\boldsymbol{x},\boldsymbol{v})\in[0,T]\times\mathbb{R}^{2d}$;

(2) $\|f(t)\|_{L^p(\mathbb{R}^{2d})}=e^{\frac{\lambda d(p-1)t}{p}}\|f_0\|_{L^p(\mathbb{R}^{2d})},\ 1\le p<\infty, \quad \forall t\in[0,T]$.

Proof. Since $C_0^\infty(\mathbb{R}^{2d})$ is dense in $L^1(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ with respect to the topology in $L^1(\mathbb{R}^{2d})$, we can take a sequence $\{f_0^{\varepsilon_i}\}$ in $C_0^\infty(\mathbb{R}^{2d})$ such that

$$\|f_0^{\varepsilon_i}-f_0\|_{L^1(\mathbb{R}^{2d})}\to 0 \text{ as } \varepsilon_i\to 0 \quad \text{and} \quad \|f_0^{\varepsilon_i}\|_{L^\infty(\mathbb{R}^{2d})}\le \|f_0\|_{L^\infty(\mathbb{R}^{2d})}.$$

Using the method of characteristics, we know

$$\begin{cases} f_t^{\varepsilon_i}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f^{\varepsilon_i}+\lambda\nabla_{\boldsymbol{v}}\cdot[(E(t,\boldsymbol{x})-\boldsymbol{v})f^{\varepsilon_i}]=0 \quad \text{in } [0,T]\times\mathbb{R}^{2d}, & (2.4a)\\ f^{\varepsilon_i}|_{t=0}=f_0^{\varepsilon_i}(\boldsymbol{x},\boldsymbol{v}), & (2.4b)\end{cases}$$

admits a unique classical solution

$$0\le f^{\varepsilon_i}(t,X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))=f_0^{\varepsilon_i}(\boldsymbol{x}_0,\boldsymbol{v}_0)e^{\lambda dt}, \quad \forall t\in[0,T]. \tag{2.5}$$

Denote by f^{ε_i} and f^{ε_j} the solutions corresponding to the initial data $f_0^{\varepsilon_i}$ and $f_0^{\varepsilon_j}$, respectively. It follows from equation (2.4) that

$$\begin{cases} (f^{\varepsilon_i}-f^{\varepsilon_j})_t+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(f^{\varepsilon_i}-f^{\varepsilon_j})+\lambda\nabla_{\boldsymbol{v}}\cdot[(E(t,\boldsymbol{x})-\boldsymbol{v})(f^{\varepsilon_i}-f^{\varepsilon_j})]=0 \\ \qquad\qquad \text{in } [0,T]\times\mathbb{R}^{2d}, & (2.6a)\\ (f^{\varepsilon_i}-f^{\varepsilon_j})|_{t=0}=f_0^{\varepsilon_i}(\boldsymbol{x},\boldsymbol{v})-f_0^{\varepsilon_j}(\boldsymbol{x},\boldsymbol{v}). & (2.6b)\end{cases}$$

Multiplying (2.6a) by $sgn(f^{\varepsilon_i}-f^{\varepsilon_j})$ and integrating the resulting equation over $[0,t]\times\mathbb{R}^{2d}$, we obtain

$$\int_{\mathbb{R}^{2d}}|f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})-f^{\varepsilon_j}(t,\boldsymbol{x},\boldsymbol{v})|d\boldsymbol{x}d\boldsymbol{v}=\int_{\mathbb{R}^{2d}}|f_0^{\varepsilon_i}(\boldsymbol{x},\boldsymbol{v})-f_0^{\varepsilon_j}(\boldsymbol{x},\boldsymbol{v})|d\boldsymbol{x}d\boldsymbol{v}, \quad \forall t\in[0,T]. \tag{2.7}$$

From equation (2.7), we know $\{f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})\}$ is a Cauchy sequence in $C([0,T],L^1(\mathbb{R}^{2d}))$. Thus, there exists $f(t,\boldsymbol{x},\boldsymbol{v})\in C([0,T],L^1(\mathbb{R}^{2d}))$ such that

$$f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})\to f(t,\boldsymbol{x},\boldsymbol{v}) \quad \text{in } C([0,T],L^1(\mathbb{R}^{2d})), \text{ as } \varepsilon_i\to 0. \tag{2.8}$$

By Riesz's theorem, there exists a subsequence still denoted by $f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})$ such that

$$f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})\to f(t,\boldsymbol{x},\boldsymbol{v}), \quad \text{a.e. } (t,\boldsymbol{x},\boldsymbol{v})\in[0,T]\times\mathbb{R}^{2d}, \text{ as } \varepsilon_i\to 0. \tag{2.9}$$

Letting $\varepsilon\to 0$, from equations (2.4) and (2.5) we get

$$f_t+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f+\lambda\nabla_{\boldsymbol{v}}\cdot[(E(t,\boldsymbol{x})-\boldsymbol{v})f]=0 \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^{2d}),$$

$$0\le f(t,\boldsymbol{x},\boldsymbol{v})\le \|f_0\|_{L^\infty(\mathbb{R}^{2d})}e^{\lambda dt}, \quad \text{a.e. } (t,\boldsymbol{x},\boldsymbol{v})\in[0,T]\times\mathbb{R}^{2d}.$$

Therefore, $f(t,\boldsymbol{x},\boldsymbol{v})$ is a weak solution and

$$0\le f(t,\boldsymbol{x},\boldsymbol{v})\in C([0,T],L^1(\mathbb{R}^{2d}))\cap L^\infty([0,T]\times\mathbb{R}^{2d}).$$

Multiplying (2.4a) by $p(f^{\varepsilon_i})^{p-1}$ $(1\le p<\infty)$ and integrating the resulting equation over $\mathbb{R}^{2d}$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^{2d}}|f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})|^p d\boldsymbol{x}d\boldsymbol{v}=\lambda d(p-1)\int_{\mathbb{R}^{2d}}|f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})|^p d\boldsymbol{x}d\boldsymbol{v}.$$

Solving the above ODE yields

$$\|f^{\varepsilon_i}(t)\|_{L^p(\mathbb{R}^{2d})} = e^{\frac{\lambda d(p-1)t}{p}} \|f_0^{\varepsilon_i}\|_{L^p(\mathbb{R}^{2d})}, \ 1 \le p < \infty, \quad \forall t \in [0,T]. \tag{2.10}$$

Combining equations (2.5) and (2.8), we infer that $\{f^{\varepsilon_i}(t,\boldsymbol{x},\boldsymbol{v})\}$ is a Cauchy sequence in $C([0,T],L^p(\mathbb{R}^{2d}))$ $(1<p<\infty)$ by interpolation. Using equation (2.9) and letting $\varepsilon_i \to 0$, we deduce that

$$\|f(t)\|_{L^p(\mathbb{R}^{2d})} = e^{\frac{\lambda d(p-1)t}{p}} \|f_0\|_{L^p(\mathbb{R}^{2d})}, \ 1 \le p < \infty, \quad \forall t \in [0,T], \tag{2.11}$$

which amounts to the uniqueness of the weak solution, due to the linearity of equation (2.1). □

The following lemma implies that f is a measure-preserving map along the characteristics. It plays an important role in our subsequent proof.

