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INTEGRABILITY AND ASYMPTOTIC BEHAVIORS OF POSITIVE SOLUTIONS FOR SOME INTEGRAL EQUATIONS*

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Abstract. In this paper, we investigate the integrability and asymptotic behaviors of positive solutions for a nonlinear integral system in a functional setting. Using the regularity lifting lemma and some delicate analysis techniques, we obtain the optimal integral intervals and the asymptotic estimates for such solutions around the origin and near infinity. Moreover, the index of regular solution is distinct from one of the previous several related systems.

Keywords. Integral system, positive solutions, integrability, asymptotic behaviors, regularity lifting lemma.

AMS subject classifications. 45G05, 45G15, 45E10, 45M99.

1. Introduction

In this paper, we investigate the following singular integral system

$$\begin{cases} u(x)=\dfrac{1}{|x|^\alpha}\displaystyle\int_{\mathbb{R}^n}\frac{v^p(y)}{|x-y|^\lambda}dy, \\ v(x)=\dfrac{1}{|x|^\alpha}\displaystyle\int_{\mathbb{R}^n}\frac{u^q(y)}{|x-y|^\lambda}dy. \end{cases} \tag{1.1}$$

Here $0<\alpha<\alpha+\lambda<n$ and $p,\ q\geq 1$.

In recent years, there has been tremendous interest in studying integral equations whose equivalence (under certain integrability conditions) with partial differential equations in $\mathbb{R}^n$ allows one to study global properties, such as integrability and asymptotic behavior, of the differential equations. First, we recount some closely related investigations and give some background. When $\alpha=0$ and $\lambda=n-2$, system (1.1) can be rewritten as

$$\begin{cases} u(x)=\displaystyle\int_{\mathbb{R}^n}\frac{v^p(y)}{|x-y|^{n-2}}dy, \\ v(x)=\displaystyle\int_{\mathbb{R}^n}\frac{u^q(y)}{|x-y|^{n-2}}dy. \end{cases} \tag{1.2}$$

Under some integrability condition, system (1.2) is equivalent to the following differential equations

$$\begin{cases} -\triangle u(x)=v^p(x), \\ -\triangle v(x)=u^q(x). \end{cases} \tag{1.3}$$

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These are the famous Lane–Emden systems, which arises in chemical, biological and physical studies. The Lane–Emden conjecture (for systems) states that system (1.3) has positive solutions if and only if those parameters (p,q) satisfy the following inequality:

$$\frac{1}{p+1}+\frac{1}{q+1}\leq\frac{n-2}{n}. \tag{1.4}$$

Serrin and Zou in [16] used the shooting method and the Pohozaev identity to show the existence of positive solutions of system (1.3) when (p,q) lies on or above the Sobolev hyperbola curves defined by inequality (1.4). A natural extension model of system (1.3) is the Hardy–Littlewood-Sobolev type integral equations

$$\begin{cases} u(x)=\displaystyle\int_{\mathbb{R}^n}\frac{v^p(y)}{|x-y|^\lambda}dy, \\ v(x)=\displaystyle\int_{\mathbb{R}^n}\frac{u^q(y)}{|x-y|^\lambda}dy. \end{cases} \tag{1.5}$$

Mitidieri [15] proved that the Lane–Emden conjecture holds with the additional assumption that (u,v) is a pair of radial solutions of equation (1.5).

The system (1.5) is closely related to the sharp constant of Hardy–Littlewood–Sobolev inequality and provides an important way to study the Lane–Emden conjecture. When $u\in L^{q+1}(\mathbb{R}^n)$, $v\in L^{p+1}(\mathbb{R}^n)$, and $1/(p+1)+1/(q+1)=\lambda/n$, Lieb [12] proved the existence of positive solutions to system (1.5). Under the same conditions, Chen, Li and Ou [2], with the help of moving plane method, showed that all positive solutions of system (1.5), are radially symmetric and monotonic decreasing about some point. System (1.5) has also the following natural extension:

$$\begin{cases} u(x)=\dfrac{1}{|x|^\alpha}\displaystyle\int_{\mathbb{R}^n}\frac{v^p(y)}{|x-y|^\lambda|y|^\beta}dy, \\ v(x)=\dfrac{1}{|x|^\beta}\displaystyle\int_{\mathbb{R}^n}\frac{u^q(y)}{|x-y|^\lambda|y|^\alpha}dy, \end{cases} \tag{1.6}$$

which is the critical point of the best constant of the weighted Hardy–Littlewood–Sobolev inequality and has been investigated by several authors. The first existence result of system (1.6) is also due to Lieb's work in [12]. Later on, Jin and Li [5] concluded that the positive solution of system (1.6) is radially symmetric and monotonic decreasing about the origin, provided that $u\in L^{q+1}(\mathbb{R}^n)$, $v\in L^{p+1}(\mathbb{R}^n)$, and

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{\lambda+\alpha+\beta}{n}, \qquad p,q>1.$$

At the same time, some qualitative properties of solutions to system (1.6), such as integrability and asymptotic behaviors, were also studied. Specifically, in [4], for system (1.6), Jin and Li obtained the optimal integrable intervals. And Lei, Li and Ma in [8–10] derived the sharp asymptotic behaviors at the origin as well as at infinity.

Another similar integral system is the following general stationary Schrödinger equations

$$\begin{cases} u(x)=\dfrac{1}{|x|^\alpha}\displaystyle\int_{\mathbb{R}^n}\frac{v^p(y)u^q(y)}{|x-y|^\lambda|y|^\beta}dy, \\ v(x)=\dfrac{1}{|x|^\beta}\displaystyle\int_{\mathbb{R}^n}\frac{u^p(y)v^q(y)}{|x-y|^\lambda|y|^\alpha}dy. \end{cases} \tag{1.7}$$

When $\alpha=\beta=0$ and $\lambda=n-2$, system (1.7) can be rewritten as

$$\begin{cases} u(x)=\displaystyle\int_{\mathbb{R}^n}\frac{v^p(y)u^q(y)}{|x-y|^{n-2}}dy,\\ v(x)=\displaystyle\int_{\mathbb{R}^n}\frac{u^p(y)v^q(y)}{|x-y|^{n-2}}dy. \end{cases} \tag{1.8}$$

The system (1.8), under some integrability condition, is also equivalent to the following system of differential equations

$$\begin{cases} -\triangle u(x)=v^p(x)u^q(x),\\ -\triangle v(x)=u^p(x)v^q(x), \end{cases} \tag{1.9}$$

which are closely related to the stationary Schrödinger system with critical exponents for the Bose–Einstein condensate. Li and Ma [11] showed that for $n\geq 3$, $1\leq p,q\leq (n+2)/(n-2)$, and $p+q=(n+2)/(n-2)$, any positive solution pair (u,v) of system (1.9) in $L^{2n/(n-2)}(\mathbb{R}^n)\times L^{2n/(n-2)}(\mathbb{R}^n)$ is radially symmetric and unique. Subsequently, Zhao and Lei [20] considered the following nonlinear system:

$$\begin{cases} u(x)=\displaystyle\int_{\mathbb{R}^n}\frac{v^p(y)u^q(y)}{|x-y|^{\lambda}|y|^{\beta}}dy,\\ v(x)=\displaystyle\int_{\mathbb{R}^n}\frac{u^p(y)v^q(y)}{|x-y|^{\lambda}|y|^{\beta}}dy, \end{cases} \tag{1.10}$$

and showed that for $\lambda>\beta\geq 0$, $\lambda+\beta<n$, and $0<p,q$ satisfying $p+q=(2n-\lambda-\beta)/(\lambda+\beta)$, any positive solutions pair (u,v) of system (1.10) in $L^{2n/(\lambda+\beta)}(\mathbb{R}^n)\times L^{2n/(\lambda+\beta)}(\mathbb{R}^n)$ is symmetric about the origin and monotone decreasing. Additionally, if $(n-\lambda)(\lambda+\beta)\geq 2n\beta$, then the positive solutions pair (u,v) of (1.10) is bounded and admits the following asymptotic behaviors:

$$\lim_{|x|\to\infty}|x|^{\lambda}u(x)=\int_{\mathbb{R}^n}\frac{u^q(y)v^p(y)}{|y|^{\beta}}dy,\ \lim_{|x|\to\infty}|x|^{\lambda}v(x)=\int_{\mathbb{R}^n}\frac{u^p(y)v^q(y)}{|y|^{\beta}}dy. \tag{1.11}$$

Soon after, Xu, Wu and Tan in [18] studied the more general equations:

$$\begin{cases} u(x)=\displaystyle\frac{1}{|x|^{\alpha}}\int_{\mathbb{R}^n}\frac{v^p(y)u^q(y)}{|x-y|^{\lambda}|y|^{\beta}}dy,\\ v(x)=\displaystyle\frac{1}{|x|^{\alpha}}\int_{\mathbb{R}^n}\frac{u^p(y)v^q(y)}{|x-y|^{\lambda}|y|^{\beta}}dy, \end{cases} \tag{1.12}$$

and established similar results. Very recently, Xu, Jiang and Wu in [19] considered system (1.7) and gave the optimal integrable interval and asymptotic behaviors at the origin as well as near infinity.

