METASTABLE DYNAMICS FOR HYPERBOLIC VARIATIONS OF THE ALLEN–CAHN EQUATION*

RAFFAELE FOLINO†, CORRADO LATTANZIO‡, AND CORRADO MASCIA§

Abstract. Metastable dynamics of a hyperbolic variation of the Allen–Cahn equation with homogeneous Neumann boundary conditions are considered. Using the "dynamical approach" proposed by Carr–Pego [J. Carr and R.L. Pego, Comm. Pure Appl. Math., 42:523–576, 1989] and Fusco–Hale [G. Fusco and J. Hale, J. Dynamics Diff. Eqs., 1:75–94, 1989] to study slow-evolution of solutions in the classic parabolic case, we prove existence and persistence of metastable patterns for an exponentially long time. In particular, we show the existence of an "approximately invariant" N-dimensional manifold $\mathcal{M}_0$ for the hyperbolic Allen–Cahn equation: if the initial datum is in a tubular neighborhood of $\mathcal{M}_0$, the solution remains in such neighborhood for an exponentially long time. Moreover, the solution has N transition layers and the transition points move with exponentially small velocity. In addition, we determine the explicit form of a system of ordinary differential equations describing the motion of the transition layers and we analyze the differences with the corresponding motion valid for the parabolic case.

Keywords. Allen–Cahn equation; metastability; singular perturbations.

AMS subject classifications. 35L72; 35B25; 35K57.

1. Introduction

1.1. Metastability in reaction-diffusion equations. Reaction-diffusion equations are widely used to describe a variety of phenomena such as pattern formation and front propagation in biological, chemical and physical systems. When the model under study is characterized by the presence of competing equilibrium states, a crucial question is to describe the interaction and the dynamics of space occupation by the equilibria. A basic prototype is the *Allen–Cahn equation*, which has the form

$$u_t + \mathcal{L}(u) = 0 \qquad \text{where} \quad \mathcal{L}(u) := -\varepsilon^2 \Delta u + f(u), \tag{1.1}$$

and it has been originally proposed in [2] to describe the motion of antiphase boundaries in iron alloys. Such an equation has an associated energy functional

$$\int \left\{ \tfrac{1}{2} \varepsilon^2 |\nabla u|^2 + F(u) \right\} dx,$$

where the *potential F* is a primitive of f and integration in space is performed in the domain under consideration. For the Allen–Cahn model, the function F is assumed to be a double-well potential with wells of equal depth. As a consequence, the reaction function f has a cubic-type behavior with two stable and one unstable equilibria, usually normalized as ± 1 and 0, respectively.

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In the absence of diffusion, viz. $\varepsilon = 0$, the space variable x becomes an external parameter and solutions generically converge pointwise to functions with values in $\{-1, +1\}$ with sharp transition layers generated at points where the initial datum changes sign. For small $\varepsilon > 0$, if the initial datum is a small perturbation of a function with values in $\{-1, +1\}$ with well-separated transition regions, diffusion determines in a short time-scale a smoothed version of the original configuration and, on a longer time-scale, layers interact giving rise to front motion. When the space variable is one-dimensional, starting from [8, 10, 20], it has been shown that, as long as layers are well-separated the interaction force is very weak and the consequent motion is very slow. The meaning of weak/slow can be quantified more precisely, as discussed in what follows. Postponing such details, in the regime $\varepsilon \to 0^+$, the original configuration is preserved for a long time and thus such behavior has been classified as *metastability*.

Many papers have been devoted to slow motion analysis for the Allen–Cahn equation providing precise description of the relation between the size of the diffusivity ε and the time-scale of the dynamics. A complete list of references would be prohibitive. Here, we only quote the analysis on generation, persistence and annihilation of metastable patterns performed in [14]. A large class of different evolution PDEs, concerning many different areas, exhibits the phenomenon of metastability. Without claiming to be complete, we list some of the principal models that have been analyzed: scalar conservation laws [19, 31, 32, 34, 38], the Cahn–Hilliard equation [1, 3, 4, 7, 36], Gierer–Meinhardt and Gray–Scott systems [40], Keller–Segel chemotaxis models [16, 37], general gradient flows [35], high-order systems [30], gradient systems with equal depth multiple-well potentials [5, 6], Cahn–Morral systems [23], the Jin–Xin system [39].

The aforementioned bibliography is confined to one-dimensional models; however, there is a vast literature of works about motion of interfaces in several space dimensions, where the effect of the curvature of the interfaces turns out to be relevant for the dynamics. In particular, for the Allen–Cahn equation, we recall the works [9, 13, 15], where it has been shown that steep interfaces are generated in a short time with subsequent motion governed by mean curvature flow.

The present paper is devoted to the analysis of metastability in a hyperbolic framework. Precisely, given $\varepsilon > 0$, $\tau_0 \in (0, +\infty]$, $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times (0, \tau_0) \to \mathbb{R}$, we consider here the *hyperbolic Allen–Cahn equation*

$$\tau u_{tt} + g(u, \tau) u_t = \varepsilon^2 \Delta u - f(u). \tag{1.2}$$

The function f is required to be the derivative of a double well potential F with non-degenerate minima of same depth, and the function g is assumed to be strictly positive, uniformly with respect to u; namely, we assume

$$F(\pm 1) = F'(\pm 1) = 0, \quad F''(\pm 1) > 0, \quad F(u) > 0 \text{ for } u \neq \pm 1, \tag{1.3}$$

$$g(u, \tau) \geq c_g > 0 \qquad \forall u, \tag{1.4}$$

where the constant c_g may depend on τ. The uniform positivity of g in (1.4) is crucial, because it guarantees the dissipative nature of the model. If, in addition, $g(u, \tau) \to 1$ as $\tau \to 0$, we formally recover equation (1.1) from equation (1.2) in the (singular) limit.

In the case $g \equiv 1$, equation (1.2) can be obtained by adding a nonlinear zero order perturbation to the damped wave operator $\tau \partial_t^2 + \partial_t - \varepsilon^2 \Delta$. For such a choice, many studies have been devoted to the stability of fronts —mainly in one space dimension— for bistable or monostable reaction term f (see [21] and references therein). Interface formation has been analyzed in [25] in the singular limit $\varepsilon \to 0$ for space dimension equal

to 2 or 3, showing that motion is governed by mean curvature flow, as is the case for the corresponding parabolic model.

The choice of the hyperbolic variation (1.2) is motivated by the observation that there are different ways for modeling transport mechanisms. The one at the base of equation (1.1) is the classical Fourier law, originally proposed for heat conduction and then extended to many other different fields, which prescribes the instantaneous proportionality between the flux v of a quantity with "density" u and its gradient, $v = -\varepsilon^2 \nabla u$. Such choice has the advantage of providing a simple equation enjoying a number of useful properties (smoothing effects, self-similarity, . . .), but, at the same time, it has a number of drawbacks, the best known being the presence of infinite speed of propagation. Still in the framework of heat conduction modeling, following some ideas developed by Maxwell in the context of kinetic theories, Cattaneo proposed in [12] a different law for the heat flux v, based on the assumption that the equilibrium between flux and gradient of the unknown is asymptotical with a time-scale measured by the relaxation parameter $\tau > 0$, that is

$$\tau v_t + v = -\varepsilon^2 \nabla u \qquad \text{(Maxwell–Cattaneo law)} \qquad (1.5)$$

(an extensive discussion is reported in [27,28]). In the one-dimensional case, the diffusion equation given by the law (1.5) has also a probabilistic interpretation, appearing in the description of *correlated random walks* (see [22,29,41]) to be compared with the standard *random walk*, which gives raise to the standard parabolic diffusion equation.

Criticisms to the application of the use of Fourier-type law have been given also in modeling reaction-diffusion phenomena (see [24,26]). In the presence of a reaction term described by the function f, application of the Maxwell–Cattaneo law gives

$$\tau u_{tt} + \left\{ u + \tau f(u) \right\}_t = \varepsilon^2 \Delta u - f(u),$$

to be considered as a modification of the standard Allen–Cahn equation when f satisfies conditions (1.3) (see also [17], for different origins of the same equation). This equation fits into equation (1.2) with the choice $g(u,\tau) := 1 + \tau f'(u)$. We refer to this specific model as the *Allen–Cahn equation with relaxation*, reminiscent of the relaxation-type law (1.5). Existence and nonlinear stability of traveling wave solutions for this equation has been analyzed in detail in [33] for general bistable reaction terms in one space dimension.

1.2. Presentation of the main result. This study is devoted to the one-dimensional case, so that the hyperbolic Allen–Cahn equation reads as

$$\tau u_{tt} + g(u,\tau) u_t + \mathcal{L}(u) = 0, \qquad (1.6)$$

corresponding to the parabolic Allen–Cahn equation

$$u_t + \mathcal{L}(u) = 0, \qquad (1.7)$$

where $\mathcal{L}(u) := -\varepsilon^2 u_{xx} + f(u)$. Specifically, we are interested in the limiting behavior of the solutions as $\varepsilon \to 0$ with the aim of extending the metastable dynamics for equation (1.7) to the hyperbolic case (1.6) and focusing the attention on eventual differences.

In [18], adapting the *energy approach* proposed by Bronsard and Kohn [8] for the parabolic equation (1.7), the first author has shown that, if the initial profile u_0 has a transition layer structure and the initial velocity v_0 is small, then the solution maintains the transition layer structure on a time scale of order ε^{-k} with k arbitrary. The energy

approach has also been applied to Cahn–Hilliard equation in [7]. Grant [23] improved this method to prove exponentially slow motion for Cahn–Morral systems.

A different procedure, proposed by Carr–Pego in [10] and Fusco–Hale in [20], permits to prove existence and persistence of metastable states for the Allen–Cahn Equation (1.7) for a time proportional to $e^{C/\varepsilon}$. This strategy provides also an explicit differential equation for the dynamics of the transition layer positions (far from collapses). The method is based on the construction of an N-dimensional base manifold $\mathcal{M}$ consisting of functions which approximate metastable states with N transition layers. The manifold is not invariant, but if the initial datum is in a small neighborhood of $\mathcal{M}$, then the solution remains near the manifold for a time proportional to $e^{C/\varepsilon}$. Based on these ideas, slow motion results have been proved for the Cahn–Hilliard equation by Alikakos et al. [1] and by Bates and Xun [3,4]. In particular, the last ones use the same manifold constructed in [10].