LEMMA 2.1. *Assume $f(t,\boldsymbol{x},\boldsymbol{v})$ is a weak solution to equation (2.1) and $(X(t;\cdot,\cdot),V(t;\cdot,\cdot))$ is the characteristic issuing from $(\boldsymbol{x}_0,\boldsymbol{v}_0)$. For any $\varphi(\boldsymbol{x},\boldsymbol{v}) \in L^1_{loc}(\mathbb{R}^{2d})$, it holds that*

$$\int_\Omega f(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{x},\boldsymbol{v})d\boldsymbol{x}d\boldsymbol{v} = \int_{\Omega_0} f_0(\boldsymbol{x}_0,\boldsymbol{v}_0)\varphi(X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))d\boldsymbol{x}_0 d\boldsymbol{v}_0,$$

where Ω and Ω_0 satisfy

$$(X(t;\cdot,\cdot),V(t;\cdot,\cdot)): \ \Omega_0 \longrightarrow \Omega.$$

Proof. We only need to prove that

$$\int_\Omega f^\varepsilon(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{x},\boldsymbol{v})d\boldsymbol{x}d\boldsymbol{v} = \int_{\Omega_0} f_0^\varepsilon(\boldsymbol{x}_0,\boldsymbol{v}_0)\varphi(X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))d\boldsymbol{x}_0 d\boldsymbol{v}_0.$$

By virtue of our previous analysis of the characteristics, we know

$$(X(t;\cdot,\cdot),V(t;\cdot,\cdot)): \ \Omega_0 \longrightarrow \Omega$$

is a Lipschitz homomorphism. Make the following coordinate transform

$$\boldsymbol{x} = X(t;\boldsymbol{x}_0,\boldsymbol{v}_0), \quad \boldsymbol{v} = V(t;\boldsymbol{x}_0,\boldsymbol{v}_0).$$

Then the Jacobian of the transform is defined by

$$J(t,\boldsymbol{x}_0,\boldsymbol{v}_0) = \begin{vmatrix} \frac{\partial X}{\partial \boldsymbol{x}_0} & \frac{\partial X}{\partial \boldsymbol{v}_0} \\ \frac{\partial V}{\partial \boldsymbol{x}_0} & \frac{\partial V}{\partial \boldsymbol{v}_0} \end{vmatrix}.$$

Since

$$\begin{aligned} &\Big|\Big(X_1(t;\boldsymbol{x}_{10},\boldsymbol{v}_{10}),V_1(t;\boldsymbol{x}_{10},\boldsymbol{v}_{10})\Big) - \Big(X_2(t;\boldsymbol{x}_{20},\boldsymbol{v}_{20}),V_2(t;\boldsymbol{x}_{20},\boldsymbol{v}_{20})\Big)\Big| \\ \le& e^{CKT}|(\boldsymbol{x}_{10},\boldsymbol{v}_{10}) - (\boldsymbol{x}_{20},\boldsymbol{v}_{20})| \quad \forall t \in [0,T], \end{aligned}$$

from Rademacher's theorem, we know $\frac{\partial X}{\partial \boldsymbol{x}_0}, \frac{\partial X}{\partial \boldsymbol{v}_0}, \frac{\partial V}{\partial \boldsymbol{x}_0}$ and $\frac{\partial V}{\partial \boldsymbol{v}_0}$ exist for a.e. $(\boldsymbol{x}_0,\boldsymbol{v}_0) \in \mathbb{R}^{2d}$. As we compute the Lebesgue integral, we can suppose that $J(t,\boldsymbol{x}_0,\boldsymbol{v}_0)$ exists for all

$(\boldsymbol{x}_0,\boldsymbol{v}_0)\in\mathbb{R}^{2d}$. Then

$$\begin{aligned}&\int_{\Omega} f^{\varepsilon}(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{x},\boldsymbol{v})d\boldsymbol{x}d\boldsymbol{v}\\ &=\int_{\Omega_0} f^{\varepsilon}(t,X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))\varphi(X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),V(t;\boldsymbol{x}_0,\boldsymbol{v}_0))J(t,\boldsymbol{x}_0,\boldsymbol{v}_0)d\boldsymbol{x}_0d\boldsymbol{v}_0.\end{aligned} \tag{2.12}$$

Next we compute $J(t,\boldsymbol{x}_0,\boldsymbol{v}_0)$. Fix $(\boldsymbol{x}_0,\boldsymbol{v}_0)\in\mathbb{R}^{2d}$. We differentiate J with respect to t and then obtain

$$\begin{aligned}\frac{dJ}{dt}&=\sum_{i=1}^{d}\begin{vmatrix}\vdots & \vdots\\ \frac{\partial}{\partial \boldsymbol{x}_0}\frac{dX^i}{dt} & \frac{\partial}{\partial \boldsymbol{v}_0}\frac{dX^i}{dt}\\ \vdots & \vdots\\ \frac{\partial V}{\partial \boldsymbol{x}_0} & \frac{\partial V}{\partial \boldsymbol{v}_0}\end{vmatrix}+\sum_{i=1}^{d}\begin{vmatrix}\frac{\partial X}{\partial \boldsymbol{x}_0} & \frac{\partial X}{\partial \boldsymbol{v}_0}\\ \vdots & \vdots\\ \frac{\partial}{\partial \boldsymbol{x}_0}\frac{dV^i}{dt} & \frac{\partial}{\partial \boldsymbol{v}_0}\frac{dV^i}{dt}\\ \vdots & \vdots\end{vmatrix}\\ &=-\lambda dJ,\end{aligned}$$

where we have used the fact that

$$\frac{dX^i}{dt}=V^i,\quad \frac{dV^i}{dt}=\lambda(E^i(t,X)-V^i)$$

and that

$$\frac{\partial E^i}{\partial \boldsymbol{x}_0}=\frac{\partial E^i}{\partial X}\frac{\partial X}{\partial \boldsymbol{x}_0},\quad \frac{\partial E^i}{\partial \boldsymbol{v}_0}=\frac{\partial E^i}{\partial X}\frac{\partial X}{\partial \boldsymbol{v}_0}.$$

Thus, $J(t,\boldsymbol{x}_0,\boldsymbol{v}_0)=e^{-\lambda dt}$ since $J_0=1$. Substituting equation (2.5) into equation (2.12), we conclude our proof. □

3. Construction of approximate solutions

This section is devoted to construction of the approximate solutions for equation (1.1). Notice that the nonlinear term in equation (1.1) is $\boldsymbol{u}(t,\boldsymbol{x})$. The difficulty mainly comes from the fact that $\rho_r(t,\boldsymbol{x})$ may be equal to 0, so we approximate $\boldsymbol{u}(t,\boldsymbol{x})$ with $\boldsymbol{u}^{\delta}(t,\boldsymbol{x})=\frac{\boldsymbol{j}_r^{\delta}(t,\boldsymbol{x})}{\delta+\rho_r^{\delta}(t,\boldsymbol{x})}$. $\boldsymbol{j}_r^{\delta}(t,\boldsymbol{x})$ and $\rho_r^{\delta}(t,\boldsymbol{x})$ are defined in the same way as before, where $f^{\delta}(t,\boldsymbol{x},\boldsymbol{v})$ is the weak solution to the following approximate equation:

$$\begin{cases} f_t^{\delta}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f^{\delta}+\lambda\nabla_{\boldsymbol{v}}\cdot[(\boldsymbol{u}^{\delta}(t,\boldsymbol{x})-\boldsymbol{v})f^{\delta}]=0,\\ f^{\delta}|_{t=0}=f_0(\boldsymbol{x},\boldsymbol{v})\in L^1(\mathbb{R}^{2d})\cap L^{\infty}(\mathbb{R}^{2d}).\end{cases} \tag{3.1}$$