In this paper, we will study some important properties of positive solutions for system (1.1), which are distinct from the related results of the previous systems (1.6), (1.7), (1.10) and(1.12) studied in [4,8,9,17–20], respectively, on integrability and asymptotic behaviors. Precisely, with the help of regularity lifting, we will establish the optimal integrable interval for the solutions and by delicate analysis techniques, give the asymptotic behaviors of solutions for system (1.1) at the origin as well as near infinity. Furthermore, by comparison with related result, we find that the synchronism of (u,v) in integral systems, such as the similar sharp integral interval and asymptotic behavior of u and v, is an available tool to characterize the closeness of systems. Our main results can be formulated as follows:

THEOREM 1.1. *Assume that $(u,v)\in L^{q+1}(\mathbb{R}^n)\times L^{p+1}(\mathbb{R}^n)$ is a pair of positive solutions of system (1.1), and that (p,q) satisfy $1/(q+1)>\alpha/n$, $1/(p+1)>\alpha/n$, and*

$$\frac{1}{p+1}+\frac{1}{q+1}=\frac{\lambda+\alpha}{n}. \tag{1.13}$$

Then for any $1<r,s<\infty$,

$$u\in L^r(\mathbb{R}^n) \text{ and } v\in L^s(\mathbb{R}^n) \text{ if and only if}$$

- *for $p\le q$,*

$$\begin{aligned} &u\in L^r(\mathbb{R}^n), \quad \frac{1}{r}\in\left(\frac{\alpha}{n},\ \min\left\{\frac{(p+1)(\alpha+\lambda)}{n}-1,\ \frac{\alpha+\lambda}{n}\right\}\right), \quad \text{and} \\ &v\in L^s(\mathbb{R}^n), \quad \frac{1}{s}\in\left(\max\left\{\frac{\alpha}{n},\ \frac{\alpha q}{n}+\frac{\lambda+\alpha}{n}-1\right\},\ \frac{\alpha+\lambda}{n}\right); \end{aligned} \tag{1.14}$$

- *for $q<p$*

$$\begin{aligned} &u\in L^r(\mathbb{R}^n), \quad \frac{1}{r}\in\left(max\left\{\frac{\alpha}{n},\ \frac{\alpha p}{n}+\frac{\lambda+\alpha}{n}-1\right\},\frac{\alpha+\lambda}{n}\right), \quad \text{and} \\ &v\in L^s(\mathbb{R}^n), \quad \frac{1}{s}\in\left(\frac{\alpha}{n},\ \min\left\{\frac{\lambda+\alpha}{n},\ \frac{(\alpha+\lambda)(q+1)-n}{n}\right\}\right). \end{aligned} \tag{1.15}$$

THEOREM 1.2. *Assume that $(u,v)\in L^{q+1}(\mathbb{R}^n)\times L^{p+1}(\mathbb{R}^n)$ is a pair of positive solutions of system (1.1), and that (p,q) satisfy (1.13). Then*

- *for $p\le q$,*

$$\lim_{|x|\to\infty}|x|^{\alpha+\lambda}v(x)=\int_{\mathbb{R}^n}u^q(y)dy, \tag{1.16}$$

$u(x)$ admits the following asymptotic behavior at infinity

$$u(x)\backsimeq\begin{cases} \dfrac{A_1}{|x|^{(\alpha+\lambda)(p+1)-n}}, & \text{if } (\lambda+\alpha)p<n, \\ \dfrac{A_2\ \ln|x|}{|x|^{\alpha+\lambda}}, & \text{if } (\lambda+\alpha)p=n, \\ \dfrac{A_3}{|x|^{\alpha+\lambda}}, & \text{if } (\lambda+\alpha)p>n, \end{cases} \tag{1.17}$$

where

$$A_1=\left(\int_{\mathbb{R}^n}u^q(y)dy\right)^p\int_{\mathbb{R}^n}\frac{dz}{|e-z|^{\lambda}|z|^{(\alpha+\lambda)p}}, \qquad A_2=\left(\int_{\mathbb{R}^n}u^q dy\right)^p|\mathbb{S}^{n-1}|.$$

$$A_3=\int_{\mathbb{R}^n}v^p(y)dy,$$

$\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$;

- *for* $q<p$,

$$\lim_{|x|\to 0}|x|^{\alpha+\lambda}u(x)=\int_{\mathbb{R}^n}v^p(y)dy, \tag{1.18}$$

and

$$u(x)\backsimeq\begin{cases}\dfrac{B_1}{|x|^{(\alpha+\lambda)(q+1)-n}}, & \text{if } (\lambda+\alpha)q<n,\\ \dfrac{B_2\ \ln|x|}{|x|^{\alpha+\lambda}}, & \text{if } (\lambda+\alpha)q=n,\\ \dfrac{B_3}{|x|^{\alpha+\lambda}}, & \text{if } (\lambda+\alpha)q>n,\end{cases} \tag{1.19}$$

where

$$B_1=\left(\int_{\mathbb{R}^n}v^p(y)dy\right)^q\int_{\mathbb{R}^n}\frac{dz}{|e-z|^{\lambda}|z|^{(\alpha+\lambda)q}},\qquad B_2=\left(\int_{\mathbb{R}^n}v^p dy\right)^q|\mathbb{S}^{n-1}|.$$

$$B_3=\int_{\mathbb{R}^n}u^q(y)dy.$$

Here e *is a unit vector in* $\mathbb{R}^n$*, and we use the notation* $w(x)\backsimeq C/|x|^t$ *to denote that* $\lim\limits_{x\to 0\,\text{or}\,\infty}|x|^t w(x)=C$ *for a function* $w(x)$*, a real number* t*, and a non-zero real number* C*.*

THEOREM 1.3. *Assume that* $(u,v)\in L^{q+1}(\mathbb{R}^n)\times L^{p+1}(\mathbb{R}^n)$ *is a pair of positive solutions of system* (1.1)*, and that* (p,q) *satisfy* (1.13)*. Then*

- *for* $p\le q$,

$$\lim_{|x|\to 0}|x|^{\alpha}u(x)=\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^{\lambda}}dy, \tag{1.20}$$

and

$$v(x)\backsimeq\begin{cases}\dfrac{C_1}{|x|^{\alpha}}, & \text{if } \lambda+\alpha q<n,\\ \dfrac{C_2\ |\ln|x||}{|x|^{\alpha}}, & \text{if } \lambda+\alpha q=n,\\ \dfrac{C_3}{|x|^{\alpha(q+1)+\lambda-n}}, & \text{if } \lambda+\alpha q>n,\end{cases} \tag{1.21}$$

where

$$C_1=\int_{\mathbb{R}^n}\frac{u^q(y)}{|y|^{\lambda}}dy,\qquad C_2=|\mathbb{S}^{n-1}|\left(\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^{\lambda}}dy\right)^q,$$

$$C_3=\left(\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^{\lambda}}dy\right)^q\int_{\mathbb{R}^n}\frac{dy}{|y|^{\alpha q}|e-y|^{\lambda}};$$

- *for* $q<p$,

$$\lim_{|x|\to 0}|x|^{\alpha}v(x)=\int_{\mathbb{R}^n}\frac{u^q(y)}{|y|^{\lambda}}dy, \tag{1.22}$$

and

$$u(x)\backsimeq\begin{cases}\dfrac{D_1}{|x|^{\alpha}}, & \text{if } \lambda+\alpha p<n,\\ \dfrac{D_2\ |\ln|x||}{|x|^{\alpha}}, & \text{if } \lambda+\alpha p=n,\\ \dfrac{D_3}{|x|^{\alpha(p+1)+\lambda-n}}, & \text{if } \lambda+\alpha p>n,\end{cases} \tag{1.23}$$

where

$$D_1=\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^{\lambda}}dy, \qquad D_2=|\mathbb{S}^{n-1}|\left(\int_{\mathbb{R}^n}\frac{u^q(y)}{|y|^{\lambda}}dy\right)^p,$$

and

$$D_3=\left(\int_{\mathbb{R}^n}\frac{u^q(y)}{|y|^{\lambda}}dy\right)^p\int_{\mathbb{R}^n}\frac{dy}{|y|^{\alpha p}|e-y|^{\lambda}}.$$

Remark 1.1. Comparing Theorems 1.1 and Lemma 3.1 with the results of (1.6), (1.7) and (1.12) considered in [4], [19] and [18] respectively, all positive solutions have the same radial symmetry and are decreasing about the origin in a certain integrable space. But, the integrable intervals of solutions for the system (1.1) are distinct from those of systems (1.6), (1.7) and (1.12). At first, the structure of system (1.1) is different from the system (1.12) including the single-side potential integral equations (1.10), which is a particular form of (1.12) and was studied in [20]. Precisely if, set $w(x)=u(x)+v(x)$, the system of equations (1.12) and (1.10) can be transformed into a single integral equation as follows

$$w(x)=\frac{c(x)}{|x|^{\alpha}}\int_{\mathbb{R}^n}\frac{w^{p+q}(y)}{|x-y|^{\lambda}|y|^{\beta}}dy,$$

where $c(x)$ is a positive and bounded function. For (1.1), although the weighted functions of $u(x)$ and $v(x)$ in system (1.1) are the same in (1.12) and (1.10), it is impossible to rewrite a single equation due to the fact that there doses not exist $c(p,q)$ such that $a^p+b^q\le c(p,q)(a+b)^{p+q}$ for all $a,b>0$ and $p,q>1$. This different structure leads to the integrability of system (1.1) being thoroughly different from that of systems (1.12), (1.10) and (1.7). Indeed, by [18–20], we know that the integrability range of the pair of solutions (u,v) for systems (1.10), (1.7) and (1.12) lies in a fixed interval, which is independent of (p,q), and this implies that systems (1.10), (1.7) and (1.12) are more stable than (1.1) to some extent. In addition, although systems (1.6) and (1.1) have similar integrability, the classification of sharp integrable intervals for equation (1.1) is different from those of system (1.6) studied in [4]. Here we give a new and natural index, which is the dividing number for the sharp integral interval of positive solutions.