Here, we adapt the method of [10] to the hyperbolic Allen–Cahn equation (1.6) embedding the base manifold $\mathcal{M}$ in an extended phase space determined by the presence of the additional unknown $v = u_t$ as suggested by the first-order form of equation (1.6) given by

$$\begin{cases} u_t = v \\ \tau v_t = -\mathcal{L}(u) - g(u,\tau)v. \end{cases} \tag{1.8}$$

System (1.8), considered here for $t > 0$ and $x \in (0,1)$, is complemented with homogeneous Neumann boundary conditions

$$u_x(0,t) = u_x(1,t) = 0, \qquad t > 0, \tag{1.9}$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in (0,1). \tag{1.10}$$

The initial-boundary value problem (1.8)-(1.9)-(1.10) is globally well-posed for positive times in $H^1 \times L^2$. In particular, if

$$(u_0,v_0) \in \mathcal{D} = \left\{ (u,v) \in H^2 \times H^1 : u_x(0) = u_x(1) = 0 \right\},$$

the solution (u,v) is classical and belongs to $C\left([0,\infty),\mathcal{D}\right) \cap C^1\left([0,\infty),H^1 \times L^2\right)$ (among others, see [18, Appendix A]). Then, our aim is to describe the dynamics of such a globally defined solution, at least for a class of "well-prepared" initial data.

Under assumptions (1.3)–(1.4), the hyperbolic equation (1.6) supports traveling wave solutions connecting the equilibria -1 and 1, i.e. solutions of the form $u(x,t) = \Phi(x - ct)$ such that $\Phi(\pm\infty) = \pm 1$, if and only if $c = 0$. Indeed, substituting the traveling wave ansatz in the equation, we obtain

$$(\varepsilon^2 - c^2\tau)\Phi'' + cg(\Phi,\tau)\Phi' - f(\Phi) = 0,$$

and thus, multiplying by Φ' and integrating over $\mathbb{R}$, we get

$$c \int_{\mathbb{R}} g(\Phi,\tau)(\Phi')^2 \, d\xi = F(+1) - F(-1),$$

from which we deduce, under assumptions (1.3) and (1.4), that the velocity c is zero. With such choice, it is well-known that, up to translation, there is a unique solution to the problem

$$\varepsilon^2 \Phi'' - f(\Phi) = 0, \qquad \Phi(x) \to \pm 1 \quad \text{as} \quad x \to \pm\infty. \tag{1.11}$$

Normalizing Φ by adding the condition $\Phi(0) = 0$, transitions layer from -1 to $+1$ (or viceversa) are described by $\Phi(\pm(x-\bar{x}))$ for both equations (1.7) and (1.6).

Steady states Φ are at the base of the construction of the base manifold $\mathcal{M}$ which we sketch here (for precise definitions, see Section 2). Fix $N \in \mathbb{N}$ and $\varepsilon > 0$. Given a configuration $\boldsymbol{h} = (h_1, \ldots, h_N)$ of N layer positions (with $h_j < h_{j+1}$), we construct a function $u^{\boldsymbol{h}}$ which approximates a metastable state with transition points at $h_1, \ldots, h_N$, by piecing together approximated versions of Φ; that is, $u^{\boldsymbol{h}}(x) \approx \Phi(x - h_j)$ or $\Phi(h_j - x)$ for $x \approx h_j$ (see Figure 1.1).
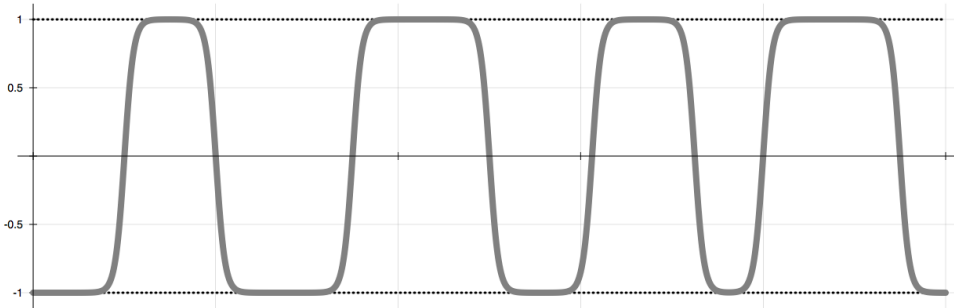

FIG. 1.1. *Example of a function $u^{\boldsymbol{h}}(x)$ with $N = 8$.*

Then, we consider the slow evolution of solutions when the transition points are well separated one from the other and bounded away from the boundary points 0 and 1. For fixed (small) $\rho > 0$, the admissible layer positions lie in the set

$$\Omega_\rho := \left\{ \boldsymbol{h} \in \mathbb{R}^N : 0 < h_1 < \cdots < h_N < 1, \, h_{j+1} - h_j > \varepsilon/\rho \text{ for } j = 0, \ldots, N \right\},$$

where $h_0 := -h_1$, $h_{N+1} := 2 - h_N$ and the *base manifold* is $\mathcal{M} := \{ u^{\boldsymbol{h}} : \boldsymbol{h} \in \Omega_\rho \}$.

In what follows, we fix a minimal distance $\delta > 0$ with $\delta < 1/N$ and we consider the parameters ε and ρ such that

$$0 < \varepsilon < \varepsilon_0 \qquad \text{and} \qquad \delta < \frac{\varepsilon}{\rho} < \frac{1}{N}, \tag{1.12}$$

for some $\varepsilon_0 > 0$ to be chosen appropriately small. In such a way, the parameters ρ and ε have the same order of magnitude. All of the subsequent estimates depend on N and δ.

Denoted by $\langle \cdot, \cdot \rangle$ the inner product in $L^2(0,1)$, to restrict the attention to a neighborhood of $\mathcal{M}$, we introduce the decomposition $u = u^{\boldsymbol{h}} + w$, where w are such that the following *orthogonality condition* holds

$$\langle w, k_j^{\boldsymbol{h}} \rangle = 0, \qquad \text{for} \quad j = 1, \ldots, N, \tag{1.13}$$

for some appropriate approximate tangent vectors $k_j^{\boldsymbol{h}}$. Then, setting

$$H_N^2 := \left\{ w \in H^2(0,1) : w_x(0) = w_x(1) = 0, \langle w, k_j^{\boldsymbol{h}} \rangle = 0 \quad \text{for } j = 1, \ldots, N \right\},$$

we consider triples $(\boldsymbol{h}, w, v)$ in the set $\Omega_\rho \times H_N^2 \times L^2(0,1)$ and the corresponding *extended base manifold*

$$\mathcal{M}_0 := \mathcal{M} \times \{0\} = \left\{ (u^{\boldsymbol{h}}, 0) : u^{\boldsymbol{h}} \in \mathcal{M} \right\}.$$

Next, we choose a tubular neighborhood of $\mathcal{M}_0$: given $\Gamma, \rho > 0$, we set

$$\mathcal{Z}_{\Gamma,\rho} := \big\{ (u,v) : u = u^{\boldsymbol{h}} + w, (\boldsymbol{h}, w, v) \in \overline{\Omega}_\rho \times H_N^2 \times L^2(0,1), \mathcal{E}^{\boldsymbol{h}}[w,v] \leq \Gamma \Psi(\boldsymbol{h}) \big\},$$

with the *energy functional* $\mathcal{E}^{\boldsymbol{h}}$ and the *barrier function* Ψ defined by

$$\mathcal{E}^{\boldsymbol{h}}[w,v] := \tfrac{1}{2} \int_0^1 \big\{ \varepsilon^2 w_x^2 + f'(u^{\boldsymbol{h}}) w^2 \big\} dx + \tfrac{1}{2} \tau \|v\|^2 + \varepsilon \tau \langle w, v \rangle, \tag{1.14}$$

$$\Psi(\boldsymbol{h}) := \sum_{j=1}^N \big\langle \mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}} \big\rangle^2, \tag{1.15}$$

where $\|\cdot\|$ is the L^2-norm. Our main result states that the channel $\mathcal{Z}_{\Gamma,\rho}$ is invariant for an exponentially long time if the parameters Γ and ρ are appropriately chosen. In other words, the manifold $\mathcal{M}_0$ is approximately invariant for the hyperbolic system (1.8).

THEOREM 1.1. *Let $f \in C^2$ and $g(\cdot, \tau) \in C^1$ with $\tau \in (0, \tau_0)$ be such that $f = F'$ and properties (1.3)-(1.4) hold. Given $N \in \mathbb{N}$ and $\delta \in (0, 1/N)$, there exist $\Gamma_2 > \Gamma_1 > 0$ and $\varepsilon_0 > 0$ (possibly depending on τ) such that, if ε, ρ satisfy conditions (1.12), $\Gamma \in [\Gamma_1, \Gamma_2]$ and the initial datum satisfies*

$$(u_0, v_0) \in \overset{\circ}{\mathcal{Z}}_{\Gamma,\rho} = \big\{ (u,v) \in \mathcal{Z}_{\Gamma,\rho} : \boldsymbol{h} \in \Omega_\rho \text{ and } \mathcal{E}^{\boldsymbol{h}}[w,v] < \Gamma \Psi(\boldsymbol{h}) \big\},$$

then the solution (u,v) to the initial-boundary value problem (1.8)-(1.9)-(1.10) remains in $\mathcal{Z}_{\Gamma,\rho}$ for a time $T_\varepsilon > 0$, and there exists $C > 0$ (possibly depending on τ) such that for any $t \in [0, T_\varepsilon]$

$$\varepsilon^{1/2} \|w\|_{L^\infty} + \|w\| + \tau^{1/2} \|v\| \leq C \exp(-A \ell^{\boldsymbol{h}}/\varepsilon), \tag{1.16}$$

$$|\boldsymbol{h}'|_\infty \leq C (\varepsilon/\tau)^{1/2} \exp(-A \ell^{\boldsymbol{h}}/\varepsilon), \tag{1.17}$$

where $A := \sqrt{\min\{f'(-1), f'(1)\}}$, $\ell^{\boldsymbol{h}} := \min\{h_j - h_{j-1}\}$ and $|\cdot|_\infty$ denotes the maximum norm in $\mathbb{R}^N$. Moreover,

$$T_\varepsilon \geq C (\tau/\varepsilon)^{1/2} (\ell^{\boldsymbol{h}(0)} - \varepsilon/\rho) \exp(A\delta/\varepsilon).$$