We use the Schauder fixed point theorem to establish the existence of approximate solutions. Take

$$\begin{aligned}\mathcal{X}:=\Big\{E(t,\boldsymbol{x}):\ &E(t,\boldsymbol{x})\in C([0,T]\times\mathbb{R}^d),\ \|E(t,\boldsymbol{x})\|_{L^{\infty}([0,T]\times\mathbb{R}^d)}\le M_0 \text{ and}\\ &E(t,\cdot) \text{ is uniformly Lipschitz continuous in } t\in[0,T]\Big\},\end{aligned} \tag{3.2}$$

where M_0 is the bound of the $\boldsymbol{v}$-support of $f_0(\boldsymbol{x},\boldsymbol{v})$. For any $E(t,\boldsymbol{x})\in\mathcal{X}$, we know there is a unique weak solution $f(t,\boldsymbol{x},\boldsymbol{v})$ to equation (2.1) according to Proposition 2.1. Define

$$\mathcal{F}[E](t,\boldsymbol{x})=\frac{\int_{|\boldsymbol{x}-\boldsymbol{y}|<r}\int_{\mathbb{R}^d}f(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}}{\delta+\int_{|\boldsymbol{x}-\boldsymbol{y}|<r}\int_{\mathbb{R}^d}f(t,\boldsymbol{y},\boldsymbol{w})d\boldsymbol{w}d\boldsymbol{y}}.$$

In the following, we suppose the weak solution $f(t,\boldsymbol{x},\boldsymbol{v})\in C_0^1([0,T]\times\mathbb{R}^{2d})$. If not, we approximate f_0 with f_0^ε and use the classical solution $f^\varepsilon(t,\boldsymbol{x},\boldsymbol{v})\in C_0^1([0,T]\times\mathbb{R}^{2d})$ to substitute for $f(t,\boldsymbol{x},\boldsymbol{v})$.

We will show that $\mathcal{F}$ satisfies the framework of the Schauder fixed point theorem and yields the following proposition. We denote the approximate solution by $f^\delta(t,\boldsymbol{x},\boldsymbol{v})$, while $M^\delta(t)$ denotes the bound of its $\boldsymbol{v}$-support at the instant t.

PROPOSITION 3.1. *Assume* $0\le f_0(\boldsymbol{x},\boldsymbol{v})\in L^1(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$ *and* $supp_{\boldsymbol{v}} f_0(\boldsymbol{x},\cdot)\subseteq B(M_0)$. *Then equation* (3.1) *admits a weak solution* $f^\delta(t,\boldsymbol{x},\boldsymbol{v})\in C([0,T],L^1(\mathbb{R}^{2d}))\cap L^\infty([0,T]\times\mathbb{R}^{2d})$, $\forall T>0$. *Moreover,* $f^\delta(t,\boldsymbol{x},\boldsymbol{v})$ *and* $M^\delta(t)$ *satisfy*

(1) $0\le f^\delta(t,\boldsymbol{x},\boldsymbol{v})\le \|f_0\|_{L^\infty(\mathbb{R}^{2d})}e^{\lambda dt}$, *a.e.* $(t,\boldsymbol{x},\boldsymbol{v})\in[0,T]\times\mathbb{R}^{2d}$;

(2) $\|f^\delta(t)\|_{L^p(\mathbb{R}^{2d})}=e^{\frac{\lambda d(p-1)t}{p}}\|f_0\|_{L^p(\mathbb{R}^{2d})}$, $1\le p<\infty$, $\forall t\in[0,T]$;

(3) $M^\delta(t)\le M_0$, $\forall t\in[0,T]$.

In order to prove Proposition 3.1, we need the following three lemmas.

LEMMA 3.1. *Assume* $E(t,\boldsymbol{x})\in\mathcal{X}$. *Then* $\mathcal{F}[E](t,\boldsymbol{x})\in\mathcal{X}$.

Proof. The proof is divided into three steps.

Step 1: $\|\mathcal{F}[E](t,\boldsymbol{x})\|_{L^\infty([0,T]\times\mathbb{R}^d)}\le M_0$.
According to Lemma 2.1, we know

$$\mathrm{supp} f(t,\cdot,\cdot)=\Big\{(\boldsymbol{x},\boldsymbol{v}):\boldsymbol{x}=X(t;\boldsymbol{x}_0,\boldsymbol{v}_0),\ \boldsymbol{v}=V(t;\boldsymbol{x}_0,\boldsymbol{v}_0), where\ (\boldsymbol{x}_0,\boldsymbol{v}_0)\in\mathrm{supp} f_0\Big\}.$$

Since

$$\frac{dV}{dt}=\lambda(E(t,X)-V)\quad\text{and}\quad\|E(t,X)\|_{L^\infty([0,T]\times\mathbb{R}^d)}\le M_0,$$

solving the above Gronwall inequality yields

$$|V(t;\boldsymbol{x}_0,\boldsymbol{v}_0)|\le M_0,\quad\forall(\boldsymbol{x}_0,\boldsymbol{v}_0)\in\mathrm{supp} f_0.$$

Thus,

$$\|\mathcal{F}[E](t,\boldsymbol{x})\|_{L^\infty([0,T]\times\mathbb{R}^d)}\le M_0\left\|\frac{\rho_r(t,\boldsymbol{x})}{\delta+\rho_r(t,\boldsymbol{x})}\right\|_{L^\infty([0,T]\times\mathbb{R}^d)}\le M_0.$$

Step 2: $|\mathcal{F}[E](t,\boldsymbol{x}_2)-\mathcal{F}[E](t,\boldsymbol{x}_1)|\le C|\boldsymbol{x}_2-\boldsymbol{x}_1|$, $\forall t\in[0,T]$.
It is sufficient to prove

$$|\boldsymbol{j}_r(t,\boldsymbol{x}_2)-\boldsymbol{j}_r(t,\boldsymbol{x}_1)|\le C|\boldsymbol{x}_2-\boldsymbol{x}_1|\quad\text{and}\quad|\rho_r(t,\boldsymbol{x}_2)-\rho_r(t,\boldsymbol{x}_1)|\le C|\boldsymbol{x}_2-\boldsymbol{x}_1|.$$

Define

$$\Delta(\boldsymbol{x}_1,\boldsymbol{x}_2)=\Big(B(\boldsymbol{x}_1,r)\setminus B(\boldsymbol{x}_2,r)\Big)\cup\Big(B(\boldsymbol{x}_2,r)\setminus B(\boldsymbol{x}_1,r)\Big),$$

where

$$B(\boldsymbol{x}_i,r)=\{\boldsymbol{y}:\ |\boldsymbol{y}-\boldsymbol{x}_i|<r\},\quad i=1,2.$$

(1) If $|\boldsymbol{x}_1-\boldsymbol{x}_2|<2r$, we have

$$
\begin{aligned}
|\Delta(\boldsymbol{x}_1,\boldsymbol{x}_2)| &\le C\left[\left(r+\frac{|\boldsymbol{x}_1-\boldsymbol{x}_2|}{2}\right)^d-\left(r-\frac{|\boldsymbol{x}_1-\boldsymbol{x}_2|}{2}\right)^d\right]\\
&\le C(r)|\boldsymbol{x}_1-\boldsymbol{x}_2|,
\end{aligned}
$$

due to

$$
\Delta(\boldsymbol{x}_1,\boldsymbol{x}_2)\subset B\left(\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2},r+\frac{|\boldsymbol{x}_1-\boldsymbol{x}_2|}{2}\right)\setminus B\left(\frac{\boldsymbol{x}_1+\boldsymbol{x}_2}{2},r-\frac{|\boldsymbol{x}_1-\boldsymbol{x}_2|}{2}\right).
$$