Remark 1.2. Another significant difference among systems (1.1), (1.6), (1.7) and (1.12) is the corresponding asymptotic behavior. Indeed, by [20] and [18], we see that the pair of solutions (u,v) in (1.10) and (1.12) has the same decay estimate at the origin as well as at infinity. At the same time, by [19], we have learned that system (1.7) admits the following asymptotic estimates:

$$\lim_{|x|\to 0}|x|^{\alpha}u(x)=\int_{\mathbb{R}^n}\frac{u^q(y)v^p(y)}{|y|^{\beta+\lambda}}dy,\quad \lim_{|x|\to\infty}|x|^{\alpha+\lambda}u(x)=\int_{\mathbb{R}^n}\frac{u^q(y)v^p(y)}{|y|^{\beta}}dy; \tag{1.24}$$

and

$$\lim_{|x|\to 0}|x|^{\beta}v(x)=\int_{\mathbb{R}^n}\frac{v^q(y)u^p(y)}{|y|^{\alpha+\lambda}}dy,\quad \lim_{|x|\to\infty}|x|^{\beta+\lambda}v(x)=\int_{\mathbb{R}^n}\frac{v^q(y)u^p(y)}{|y|^{\alpha}}dy. \tag{1.25}$$

However, for system (1.1), from equations (1.16) – (1.23), it is easy to see that u and v have different asymptotic behaviors. This implies that those (u,v) that are among the Schrodinger type integral equations have more consistency than (1.1). This study also indicates that the similar properties of (u,v) in integral systems, such as the similar sharp integral interval and aymptotic behavior of u and v at the origin as well as at infinity, is an available tool to characterize the tightness of systems. As for system (1.6), the difference with (1.1) is large. System (1.6) is invariant under Kelvin's transformation, namely, if (u,v) is a solution of system (1.6), then for any $x\in\mathbb{R}^n$, system (1.6) admits another positive solution $(\overline{u}(x),\ \overline{v}(x))$ defined by

$$\begin{cases} \overline{u}(x)=\dfrac{1}{|x|^{\frac{2n}{q+1}}}u\left(\dfrac{x}{|x|^2}\right),\\ \overline{v}(x)=\dfrac{1}{|x|^{\frac{2n}{p+1}}}v\left(\dfrac{x}{|x|^2}\right). \end{cases}$$

Hence, the authors in [8] utilized Kelvin type transform to change the asymptotic estimates at the center into one at the infinity. However, for (1.1) the invariance of Kelvin's transformation does not holds. Hence, we have to look for a new way to obtain the sharp asymptotic estimates at the center as well as at the infinity.

2. Optimal integrability

In this section, we will apply the regularity lifting lemma to obtain the optimal integrable intervals of the solutions of system (1.1).

Proof. **(Proof of Theorem 1.1.)** For convenience, we introduce some necessary notations. First, for simplicity, we denote $\|\cdot\|_{L^p(\mathbb{R}^n)}$ by $\|\cdot\|_p$ and set

$$\mathscr{X}(\tau,r)\triangleq\left\{(f,g)\in L^{\tau}(\mathbb{R}^n)\times L^{r}(\mathbb{R}^n) \text{ with } \|(f,g)\|_{\mathscr{X}(\tau,r)}\triangleq\|f\|_{\tau}+\|g\|_{r}\right\}.$$

Define

$$u_A(x)\triangleq\begin{cases} u(x), & \text{if } u(x)\geq A, or\ |x|\geq A,\\ 0, & \text{otherwise;}\end{cases}$$

and

$$\begin{aligned}\mathscr{T}(f,g)&\triangleq T(f,g)+(M_{A,2}(x),M_{A,1}(x))\\ &\triangleq(T_{A,2}\ g(x),\ \ T_{A,1}f(x))+(M_{A,2}(x),M_{A,1}(x)),\end{aligned}$$

where $T_{A,1}f(x)$, $T_{A,2}\ g(x)$, $M_{A,1}(x)$ and $M_{A,2}(x)$ are written as follows

$$T_{A,1}f(x) \triangleq \frac{1}{|x|^\alpha}\int_{\mathbb{R}^n} \frac{u_A^{q-1}\ f(y)}{|x-y|^\lambda}\ dy,\ \ T_{A,2}\ g(x) \triangleq \frac{1}{|x|^\alpha}\int_{\mathbb{R}^n} \frac{v_A^{p-1}\ g(y)}{|x-y|^\lambda} dy;$$

and

$$M_{A,1}(x) \triangleq \frac{1}{|x|^\alpha}\int_{\mathbb{R}^n} \frac{(v-v_A)^p(y)}{|x-y|^\lambda} dy,\ M_{A,2}(x) \triangleq \frac{1}{|x|^\alpha}\int_{\mathbb{R}^n} \frac{(u-u_A)^q(y)}{|x-y|^\lambda} dy.$$

Since $u-u_A$, $v-v_A$ are bounded with compact support, by the weighted Hardy–Littlewood–Sobolev inequality, it is easy to verify that for $r \in (n/(\alpha+\lambda),\ n/\alpha)$,

$$\|M_{A,1}\|_r \le \|(v-v_A)^p\|_{\frac{nr}{n+(n-\lambda-\alpha)r}},\quad \|M_{A,2}\|_r \le \|(u-u_A)^q\|_{\frac{nr}{n+(n-\lambda-\alpha)r}}. \tag{2.1}$$

Next, we turn to $(T_{A,2}g,\ T_{A,1}f)$. With the help of the weighted Hardy–Littlewood–Sobolev inequality and Hölder's inequality, we derive that for $f \in L^r(\mathbb{R}^n), r \in (n/(\alpha+\lambda),\ n/\alpha)$

$$\begin{aligned}\|T_{A,1}f\|_s &\le C(n,\lambda,q,p)\|u_A^{q-1}f\|_{\frac{ns}{n+(n-\lambda-\alpha)s}}\\ &\le C(n,\lambda,p,q)\|u_A^{q-1}\|_{\frac{q+1}{q-1}}\|f\|_r,\end{aligned} \tag{2.2}$$

and for $g \in L^s(\mathbb{R}^n), s \in (n/(\alpha+\lambda),\ n/\alpha)$,

$$\begin{aligned}\|T_{A,2}g\|_r &\le C(n,\lambda,q,p)\|v_A^{p-1}g\|_{\frac{nr}{n+(n-\lambda-\alpha)r}}\\ &\le C(n,\lambda,p,q)\|v_A^{p-1}\|_{\frac{p+1}{p-1}}\|g\|_s.\end{aligned} \tag{2.3}$$

Here (r,s) satisfy

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{p+1} - \frac{1}{q+1}.$$

By the above estimates, we claim that $T(f,g)$ is a contraction mapping from $\mathscr{X}(r,s)$ to itself. Indeed, choosing A sufficiently large, we have

$$\begin{aligned}\|T(f,g)\|_{\mathscr{X}(r,s)} &= \|T_{A,2}g\|_r + \|T_{A,1}f\|_s\\ &\le C(n,\lambda,p,q)(\|u_A^{q-1}\|_{\frac{q+1}{q-1}} + \|v_A^{p-1}\|_{\frac{p+1}{p-1}})(\|f\|_r + \|g\|_s)\\ &\le \frac{1}{2}\|(f,g)\|_{\mathscr{X}(r,s)}.\end{aligned}$$

To obtain the sharp integrability, we will discuss the following two cases.

Case 1 $p \le q$: By equation (1.13), we have $2/(q+1) \le (\alpha+\lambda)/n$ and $1/(p+1) \ge 1/(q+1)$. Taking $1/r \in (\alpha/n,\ \ 2/(q+1))$, the corresponding interval is

$$\frac{1}{s} = \frac{1}{r} + \frac{1}{p+1} - \frac{1}{q+1} \in (\frac{\alpha}{n} + \frac{1}{p+1} - \frac{1}{q+1},\ \ \frac{\alpha+\lambda}{n}).$$

Now choosing $\mathscr{X} = \mathscr{X}(r,s)$ and $\mathscr{Y} = \mathscr{X}(q+1,p+1)$ in Lemma 2.1 in [4], we conclude that $(u,v) \in \mathscr{X}(r,s) \cap \mathscr{X}(q+1,p+1)$. Namely,

$$u \in L^r,\ \frac{1}{r} \in (\frac{\alpha}{n}, \frac{2}{q+1});\quad v \in L^s,\ \frac{1}{s} \in (\frac{\alpha}{n} + \frac{1}{p+1} - \frac{1}{q+1},\ \ \frac{\alpha+\lambda}{n}). \tag{2.4}$$