REMARK 1.1. It is worth to observe that in the above theorem, and in general in the whole paper, τ should be viewed as a *fixed* parameter in $(0, \tau_0)$, and, as clearly stated, the constants may depend on it. However, we prefer to make the ratio ε/τ appear in the estimates above because the constants may be chosen uniform with respect to τ in many cases, as for the relaxation limit $\tau \to 0$ from the hyperbolic equation (1.6) to the parabolic Allen–Cahn equation (1.7), namely for $g(u, \tau) \to 1$ as $\tau \to 0$; the main examples in this framework we have already introduced above are $g \equiv 1$ and $g(u, \tau) = 1 + \tau f'(u)$. More precisely, if $g(u, \tau) \to 1$ as $\tau \to 0$ in any reasonable way and u is bounded, then (1.4) implies $0 < c_g \leq g(u, \tau) \leq C_g$ with c_g and C_g independent from τ in a (right) neighborhood of zero. With this extra (uniform in τ) control at our disposal, one can follow the proofs needed to obtain our main result and see that the only dependence in τ in the bounds for $\boldsymbol{h}'$ and T_ε is through the aforementioned ratio ε/τ, which can be used to study the interplay between the two small parameters ε and τ while performing this relaxation limit.

The strategy to prove Theorem 1.1 is the following. Firstly, plugging the decomposition $u = u^{\boldsymbol{h}} + w$ into system (1.8) and using conditions (1.13), we obtain an ODE-PDE

coupled system describing the dynamics for $(\boldsymbol{h}, w, v)$, see system (3.6). Then, we show that, if the solution (u, v) belongs to $\mathcal{Z}_{\Gamma, \rho}$, the estimates (1.16) and (1.17) hold. Next, we estimate the time T_ε taken for the solution (u, v) to leave the channel $\mathcal{Z}_{\Gamma, \rho}$. The boundary of $\mathcal{Z}_{\Gamma, \rho}$ is the union of two parts: the "ends" where $\boldsymbol{h} \in \partial \Omega_\rho$, meaning $h_j - h_{j-1} = \varepsilon/\rho$ for some j and "sides" where $\mathcal{E}^{\boldsymbol{h}}[w, v] = \Gamma \Psi(\boldsymbol{h})$. Using an energy estimate, we infer that the solution can leave $\mathcal{Z}_{\Gamma, \rho}$ only through the ends. Since, for estimate (1.17), the transition points move with exponentially small velocity, the solution (u, v) stays in the channel for an exponentially long time.

As long as the solution (u, v) remains in the channel $\mathcal{Z}_{\Gamma, \rho}$, u is a function with N transition layers. The estimate (1.17) ensures the slow motion of solutions and gives a lower bound on the lifetime of the metastable states. In order to give further information on the motion of the transition layers and an upper bound on such lifetime, we study in detail an approximation of the equation for $\boldsymbol{h}$, determined formally by the requirement that $u(x, t) = u^{\boldsymbol{h}(t)}(x)$ is an exact solution. Such a requirement is expected to be appropriate in the limit $\varepsilon \to 0$. In this way, we obtain a system of ordinary differential equations for $\boldsymbol{h}$ which does not depend on w and v and has the form

$$\tau \boldsymbol{h}'' + \gamma_\tau \boldsymbol{h}' = \mathcal{P}^*(\boldsymbol{h}), \tag{1.18}$$

where $\gamma_\tau := \overline{g}(\cdot, \tau)$ and the (weighted) average $\overline{g}$ of the continuous function g is given by

$$\overline{g} := \frac{1}{\|\sqrt{F}\|_{L^1}} \int_{-1}^{1} \sqrt{F(s)}\, g(s)\, ds,$$

and $\mathcal{P}^*$ is a function, depending on F. Equation (1.18) has to be compared with the corresponding one for the parabolic case (1.7), which is $\boldsymbol{h}' = \mathcal{P}^*(\boldsymbol{h})$. For the nonlinear damped wave equation $g \equiv 1$, we have $\gamma_\tau = 1$, while for the Allen–Cahn equation with relaxation we obtain $\gamma_\tau = 1 + \tau \overline{f'}$. Since $\overline{f'}$ is negative, the (physical relevant) relaxation case exhibits smaller friction effects with respect to the damped one (details in Section 4).

System (1.18) has a unique equilibrium point $(\boldsymbol{h}^e, 0)$ where $\boldsymbol{h}^e$ is the unique zero of $\mathcal{P}^*$, that corresponds to the unique stationary solution u^e of equation (1.6) with N transition layers, normalized by the condition $u(0) < 0$, without loss of generality. In the parabolic case, $\boldsymbol{h}^e$ is an unstable equilibrium point with N positive eigenvalues; whereas, for the hyperbolic model, $(\boldsymbol{h}^e, 0)$ is an unstable equilibrium point for system (1.18) with N positive eigenvalues and N negative eigenvalues.

The rest of the paper is organized as follows. In Section 2 we give all the definitions, preliminaries and the construction of the manifold $\mathcal{M}$. Furthermore, we recall all the results of Carr and Pego [10] needed to prove Theorem 1.1. Section 3 is devoted to the derivation of the equation of motion for the triple $(\boldsymbol{h}, w, v)$ and to the proof of Theorem 1.1. In Section 4, we deduce the approximating equation for $\boldsymbol{h}$, we prove that there is a unique equilibrium point $(\boldsymbol{h}^e, 0)$ and we study its stability. Finally, using singular perturbation theory, we show that, for τ small, if g is uniformly bounded and $g(u, \tau) \to 1$ a.e. as $\tau \to 0$, the behavior of the solution to system (1.18) is the same of the parabolic case (see Theorem 4.1).

2. Preliminaries

Following [10], we construct the base manifold and collect estimates that we will use in the proof of our results. For fixed $\rho > 0$, we recall the definition

$$\Omega_\rho := \big\{ \boldsymbol{h} \in \mathbb{R}^N : 0 < h_1 < \cdots < h_N < 1, \quad h_j - h_{j-1} > \varepsilon/\rho \text{ for } j = 1, \ldots, N+1 \big\},$$

where $h_0 := -h_1$ and $h_{N+1} := 2 - h_N$. By construction, if $\rho_1 < \rho_2$, then $\Omega_{\rho_1} \subset \Omega_{\rho_2}$.

The idea is to associate to any $\boldsymbol{h} \in \Omega_\rho$ a function $u^{\boldsymbol{h}} = u^{\boldsymbol{h}}(x)$ which approximates a metastable state with N transition points at $h_1, \ldots, h_N$ by matching appropriate steady states of equation (1.7). The collection of $u^{\boldsymbol{h}}$ determines a N-dimensional manifold. In order to describe the dynamics in a neighborhood of such manifold, the framework has to be complemented with a projection which permits to separate the solution into a component on the manifold and a corresponding remainder. For the Allen–Cahn equation (1.7), two different constructions have been proposed in [10] and [20].

In [20], Fusco and Hale use functions $\Phi(\pm(x - \bar{x}))$ with Φ the solution of problem (1.11) previously defined, and set

$$U^{\boldsymbol{h}}(x) := \Phi\big((x - h_j)(-1)^{j+1}\big), \quad x \in [h_{j-1/2}, h_{j+1/2}], \quad j = 1, \ldots, N,$$

where

$$h_{j+1/2} := \tfrac{1}{2}(h_j + h_{j+1}) \qquad j = 0, \ldots, N,$$

(note that $h_{1/2} = 0$, $h_{N+1/2} = 1$). Hence, they obtain a manifold $\mathcal{M}^{\mathrm{FH}}$ composed by continuous functions $U^{\boldsymbol{h}}$ with a piecewise continuous first order derivative that jumps at $h_{j+1/2}$, $j = 1, \ldots, N-1$. In particular, the elements of the manifold belong to H^1 and not to H^2 (if $N > 1$). In addition, they construct a tubular neighborhood of $\mathcal{M}^{\mathrm{FH}}$ with coordinates $(\boldsymbol{h}, V)$ by setting

$$u = U^{\boldsymbol{h}} + V \qquad \text{with} \quad \langle V, U_j^{\boldsymbol{h}} \rangle = 0 \quad j = 1, \ldots, N,$$

where $U_j^{\boldsymbol{h}}$ are the derivatives of $U^{\boldsymbol{h}}$ with respect to h_j. By construction, $U_j^{\boldsymbol{h}}$ have disjoint supports and $U_j^{\boldsymbol{h}}(x) = -U_x^{\boldsymbol{h}}(x)$ for all $x \in (h_{j-1/2}, h_{j+1/2})$. In [20], it is also conjectured that equation (1.7) has an invariant manifold $\mathcal{M}_*^{\mathrm{FH}}$ near $\mathcal{M}^{\mathrm{FH}}$ and that this manifold $\mathcal{M}_*^{\mathrm{FH}}$ is a graph over $\mathcal{M}^{\mathrm{FH}}$. Fusco and Hale did not prove the existence of the invariant manifold, but assuming existence, they calculated a first approximation for $\mathcal{M}_*^{\mathrm{FH}}$ and for the differential equation for $\boldsymbol{h}$ describing the reduced flow. They also conjectured that metastable states with N transitions are associated with the unstable manifold of stationary solutions of equation (1.7) having N layers.