Then

$$
\begin{aligned}
|\boldsymbol{j}_r(t,\boldsymbol{x}_2)-\boldsymbol{j}_r(t,\boldsymbol{x}_1)| &= \left|\int_{B(\boldsymbol{x}_1,r)}\int_{\mathbb{R}^d} f(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}-\int_{B(\boldsymbol{x}_2,r)}\int_{\mathbb{R}^d} f(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}\right|\\
&\le \int_{\Delta(\boldsymbol{x}_1,\boldsymbol{x}_2)}\left|\int_{\mathbb{R}^d} f(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}d\boldsymbol{w}\right|d\boldsymbol{y}\\
&\le C\|f_0\|_{L^\infty(\mathbb{R}^{2d})}M_0^{d+1}|\Delta(\boldsymbol{x}_1,\boldsymbol{x}_2)|\\
&\le C|\boldsymbol{x}_2-\boldsymbol{x}_1|,
\end{aligned}
$$

where we have used the fact that the $\boldsymbol{v}$-support of $f(t,\boldsymbol{x},\boldsymbol{v})$ is uniformly bounded in $[0,T]$. Similarly, we have

$$
|\rho_r(t,\boldsymbol{x}_2)-\rho_r(t,\boldsymbol{x}_1)|\le C|\boldsymbol{x}_2-\boldsymbol{x}_1|.
$$

(2) If $|\boldsymbol{x}_1-\boldsymbol{x}_2|\ge 2r$, we have

$$
\begin{aligned}
|\boldsymbol{j}_r(t,\boldsymbol{x}_2)-\boldsymbol{j}_r(t,\boldsymbol{x}_1)| &\le |\boldsymbol{j}_r(t,\boldsymbol{x}_2)|+|\boldsymbol{j}_r(t,\boldsymbol{x}_1)|\\
&\le C\|f_0\|_{L^\infty(\mathbb{R}^{2d})}M_0^{d+1}r^d\\
&\le C|\boldsymbol{x}_2-\boldsymbol{x}_1|.
\end{aligned}
$$

Similarly, we get

$$
|\rho_r(t,\boldsymbol{x}_2)-\rho_r(t,\boldsymbol{x}_1)|\le C|\boldsymbol{x}_2-\boldsymbol{x}_1|.
$$

Combining (1) and (2) yields the conclusion of Step 2.

Step 3: $|\mathcal{F}[E](t_2,\boldsymbol{x})-\mathcal{F}[E](t_1,\boldsymbol{x})|\le C|t_2-t_1|,\ \forall t_1,t_2\in[0,T]$

We only need to prove

$$
|\boldsymbol{j}_r(t_2,\boldsymbol{x})-\boldsymbol{j}_r(t_1,\boldsymbol{x})|\le C|t_2-t_1| \quad \text{and} \quad |\rho_r(t_2,\boldsymbol{x})-\rho_r(t_1,\boldsymbol{x})|\le C|t_2-t_1|.
$$

Employing equation (2.1), we have

$$
\begin{aligned}
&|\boldsymbol{j}_r(t_2,\boldsymbol{x})-\boldsymbol{j}_r(t_1,\boldsymbol{x})|\\
=&\left|\int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d}[f(t_2,\boldsymbol{y},\boldsymbol{w})-f(t_1,\boldsymbol{y},\boldsymbol{w})]\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}\right|\\
=&\left|\int_{t_1}^{t_2}\int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d}\frac{\partial f}{\partial t}\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}dt\right|
\end{aligned}
$$

$$
\begin{aligned}
&=\left|\int_{t_1}^{t_2}\int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d}\left\{-\boldsymbol{w}\cdot\nabla_{\boldsymbol{y}}f-\lambda\nabla_{\boldsymbol{w}}\cdot\left[\Big(E(t,\boldsymbol{y})-\boldsymbol{w}\Big)f\right]\right\}\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}dt\right|\\
&=\left|\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\boldsymbol{w}\int_{\partial B(\boldsymbol{x},r)}-f\boldsymbol{w}\cdot nd\sigma d\boldsymbol{w}dt+\lambda d\int_{t_1}^{t_2}\int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d}f\Big[E(t,\boldsymbol{y})-\boldsymbol{w}\Big]d\boldsymbol{w}d\boldsymbol{y}dt\right|\\
&\le C|t_2-t_1|.
\end{aligned}
$$

by direct computation. Similarly,

$$|\rho_r(t_2,\boldsymbol{x})-\rho_r(t_1,\boldsymbol{x})|\le C|t_2-t_1|.$$

Combining Step 2 and Step 3, we know

$$
\begin{aligned}
&|\mathcal{F}[E](t_2,\boldsymbol{x}_2)-\mathcal{F}[E](t_1,\boldsymbol{x}_1)|\\
\le&|\mathcal{F}[E](t_2,\boldsymbol{x}_2)-\mathcal{F}[E](t_2,\boldsymbol{x}_1)|+|\mathcal{F}[E](t_2,\boldsymbol{x}_1)-\mathcal{F}[E](t_1,\boldsymbol{x}_1)|\\
\le&C|\boldsymbol{x}_2-\boldsymbol{x}_1|+C|t_2-t_1|.
\end{aligned}
$$

Therefore, $\mathcal{F}[E](t,\boldsymbol{x})\in C([0,T]\times\mathbb{R}^d)$. According to the definition of $\mathcal{X}$, it follows that $\mathcal{F}[E]\in\mathcal{X}$ from Steps 1, 2 and 3. ∎

The next lemma implies that $\mathcal{F}$ is a continuous functional in $\mathcal{X}$.

LEMMA 3.2. *Assume* $\{E_n\}\in\mathcal{X}$ *satisfy* $\|E_n-E\|_{L^\infty([0,T]\times\mathbb{R}^d)}\to 0$, *as* $n\to\infty$. *Then* $\|\mathcal{F}[E_n]-\mathcal{F}[E]\|_{L^\infty([0,T]\times\mathbb{R}^d)}\to 0$, *as* $n\to\infty$.

Proof. We only need to prove

$$\|\boldsymbol{j}_r^n(t,\boldsymbol{x})-\boldsymbol{j}_r(t,\boldsymbol{x})\|_{L^\infty([0,T]\times\mathbb{R}^d)}\to 0,\quad \text{as } n\to\infty$$

and

$$\|\rho_r^n(t,\boldsymbol{x})-\rho_r(t,\boldsymbol{x})\|_{L^\infty([0,T]\times\mathbb{R}^d)}\to 0,\quad \text{as } n\to\infty.$$

Define

$$U_r^n(t,\boldsymbol{x})=\{(\boldsymbol{y}_0,\boldsymbol{w}_0):\ \Big(X_n(t;\boldsymbol{y}_0,\boldsymbol{w}_0),V_n(t;\boldsymbol{y}_0,\boldsymbol{w}_0)\Big)\subseteq B(\boldsymbol{x},r)\times\mathrm{supp}_{\boldsymbol{v}}f^n(t,\cdot,\cdot)\},$$

$$U_r(t,\boldsymbol{x})=\{(\boldsymbol{y}_0,\boldsymbol{w}_0):\ \Big(X(t;\boldsymbol{y}_0,\boldsymbol{w}_0),V(t;\boldsymbol{y}_0,\boldsymbol{w}_0)\Big)\subseteq B(\boldsymbol{x},r)\times\mathrm{supp}_{\boldsymbol{v}}f(t,\cdot,\cdot)\},$$

and

$$\Delta(U_r^n,U_r)=(U_r^n\setminus U_r)\cup(U_r\setminus U_r^n),$$

where (X_n,V_n) is defined similarly to (X,V) in equation (2.3). Using Lemma 2.1, we obtain