On the other hand, we apply weighted Hardy–Littlewood–Sobolev inequality to system (1.1) and obtain that

$$\|u\|_\tau \le \|v\|^p_{\frac{n\tau p}{n+(n-\lambda-\alpha)\tau}}, \quad \tau \in (\frac{n}{\lambda+\alpha}, \frac{n}{\alpha}),$$

and

$$\frac{n+(n-\lambda-\alpha)\tau}{n\tau p} < \frac{\alpha+\lambda}{n}.$$

This implies that

$$\frac{1}{\tau} < \frac{(p+1)(\alpha+\lambda)}{n} - 1 = (p+1)(\frac{1}{p+1} + \frac{1}{q+1}) - 1 = \frac{p+1}{q+1},$$

and

$$\frac{(p+1)(\alpha+\lambda)}{n} - 1 - \frac{2}{q+1} = \frac{p-1}{q+1} > 0.$$

Therefore, the integrability of u is extended to the following interval

$$u \in L^r(\mathbb{R}^n), \quad \frac{1}{r} \in \left(\frac{\alpha}{n}, \quad \min\left\{\frac{(p+1)(\alpha+\lambda)}{n} - 1, \ \frac{\alpha+\lambda}{n}\right\}\right). \tag{2.5}$$

Similarly, we conclude that

$$\|v\|_\tau \le \|u\|^q_{\frac{n\tau q}{n+(n-\lambda-\alpha)\tau}}, \quad \frac{1}{\tau} > \frac{\alpha q}{n} + \frac{\lambda+\alpha}{n} - 1.$$

It is easy to check that

$$\frac{\alpha}{n} + \frac{1}{p+1} - \frac{1}{q+1} - \left(\frac{\alpha q}{n} + \frac{\lambda+\alpha}{n} - 1\right) = \frac{(q-1)}{n(q+1)}[n - \alpha(q+1)] > 0,$$

This together with equation (2.5) implies that

$$v \in L^s(\mathbb{R}^n), \qquad \frac{1}{s} \in \left(\max\left\{\frac{\alpha}{n}, \ \frac{\alpha q}{n} + \frac{\lambda+\alpha}{n} - 1\right\}, \ \frac{\alpha+\lambda}{n}\right);$$

and

$$u \in L^r(\mathbb{R}^n), \quad \frac{1}{r} \in \left(\frac{\alpha}{n}, \quad \min\left\{\frac{(p+1)(\alpha+\lambda)}{n} - 1, \ \frac{\alpha+\lambda}{n}\right\}\right).$$

Case 2 $q < p$: It is easy to check that $2/(p+1) < (\alpha+\lambda)/n$ and $1/(q+1) > 1/(p+1) > \alpha/n$. Meanwhile, for $1/r \in (\alpha/n + 1/(q+1) - 1/(p+1), \ (\alpha+\lambda)/n)$, we find that s satisfies

$$\frac{1}{s} \in (\frac{\alpha}{n}, \ \frac{\lambda+\alpha}{n} + \frac{1}{p+1} - \frac{1}{q+1}).$$

Therefore, with the same method as Case 1, we can conclude that

$$u \in L^r(\mathbb{R}^n), \quad \frac{1}{r} \in \left(\max\left\{\frac{\alpha}{n}, \ \frac{\alpha p}{n} + \frac{\lambda+\alpha}{n} - 1\right\}, \frac{\alpha+\lambda}{n}\right);$$

and

$$v\in L^s(\mathbb{R}^n),\quad \frac{1}{s}\in\left(\frac{\alpha}{n},\ \min\left\{\frac{\lambda+\alpha}{n},\ \frac{(\alpha+\lambda)(q+1)-n}{n}\right\}\right).$$

Now we discuss that the integrable interval is sharp. Indeed, as $|x|<1$ and $2\le|y|\le 4$, by equation (1.1), then

$$\begin{aligned}u(x)&=|x|^{-\alpha}\int_{\mathbb{R}^n}\frac{v^p(y)}{|x-y|^\lambda}dy\\&\ge|x|^{-\alpha}\int_{2\le|y|\le4}\frac{v^p(y)}{(|x|+y|)^\lambda}dy=C(\lambda,p,\|v\|_p)|x|^{-\alpha},\end{aligned}\tag{2.6}$$

and

$$v(x)\ge C(\lambda,p,\|u\|_q)|x|^{-\alpha},\quad |x|\le 1.\tag{2.7}$$

This implies that

$$\int_{\mathbb{R}^n}u^r(y)dy=\|u\|_r^r=\infty,\quad \int_{\mathbb{R}^n}v^r(y)dy=\|v\|_r^r=\infty,\quad \frac{1}{r}\le\frac{\alpha}{n}.$$

And for $|x|<\frac{1}{2}$,

$$v(x)=\frac{1}{|x|^\alpha}\int\frac{u^q(y)}{|x-y|^\lambda}dy\ge\frac{C(n)}{|x|^{\alpha+\alpha q+\lambda}}\int_{|x|\le|y|\le2|x|\le1}dy=\frac{C(n)}{|x|^{\alpha+\alpha q+\lambda-n}}.$$

which leads to that

$$\|v\|_s^s\ge C(n)\int_{|x|\le\frac{1}{2}}|x|^{-[\alpha+\alpha q+\lambda-n]s}dx=\infty,\qquad \frac{1}{s}\le\frac{\alpha+\alpha q+\lambda-n}{n}.\tag{2.8}$$

Similarly, for $|y|<1$ and $|x|\ge2$, we conclude that

$$u(x)\ge C(\lambda,p,\|v\|_p)|x|^{-\lambda-\alpha}\quad\text{and}\quad v(x)\ge C(\lambda,q,\|u\|_q)|x|^{-\lambda-\alpha}.\tag{2.9}$$

This implies that

$$\min\left\{\|u\|_s^s,\ \|v\|_s^s\right\}\ge\int_{|x|\ge2}|x|^{-\alpha s-\lambda s}dx=\infty,\qquad \frac{1}{s}\ge\frac{\alpha+\lambda}{n}.\tag{2.10}$$

Moreover, for $|x|\ge2$

$$\begin{aligned}v(x)&=\frac{1}{|x|^\alpha}\int\frac{u^q(y)}{|x-y|^\lambda}dy\\&\ge\frac{C(n)}{|x|^{(\alpha+\lambda)(q+1)}}\int_{|x|\le|y|\le2|x|}dy=\frac{C(n)}{|x|^{(\alpha+\lambda)(q+1)-n}},\end{aligned}$$

and

$$\|v\|_s^s\ge C(n)\int_{|x|>2}\frac{1}{|x|^{[(\alpha+\lambda)(q+1)-n]s}}dx=\infty,\quad \frac{1}{s}\ge\frac{(p+1)(\alpha+\lambda)-n}{n}.$$

This completes the proof of Theorem 1.1. ∎

3. Asymptotic behaviors near infinity

This section is devoted to the proof of Theorem 1.2. First, we give an auxiliary lemma.

LEMMA 3.1. *Assume that* $(u(x),v(x))\in L^{q+1}(\mathbb{R}^n)\times L^{p+1}(\mathbb{R}^n)$ *is the solution of system* (1.1). *Then it must be radially symmetric and decreasing about the origin point.*

Proof. For the radial symmetry of positive solutions of system (1.1), we can apply the classical moving plane method in integral forms recently introduced by Chen, Li and Ou in [2,3]. We omit the details here.

Proof. **(Proof of Theorem 1.2.)** We consider only the case $p\le q$, since the result in the case $p>q$ can be obtained by the similar arguments.

To obtain the sharp asymptotic behaviors of $v(x)$, we rewrite (1.1) as follows:

$$\begin{aligned}|x|^{\alpha+\lambda}v(x) &= \int_{\mathbb{R}^n}\frac{|x|^\lambda}{|x-y|^\lambda}u^q(y)dy\\ &= \left(\int_{\mathbb{R}^n\setminus B_R(0)\setminus B_{\frac{|x|}{2}}(x)}+\int_{B_R(0)}+\int_{B_{\frac{|x|}{2}}(x)}\right)\frac{|x|^\lambda}{|x-y|^\lambda}u^q(y)dy\\ &\triangleq \mathscr{A}_1+\mathscr{A}_2+\mathscr{A}_3.\end{aligned}$$

Noting that $\frac{2}{q+1}\le\frac{\alpha+\lambda}{n}$ and

$$\frac{(p+1)(\alpha+\lambda)}{n}-1-\frac{2}{q+1}=\frac{p-1}{q+1}>0,$$

it is easy to check that

$$\frac{1}{q}\in\left(\frac{1}{q+1},\frac{2}{q+1}\right)\subset\left(\frac{\alpha}{n},\ \min\left\{\frac{(p+1)(\alpha+\lambda)}{n}-1,\ \frac{\alpha+\lambda}{n}\right\}\right). \tag{3.1}$$

This together with Theorem 1.1 implies that

$$\int_{\mathbb{R}^n}u^q(y)dy<\infty. \tag{3.2}$$

As $|x|\ge 2R$ and $y\in B_R(0)$, $|x-y|\ge|x|-|y|>|x|/2$, then

$$\left|\frac{|x|^\lambda}{|x-y|^\lambda}-1\right|u^q(y)\le C(\lambda)u^q(y).$$

Therefore

$$\lim_{R\to\infty}\lim_{|x|\to\infty}\mathscr{A}_1=\int_{\mathbb{R}^n}u^q(y)dy, \tag{3.3}$$

and

$$\lim_{R\to\infty}\lim_{|x|\to\infty}\left|\mathscr{A}_3\right|\le\lim_{R\to\infty}\lim_{|x|\to\infty}\int_{\mathbb{R}^n\setminus B_R(0)}2^\lambda u^q(y)dy=0. \tag{3.4}$$

Now we turn to $\mathscr{A}_2$. Observing that for $\frac{1}{s}\in(\frac{\alpha}{n},\min\{\frac{(p+1)(\alpha+\lambda)}{n}-1,\ \frac{\alpha+\lambda}{n}\})$, by Theorem 1.1, we conclude that

$$C(n,s)u^s(x)|x|^n\le\int_{B_{|x|/2}(x)}u^s(y)dy\le M<\infty.$$

With $\frac{1}{s}=\min\{\frac{(p+1)(\alpha+\lambda)}{n}-1,\ \frac{\alpha+\lambda}{n}\}-\varepsilon$, the above inequality implies that

$$\begin{aligned}|\mathscr{A}_2| &= \int_{B_{|x|/2}(x)} \frac{|x|^\lambda}{|x-y|^\lambda} u^q(y)dy \le |x|^\lambda u^q(\frac{|x|}{2}) \int_{B_{|x|/2}(x)} |x-y|^{-\lambda}dy \\ &\le C(\lambda,q)|x|^{n(1-q/s)} \to 0, \quad \text{as} \quad |x|\to\infty.\end{aligned}$$

This, combining with results (3.3) and (3.4) leads to

$$\lim_{|x|\to\infty} |x|^{\alpha+\lambda} v(x) = \int_{\mathbb{R}^n} u^q(y)dy \triangleq B_3. \tag{3.5}$$

Now we will discuss the asymptotic behavior of $u(x)$ at infinity in the following two cases.