Both conjectures have been proved in [11] using a different base manifold, previously constructed in [10]. The approach used by Carr and Pego is based on a different choice and matching of steady states, which provides functions $u^{\boldsymbol{h}}$, composing the base manifold $\mathcal{M}^{\mathrm{CP}}$, which are smooth in both x and $\boldsymbol{h}$. The crucial difference with respect to the Fusco–Hale approach, resides in the fact that, for $\mathcal{L}(u) := -\varepsilon^2 u_{xx} + f(u)$,

$$\mathcal{L}(U^{\boldsymbol{h}}) = 0 \qquad \text{and} \quad \mathcal{L}(u^{\boldsymbol{h}}) \neq 0 \qquad \text{for} \quad x \approx h_j,$$
$$\mathcal{L}(U^{\boldsymbol{h}}) \neq 0 \qquad \text{and} \quad \mathcal{L}(u^{\boldsymbol{h}}) = 0 \qquad \text{for} \quad x \approx h_{j+1/2},$$

with major consequences on the location of $\mathcal{L}(u^{\boldsymbol{h}})$ with respect to the tangent space to $\mathcal{M}^{\mathrm{CP}}$ at $u^{\boldsymbol{h}}$, as will be clear in the following presentation.

In this paper, we follow the framework established by Carr and Pego adapting it to the case of the hyperbolic Allen–Cahn Equation (1.6). Since the equation we consider corresponds to the system (1.8), the dynamics is determined by an additional unknown, the time derivative $v = u_t$, and thus the base manifold $\mathcal{M}^{\mathrm{CP}}$ has to be embedded in a extended vector space. Here, taking advantage of the fact that we are looking for a manifold that is only approximately invariant, we perform this extension in a trivial way, considering the *extended base manifold* $\mathcal{M}_0^{\mathrm{CP}} := \mathcal{M}^{\mathrm{CP}} \times \{0\}$.

From now on, we drop the letters CP in the symbol used for the manifolds.

2.1. Carr–Pego base manifold.   Given $L > 0$, let $\varphi(\cdot, L, +1)$ be the solution to

$$-\varphi_{xx} + f(\varphi) = 0, \qquad \varphi\left(-\tfrac{1}{2}L\right) = \varphi\left(\tfrac{1}{2}L\right) = 0, \qquad (2.1)$$

with $\varphi > 0$ in $\left(-\tfrac{1}{2}L, \tfrac{1}{2}L\right)$, and let $\varphi(\cdot, L, -1)$ be the solution to equation (2.1) with $\varphi < 0$ in $\left(-\tfrac{1}{2}L, \tfrac{1}{2}L\right)$. Observe that if φ satisfies equation (2.1), then

$$\varphi_x^2 = 2\{F(\varphi) - F(\varphi(0))\}. \qquad (2.2)$$

Using this formula, we can prove existence and uniqueness of the solutions $\varphi(\cdot, L, \pm 1)$.

LEMMA 2.1.   *Let $f = F'$, with F a smooth function satisfying conditions (1.3). There exists $L_0 > 0$ such that, if $L > L_0$, then the functions $\varphi(\cdot, L, \pm 1)$ are well-defined and, denoting by*

$$M_\pm(L) := \max_x |\varphi(x, L, \pm 1)| = |\varphi(0, L, \pm 1)|,$$

we have that $M_\pm$ is an increasing function of L and $M_\pm(+\infty) = 1$.

This lemma is consequence of the fact that ± 1 are absolute minima of F and so, there exist periodic solutions of equation (2.2) oscillating around 0. Indeed, the existence of such solutions is guaranteed if there exist $M_\pm \in (0,1)$ such that $F(M_+) = F(-M_-)$, $F'(M_\pm) \neq 0$ and $F(s) > F(M_+)$ for all $s \in (-M_-, M_+)$. This condition is certainly satisfied if $M_\pm$ are close to $+1$. Let us consider the positive case $\varphi(\cdot, L, +1)$ and $M_+(L) = \varphi(0, L, +1)$. By integrating equation (2.2) in $\left(-\tfrac{1}{2}L, 0\right)$ and using the boundary conditions in equation (2.1), we obtain

$$L = \sqrt{2} \int_0^{M_+} \frac{ds}{\sqrt{F(s) - F(M_+)}}. \qquad (2.3)$$

The integral in equation (2.3) tends to infinity as $M_+ \to 1^-$ and it is an increasing function of M_+ for M_+ close to $+1$. Hence, for L sufficiently large, there exists a unique M_+ such that equation (2.3) is satisfied and so the function $\varphi(\cdot, L, +1)$ is well-defined. The negative case $\varphi(\cdot, L, -1)$ and $M_-(L) = -\varphi(0, L, -1)$ is similar.

Now, given $\ell > 0$, let us define $\phi(x, \ell, \pm 1) := \varphi\left(\tfrac{x}{\varepsilon}, \tfrac{\ell}{\varepsilon}, \pm 1\right)$. By definition, it follows that $\phi(\cdot, \ell, +1)$ is the solution to

$$\mathcal{L}(\phi) := -\varepsilon^2 \phi_{xx} + f(\phi) = 0, \qquad \phi\left(-\tfrac{1}{2}\ell\right) = \phi\left(\tfrac{1}{2}\ell\right) = 0, \qquad (2.4)$$

with $\phi > 0$ in $\left(-\tfrac{1}{2}\ell, \tfrac{1}{2}\ell\right)$, and $\phi(\cdot, \ell, -1)$ is the solution to equation (2.4) with $\phi < 0$ in $\left(-\tfrac{1}{2}\ell, \tfrac{1}{2}\ell\right)$. Moreover, the functions $\phi(\cdot, \ell, \pm 1)$ are well defined if $\ell > \varepsilon L_0$, they depend on ε and ℓ only through the ratio ε/ℓ. Finally,

$$\max_x |\phi(\cdot, \ell, \pm 1)| = M_\pm(\ell/\varepsilon) \qquad \text{and} \qquad \max_x |\phi_x(\cdot, \ell, \pm 1)| \leq C\varepsilon^{-1},$$

where $C > 0$ is a constant depending only on F. In particular, $M_\pm$ tends to $+1$ as $\varepsilon/\ell \to 0$ (more details in Proposition 2.3).

For $\boldsymbol{h} \in \Omega_\rho$ with $\rho < 1/L_0$, we define the function $u^{\boldsymbol{h}}$ with N transition points at $h_1, \dots, h_N$ by matching together steady states to equation (1.6) with layer distance equal to ℓ, using smooth cut-off functions. Given $\chi : \mathbb{R} \to [0,1]$ a C^∞ function with $\chi(x) = 0$ for $x \leq -1$ and $\chi(x) = 1$ for $x \geq 1$, set

$$\chi^j(x) := \chi\left(\frac{x - h_j}{\varepsilon}\right) \qquad \text{and} \qquad \phi^j(x) := \phi\left(x - h_{j-1/2}, h_j - h_{j-1}, (-1)^j\right).$$

Then the function $u^{\boldsymbol{h}}$ is given by the convex combination

$$u^{\boldsymbol{h}} := \left(1 - \chi^j\right)\phi^j + \chi^j\phi^{j+1} \qquad \text{in} \quad I_j := [h_{j-1/2}, h_{j+1/2}], \tag{2.5}$$

and the base manifold for the equation (1.7) is defined as

$$\mathcal{M} := \{u^{\boldsymbol{h}} : \boldsymbol{h} \in \Omega_\rho\}.$$

If $\rho > 0$ is sufficiently small and $\boldsymbol{h} \in \Omega_\rho$, then $u^{\boldsymbol{h}}(x) \approx \Phi\left((x - h_j)(-1)^{j-1}\right)$ for x near h_j and $u^{\boldsymbol{h}}(x) \approx \pm 1$ away from h_j for $j = 1, \ldots, N$. Therefore, states $u^{\boldsymbol{h}}$ on the base manifold are well approximated near transition layers by $U^{\boldsymbol{h}}$.

By definition, $u^{\boldsymbol{h}}$ is a smooth function of x and $\boldsymbol{h}$ and enjoys the properties

$$\begin{aligned}
u^{\boldsymbol{h}}(0) &= \phi(0, 2h_1, -1) < 0, & u^{\boldsymbol{h}}(h_{j+1/2}) &= \phi\left(0, h_{j+1} - h_j, (-1)^{j+1}\right) \\
u^{\boldsymbol{h}}(h_j) &= 0, & \mathcal{L}(u^{\boldsymbol{h}}(x)) &= 0 \quad \text{for } |x - h_j| \geq \varepsilon,
\end{aligned}$$

for any $j = 1, \ldots, N$. In what follows, we use the notation

$$u_j^{\boldsymbol{h}} := \partial_{h_j} u^{\boldsymbol{h}}, \qquad \nabla_{\boldsymbol{h}} u^{\boldsymbol{h}} := \left(u_1^{\boldsymbol{h}}, \ldots, u_N^{\boldsymbol{h}}\right),$$

and we denote the tangent space to $\mathcal{M}$ at $u^{\boldsymbol{h}}$ by $T\mathcal{M}(u^{\boldsymbol{h}}) = \text{span}\{u_j^{\boldsymbol{h}} : j = 1, \ldots, N\}$. At this point, the natural idea would be to construct a tubular neighborhood of $\mathcal{M}$, with coordinates $(\boldsymbol{h}, w)$ where w is orthogonal to $T\mathcal{M}(u^{\boldsymbol{h}})$. Since $\mathcal{M}$ is not invariant, there is higher flexibility in the construction of its neighborhood and tubular co-ordinates near $\mathcal{M}$ can be defined using approximate tangent vectors to $\mathcal{M}$. For $j = 1, \ldots, N$, introduce the cutoff function γ^j given by

$$\gamma^j(x) := \chi\left(\frac{x - h_{j-1/2} - \varepsilon}{\varepsilon}\right)\left[1 - \chi\left(\frac{x - h_{j+1/2} + \varepsilon}{\varepsilon}\right)\right].$$

Then, the *approximate tangent vectors* $k_j^{\boldsymbol{h}}$ are defined by

$$k_j^{\boldsymbol{h}}(x) := -\gamma^j(x)u_x^{\boldsymbol{h}}(x).$$

By construction, $k_j^{\boldsymbol{h}}$ are smooth functions of x and $\boldsymbol{h}$ and are such that

$$\begin{aligned}
k_j^{\boldsymbol{h}}(x) &= 0 & \text{for} \quad & x \notin [h_{j-1/2}, h_{j+1/2}], \\
k_j^{\boldsymbol{h}}(x) &= -u_x^{\boldsymbol{h}}(x) & \text{for} \quad & x \in [h_{j-1/2} + 2\varepsilon, h_{j+1/2} - 2\varepsilon].
\end{aligned}$$

As above, we use the notation

$$k_{ji}^{\boldsymbol{h}} := \partial_{h_i} k_j^{\boldsymbol{h}}, \qquad \nabla_{\boldsymbol{h}} k_j^{\boldsymbol{h}} := \left(k_{j1}^{\boldsymbol{h}}, \ldots, k_{jN}^{\boldsymbol{h}}\right).$$

The definition of the approximate tangent vectors is motivated by the relations

$$u_j^{\boldsymbol{h}} = \partial_{h_j} u^{\boldsymbol{h}} \approx \partial_{h_j} \Phi\left((x - h_j)(-1)^{j-1}\right) = (-1)^j \Phi'\left((x - h_j)(-1)^{j-1}\right) \approx -u_x^{\boldsymbol{h}}$$

for $x \in [h_{j-1/2}, h_{j+1/2}]$. In addition, the multiplication by the cutoff term γ^j is reminiscent of the fact that the tangent space of $\mathcal{M}^{\text{FH}}$ is spanned by $U_j^{\boldsymbol{h}}$ that have disjoint supports.