$$
\begin{aligned}
&|\boldsymbol{j}_r^n(t,\boldsymbol{x})-\boldsymbol{j}_r(t,\boldsymbol{x})|\\
=&\left|\int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d}f^n(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}-\int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d}f(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}\right|\\
\le&\int_{\Delta(U_r^n,U_r)}f_0(\boldsymbol{y}_0,\boldsymbol{w}_0)|V_n(t;\boldsymbol{y}_0,\boldsymbol{w}_0)|d\boldsymbol{w}_0d\boldsymbol{y}_0
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{U_r} f_0(\boldsymbol{y}_0,\boldsymbol{w}_0)|V_n(t;\boldsymbol{y}_0,\boldsymbol{w}_0)-V(t;\boldsymbol{y}_0,\boldsymbol{w}_0)|d\boldsymbol{w}_0 d\boldsymbol{y}_0\\
&\le C|\Delta(U_r^n,U_r)|+C\sup_{0\le t\le T}|V_n(t;\boldsymbol{y}_0,\boldsymbol{w}_0)-V(t;\boldsymbol{y}_0,\boldsymbol{w}_0)| \qquad (3.3)
\end{aligned}
$$

for any $(\boldsymbol{y},\boldsymbol{w})\in B(\boldsymbol{x},r)\times B(M_0)$. Employing the characteristic equation (2.3), we have

$$
\begin{cases}
\dfrac{d}{ds}[X_n(s;t,\boldsymbol{y},\boldsymbol{w})-X(s;t,\boldsymbol{y},\boldsymbol{w})]=V_n(s;t,\boldsymbol{y},\boldsymbol{w})-V(s;t,\boldsymbol{y},\boldsymbol{w}),\\
\dfrac{d}{ds}[V_n(s;t,\boldsymbol{y},\boldsymbol{w})-V(s;t,\boldsymbol{y},\boldsymbol{w})]=\lambda[E_n(s,X_n(s))-E(s,X(s))-(V_n(s;t,\boldsymbol{y},\boldsymbol{w})\\
\qquad\qquad -V(s;t,\boldsymbol{y},\boldsymbol{w}))],
\end{cases}
$$

$$
(X_n(s;t,\boldsymbol{y},\boldsymbol{w})-X(s;t,\boldsymbol{y},\boldsymbol{w}))|_{s=t}=0,\quad (V_n(s;t,\boldsymbol{y},\boldsymbol{w})-V(s;t,\boldsymbol{y},\boldsymbol{w}))|_{s=t}=0.
$$

For any $\varepsilon>0$, if $\|E_n-E\|_{L^\infty([0,T]\times\mathbb{R}^d)}<\varepsilon$, then a simple computation yields

$$
\begin{aligned}
|V_n(0;t,\boldsymbol{y},\boldsymbol{w})-V(0;t,\boldsymbol{y},\boldsymbol{w})|&\le(e^{\lambda T}-1)\varepsilon,\\
|X_n(0;t,\boldsymbol{y},\boldsymbol{w})-X(0;t,\boldsymbol{y},\boldsymbol{w})|&\le T(e^{\lambda T}-1)\varepsilon.
\end{aligned} \qquad (3.4)
$$

Since $E_n\le M_0$ for all $n\in\mathbb{N}$, one can infer that U_r and U_r^n for all n are uniformly bounded from the same type of equations as in equation (2.3). Thus

$$
|\Delta(U_r^n,U_r)|\le(|\partial U_r^n|+|\partial U_r|)\cdot T(e^{\lambda T}-1)\varepsilon\le C\varepsilon. \qquad (3.5)
$$

We can show

$$
|V_n(t;\boldsymbol{y}_0,\boldsymbol{w}_0)-V(t;\boldsymbol{y}_0,\boldsymbol{w}_0)|\le C\varepsilon\quad \text{for } t\in[0,T] \qquad (3.6)
$$

in the same way as in equation (3.4). Combining equations (3.3), (3.4), and (3.5) with equation (3.6), we obtain

$$
\|\boldsymbol{j}_r^n(t,\boldsymbol{x})-\boldsymbol{j}_r(t,\boldsymbol{x})\|_{L^\infty([0,T]\times\mathbb{R}^d)}\le C\varepsilon.
$$

Similarly, we get

$$
\|\rho_r^n(t,\boldsymbol{x})-\rho_r(t,\boldsymbol{x})\|_{L^\infty([0,T]\times\mathbb{R}^d)}\le C\varepsilon.
$$

This completes the proof. ∎

The following velocity averaging lemma is due to [13] (Theorem 5 and Remark 3 of Theorem 3). It plays a key role in the proof of Theorem 1.2.

LEMMA 3.3 (DiPerna and Lions). *Let $m\ge 0$, $f,g\in L^2(\mathbb{R}\times\mathbb{R}^{2d})$ and $f(t,\boldsymbol{x},\boldsymbol{v}),g(t,\boldsymbol{x},\boldsymbol{v})$ satisfy*

$$
\frac{\partial f}{\partial t}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f=\nabla_{\boldsymbol{v}}^{\xi} g\quad in\ \mathcal{D}'\left(\mathbb{R}\times\mathbb{R}^{2d}\right),
$$

where $\nabla_{\boldsymbol{v}}^{\xi}=\partial_{\boldsymbol{v}^1}^{\xi^1}\partial_{\boldsymbol{v}^2}^{\xi^2}\cdots\partial_{\boldsymbol{v}^d}^{\xi^d}$ and $|\xi|=\sum_{i=1}^d\xi^i=m$. Then for any $\varphi(\boldsymbol{v})\in C_c^\infty(\mathbb{R}^d)$, it holds that

$$
\left\|\int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{v})d\boldsymbol{v}\right\|_{H^s(\mathbb{R}\times\mathbb{R}^d)}\le C\left(\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^{2d})}+\|g\|_{L^2(\mathbb{R}\times\mathbb{R}^{2d})}\right),
$$

where $s = \frac{1}{2(1+m)}$ *and* C *is a positive constant.*

This lemma is used to prove that $\mathcal{F}$ is compact. Using the fact that the $\boldsymbol{v}$-support of the solution $f(t,\boldsymbol{x},\boldsymbol{v})$ to equation (2.1) is uniformly bounded in $[0,T]$, we remove the constraint $|\boldsymbol{x}|^2 f_0(\boldsymbol{x},\boldsymbol{v}) \in L^1(\mathbb{R}^{2d})$ in [23].

LEMMA 3.4. *Assume* $\{E^n\} \subseteq \mathcal{X}$. *Then there exists a subsequence still denoted by* $\{E^n\}$ *such that* $\|\mathcal{F}[E^n] - \mathcal{F}[E]\|_{L^\infty([0,T]\times\mathbb{R}^d)} \to 0$, *as* $n \to \infty$.

Proof. We only need to prove

$$\boldsymbol{j}_r^n(t,\boldsymbol{x}) \to \boldsymbol{j}_r(t,\boldsymbol{x}) \text{ and } \rho_r^n(t,\boldsymbol{x}) \to \rho_r(t,\boldsymbol{x}) \quad \text{uniformly in } [0,T]\times\mathbb{R}^d,$$

as $n \to \infty$.