Case 1. $\frac{\alpha}{n} < \frac{1}{q+1} < \frac{\alpha}{n+\alpha-\lambda}$:

Observing that

$$\frac{\alpha(q+1)}{n} + \frac{\lambda}{n} - 1 > \frac{\alpha}{n},$$

and by Theorem 1.1, it is easy to check that

$$v \in L^s(\mathbb{R}^n), \qquad \frac{1}{s} \in \Big(\frac{\alpha q}{n} + \frac{\lambda+\alpha}{n} - 1,\ \frac{\alpha+\lambda}{n}\Big).$$

In what follows, for this case, we discuss the sharp asymptotic behavior of $u(x)$ in the following three subcases:

(i). $\frac{1}{p+1} > \frac{\alpha+\lambda}{n+\lambda+\alpha}$, i.e., $(\alpha+\lambda)p < n$:

By (1.1), we can write

$$\begin{aligned}|x|^{(\alpha+\lambda)(p+1)-n} u(x) &= |x|^{\lambda+(\alpha+\lambda)p-n} \int_{\mathbb{R}^n} \frac{v^p(y)}{|x-y|^\lambda} dy \\ &= |x|^{\lambda+(\alpha+\lambda)p-n} \Big(\int_{B_R(0)} + \int_{\mathbb{R}^n\setminus B_R(0)}\Big) \frac{v^p(y)}{|x-y|^\lambda} dy \\ &\triangleq \mathscr{B}_1(x) + \mathscr{B}_2(x).\end{aligned} \tag{3.6}$$

Firstly, we conclude that the following improper integral is convergent

$$\begin{aligned}\mathscr{H} &= \int_{\mathbb{R}^n} \frac{1}{|e-z|^\lambda} \frac{1}{|z|^{(\alpha+\lambda)p}} dz \\ &= \Big(\int_{|z|\le\frac{1}{2}} + \int_{\frac{1}{2}\le|z|\le 2} + \int_{|z|\ge 2}\Big) \frac{1}{|e-z|^\lambda} \frac{1}{|z|^{(\alpha+\lambda)p}} dz \\ &\triangleq \mathscr{H}_1 + \mathscr{H}_2 + \mathscr{H}_3.\end{aligned}$$

Noting that as $\frac{1}{p+1} > \frac{\alpha+\lambda}{n+\lambda+\alpha}$ $(p(\alpha+\lambda) < n)$ and $\lambda < n$, we have

$$\begin{aligned}\mathscr{H}_1 + \mathscr{H}_2 &= \Big(\int_{|z|\le\frac{1}{2}} + \int_{\frac{1}{2}\le|z|\le 2}\Big) \frac{1}{|e-z|^\lambda} \frac{1}{|z|^{(\alpha+\lambda)p}} dz \\ &\le \int_{|z|\le\frac{1}{2}} \frac{2^\lambda}{|z|^{(\alpha+\lambda)p}} dz + \int_{|z-e|\le 3} \frac{2^{(\alpha+\lambda)p}}{|z-e|^\lambda} dz < \infty,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\mathscr{B}_1(x) &= |x|^{\lambda+(\alpha+\lambda)p-n}\int_{B_R(0)}\frac{v^p(y)}{|x-y|^\lambda}dy\\ &\le |x|^{(\alpha+\lambda)p-n}\int_{B_R(0)}v^p(y)dy\\ &\le C(R,\|v\|_{p+1})|x|^{(\alpha+\lambda)p-n}\to 0, \quad \text{as} \quad |x|\to\infty. \end{aligned}\tag{3.8}$$

Due to $\frac{\alpha}{n}<\frac{1}{q+1}$ and $\frac{1}{p+1}+\frac{1}{q+1}=\frac{\lambda+\alpha}{n}$, we have $\frac{1}{p+1}<\frac{\lambda}{n}$ and

$$\frac{1}{p+1}<\frac{\lambda+\alpha}{n+\alpha},$$

which implies that

$$p(\alpha+\lambda)+\lambda>n,$$

and

$$\mathscr{H}_3=\int_{|z|\ge 2}\frac{1}{|e-z|^\lambda}\frac{1}{|z|^{(\alpha+\lambda)p}}dz\le C(\lambda)\int_{|z|\ge 2}\frac{1}{|z|^{(\alpha+\lambda)p+\lambda}}dz<\infty.$$

This together with estimate (3.7) shows that $\mathscr{H}$ is well-posed. When $|x|>R$ with R large enough, by equation (3.5), we have

$$v(x)\approx\frac{1}{|x|^{\alpha+\lambda}}\int_{\mathbb{R}^n}u^q(y)dy\triangleq\frac{B_3}{|x|^{\alpha+\lambda}}.$$

So, it is easy to verify that

$$\begin{aligned}\mathscr{B}_2(x) &\approx |x|^{\lambda+(\alpha+\lambda)p-n}\int_{\mathbb{R}^n\setminus B_R(0)}\frac{1}{|x-y|^\lambda}\frac{B_3^p}{|y|^{(\alpha+\lambda)p}}dy\\ &= B_3^p\int_{|z|\ge R/|x|}\frac{1}{|e-z|^\lambda}\frac{1}{|z|^{(\alpha+\lambda)p}}dz,\end{aligned}\tag{3.9}$$

and

$$\lim_{|x|\to\infty}\mathscr{B}_2=B_3^p\int_{\mathbb{R}^n}\frac{1}{|e-z|^\lambda}\frac{1}{|z|^{(\alpha+\lambda)p}}dz.$$

Hence, by results (3.6), (3.8) and (3.9), we deduce that

$$\lim_{|x|\to\infty}|x|^{(\alpha+\lambda)(p+1)-n}u(x)=\left[\int_{\mathbb{R}^n}u^q(y)dy\right]^p\int_{\mathbb{R}^n}\frac{1}{|e-z|^\lambda}\frac{1}{|z|^{(\alpha+\lambda)p}}dz.$$

(ii). $\frac{1}{p+1}=\frac{\alpha+\lambda}{n+\lambda+\alpha}$, i.e., $(\alpha+\lambda)p=n$:

We first write

$$\begin{aligned}\frac{|x|^{\alpha+\lambda}}{\ln|x|}u(x) &= \frac{|x|^\lambda}{\ln|x|}\int_{\mathbb{R}^n}\frac{v^p(y)}{|x-y|^\lambda}dy\\ &= \frac{|x|^\lambda}{\ln|x|}\left(\int_{B_R(0)}+\int_{\mathbb{R}^n\setminus B_R(0)}\right)\frac{v^p(y)}{|x-y|^\lambda}dy\\ &\triangleq \mathscr{C}_1(x)+\mathscr{C}_2(x).\end{aligned}$$

As $|x|>2R$ and $|y|\le R$, we have

$$\begin{aligned}\mathscr{C}_1(x) &= \frac{|x|^\lambda}{\ln|x|}\int_{B_R(0)}\frac{v^p(y)}{|x-y|^\lambda}dy\\ &\le \frac{C(n)}{\ln|x|}\int_{B_R(0)}v^p(y)dy \le \frac{C(n)}{\ln|x|}\Big(\int_{B_R(0)}v^{p+1}(y)dy\Big)^{\frac{p}{p+1}}\ R^{\frac{n}{p+1}},\end{aligned}$$

and

$$\lim_{|x|\to\infty}\mathscr{C}_1(x)=0.$$

Meanwhile, by equation (3.5), we have

$$\begin{aligned}\mathscr{C}_2(x) &\backsimeq \frac{|x|^\lambda}{\ln|x|}\int_{\mathbb{R}^n\setminus B_R(0)}\frac{1}{|x-y|^\lambda}\frac{B_3^p}{|y|^{(\alpha+\lambda)p}}dy\\ &= \frac{B_3^p}{\ln|x|}\int_{|z|>\frac{R}{|x|}}\frac{1}{|e-z|^\lambda}\frac{1}{|z|^n}dz\\ &= \frac{B_3^p}{\ln|x|}\Big(\int_{\frac12\le|z|\le2}+\int_{\frac12>|z|>\frac{R}{|x|}}+\int_{|z|\ge2}\Big)\frac{1}{|e-z|^\lambda}\frac{1}{|z|^n}dz\\ &\triangleq \mathscr{C}_{2.1}(x)+\mathscr{C}_{2.2}(x)+\mathscr{C}_{2.3}(x),\end{aligned}$$