The following estimates will be useful in the sequel.

PROPOSITION 2.1 (Carr–Pego [10]).   Let $f = F'$ with F satisfying conditions (1.3). Given $N \in \mathbb{N}$ and $\delta \in (0, 1/N)$, there exist $\varepsilon_0, C, A_0 > 0$, and a function $\omega = \omega(s)$ with $\omega \to 0$ as $s \to 0^+$ such that if ε and ρ are chosen so that conditions (1.12) hold and $\boldsymbol{h} \in \Omega_\rho$, then

$$\|u_j^{\boldsymbol{h}}\|_{L^\infty} + \varepsilon^{1/2}\|k_{jj}^{\boldsymbol{h}}\| + \|k_{jj}^{\boldsymbol{h}}\|_{L^1} \leq C\varepsilon^{-1},$$

$$A_0 - \omega(\rho) \leq \varepsilon^{1/2}\|u_j^{\boldsymbol{h}}\| \leq A_0 + \omega(\rho),$$
$$A_0 - \omega(\rho) \leq \varepsilon^{1/2}\|k_j^{\boldsymbol{h}}\| \leq A_0 + \omega(\rho),$$
$$\left\{A_0 - \omega(\rho)\right\}^2 \leq \varepsilon\langle u_j^{\boldsymbol{h}}, k_j^{\boldsymbol{h}}\rangle \leq \left\{A_0 + \omega(\rho)\right\}^2,$$

for $\boldsymbol{h} \in \Omega_\rho$ and $j = 1, \ldots, N$. Moreover, if $j \neq i$, we have

$$|\langle u_j^{\boldsymbol{h}}, k_i^{\boldsymbol{h}}\rangle| + \varepsilon^{1/2}\|k_{ij}^{\boldsymbol{h}}\| + \|k_{ij}^{\boldsymbol{h}}\|_{L^1} \leq \omega(\rho)\varepsilon^{-1}.$$

Heuristically, the exponent of ε can be obtained by replacing $u_j^{\boldsymbol{h}}$ and $k_j^{\boldsymbol{h}}$ with $-\Phi'(x - h_j)$.

A function u near $\mathcal{M}$ may be written in terms of coordinates $(\boldsymbol{h}, w)$ as $u = u^{\boldsymbol{h}} + w$, with w satisfying the orthogonality condition (1.13). To state this result, let us set

$$\mathcal{B}_{\rho,\sigma} := \left\{u \in L^\infty : \inf_{\boldsymbol{h} \in \Omega_\rho} \|u - u^{\boldsymbol{h}}\|_{L^\infty} < \sigma\right\},$$

$$\hat{\mathcal{S}}_{\rho,\sigma} := \left\{(\boldsymbol{h}, w) \in \Omega_\rho \times L^\infty : \|w\|_{L^\infty} < \sigma, \langle w, k_j^{\boldsymbol{h}}\rangle = 0 \quad \text{for } j = 1, \ldots, N\right\}.$$

PROPOSITION 2.2 (Carr–Pego [10]).   There exist $\rho_1, \rho_2, \sigma, C > 0$ with $\rho_1 < \rho_2$ and a smooth function $\mathcal{H} : \mathcal{B}_{\rho_1,\sigma} \to \Omega_{\rho_2}$ such that, whenever $\boldsymbol{h} = \mathcal{H}(u)$, we have

$$\langle u - u^{\boldsymbol{h}}, k_j^{\boldsymbol{h}}\rangle = 0 \qquad \text{for} \quad j = 1, \ldots, N,$$

and

$$\|u - u^{\boldsymbol{h}}\|_{L^\infty} \leq C \inf\left\{\|u - u^{\boldsymbol{l}}\|_{L^\infty} : \boldsymbol{l} \in \Omega_{\rho_1}\right\} < C\sigma.$$

Moreover, defining $\mathcal{U} : \hat{\mathcal{S}}_{\rho_1,\sigma} \to \mathcal{B}_{\rho_1,\sigma}$ by setting

$$\mathcal{U}(\boldsymbol{h}, w) := u^{\boldsymbol{h}} + w \quad \text{and} \quad \mathcal{S}_{\rho_1,\sigma} := \mathcal{U}(\hat{\mathcal{S}}_{\rho_1,\sigma}),$$

the function $\mathcal{U}$ is injective, $(\mathcal{H} \circ \mathcal{U})(\boldsymbol{h}, w) = \boldsymbol{h}$ for all $(\boldsymbol{h}, w) \in \hat{\mathcal{S}}_{\rho_1,\sigma}$ and the set $\mathcal{S}_{\rho_1,\sigma}$ is open in $L^\infty(0, 1)$.

In the last statement, constants $\rho_1, \rho_2, \sigma, C$ can be chosen independent on ε.

2.2. Energy functional $\mathcal{E}^{\boldsymbol{h}}$ and barrier function Ψ.   As stated in the introduction, the neighborhood $\mathcal{Z}_{\Gamma,\rho}$ of the extended base manifold $\mathcal{M}_0$ is defined in terms of the *energy functional* $\mathcal{E}^{\boldsymbol{h}}$ and the *barrier function* Ψ, see definitions (1.14) and (1.15). The positivity of the first term in $\mathcal{E}^{\boldsymbol{h}}$ holds for ρ small and w satisfying the orthogonality condition (1.13).

THEOREM 2.1 (Carr–Pego [10]).   Let $f = F'$, with F satisfying conditions (1.3). Given $N \in \mathbb{N}$ and $\delta \in (0, 1/N)$, there exist $\varepsilon_0, \Lambda > 0$, such that if ε and ρ are chosen so that conditions (1.12) hold and $\boldsymbol{h} \in \Omega_\rho$, then

$$\Lambda \int_0^1 \left\{\varepsilon^2 w_x^2 + w^2\right\} dx \leq \int_0^1 \left\{\varepsilon^2 w_x^2 + f'(u^{\boldsymbol{h}})w^2\right\} dx,$$

for any $w \in H^1(0,1)$ *satisfying* $\langle w, k_j^{\boldsymbol{h}} \rangle = 0$ *for* $j = 1, \ldots, N$.

Given $\boldsymbol{h} \in \Omega_\rho$, we consider the operator $L^{\boldsymbol{h}}$, linearization of $\mathcal{L}(u)$ about $u^{\boldsymbol{h}}$, i.e.

$$L^{\boldsymbol{h}} w := -\varepsilon^2 w_{xx} + f'(u^{\boldsymbol{h}}) w. \tag{2.6}$$

If $w \in H^2$ and $w_x(0) = w_x(1) = 0$, integrating by parts, we infer

$$\langle w, L^{\boldsymbol{h}} w \rangle = \int_0^1 \left\{ \varepsilon^2 w_x^2 + f'(u^{\boldsymbol{h}}) w^2 \right\} dx.$$

Hence, in this case, the energy functional can be written as

$$\mathcal{E}^{\boldsymbol{h}}[w,v] = \tfrac{1}{2} \langle w, L^{\boldsymbol{h}} w \rangle + \tfrac{1}{2} \tau \|v\|^2 + \varepsilon \tau \langle w, v \rangle, \tag{2.7}$$

and from Theorem 2.1 it follows that

$$\Lambda \|w\|^2 \leq \langle w, L^{\boldsymbol{h}} w \rangle. \tag{2.8}$$

Moreover, let $x_2 \in [0,1]$ be such that $|w(x_2)| = \|w\|_{L^\infty}$ and let $x_1 \in [0,1]$ be such that $|w(x_1)| = \min\{|w(x)| : x \in [0,1]\}$. Assume without loss of generality $x_2 > x_1$ (otherwise replace $w(x)$ by $w(1-x)$). We have

$$\varepsilon w(x_2)^2 - \varepsilon w(x_1)^2 = \int_{x_1}^{x_2} 2\varepsilon w(x) w_x(x) \, dx \leq \int_0^1 \left\{ \varepsilon^2 w_x^2 + w^2 \right\} dx,$$

and so,

$$\varepsilon \|w\|_{L^\infty}^2 \leq \varepsilon w(x_1)^2 + \int_0^1 \left\{ \varepsilon^2 w_x^2 + w^2 \right\} dx \leq (1 + \varepsilon) \int_0^1 \left\{ \varepsilon^2 w_x^2 + w^2 \right\}.$$

By applying Theorem 2.1 and taking into account the latter bound with $\varepsilon \leq 1$, we deduce also the estimate

$$\tfrac{1}{2} \Lambda \varepsilon \|w\|_{L^\infty}^2 \leq \int_0^1 \left\{ \varepsilon^2 w_x^2 + f'(u^{\boldsymbol{h}}) w^2 \right\} dx = \langle w, L^{\boldsymbol{h}} w \rangle. \tag{2.9}$$

In order to provide representations of the barrier Ψ, given in definition (1.15), we introduce some auxiliary functions. Since $\phi(0, \ell, \pm 1)$ depends only on the ratio $r = \varepsilon / \ell$, we can define

$$\alpha_\pm(r) := F(\phi(0, \ell, \pm 1)), \qquad \beta_\pm(r) := 1 \mp \phi(0, \ell, \pm 1).$$

By definition, $\phi(0, \ell, \pm 1)$ is close to $+1$ or -1 and so, $\alpha_\pm(r), \beta_\pm(r)$ are close to 0. The next result characterizes the leading terms in $\alpha_\pm$ and $\beta_\pm$ as $r \to 0$.