For any $\varepsilon > 0$, there exists a ball $B(R) \in \mathbb{R}^d_{\boldsymbol{x}}$ such that

$$\int_{\mathbb{R}^d\setminus B(R)} \int_{\mathbb{R}^d} f_0(\boldsymbol{x},\boldsymbol{v}) d\boldsymbol{v} d\boldsymbol{x} < \varepsilon.$$

Employing Lemma 2.1, we have

$$\begin{aligned} &\left| \int_0^T \int_{\mathbb{R}^d\setminus B(R+M_0T)} \int_{\mathbb{R}^d} f^n(t,\boldsymbol{x},\boldsymbol{v}) \boldsymbol{v} d\boldsymbol{v} d\boldsymbol{x} dt \right| \\ \le & M_0 \left| \int_0^T \int_{\mathbb{R}^d\setminus B(R)} \int_{\mathbb{R}^d} f_0(\boldsymbol{x},\boldsymbol{v}) d\boldsymbol{v} d\boldsymbol{x} dt \right| \\ \le & M_0 T \varepsilon. \end{aligned} \tag{3.7}$$

Since

$$\frac{\partial f^n}{\partial t} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} f^n = -\lambda \nabla_{\boldsymbol{v}} \cdot [(E^n(t,\boldsymbol{x}) - \boldsymbol{v}) f^n] \quad \text{in } \mathcal{D}'((0,T)\times\mathbb{R}^{2d}),$$

and

$$\|f^n\|_{L^2([0,T]\times\mathbb{R}^{2d})} \le C, \quad \|(E^n(t,\boldsymbol{x}) - \boldsymbol{v}) f^n\|_{L^2([0,T]\times\mathbb{R}^{2d})} \le C \quad \forall n. \tag{3.8}$$

Using Lemma 3.3, we get

$$\left\| \int_{\mathbb{R}^d} f^n(t,\boldsymbol{x},\boldsymbol{v}) \boldsymbol{v} d\boldsymbol{v} \right\|_{H^{\frac{1}{4}}([0,T]\times\mathbb{R}^d)} \le C,$$

where we have used the fact that the $\boldsymbol{v}$-support of f^n is uniformly bounded for $t \in [0,T]$. Since

$$H^{\frac{1}{4}}([0,T]\times B(R+M_0T)) \hookrightarrow\hookrightarrow L^1([0,T]\times B(R+M_0T)), \tag{3.9}$$

equation (3.7), indicates that there exists a subsequence still denoted by $\boldsymbol{j}^n$ such that

$$\int_0^T \int_{\mathbb{R}^d} |\boldsymbol{j}^n(t,\boldsymbol{x}) - \boldsymbol{j}(t,\boldsymbol{x})| d\boldsymbol{x} dt \to 0, \quad \text{as } n \to \infty, \tag{3.10}$$

where

$$\boldsymbol{j}^n(t,\boldsymbol{x}) = \int_{\mathbb{R}^d} f^n(t,\boldsymbol{x},\boldsymbol{v}) \boldsymbol{v} d\boldsymbol{v}.$$

From equation (3.8), we infer that there exists a subsequence still denoted by $\{f^n\}$ such that

$$f^n(t,\boldsymbol{x},\boldsymbol{v}) \rightharpoonup f(t,\boldsymbol{x},\boldsymbol{v}) \quad \text{weakly in } L^2([0,T]\times\mathbb{R}^{2d}), \text{ as } n\to\infty.$$

Since $\{f^n\}$ is compactly supported in $\boldsymbol{v}$ uniformly, we can easily show that

$$\int_{\mathbb{R}^d} f^n(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}d\boldsymbol{v} \rightharpoonup \int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}d\boldsymbol{v} \quad \text{weakly in } L^2([0,T]\times\mathbb{R}^d), \text{ as } n\to\infty.$$

Using the uniqueness of the weak limit and of the strong limit, we deduce that

$$\boldsymbol{j}(t,\boldsymbol{x}) = \int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{v}d\boldsymbol{v}.$$

By the definition of $\boldsymbol{j}_r^n(t,\boldsymbol{x})$ and $\boldsymbol{j}_r(t,\boldsymbol{x})$, we have

$$\begin{aligned}|\boldsymbol{j}_r^n(t,\boldsymbol{x})-\boldsymbol{j}_r(t,\boldsymbol{x})| &\le \int_{B(\boldsymbol{x},r)} |\boldsymbol{j}^n(t,\boldsymbol{y})-\boldsymbol{j}(t,\boldsymbol{y})|d\boldsymbol{y} \\ &\le \int_{\mathbb{R}^d} |\boldsymbol{j}^n(t,\boldsymbol{y})-\boldsymbol{j}(t,\boldsymbol{y})|d\boldsymbol{y}, \quad \forall \boldsymbol{x}\in\mathbb{R}^d.\end{aligned}$$

By equation (3.10), it follows from Riesz's theorem that there exists a further subsequence, still denoted by $\{\boldsymbol{j}_r^n\}$, such that

$$|\boldsymbol{j}_r^n(t,\boldsymbol{x})-\boldsymbol{j}_r(t,\boldsymbol{x})| \to 0 \quad \text{for } a.e.\ t\in[0,T]$$

uniformly with respect to $\boldsymbol{x}$, as $n\to\infty$. Using the fact that

$$|\boldsymbol{j}_r^n(t_2,\boldsymbol{x})-\boldsymbol{j}_r^n(t_1,\boldsymbol{x})| \le C|t_2-t_1| \quad \text{and} \quad |\boldsymbol{j}_r(t_2,\boldsymbol{x})-\boldsymbol{j}_r(t_1,\boldsymbol{x})| \le C|t_2-t_1|,$$

we know that

$$\boldsymbol{j}_r^n(t,\boldsymbol{x}) \to \boldsymbol{j}_r(t,\boldsymbol{x}) \quad \text{uniformly in } [0,T]\times\mathbb{R}^d, \text{ as } n\to\infty.$$

Similarly, we get

$$\rho_r^n(t,\boldsymbol{x}) \to \rho_r(t,\boldsymbol{x}) \quad \text{uniformly in } [0,T]\times\mathbb{R}^d, \text{ as } n\to\infty,$$

which concludes the proof. ∎

Using the above lemmas, we can easily present the proof of Proposition 3.1, by using the Schauder fixed point theorem and Proposition 2.1.

Proof. **(Proof of Proposition 3.1.)** $\mathcal{X}$ is convex, bounded, and closed. From Lemma 3.1 and Lemma 3.4, we know $\mathcal{F}$ is continuous and compact in $\mathcal{X}$, and $\mathcal{F}\mathcal{X}\subseteq\mathcal{X}$. Using the Schauder fixed point theorem, we infer that there is a fixed point in $\mathcal{X}$. Therefore, equation (3.1) has a weak solution.

Based on our analysis to the linear equation, we know Proposition 2.1 (1) and (2) hold for every $E(t,\boldsymbol{x})\in\mathcal{X}$. Specifically, for the fixed point of $\mathcal{F}$, we have Proposition 3.1 (1) and (2). From Step 1 of Lemma 3.1, we know Proposition 3.1 (3) holds. Thus we complete the proof of Proposition 3.1. ∎

4. Proof of the Theorem

In this section, we will recover the weak solution to equation (1.1) by passing to the weak limit of the approximate solutions to equation (3.1). In Section 3, $f(t,\boldsymbol{x},\boldsymbol{v})$ represents the weak limit of $f^n(t,\boldsymbol{x},\boldsymbol{v})$. Hereafter, we will use $f(t,\boldsymbol{x},\boldsymbol{v})$ to represent the weak limit of $f^\delta(t,\boldsymbol{x},\boldsymbol{v})$.