where $e=\frac{x}{|x|},\ \ x\neq0$. Noting that

$$\Big(\int_{\frac12\le|z|\le2}+\int_{|z|\ge2}\Big)\frac{1}{|e-z|^\lambda}\frac{1}{|z|^n}dz\le\int_{|z-e|\le3}\frac{2^n}{|e-z|^\lambda}dz+\int_{|z|\ge2}\frac{2^\lambda}{|z|^{n+\lambda}}dz<\infty,$$

which implies that

$$\lim_{|x|\to\infty}(\mathscr{C}_{2.1}(x)+\mathscr{C}_{2.3}(x))=0.$$

Next we discuss $\mathscr{C}_{2.2}(x)$. We write

$$\begin{aligned}\mathscr{C}_{2.2}(x) &= \frac{B_3^p}{\ln|x|}\int_{1/2>|z|>R/|x|}\frac{1}{|e-z|^\lambda}\frac{1}{|z|^n}dz\\ &= \frac{B_3^p}{\ln|x|}\int_{R/|x|}^{1/2}\frac1r\int_{\mathbb{S}^{n-1}}\frac{ds(w)}{|e-rw|^\lambda}dr.\end{aligned}$$

Then

$$\begin{aligned}\lim_{|x|\to\infty}\mathscr{C}_{2.2}(x) &= \lim_{|x|\to\infty}\frac{B_3^p}{\ln|x|}\int_{\frac{R}{|x|}}^{\frac12}\frac1r\int_{\mathbb{S}^{n-1}}\frac{ds(w)}{|e-rw|^\lambda}dr\\ &= \lim_{|x|\to\infty}\frac{\dfrac{B_3^p}{\frac{R}{|x|}}\displaystyle\int_{\mathbb{S}^{n-1}}\frac{\frac{R}{|x|^2}\ ds(w)}{|e-\frac{R}{|x|}w|^\lambda}}{\frac{1}{|x|}}=B_3^p\ |\mathbb{S}^{n-1}|.\end{aligned}$$

Therefore

$$\lim_{|x|\to\infty}\frac{|x|^{\alpha+\lambda}}{\ln|x|}u(x)=\Big(\int_{\mathbb{R}^n}u^q(y)dy\Big)^p\ \ |\mathbb{S}^{n-1}|.$$

(iii). $\frac{1}{p+1} < \frac{\alpha+\lambda}{n+\lambda+\alpha}$, i.e., $(\alpha+\lambda)p > n$:
Since $v \in L^{p+1}(\mathbb{R}^n)$ and by equation (3.5), we have

$$\begin{aligned}\int_{\mathbb{R}^n} v^p(y)dy &= \int_{|y|\le R} v^p(y)dy + \int_{|y|\ge R} v^p(y)dy\\ &\le C(R,n)\left(\|v\|_{p+1}^p + \int_{|y|>R} \frac{1}{|y|^{(\alpha+\lambda)p}}dy\right) < \infty.\end{aligned}$$

We decompose $|x|^{\alpha+\lambda}u(x)$ as follows.

$$\begin{aligned}|x|^{\alpha+\lambda}u(x) &= |x|^\lambda \int_{\mathbb{R}^n} \frac{v^p(y)}{|x-y|^\lambda}dy\\ &= |x|^\lambda \left(\int_{B_R(0)} + \int_{B_{|x|/2}(x)} + \int_{\mathbb{R}^n\setminus B_{|x|/2}(x)\setminus B_R(0)}\right)\frac{v^p(y)}{|x-y|^\lambda}dy\\ &\triangleq \mathscr{D}_1(x) + \mathscr{D}_2(x) + \mathscr{D}_3(x).\end{aligned}$$

It is easy to check that for $|x| > 2R$,

$$\lim_{R\to\infty}\lim_{|x|\to\infty} \mathscr{D}_1(x) = \lim_{R\to\infty}\lim_{|x|\to\infty}\int_{B_R(0)} \frac{|x|^\lambda v^p(y)}{|x-y|^\lambda}dy = \int_{\mathbb{R}^n} v^p(y)dy,$$

and

$$\lim_{R\to\infty}\lim_{|x|\to\infty} \mathscr{D}_3(x) \le \lim_{R\to\infty}\lim_{|x|\to\infty}\int_{\mathbb{R}^n\setminus B_R(0)} v^p(y)dy = 0.$$

Noting that for $v \in L^s(\mathbb{R}^n)$, $\frac{1}{s} \in (\frac{\alpha q}{n} + \frac{\lambda+\alpha}{n} - 1,\ \frac{\alpha+\lambda}{n})$, we have

$$Cv^s(|x|)|x|^n \le \int_{B_{|x|/2}(x)} v^s(y)dy < \infty,$$

and

$$\begin{aligned}\mathscr{D}_2(x) &= |x|^\lambda \int_{B_{|x|/2}(x)} \frac{v^p(y)}{|x-y|^\lambda}dy\\ &\le C|x|^n v^p(\tfrac{|x|}{2}) = C\left(|x|^n v^s(\tfrac{|x|}{2})\right)^{p/s} |x|^{n(1-p/s)}.\end{aligned}$$

On the other hand, observe that

$$\frac{p}{\frac{\alpha+\lambda}{n}} = \frac{n\ p}{\alpha+\lambda} > (\frac{n}{\alpha+\lambda})^2 > 1,$$

and, taking $\frac{1}{s} = \frac{(\alpha+\lambda)(1-\varepsilon)}{n}$ with $\varepsilon > 0$ small enough, we get

$$\lim_{|x|\to\infty} \mathscr{D}_2(x) = 0,$$

and

$$\lim_{|x|\to\infty} |x|^{\alpha+\lambda}u(x) = \int_{\mathbb{R}^n} v^p(y).$$

Case 2. $\frac{\lambda+\alpha}{2n} \geq \frac{1}{q+1} \geq \frac{\alpha}{n+\alpha-\lambda}$:

(i). $\frac{1}{p+1} < \frac{\alpha+\lambda}{n+\alpha+\lambda}$, i.e., $(\alpha+\lambda)p > n$:

Obviously,

$$\frac{\alpha}{n} < \frac{1}{p+1} < \frac{1}{p} < \frac{\lambda+\alpha}{n},$$

which combining with Theorem 1.1, implies that

$$\int_{\mathbb{R}^n} v^p(y)dy < \infty.$$

Noting that

$$\begin{aligned} \frac{p}{\frac{\alpha+\lambda}{n}} &= \frac{n(p+1)}{\alpha+\lambda} - \frac{n}{\alpha+\lambda} \\ &> \frac{n(n+\alpha+\lambda)-n(\alpha+\lambda)}{(\alpha+\lambda)^2} = \frac{n^2}{(\alpha+\lambda)^2} > 1 \end{aligned}$$

and using the same method in Case 1's (iii), we deduce that

$$\lim_{|x|\to\infty} |x|^{\alpha+\lambda} u(x) = \int_{\mathbb{R}^n} v^p(y)dy.$$

(ii). $\frac{\alpha+\lambda}{n+\alpha+\lambda} = \frac{1}{p+1}$, i.e., $(\alpha+\lambda)p = n$:

Similarly to Case 1's (ii), we can deduce that

$$\lim_{|x|\to\infty} \frac{|x|^{\alpha+\lambda}}{\ln|x|} u(x) = \left[\int_{\mathbb{R}^n} u^q(y)dy\right]^p |\mathbb{S}^{n-1}|.$$

(iii). $\frac{\alpha+\lambda}{n+\alpha+\lambda} < \frac{1}{p+1}$, i.e., $(\alpha+\lambda)p < n$:

By (1.13), it is easy to check that

$$\frac{\alpha+\lambda}{n+\alpha+\lambda} < \frac{1}{p+1} \leq \frac{\alpha+\lambda}{n} - \frac{\alpha}{n+\alpha-\lambda} < \frac{\alpha+\lambda}{n+\alpha},$$

which leads to that

$$(\lambda+\alpha)p+\lambda > n \quad \text{and} \quad (\lambda+\alpha)p < n.$$

and

$$\int_{\mathbb{R}^n} \frac{1}{|z|^{(\lambda+\alpha)p}|e-z|^\lambda} dz < \infty.$$

Hence, similarly to the Case 1's (i), we conclude that

$$\lim_{|x|\to\infty} |x|^{(\alpha+\lambda)(p+1)-n} u(x) = \left[\int_{\mathbb{R}^n} u^q(y)dy\right]^p \int_{\mathbb{R}^n} \frac{1}{|e-z|^\lambda} \frac{1}{|z|^{(\lambda+\alpha)p}} dy.$$

This completes the proof of Theorem 1.2. ∎

4. Asymptotic behaviors at origin

In this section, we will prove Theorem 1.3.

Proof. **(Proof of Theorem 1.3.)** Due to the symmetry of system (1.1), to obtain the asymptotic behavior of (u,v) at the origin, it suffices to show that the case $p\leq q$ holds. Next, we give the proof in the following three cases:

Case 1. $\frac{1}{q+1}>\frac{\alpha}{n+\alpha-\lambda}$, i.e., $\alpha q+\lambda<n$.