PROPOSITION 2.3 (Carr–Pego [10]). *Let F be such that conditions (1.3) hold and set $A_\pm^2 := F''(\pm 1)$. There exists $r_0, K_\pm > 0$ such that if $0 < r < r_0$, then*

$$\alpha_\pm(r) = \tfrac{1}{2} K_\pm^2 A_\pm^2 \exp(-A_\pm/r) \left\{ 1 + O\left(r^{-1} \exp(-A_\pm/2r) \right) \right\},$$
$$\beta_\pm(r) = K_\pm \exp\left(-A_\pm/2r \right) \left\{ 1 + O\left(r^{-1} \exp(-A_\pm/2r) \right) \right\},$$

with corresponding asymptotic formulae for the derivatives of $\alpha_\pm$ and $\beta_\pm$.

Explicit expressions of $K_\pm$ in terms of F can be found in [10].

For $j = 0, \ldots, N$, we set

$$r_{j+1/2} := \frac{\varepsilon}{h_{j+1} - h_j},$$

and

$$\alpha^{j+1/2} := \begin{cases} \alpha_+(r_{j+1/2}) & j \text{ odd}, \\ \alpha_-(r_{j+1/2}) & j \text{ even}, \end{cases} \qquad \beta^{j+1/2} := \begin{cases} \beta_+(r_{j+1/2}) & j \text{ odd}, \\ \beta_-(r_{j+1/2}) & j \text{ even}, \end{cases}$$

For $\boldsymbol{h} \in \Omega_\rho$, since

$$\mathcal{L}(u^{\boldsymbol{h}}(x)) = 0 \qquad \text{if } |x - h_j| \geq \varepsilon,$$
$$k_j^{\boldsymbol{h}}(x) = -u_x^{\boldsymbol{h}}(x) \quad \text{if } |x - h_j| \leq \varepsilon,$$

direct integration gives

$$\langle \mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}} \rangle = \int_{h_j - \varepsilon}^{h_j + \varepsilon} \left\{ \varepsilon^2 u_{xx}^{\boldsymbol{h}} - f(u^{\boldsymbol{h}}) \right\} u_x^{\boldsymbol{h}} \, dx = \alpha^{j-1/2} - \alpha^{j+1/2}.$$

Thus, the barrier function Ψ, defined in (1.15), can be written as

$$\Psi(\boldsymbol{h}) = \sum_{j=1}^{N} \left(\alpha^{j-1/2} - \alpha^{j+1/2} \right)^2. \tag{2.10}$$

The next statement collects some estimates we will use later on.

PROPOSITION 2.4.   *Let $f = F'$, with F satisfying conditions (1.3). Given $N \in \mathbb{N}$ and $\delta \in (0, 1/N)$, there exist $\varepsilon_0, C > 0$, such that if ε and ρ are chosen so that conditions (1.12) hold and $\boldsymbol{h} \in \Omega_\rho$, then*

$$\|\mathcal{L}(u^{\boldsymbol{h}})\| \leq C \varepsilon^{1/2} \sum_{j=1}^{N} |\alpha^{j+1/2} - \alpha^{j-1/2}| \leq C \varepsilon^{1/2} \exp(-A\ell^{\boldsymbol{h}}/\varepsilon), \tag{2.11}$$

$$\|L^{\boldsymbol{h}} u_j^{\boldsymbol{h}}\| \leq C \varepsilon^{-1/2} \max\{\alpha^{j-1/2}, \alpha^{j+1/2}\} \leq C \varepsilon^{-1/2} \exp(-A\ell^{\boldsymbol{h}}/\varepsilon), \tag{2.12}$$

where $\ell^{\boldsymbol{h}} := \min\{h_j - h_{j-1} : j = 1, \ldots, N+1\}$ and $A := \min A_\pm$.

Here, we give only an idea of the proofs, the complete ones can be found in [10] for inequality (2.11) and in [11] for inequality (2.12). Recalling the definition (2.5), for $x \in I_j$, we have

$$\mathcal{L}(u^{\boldsymbol{h}}) = \varepsilon^2 \chi_{xx}^j \left(\phi^j - \phi^{j+1} \right) + 2\varepsilon^2 \chi_x^j \left(\phi_x^j - \phi_x^{j+1} \right) - G, \tag{2.13}$$

where the remainder G is given by

$$G = \left(1 - \chi^j \right) f(\phi^j) + \chi^j f(\phi^{j+1}) - f \left((1 - \chi^j)\phi^j + \chi^j \phi^{j+1} \right).$$

Using the Lagrange interpolation formula,

$$G = \left(\phi^{j+1} - \phi^j \right)^2 \left\{ (1 - \chi^j) \int_0^{\chi^j} s f''(\theta) ds + \chi^j \int_{\chi^j}^1 (1 - s) f''(\theta) ds \right\},$$

with $\theta(s) = (1-s)\phi^j(x) + s\phi^{j+1}(x)$. It can be shown (see [10, Lemma 8.2]) that there exist $C > 0$ such that for $x \in [h_j - \varepsilon, h_j + \varepsilon]$, $j \in \{1, \ldots, N\}$, we have

$$\left|\phi^j(x) - \phi^{j+1}(x)\right| + \varepsilon\left|\phi_x^j(x) - \phi_x^{j+1}(x)\right| < C\left|\alpha^{j-1/2} - \alpha^{j+1/2}\right|, \tag{2.14}$$

provided $r_j, r_{j+1} < r_0$ and r_0 is sufficiently small. Using these estimates, the fact that $\mathcal{L}(u^{\boldsymbol{h}}(x)) = 0$ if $|x - h_j| > \varepsilon$ and that the m-th derivative of $\varepsilon^m\chi^j$ is uniformly bounded (independently on ε), we obtain

$$|\mathcal{L}(u^{\boldsymbol{h}}(x))| \le C\left|\alpha^{j+1/2} - \alpha^{j-1/2}\right| \quad \text{for } x \in I_j.$$

Then, the L^2-bound (2.11) follows since $\mathcal{L}(u^{\boldsymbol{h}})$ has support of length 2ε in I_j. The estimate (2.12) is obtained in a similar way, by differentiating equation (2.13) with respect to $\boldsymbol{h}$.

3. Dynamics near the base manifold

In this section we study the dynamics of equation (1.8)-(1.9) in a neighborhood of $\mathcal{M}_0$ using the decomposition $u = u^{\boldsymbol{h}} + w$ and deriving the system of equations for $(\boldsymbol{h}, w, v)$. Such description will be used to prove Theorem 1.1.

3.1. Equations for the motion. Let (u, v) be a classical solution of equation (1.8)-(1.9), with u lying in the tubular neighborhood $\mathcal{S}_{\rho,\sigma}$ for $t \in [0, T]$ with $T > 0$. Let $\boldsymbol{h}(t) = \mathcal{H}(u(\cdot, t))$ and $w(x, t) = u(x, t) - u^{\boldsymbol{h}(t)}$, where $u^{\boldsymbol{h}}$ is defined by (2.5). We recall that $u(\cdot, t) \in \mathcal{S}_{\rho,\sigma}$ for $t \in [0, T]$ means that $\boldsymbol{h}(t) \in \Omega_\rho$, $w(\cdot, t) \in H_N^2$ and $\|w(\cdot, t)\|_{L^\infty} < \sigma$ for $t \in [0, T]$. Moreover, $v(\cdot, t) \in L^2(0, 1)$ for $t \in [0, T]$.

From system (1.8) it follows that the pair (w, v) satisfies

$$\begin{cases} w_t = v - \nabla_{\boldsymbol{h}} u^{\boldsymbol{h}} \cdot \boldsymbol{h}', \\ \tau v_t = -\mathcal{L}(u^{\boldsymbol{h}} + w) - g(u^{\boldsymbol{h}} + w, \tau)v, \end{cases} \tag{3.1}$$

where $\cdot$ denotes the inner product in $\mathbb{R}^N$. Expanding, we get

$$\mathcal{L}(u^{\boldsymbol{h}} + w) = \mathcal{L}(u^{\boldsymbol{h}}) + L^{\boldsymbol{h}}w - f_2 w^2, \qquad \text{where} \quad f_2 := \int_0^1 (1-s)f''(u^{\boldsymbol{h}} + sw)\,ds,$$

and $L^{\boldsymbol{h}}$ is the differential operator of definition (2.6).