Proof. **(Proof of Theorem 1.2.)** From Proposition 3.1, we know there exists a sequence $\{f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})\}$ such that

$$f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v}) \rightharpoonup f(t,\boldsymbol{x},\boldsymbol{v}) \quad \text{weakly in } L^2([0,T]\times\mathbb{R}^{2d}), \quad \text{as } \delta_i \to 0. \tag{4.1}$$

In addition, the $\boldsymbol{v}$-supports of $f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})$ and $f(t,\boldsymbol{x},\boldsymbol{v})$ are uniformly bounded in $t\in[0,T]$. Since $\|f^{\delta_i}\boldsymbol{u}^{\delta_i}\|_{L^2([0,T]\times\mathbb{R}^{2d})}\le C$ for all $\delta_i>0$, there also exists a subsequence, still denoted by $\{f^{\delta_i}\boldsymbol{u}^{\delta_i}\}$ such that

$$f^{\delta_i}\boldsymbol{u}^{\delta_i} \rightharpoonup \boldsymbol{m} \quad \text{weakly in } L^2([0,T]\times\mathbb{R}^{2d}), \text{ as } \delta_i\to 0.$$

We only need to prove $\boldsymbol{m}=f\boldsymbol{u}$. Using the same proof as was used for Lemma 3.4, we also have

$$\int_{\mathbb{R}^d} f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{v})d\boldsymbol{v} \to \int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{v})d\boldsymbol{v}, \quad \forall\varphi(\boldsymbol{v})\in C_c^\infty(\mathbb{R}^d) \tag{4.2}$$

for a.e. $(t,\boldsymbol{x})\in[0,T]\times\mathbb{R}^d$, as $\delta_i\to 0$. With $\boldsymbol{j}_r^{\delta_i}$ and $\rho_r^{\delta_i}$ defined by

$$\boldsymbol{j}_r^{\delta_i}(t,\boldsymbol{x}) = \int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d} f^{\delta_i}(t,\boldsymbol{y},\boldsymbol{w})\boldsymbol{w}d\boldsymbol{w}d\boldsymbol{y}, \quad \rho_r^{\delta_i}(t,\boldsymbol{x}) = \int_{B(\boldsymbol{x},r)}\int_{\mathbb{R}^d} f^{\delta_i}(t,\boldsymbol{y},\boldsymbol{w})d\boldsymbol{w}d\boldsymbol{y},$$

the Lebesgue dominated convergence theorem yields

$$\boldsymbol{j}_r^{\delta_i}(t,\boldsymbol{x}) \to \boldsymbol{j}_r(t,\boldsymbol{x}) \quad a.e. \text{ in } [0,T]\times\mathbb{R}^d, \text{ as } \delta_i\to 0, \tag{4.3}$$

and

$$\rho_r^{\delta_i}(t,\boldsymbol{x}) \to \rho_r(t,\boldsymbol{x}) \quad a.e. \text{ in } [0,T]\times\mathbb{R}^d, \text{ as } \delta_i\to 0, \tag{4.4}$$

where we have used the fact that the $\boldsymbol{v}$-support of $f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})$ is uniformly bounded in $[0,T]$. Define

$$A=\{(t,\boldsymbol{x}):\ \rho_r(t,\boldsymbol{x})=0\}, \quad B=\{(t,\boldsymbol{x}):\ \rho_r(t,\boldsymbol{x})>0\}.$$

By the definition of A, we know $A\subseteq[0,T]\times\mathbb{R}^d\setminus \mathrm{supp} f(\cdot,\cdot,\boldsymbol{v})$ for any $\boldsymbol{v}\in\mathbb{R}^d$. By the definition of $\boldsymbol{u}^{\delta_i}$, it follows from Proposition 3.1 (3) that $|\boldsymbol{u}^{\delta_i}|\le M_0$. Combining equation (4.2) and the Lebesgue dominated convergence theorem yields

$$\int_A\int_{\mathbb{R}^d} f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{v})d\boldsymbol{v}\phi(t,\boldsymbol{x})\boldsymbol{u}^{\delta_i}d\boldsymbol{x}dt\to 0, \tag{4.5}$$

for any $\varphi(\boldsymbol{v})\in C_c^\infty(\mathbb{R}^d)$ and $\phi(t,\boldsymbol{x})\in C_c^\infty((0,T)\times\mathbb{R}^d)$, as $\delta_i\to 0$. Using the definition of $\boldsymbol{u}(t,\boldsymbol{x})$ in equation (1.2), we also have

$$\int_A\int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\varphi(\boldsymbol{v})d\boldsymbol{v}\phi(t,\boldsymbol{x})\boldsymbol{u}d\boldsymbol{x}dt = 0, \tag{4.6}$$

for any $\varphi(\boldsymbol{v})\in C_c^\infty(\mathbb{R}^d)$ and $\phi(t,\boldsymbol{x})\in C_c^\infty((0,T)\times\mathbb{R}^d)$. Thus,

$$\lim_{\delta_i\to 0}\int_A\int_{\mathbb{R}^d} f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{u}^{\delta_i}\varphi(\boldsymbol{v})\phi(t,\boldsymbol{x})d\boldsymbol{v}d\boldsymbol{x}dt=\int_A\int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{u}\varphi(\boldsymbol{v})\phi(t,\boldsymbol{x})d\boldsymbol{v}d\boldsymbol{x}dt, \tag{4.7}$$

for all $\varphi(\boldsymbol{v})\in C_c^\infty(\mathbb{R}^d)$ and $\phi(t,\boldsymbol{x})\in C_c^\infty((0,T)\times\mathbb{R}^d)$.

For any $(t,\boldsymbol{x})\in B$, combining equation (4.3), equation (4.4), and the definition of $\boldsymbol{u}$ gives

$$\boldsymbol{u}^{\delta_i}(t,\boldsymbol{x})\to\boldsymbol{u}(t,\boldsymbol{x})\quad a.e.\text{ in } B,\text{ as } \delta_i\to 0.$$

Then the Lebesgue dominated convergence theorem leads to

$$\lim_{\delta_i\to 0}\int_B\int_{\mathbb{R}^d} f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{u}^{\delta_i}\varphi(\boldsymbol{v})\phi(t,\boldsymbol{x})d\boldsymbol{v}d\boldsymbol{x}dt=\int_B\int_{\mathbb{R}^d} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{u}\varphi(\boldsymbol{v})\phi(t,\boldsymbol{x})d\boldsymbol{v}d\boldsymbol{x}dt, \tag{4.8}$$

for all $\varphi(\boldsymbol{v})\in C_c^\infty(\mathbb{R}^d)$ and $\phi(t,\boldsymbol{x})\in C_c^\infty((0,T)\times\mathbb{R}^d)$. Combining equations (4.7) and (4.8), we have

$$\lim_{\delta_i\to 0}\int_0^T\int_{\mathbb{R}^{2d}} f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{u}^{\delta_i}\varphi(\boldsymbol{v})\phi(t,\boldsymbol{x})d\boldsymbol{v}d\boldsymbol{x}dt=\int_0^T\int_{\mathbb{R}^{2d}} f(t,\boldsymbol{x},\boldsymbol{v})\boldsymbol{u}\varphi(\boldsymbol{v})\phi(t,\boldsymbol{x})d\boldsymbol{v}d\boldsymbol{x}dt, \tag{4.9}$$

for all $\varphi(\boldsymbol{v})\in C_c^\infty(\mathbb{R}^d)$ and $\phi(t,\boldsymbol{x})\in C_c^\infty((0,T)\times\mathbb{R}^d)$. Using the density of the sums and products of the form $\varphi(\boldsymbol{v})\phi(t,\boldsymbol{x})$ in $C_c^\infty((0,T)\times\mathbb{R}^{2d}$, we get

$$f^{\delta_i}\boldsymbol{u}^{\delta_i}\to f\boldsymbol{u}\quad\text{in } \mathcal{D}'((0,T)\times\mathbb{R}^{2d}),\text{ as } \delta_i\to 0.$$

For any $\varepsilon>0$, we can prove

$$\begin{aligned}&\left|\int_{\mathbb{R}^d\setminus B(R+M_0T)}\int_{\mathbb{R}^d} f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})d\boldsymbol{v}d\boldsymbol{x}\right|\\ \le&\left|\int_{\mathbb{R}^d\setminus B(R)}\int_{\mathbb{R}^d} f_0(\boldsymbol{x},\boldsymbol{v})d\boldsymbol{v}d\boldsymbol{x}\right|\\ \le&\varepsilon,\end{aligned} \tag{4.10}$$

for sufficiently large R. Combining Proposition 3.1 (1) and the Dunford–Pettis theorem, we deduce that

$$f^{\delta_i}(t,\boldsymbol{x},\boldsymbol{v})\rightharpoonup f(t,\boldsymbol{x},\boldsymbol{v})\qquad\text{weakly-}\star\text{ in } L^\infty(0,T;L^1(\mathbb{R}^{2d}))),\text{ as } \delta_i\to 0.$$

From equation (1.1), we know $f\in C([0,T],L^1(\mathbb{R}^{2d})-W)$. Thus f is a weak solution to equation (1.1). Employing equation (4.1) and Proposition 3.1, it is easy to see that Theorem 1.2 (1), (2), and (3) hold. This completes the proof. □

5. Conclusion

In this paper, we prove the existence of weak solutions to a non-symmetric kinetic flocking model with cut-off interaction function. The difficulty mainly arises from the singularity at the denominator of the nonlinear term $\boldsymbol{u}$, which causes it not to possess Lipschitz regularity. Thus the existence theories in [4] (Theorem 3.10), [20] (Theorem 3.3), and [19] (Theorem 6.2) are not valid in this case because their analyses depend on the Lipschitz continuity of the interaction kernel.