By inequality (3.2) and $0<\lambda<n$, we have

$$\int_{|y|\geq R}\frac{u^q(y)}{|y|^\lambda}dy\leq\int_{|y|\geq R}u^q(y)dy<\infty.$$

Now we turn to consider the convergence of $\int_{|y|\leq 1}\frac{u^q(y)}{|y|^\lambda}dy$. Noting that as $\frac{1}{q+1}>\frac{\alpha}{n+\alpha-\lambda}$, then $\frac{\alpha q}{n-\lambda}<1$. Hence there exists $0<\varepsilon<\min\{\frac{\lambda}{\alpha+\lambda},\ \frac{(p+1)(\alpha+\lambda)}{n}-1\}$ small enough such that $\frac{\alpha q}{n-\lambda}<1-\varepsilon$. Set

$$s\triangleq\frac{n(1-\varepsilon)}{\alpha q}.$$

Obviously, $s=\frac{n(1-\varepsilon)}{\alpha q}>\frac{n}{n-\lambda}$, we deduce that

$$sq=\frac{n(1-\varepsilon)}{\alpha}<\frac{n}{\alpha}\quad\text{and}\quad s'\lambda=\frac{s\lambda}{s-1}<n. \tag{4.1}$$

This together with Hölder's inequality and Theorem 1.2, implies that

$$\int_{|y|\leq R}u^q(y)dy\leq\int_{|y|\leq R}\frac{u^q(y)}{|y|^\lambda}dy\leq\left(\int_{|y|\leq R}|y|^{-s'\lambda}dy\right)^{\frac{1}{s'}}\left(\int_{\mathbb{R}^n}u^{sq}(y)dy\right)^{\frac{1}{s}},$$

and

$$\int_{\mathbb{R}^n}\left(\frac{u^q(y)}{|y|^\lambda}+u^q(y)\right)dy<\infty. \tag{4.2}$$

To obtain the asymptotic behaviors of $v(x)$ at the origin, we now consider the following formula

$$\int_{\mathbb{R}^n}\frac{u^q(y)}{|x-y|^\lambda}dy=\left(\int_{|y|<\delta}+\int_{|y|\geq\delta}\right)\frac{u^q(y)}{|x-y|^\lambda}dy\triangleq I(x)+II(x).$$

Noting that as $|x|\leq\delta/2$, we have

$$\begin{aligned}I(x)&=\int_{|y|\leq\delta}\frac{u^q(y)}{|x-y|^\lambda}dy\\&\leq\left(\int_{|x-y|\leq2\delta}|x-y|^{-s'\lambda}dy\right)^{\frac{1}{s'}}\left(\int_{\mathbb{R}^n}u^{sq}(y)dy\right)^{\frac{1}{s}}\\&\leq C(\|u\|_{sq}^q,n,\lambda,s)\delta^{n-s'\lambda}\to0,\quad\delta\to0.\end{aligned}$$

Here s takes the same value as in expression (4.1). What's more, as $|x|\leq\delta/2$ and $|y|\geq\delta$, we have

$$II(x)=\int_{|y|\geq\delta}\frac{u^q(y)}{|x-y|^\lambda}dy\leq C(\lambda)\int_{\mathbb{R}^n}\frac{u^q(y)}{|y|^\lambda}dy,$$

and with estimate (4.2)

$$\begin{aligned}\lim_{|x|\to 0}|x|^{\alpha}v(x) &= \lim_{\delta\to 0}\lim_{|x|\to 0} II(x)\\ &= \lim_{\delta\to 0}\lim_{|x|\to 0}\int_{|y|\geq\delta}\frac{u^q(y)}{|x-y|^{\lambda}}dy = \int_{\mathbb{R}^n}\frac{u^q(y)}{|y|^{\lambda}}dy \triangleq C_1. \end{aligned}\tag{4.3}$$

Next, we turn to $u(x)$. By $\frac{1}{q+1}\in(\frac{\alpha}{n},\ \frac{\alpha+\lambda}{2n}]$ and $p\leq q$, we see that

$$\frac{1}{p+1}\in\Big[\frac{\alpha+\lambda}{2n},\quad \frac{\alpha+\lambda}{n}-\frac{\alpha}{n+\alpha-\lambda}\Big).$$

Noting that

$$\frac{\alpha}{n-\lambda+\alpha}<\frac{\alpha+\lambda}{2n}\ \text{and}\ \frac{\alpha+\lambda}{n}-\frac{\alpha}{n+\alpha-\lambda}<\frac{\alpha+\lambda}{n+\alpha},$$

we have

$$\frac{\alpha}{n-\lambda+\alpha}<\frac{1}{p+1}\ \text{and}\ \frac{1}{p+1}<\frac{\alpha+\lambda}{n+\alpha},$$

which from equations (4.3) and (1.16) implies that

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^{\lambda}}dy &= \left(\int_{|y|\leq R}+\int_{|y|\geq R}\right)\frac{v^p(y)}{|y|^{\lambda}}dy\\ &\leq C(n,\lambda,p)\left(\int_{|y|\leq R}|y|^{-\lambda-\alpha p}dy+\int_{|y|\geq R}|y|^{-\lambda-(\alpha+\lambda)p}dy\right)<\infty,\end{aligned}$$

and

$$\lim_{|x|\to 0}|x|^{\alpha}u(x)=\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^{\lambda}}dy.$$

Case 2. $\frac{1}{q+1}=\frac{\alpha}{n+\alpha-\lambda}$, i.e., $\alpha q+\lambda=n$:
In this case, it is easy to check that

$$\frac{\alpha q}{n}+\frac{\lambda+\alpha}{n}-1=\frac{\alpha}{n},$$

and together with (1.14)

$$v\in L^s(\mathbb{R}^n),\quad \frac{1}{s}\in(\frac{\alpha}{n},\ \frac{\alpha+\lambda}{n}).\tag{4.4}$$

At the same time, noting that

$$\frac{\alpha}{n+\alpha-\lambda}=\frac{1}{q+1}<\frac{1}{p+1},$$

and

$$\frac{1}{p+1}=\frac{\lambda+\alpha}{n}-\frac{\alpha}{n+\alpha-\lambda}<\frac{\lambda+\alpha}{n+\alpha},\tag{4.5}$$

we have

$$\frac{\alpha p}{n} < \frac{n-\lambda}{n} < \frac{(\alpha+\lambda)p}{n}.$$

Now, taking $\frac{1}{\tau} = \frac{n-\lambda}{n} - \varepsilon$ with $\varepsilon > 0$ small enough, we deduce that

$$\int_{|y|\le R} \frac{v^p(y)}{|y|^\lambda} dy \le \left(\int_{|y|\le R} v^{p\tau}(y)dy\right)^{\frac{1}{\tau}} \left(\int_{|y|\le R} |y|^{-\tau'\lambda} dy\right)^{\frac{1}{\tau'}}. \tag{4.6}$$

Here $\frac{1}{\tau} + \frac{1}{\tau'} = 1$. This together with (4.4), (4.5) and (1.16) implies

$$\int_{\mathbb{R}^n} \frac{v^p(y)}{|y|^\lambda} dy = \int_{|y|\le R} \frac{v^p(y)}{|y|^\lambda} dy + C\int_{|y|>R} |y|^{-\lambda-(\lambda+\alpha)p} dy < \infty.$$

With the a priori estimate of $v(x)$, we claim that

$$\lim_{|x|\to 0} |x|^\alpha u(x) = \int_{\mathbb{R}^n} \frac{v^p(y)}{|y|^\lambda} dy \triangleq D_1. \tag{4.7}$$

Indeed,

$$\int_{\mathbb{R}^n} \frac{v^p(y)}{|x-y|^\lambda} dy = \left(\int_{B_\delta(0)} + \int_{\mathbb{R}^n\setminus B_\delta(0)}\right) \frac{v^p(y)}{|x-y|^\lambda} dy = J_1(x) + J_2(x).$$

As $|x| < \delta/2$, we have

$$J_2(x) \le C(\lambda) \int_{\mathbb{R}^n} \frac{v^p(y)}{|y|^\lambda} dy < \infty,$$

and

$$J_1(x) = \int_{B_\delta(0)} \frac{v^p(y)}{|x-y|^\lambda} dy \le \left(\int_{\mathbb{R}^n} v^{p\tau}(y)dy\right)^{\frac{1}{\tau}} \left(\int_{|y|\le 2\delta} |y|^{-\tau'\lambda} dy\right)^{\frac{1}{\tau'}} < \infty.$$

where τ takes same value as (4.6). Hence, $\lim\limits_{\delta\to 0} J_1(x) = 0$ and by the Lebesgue convergence theorem, we can obtain the desired result.