Differentiating with respect to t the orthogonality condition (1.13), we obtain

$$\sum_{j=1}^N \left\{\langle u_j^{\boldsymbol{h}}, k_i^{\boldsymbol{h}}\rangle - \langle w, k_{ij}^{\boldsymbol{h}}\rangle\right\} h_j' = \langle v, k_i^{\boldsymbol{h}}\rangle, \qquad i = 1, \ldots, N. \tag{3.2}$$

Using the notation

$$D_{ij}(\boldsymbol{h}) := \langle u_j^{\boldsymbol{h}}, k_i^{\boldsymbol{h}}\rangle, \qquad \hat{D}_{ij}(\boldsymbol{h}, w) := \langle w, k_{ij}^{\boldsymbol{h}}\rangle, \qquad Y_i(\boldsymbol{h}, v) := \langle v, k_i^{\boldsymbol{h}}\rangle,$$

equation (3.2) becomes

$$\left\{D(\boldsymbol{h}) - \hat{D}(\boldsymbol{h}, w)\right\}\boldsymbol{h}' = Y(\boldsymbol{h}, v). \tag{3.3}$$

From equations (3.1) and (3.3), we obtain the ODE-PDE coupled system

$$\begin{cases} w_t = v - \nabla_{\boldsymbol{h}} u^{\boldsymbol{h}} \cdot \boldsymbol{h}', \\ \tau v_t = -\mathcal{L}(u^{\boldsymbol{h}}) - L^{\boldsymbol{h}}w + f_2 w^2 - g(u^{\boldsymbol{h}} + w, \tau)v, \\ \left\{D(\boldsymbol{h}) - \hat{D}(\boldsymbol{h}, w)\right\}\boldsymbol{h}' = Y(\boldsymbol{h}, v). \end{cases}$$

The matrix $D(\boldsymbol{h})$ is diagonally dominant, because, for any $\eta \in (0,1)$ there exists $\rho_0 > 0$ such that if $\rho < \rho_0$, then

$$D_{ii}(\boldsymbol{h}) - \sum_{j \neq i} |D_{ij}(\boldsymbol{h})| = \langle u_i^{\boldsymbol{h}}, k_i^{\boldsymbol{h}} \rangle - \sum_{j \neq i} |\langle u_j^{\boldsymbol{h}}, k_i^{\boldsymbol{h}} \rangle|$$

$$\geq \left\{ \left(A_0 - \omega(\rho) \right)^2 - (N-1)\omega(\rho) \right\} \varepsilon^{-1} > \eta A_0^2 \varepsilon^{-1}, \qquad (3.4)$$

thanks to Proposition 2.1. Also, for a known property of inverses of diagonally dominant matrices (see [42]), $D(\boldsymbol{h})$ is invertible and it holds

$$\| D^{-1}(\boldsymbol{h}) \|_\infty \leq \eta^{-1} A_0^{-2} \varepsilon,$$

where $\| \cdot \|_\infty$ denotes the operator norm induced by the norm $| \cdot |_\infty$. In Section 4, it is determined the explicit expression for the principal term in the expansion of the inverse $D^{-1}(\boldsymbol{h})$ as $\varepsilon \to 0$.

The invertibility of the matrix $D(\boldsymbol{h}) - \hat{D}(\boldsymbol{h}, w)$ descends from the smallness of $\hat{D}(\boldsymbol{h}, w)$ for $w \to 0$ and inequality (3.4). Indeed, for $(\boldsymbol{h}, w) \in \hat{\mathcal{S}}_{\rho,\sigma}$, applying Proposition 2.1, we infer

$$\sum_j |\hat{D}_{ij}(\boldsymbol{h}, w)| \leq \| w \|_{L^\infty} \left\{ \| k_{ii}^{\boldsymbol{h}} \|_{L^1} + \sum_{j \neq i} \| k_{ij}^{\boldsymbol{h}} \|_{L^1} \right\}$$

$$\leq \sigma \left\{ C + (N-1)\omega(\rho) \right\} \varepsilon^{-1},$$

and thus

$$D_{ii}(\boldsymbol{h}) - \hat{D}_{ii}(\boldsymbol{h}, w) - \sum_{j \neq i} |D_{ij}(\boldsymbol{h}) - \hat{D}_{ij}(\boldsymbol{h}, w)| \geq \eta A_0^2 \varepsilon^{-1} - \sigma \left\{ C + (N-1)\omega(\rho) \right\} \varepsilon^{-1}.$$

Therefore, the matrix $D - \hat{D}$ is invertible for ρ and σ sufficiently small and

$$\| \{ D(\boldsymbol{h}) - \hat{D}(\boldsymbol{h}, w) \}^{-1} \|_\infty \leq 2\eta A_0^{-2} \varepsilon. \qquad (3.5)$$

Applying $\{ D(\boldsymbol{h}) - \hat{D}(\boldsymbol{h}, w) \}^{-1}$ in the equation for $\boldsymbol{h}$, we obtain the final form of the system

$$\begin{cases} w_t = v - \nabla_{\boldsymbol{h}} u^{\boldsymbol{h}} \cdot \boldsymbol{h}', \\ \tau v_t = -\mathcal{L}(u^{\boldsymbol{h}}) - L^{\boldsymbol{h}} w + f_2 w^2 - g(u^{\boldsymbol{h}} + w, \tau) v, \\ \boldsymbol{h}' = \{ D(\boldsymbol{h}) - \hat{D}(\boldsymbol{h}, w) \}^{-1} Y(\boldsymbol{h}, v). \end{cases} \qquad (3.6)$$

The proof of our main result consists in providing estimates for the solutions to system (3.6).

3.2. Proof of the main result.   To start with, we observe that if $(\boldsymbol{h}, w) \in \hat{\mathcal{S}}_{\rho,\sigma}$ for ρ, σ small then there exists $C > 0$ such that

$$|\boldsymbol{h}'|_\infty \leq 2\eta A_0^{-2} \varepsilon |Y(\boldsymbol{h}, v)|_\infty \leq C \varepsilon^{1/2} \| v \|, \qquad (3.7)$$

using inequality (3.5) and the third estimate in Proposition 2.1.

In order to prove Theorem 1.1, we restrict the attention to the set

$$\hat{\mathcal{Z}}_{\Gamma, \rho} := \left\{ (\boldsymbol{h}, w, v) \in \overline{\Omega}_\rho \times H_N^2 \times L^2(0,1) : \mathcal{E}^{\boldsymbol{h}}[w, v] \leq \Gamma \Psi(\boldsymbol{h}) \right\}.$$

The aim of the next result is twofold. Firstly, it states that if the triple $(\boldsymbol{h}, w, v)$ belongs to $\hat{\mathcal{Z}}_{\Gamma,\rho}$ then the bound on (w, v) stated in Theorem 1.1, estimate (1.16), holds true. Secondly, assuming in addition that $(\boldsymbol{h}, w, v)$ is a solution to system (3.6), then also the bound on $\boldsymbol{h}'$ in Theorem 1.1, estimate (1.17), is valid.

PROPOSITION 3.1. *Let $F \in C^3$ be such that conditions (1.3) hold and $g(\cdot, \tau) \in C^1$. Given $N \in \mathbb{N}$ and $\delta \in (0, 1/N)$, there exist $\varepsilon_0, C > 0$, such that for ε and ρ satisfying conditions (1.12),*

(i) if $(\boldsymbol{h}, w, v) \in \hat{\mathcal{Z}}_{\Gamma,\rho}$, then

$$\begin{aligned}
\tfrac{1}{8}\Lambda \varepsilon \|w\|_{L^\infty}^2 + \tfrac{1}{4}\tau\|v\|^2 &\leq \mathcal{E}^{\boldsymbol{h}}[w, v], \\
\tfrac{1}{4}\Lambda \|w\|^2 + \tfrac{1}{4}\tau\|v\|^2 &\leq \mathcal{E}^{\boldsymbol{h}}[w, v], \\
\mathcal{E}^{\boldsymbol{h}}[w, v] &\leq C\Gamma\exp(-2A\ell^{\boldsymbol{h}}/\varepsilon),
\end{aligned} \tag{3.8}$$

where Λ is the positive constant introduced in Theorem 2.1;

(ii) if $(\boldsymbol{h}, w, v) \in \hat{\mathcal{Z}}_{\Gamma,\rho}$ is a solution of system (3.6) for $t \in [0, T]$, then

$$|\boldsymbol{h}'|_\infty \leq C(\varepsilon/\tau)^{1/2}\exp(-A\ell^{\boldsymbol{h}}/\varepsilon). \tag{3.9}$$

Proof. Let us prove the first inequality in (3.8). Using Young's inequality, we have

$$\varepsilon|\langle w, v\rangle| \leq \varepsilon^2\|w\|^2 + \tfrac{1}{4}\|v\|^2 \leq \varepsilon^2\|w\|_{L^\infty}^2 + \tfrac{1}{4}\|v\|^2,$$

and so, recalling the expression for the energy $\mathcal{E}^{\boldsymbol{h}}$ given in equation (2.7),

$$\mathcal{E}^{\boldsymbol{h}}[w, v] \geq \tfrac{1}{2}\langle w, L^{\boldsymbol{h}}w\rangle + \tfrac{1}{4}\tau\|v\|^2 - \varepsilon^2\tau\|w\|_{L^\infty}^2.$$

Using estimate (2.9), we obtain, for $\varepsilon < \Lambda/8\tau$,

$$\mathcal{E}^{\boldsymbol{h}}[w, v] \geq \left(\tfrac{1}{4}\Lambda - \varepsilon\tau\right)\varepsilon\|w\|_{L^\infty}^2 + \tfrac{1}{4}\tau\|v\|^2 \geq \tfrac{1}{8}\Lambda\varepsilon\|w\|_{L^\infty}^2 + \tfrac{1}{4}\tau\|v\|^2.$$

Moreover, from (2.8), for $\varepsilon^2 < \Lambda/4\tau$ one has

$$\mathcal{E}^{\boldsymbol{h}}[w, v] \geq \left(\tfrac{1}{2}\Lambda - \varepsilon^2\tau\right)\|w\|^2 + \tfrac{1}{4}\tau\|v\|^2 \geq \tfrac{1}{4}\Lambda\|w\|^2 + \tfrac{1}{4}\tau\|v\|^2,$$

concluding the first two inequalities of (3.8). The upper bound for $\mathcal{E}^{\boldsymbol{h}}[w, v]$ follows from the definition of $\hat{\mathcal{Z}}_{\Gamma,\rho}$, the expression of the barrier Ψ given in equation (2.10) and Proposition 2.3.

To prove part (ii), using the inequalities (3.8), we deduce the estimate

$$\|w\|_{L^\infty} \leq C\varepsilon^{-1/2}\exp(-A\ell^{\boldsymbol{h}}/\varepsilon) \leq C\varepsilon^{-1/2}\exp(-A\delta/\varepsilon) =: \sigma$$

since, by definition and conditions (1.12), $\ell^{\boldsymbol{h}} > \varepsilon/\rho > \delta$. Hence, for ε sufficiently small, both ρ and σ are small, and thus, for $(\boldsymbol{h}, w) \in \hat{\mathcal{S}}_{\rho,\sigma}$, estimate (3.7) holds. Therefore, if $(\boldsymbol{h}, w, v) \in \hat{\mathcal{Z}}_{\Gamma,\rho}$, then $(\boldsymbol{h}, w) \in \hat{\mathcal{S}}_{\rho,\sigma}$ and the estimate (3.9) is obtained by applying the inequalities (3.8) in inequality (3.7). $\square$

Now, we estimate the time T taken for the solution (u, v) to leave $\mathcal{Z}_{\Gamma,\rho}$. To do this, we study the system (3.6) in the set $\hat{\mathcal{Z}}_{\Gamma,\rho}$ by using energy estimates.