So far, nearly all the literature is concentrated on the Cucker–Smale model with a global interaction function. There are few results related to the model we considered, with the exception of [23]. Under the natural condition that the initial distribution function $f_0(\boldsymbol{x}, \boldsymbol{v})$ has compact $\boldsymbol{v}$-support, we improve the result in [23] and simplify its proof, which can also be viewed as progress on this difficult problem.

In fact, we provide a framework that can be used to establish the existence of weak solutions to a kind of kinetic flocking model with non-Lipschitz continuous interaction kernels. Our proof is based on the weak convergence method. The velocity averaging lemma plays an important role in our analysis. It provides the compactness in our framework. For its further application in kinetic theory, we refer the reader to [13, 14].

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REFERENCES

[1] H.-O. Bae, Y.-P. Choi, S.-Y. Ha, and M.-J. Kang, *Time-asymptotic interaction of flocking particles and an incompressible viscous fluid*, Nonlinearity, 25(4):1155–1177, 2012.

[2] H.-O. Bae, Y.-P. Choi, S.-Y. Ha, and M.-J. Kang, *Asymptotic flocking dynamics of Cucker-Smale particles immersed in compressible fluids*, Discrete Contin. Dyn. Syst., 34(11):4419–4458, 2014. 1

[3] H.-O. Bae, Y.-P. Choi, S.-Y. Ha, and M.-J. Kang, *Global existence of strong solution for the Cucker-Smale-Navier-Stokes system*, J. Diff. Eqs., 257(6):2225–2255, 2014. 1

[4] J.A. Canizo, J.A Carrillo, and J. Rosado, *A well-posedness theory in measures for some kinetic models of collective motion*, Math. Models Meth. Appl. Sci., 21(03):515–539, 2011. 1, 5

[5] J.A. Carrillo, M. Fornasier, J. Rosado, and G. Toscani, *Asymptotic flocking dynamics for the kinetic Cucker-Smale model*, SIAM J. Math. Anal., 42(1):218–236, 2010. 1

[6] J.A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil, *Particle, kinetic, and hydrodynamic models of swarming*, Mathematical Modeling of Collective Behavior in Socio-economic and Life Sciences, Springer, 297–336, 2010.

[7] J.A. Carrillo, Y.-P. Choi, and M. Hauray, *The derivation of swarming models: mean-field limit and wasserstein distances*, in Collective Dynamics from Bacteria to Crowds, Springer, 1–46, 2014.

[8] Y.-P. Choi, S.-Y. Ha, and S.-B. Yun, *Complete synchronization of Kuramoto oscillators with finite inertia*, Phys. D, 239(1):32–44, 2011.

[9] F. Cucker and S. Smale, *Emergent behavior in flocks*, IEEE Trans. Automat. Control, 52(5):852–862, 2007. 1

[10] P. Degond, *Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in* 1 *and* 2 *space dimensions*, Ann. Sci. Éc. Norm. supér., 19:519–542, 1986.

[11] P. Degond and S. Motsch, *Continuum limit of self-driven particles with orientation interaction*, Math. Models Meth. Appl. Sci., 18(01):1193–1215, 2008.

[12] P. Degond and T. Yang, *Diffusion in a continuum model of self-propelled particles with alignment interaction*, Math. Models Meth. Appl. Sci., 20(01):1459–1490, 2010.

[13] R.J. DiPerna and P.-L. Lions, *Global weak solutions of Vlasov-Maxwell systems*, Comm. Pure Appl. Math., 42(6):729–757, 1989. 3, 5

[14] R.J. DiPerna and P.-L. Lions, *On the Cauchy problem for Boltzmann equations: global existence and weak stability*, Ann. of Math., 130(2):321–366, 1989. 5

[15] R. Duan, M. Fornasier, and G. Toscani, *A kinetic flocking model with diffusion*, Comm. Math. Phys., 300(1):95–145, 2010.

[16] R. Eftimie, *Hyperbolic and kinetic models for self-organized biological aggregations and movement: a brief review*, J. Math. Biol., 65(1):35–75, 2012.

[17] S.-Y. Ha, F. Huang, and Y. Wang, *A global unique solvability of entropic weak solution to the one-dimensional pressureless Euler system with a flocking dissipation*, J. Diff. Eqs., 257(5):1333–1371, 2014.

[18] S.-Y. Ha, M.-J. Kang, and B. Kwon, *A hydrodynamic model for the interaction of Cucker-Smale particles and incompressible fluid*, Math. Models Meth. Appl. Sci., 24(11):2311–2359, 2014. 1

[19] S.-Y. Ha and J.-G. Liu, *A simple proof of the Cucker-Smale flocking dynamics and mean-field limit*, Commun. Math. Sci., 7(2):297–325, 2009. 5

[20] S.-Y. Ha and E. Tadmor, *From particle to kinetic and hydrodynamic descriptions of flocking*, Kinet. Relat. Models, 1(3):415–435, 2008. 1, 5

[21] J. Haskovec, *Flocking dynamics and mean-field limit in the Cucker-Smale type model with topological interactions*, Phys. D, 261(15):42–51, 2013.

[22] A. Jadbabaie and J. Lin, *Coordination of groups of mobile autonomous agents using nearest neighbor rules*, IEEE Trans. Automat. Control, 48(6):988–1001, 2003. 1

[23] T.K. Karper, A. Mellet, and K. Trivisa, *Existence of weak solutions to kinetic flocking models*, SIAM J. Math. Anal., 45(1):215–243, 2013. 1, 1, 3, 5

[24] T.K. Karper, A. Mellet, and K. Trivisa, *Hydrodynamic limit of the kinetic Cucker-Smale flocking model*, Math. Models Meth. Appl. Sci., 25(01):131–163, 2015.

[25] Z. Liu and L. Guo, *Connectivity and synchronization of vicsek model*, Sci. China Inf. Sci., 51(7):848–858, 2008. 1

[26] Z. Liu and L. Guo, *Synchronization of multi-agent systems without connectivity assumptions*, Automatica, 45(12):2744–2753, 2009.

[27] S. Motsch and E. Tadmor, *A new model for self-organized dynamics and its flocking behavior*, J. Stat. Phys., 144(5):923–947, 2011. 1

[28] B. Perthame and P.E. Souganidis, *A limiting case for velocity averaging*, Ann. Sci. Éc. Norm. supér., 31:591–598, 1998.

[29] C. Reynolds, *Flocks, herds and schools: A distributed behavioral model*, in ACM Siggraph Computer Graphics, ACM, 21:25–34, 1987.

[30] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, and O. Shochet, *Novel type of phase transition in a system of self-driven particles*, Phys. Rev. Lett., 75(6):1226–1229, 1995. 1