Now we turn to discuss $v(x)$. At first, we write

$$\begin{aligned} \frac{|x|^\alpha}{-\ln|x|} v(x) &= \frac{1}{-\ln|x|} \int \frac{u^q(y)}{|x-y|^\lambda} dy \\ &= \frac{1}{-\ln|x|} \int_{|y|\le\delta} \frac{u^q(y)}{|x-y|^\lambda} dy + \frac{1}{-\ln|x|} \int_{|y|\ge\delta} \frac{u^q(y)}{|x-y|^\lambda} dy \\ &= \mathbb{A}_1(x) + \mathbb{A}_2(x). \end{aligned}$$

As $|x| < \delta/2$, by (3.2), we have

$$\begin{aligned} \lim_{|x|\to 0} \Big|\mathbb{A}_2(x)\Big| &= \lim_{|x|\to 0} \left|\frac{1}{-\ln|x|} \int_{|y|\ge\delta} \frac{u^q(y)}{|x-y|^\lambda} dy\right| \\ &\le \lim_{|x|\to 0} \left|\frac{C(\lambda)}{-\ln|x|} \int_{\mathbb{R}^n} \frac{u^q(y)}{|\delta|^\lambda} dy\right| = 0. \end{aligned}$$

To obtain the asymptotic behavior of $v(x)$, it suffices to build up the estimate of $\mathbb{A}_1(x)$. By (4.7), we deduce that

$$\begin{aligned}\mathbb{A}_1(x) &= \frac{1}{-\ln|x|}\int_{|y|\le\delta}\frac{u^q(y)}{|x-y|^\lambda}dy \cong \frac{1}{-\ln|x|}\int_{|y|\le\delta}\frac{|y|^{-\alpha q}}{|x-y|^\lambda}dy\\ &= \frac{1}{-\ln|x|}\int_{|z|\le\frac{\delta}{|x|}}\frac{|z|^{-\alpha q}}{|e-z|^\lambda}dz = \frac{1}{-\ln|x|}\int_0^{\frac{\delta}{|x|}} r^{n-1-\alpha q}\int_{\mathbb{S}^{n-1}}\frac{ds(\omega)}{|r\omega-e|^\lambda}dr\\ &= \frac{1}{-\ln|x|}\int_{\mathbb{S}^{n-1}}\left(\int_0^R+\int_R^{\frac{\delta}{|x|}}\right)\left[\frac{r^{\lambda-1}}{|r\omega-e|^\lambda}\right]drds(\omega) \triangleq \mathbb{A}_{1,1}(x)+\mathbb{A}_{1,2}(x),\end{aligned}$$

where we use that fact $\frac{1}{q+1}=\frac{\alpha}{n+\alpha-\lambda}$., i.e., $n-1-\alpha q=\lambda-1$.

Next we discuss $\mathbb{A}_{1,1}(x)$ and $\mathbb{A}_{1,2}(x)$, respectively. A straight computation shows that

$$\begin{aligned}\lim_{|x|\to 0}\mathbb{A}_{1,1}(x) &= \lim_{|x|\to 0}\left\{\frac{1}{-ln|x|}\int_{S^{n-1}}\left(\int_0^{\frac12}+\int_{\frac12}^2+\int_2^R\right)\left[\frac{r^{\lambda-1}}{|r\omega-e|^\lambda}\right]drds(\omega)\right\}\\ &\le \lim_{|x|\to 0}\frac{C(R,n)}{-ln|x|}=0.\end{aligned}$$

At the same time, as $r\ge R$, we have $r(1-\dfrac{1}{R})\le|e-r\omega|\le r(1+\dfrac{1}{R})$. Therefore,

$$\begin{aligned}\mathbb{A}_{1,2}(x) &= \frac{D_1^q}{-ln|x|}\int_R^{\frac{\delta}{|x|}}\int_{S^{n-1}}\left[\frac{r^{\lambda-1}}{|r\omega-e|^\lambda}\right]drds(\omega)\\ &\approx \frac{D_1^q}{-ln|x|}\int_{S^{n-1}}\int_R^{\frac{\delta}{|x|}}\frac{dr}{r}ds(\omega),\end{aligned}$$

and

$$\lim_{|x|\to 0}\frac{|x|^\alpha}{-\ln|x|}v(x)=|S_{n-1}|\left(\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^\lambda}dy\right)^q. \tag{4.8}$$

Case 3. $\frac{\alpha}{n}<\frac{1}{q+1}<\frac{\alpha}{n+\alpha-\lambda}$, i.e., $\alpha q+\lambda>n$:
Obviously,

$$\frac{\alpha q+\alpha}{n}+\frac{\lambda}{n}-1>\frac{\alpha}{n}.$$

Therefore, by Theorem 1.1

$$v\in L^s(\mathbb{R}^n),\quad \frac{1}{s}\in\left(\frac{\alpha q+\alpha}{n}+\frac{\lambda}{n}-1,\ \frac{\alpha+\lambda}{n}\right). \tag{4.9}$$

Because we have $\frac{\alpha}{n}<\frac{1}{q+1}$ and $\frac{1}{q+1}=\frac{\lambda+\alpha}{n}-\frac{1}{p+1}$, we get

$$\frac{1}{p+1}<\frac{\lambda}{n}<\frac{\alpha+\lambda}{n+\alpha}. \tag{4.10}$$

Invoking equation (1.16), we get

$$\int_{|y|\ge R}\frac{v^p(y)}{|y|^\lambda}dy\le C(n)\int_{|y|>R}|y|^{-\lambda-(\alpha+\lambda)p}dy<\infty. \tag{4.11}$$

Next, we turn to $\int_{|y|\leq R}\frac{v^p(y)}{|y|^\lambda}dy$. We claim that

$$p\Big(\frac{\alpha q+\alpha}{n}+\frac{\lambda}{n}-1\Big)<\frac{n-\lambda}{n}<\frac{(\alpha+\lambda)p}{n}. \tag{4.12}$$

Obviously, the second inequality of (4.12) directly follows from (4.10). Now, we consider the first part of the inequality. Noting that $\lambda+\alpha<n$ and $\frac{1}{q+1}>\frac{\alpha}{n}$, it is easy to check that

$$\frac{\alpha(q+1)}{n}[n-\lambda-\alpha]<n-\lambda-\alpha \quad i.e \quad \alpha(q+1)-(n-\lambda)<\frac{\alpha(\lambda+\alpha)(q+1)}{n}-\alpha.$$

This together with equation (1.13), yields

$$\begin{aligned}\frac{\alpha(\lambda+\alpha)(q+1)}{n}-\alpha&=\alpha\Big[\frac{(\lambda+\alpha)(q+1)}{n}-1\Big]\\&=\frac{(q+1)\alpha}{p+1}>\alpha(q+1)-(n-\lambda),\end{aligned}$$

which implies that

$$p\Big(\frac{\alpha q+\alpha}{n}+\frac{\lambda}{n}-1\Big)<\frac{n-\lambda}{n}.$$

Therefore this completes the proof of the claim.

Now, taking $\frac{1}{s}=\frac{n-\lambda}{n}-\varepsilon$ with $\varepsilon>0$ small enough, by Hölder's inequality, we conclude that

$$\int_{|y|\leq R}\frac{v^p(y)}{|y|^\lambda}dy\leq\Big(\int_{|y|\leq R}|y|^{-\lambda s'}dy\Big)^{1/s'}\Big(\int_{|y|\leq R}v^{ps}(y)dy\Big)^{1/s}<\infty,$$

where $\frac{1}{s}+\frac{1}{s'}=1$. Therefore,

$$\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^\lambda}dy<\infty,$$

and

$$\lim_{|x|\to0}|x|^\alpha u(x)=\int_{\mathbb{R}^n}\frac{v^p(y)}{|y|^\lambda}dy\triangleq D_1<\infty. \tag{4.13}$$

Next, we discuss $v(x)$. We write $\int_{\mathbb{R}^n}\frac{u^q(y)}{|x-y|^\lambda}dy$ as follow

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{u^q(y)}{|x-y|^\lambda}dy&=\int_{B_\delta(0)}\frac{u^q(y)}{|x-y|^\lambda}dy+\int_{\mathbb{R}^n\setminus B_\delta(0)}\frac{u^q(y)}{|x-y|^\lambda}dy\\&\triangleq\mathbb{D}_1(x)+\mathbb{D}_2(x).\end{aligned}$$

Observe that as $\frac{1}{q+1}\in(\frac{\alpha}{n},\ \frac{\alpha}{n+\alpha-\lambda})$ and $\lambda\in(0,n)$, we have

$$\int_{\mathbb{R}^n}\frac{1}{|y|^{\alpha q}\ |e-y|^\lambda}dy<\infty,$$

and with equation (4.13)

$$\lim_{|x|\to0}|x|^{\alpha q+\lambda-n}\mathbb{D}_1(x)=D_1^q\lim_{|x|\to0}\int_{|z|\leq\frac{\delta}{|x|}}\frac{1}{|z|^{\alpha q}\ |e-z|^\lambda}dy=D_1^q\int_{\mathbb{R}^n}\frac{1}{|y|^{\alpha q}\ |e-y|^\lambda}dy.$$

Similarly, as $|x| \leq \frac{\delta}{2}$, by estimate (3.2), we conclude that

$$\lim_{|x|\to 0} |x|^{\alpha q+\lambda-n}\mathbb{D}_2(x) = \lim_{|x|\to 0} |x|^{\alpha q+\lambda-n} \int_{\mathbb{R}^n \setminus B_\delta(0)} \frac{u^q(y)}{|x-y|^\lambda} dy$$
$$\leq 2^\lambda \lim_{|x|\to 0} |x|^{\alpha q+\lambda-n} \int_{\mathbb{R}^n \setminus B_\delta(0)} \frac{u^q(y)}{|y|^\lambda} dy = 0.$$

Hence

$$\lim_{|x|\to 0} |x|^{\alpha(q+1)+\lambda-n} v(x) = \left(\int_{\mathbb{R}^n} \frac{v^p(y)}{|y|^\lambda} dy \right)^q \int_{\mathbb{R}^n} \frac{1}{|y|^{\alpha q}\ |e-y|^\lambda} dy.$$

This completes the proof of Theorem 1.3. ∎

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