PROPOSITION 3.2. *Let $F \in C^3$ and $g(\cdot, \tau) \in C^1$ be such that conditions (1.3) and (1.4) hold. Given $N \in \mathbb{N}$ and $\delta \in (0, 1/N)$, there exist $\Gamma_2 > \Gamma_1 > 0$ and $\varepsilon_0 > 0$ such that*

if $\Gamma \in [\Gamma_1,\Gamma_2]$, ε,ρ *satisfy conditions* (1.12) *and* $(\boldsymbol{h},w,v) \in \hat{\mathcal{Z}}_{\Gamma,\rho}$ *is a solution of system* (3.6) *for* $t \in [0,T]$, *then for some* $\eta \in (0,1)$, *we have*

$$\frac{d}{dt}\big\{\mathcal{E}^{\boldsymbol{h}}[w,v] - \Gamma\Psi(\boldsymbol{h})\big\} \leq -\eta\varepsilon\big\{\mathcal{E}^{\boldsymbol{h}}[w,v] - \Gamma\Psi(\boldsymbol{h})\big\} \qquad for \quad t \in [0,T]. \tag{3.10}$$

Proof. In all the proof, symbols C,c,η denote generic positive constants, independent on ε, and with $\eta \in (0,1)$. Let us recall that, if $(\boldsymbol{h},w,v)$ is a solution to system (3.6), then

$$w_t = v - \nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}', \qquad \tau v_t = -\mathcal{L}(u^{\boldsymbol{h}}) - L^{\boldsymbol{h}}w + f_2 w^2 - g(u^{\boldsymbol{h}} + w,\tau)v.$$

Direct differentiation and the self-adjointness of the operator $L^{\boldsymbol{h}}$ give

$$\frac{d}{dt}\big\{\tfrac{1}{2}\langle w, L^{\boldsymbol{h}}w\rangle\big\} = \langle w_t, L^{\boldsymbol{h}}w\rangle + \tfrac{1}{2}\langle w, f''(u^{\boldsymbol{h}})\big(\nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}'\big)w\rangle$$
$$= \langle v, L^{\boldsymbol{h}}w\rangle - \langle \nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}', L^{\boldsymbol{h}}w\rangle + \tfrac{1}{2}\langle w, f''(u^{\boldsymbol{h}})\big(\nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}'\big)w\rangle$$
$$= \langle v, L^{\boldsymbol{h}}w\rangle - \langle L^{\boldsymbol{h}}\nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}', w\rangle + \tfrac{1}{2}\langle w, f''(u^{\boldsymbol{h}})\big(\nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}'\big)w\rangle.$$

Using the Cauchy–Schwarz inequality and the estimates in Proposition 2.4 and 2.1, we infer

$$\frac{d}{dt}\big\{\tfrac{1}{2}\langle w, L^{\boldsymbol{h}}w\rangle\big\} \leq \langle v, L^{\boldsymbol{h}}w\rangle + C\sum_j\Big(\|L^{\boldsymbol{h}}u_j^{\boldsymbol{h}}\| + \|u_j^{\boldsymbol{h}}\|_{L^\infty}\|w\|\Big)|\boldsymbol{h}'|_\infty\|w\|$$
$$\leq \langle v, L^{\boldsymbol{h}}w\rangle + C\varepsilon^{-1/2}\big(\exp(-A\delta/\varepsilon) + \varepsilon^{-1/2}\|w\|\big)|\boldsymbol{h}'|_\infty\|w\|.$$

For $(\boldsymbol{h},w,v) \in \hat{\mathcal{Z}}_{\Gamma,\rho}$, applying inequalities (3.7), (3.8) and using Young's inequality, we get

$$\frac{d}{dt}\big\{\tfrac{1}{2}\langle w, L^{\boldsymbol{h}}w\rangle\big\} \leq \langle v, L^{\boldsymbol{h}}w\rangle + C\big(\exp(-A\delta/\varepsilon) + \varepsilon^{-1/2}\|w\|\big)\|w\|\|v\|$$
$$\leq \langle v, L^{\boldsymbol{h}}w\rangle + C\exp(-2A\delta/\varepsilon)\big(1 + \varepsilon^{-1}\Gamma\big)\|w\|^2 + \eta\|v\|^2.$$

For what concerns the second term in the energy $\mathcal{E}^{\boldsymbol{h}}$, it holds

$$\frac{d}{dt}\big\{\tfrac{1}{2}\tau\|v\|^2\big\} = \langle \tau v_t, v\rangle = \langle -\mathcal{L}(u^{\boldsymbol{h}}) - L^{\boldsymbol{h}}w + f_2 w^2 - g(u^{\boldsymbol{h}} + w,\tau)v, v\rangle$$
$$\leq -\langle L^{\boldsymbol{h}}w, v\rangle + \|\mathcal{L}(u^{\boldsymbol{h}})\|\|v\| + C\|w\|_{L^\infty}\|w\|\|v\| - c_g\|v\|^2$$
$$\leq -\langle L^{\boldsymbol{h}}w, v\rangle + C\|w\|_{L^\infty}^2\|w\|^2 - (c_g - \eta)\|v\|^2 + C\|\mathcal{L}(u^{\boldsymbol{h}})\|^2.$$

Finally, the time derivative of the scalar product $\langle w, \tau v\rangle$ can be bounded by

$$\frac{d}{dt}\langle w, \tau v\rangle = \langle v - \nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}', \tau v\rangle + \langle w, -\mathcal{L}(u^{\boldsymbol{h}}) - L^{\boldsymbol{h}}w + f_2 w^2 - g(u^{\boldsymbol{h}} + w,\tau)v\rangle$$
$$\leq -\langle w, L^{\boldsymbol{h}}w\rangle + C(\varepsilon + \|w\|_{L^\infty})\|w\|^2 + (\tau + \eta\varepsilon^{-1})\|v\|^2 + C\tau\varepsilon^{-1/2}|\boldsymbol{h}'|_\infty\|v\|$$
$$+ \varepsilon^{-1}\|\mathcal{L}(u^{\boldsymbol{h}})\|^2$$
$$\leq -\langle w, L^{\boldsymbol{h}}w\rangle + C(\varepsilon + \|w\|_{L^\infty})\|w\|^2 + (C + \eta\varepsilon^{-1})\|v\|^2 + \varepsilon^{-1}\|\mathcal{L}(u^{\boldsymbol{h}})\|^2,$$

where, in particular, the inequalities

$$\langle w, \mathcal{L}(u^{\boldsymbol{h}})\rangle \leq \tfrac{1}{2}\varepsilon\|w\|^2 + \tfrac{1}{2}\varepsilon^{-1}\|\mathcal{L}(u^{\boldsymbol{h}})\|^2,$$
$$\langle w, g(u^{\boldsymbol{h}} + w,\tau)v\rangle \leq C\varepsilon\|w\|^2 + \eta\varepsilon^{-1}\|v\|^2$$

have been used. Collecting the estimates for the three terms composing $\mathcal{E}^{\boldsymbol{h}}$, we deduce

$$
\begin{aligned}
\frac{d\mathcal{E}^{\boldsymbol{h}}}{dt} &\leq -\varepsilon\langle w, L^{\boldsymbol{h}}w\rangle - [c_g - C\varepsilon - 3\eta]\|v\|^2 \\
&\quad + C\{\exp(-2A\delta/\varepsilon)(1+\varepsilon^{-1}\Gamma) + \varepsilon(\varepsilon + \|w\|_{L^\infty})\}\|w\|^2 + (C+1)\|\mathcal{L}(u^{\boldsymbol{h}})\|^2 \\
&\leq -\varepsilon\langle w, L^{\boldsymbol{h}}w\rangle + C\varepsilon\{\Gamma\exp(-c/\varepsilon) + \varepsilon\}\|w\|^2 - \eta c_g\|v\|^2 + C\|\mathcal{L}(u^{\boldsymbol{h}})\|^2,
\end{aligned}
$$

for ε and η small. Thus, from inequality (2.8) and

$$
\|\mathcal{L}(u^{\boldsymbol{h}})\|^2 \leq C\varepsilon\,\Psi(\boldsymbol{h}) \leq C\varepsilon\exp(-2A\ell^{\boldsymbol{h}}/\varepsilon), \tag{3.11}
$$

it follows that

$$
\frac{d\mathcal{E}^{\boldsymbol{h}}}{dt} \leq -\varepsilon\{1 - C(\Gamma\exp(-c/\varepsilon) + \varepsilon)\}\langle w, L^{\boldsymbol{h}}w\rangle - \eta c_g\|v\|^2 + C\varepsilon\Psi.
$$

Hence, for $\varepsilon \in (0, \varepsilon_0)$, with ε_0 small (and dependent on Γ), we deduce the bound

$$
1 - C(\Gamma\exp(-c/\varepsilon) + \varepsilon) \geq \eta.
$$

Substituting, we infer

$$
\begin{aligned}
\frac{d\mathcal{E}^{\boldsymbol{h}}}{dt} &\leq -\eta\varepsilon\langle w, L^{\boldsymbol{h}}w\rangle - \eta c_g\|v\|^2 + C\varepsilon\Psi \\
&\leq -\eta\varepsilon\mathcal{E}^{\boldsymbol{h}} - \tfrac{1}{2}\eta\varepsilon\langle w, L^{\boldsymbol{h}}w\rangle + \eta\varepsilon^2\tau\langle w, v\rangle - \eta(c_g - \tfrac{1}{2}\varepsilon\tau)\|v\|^2 + C\varepsilon\Psi \\
&\leq -\eta\varepsilon\mathcal{E}^{\boldsymbol{h}} - \tfrac{1}{2}\eta\varepsilon(1 - C\varepsilon\tau)\langle w, L^{\boldsymbol{h}}w\rangle - \eta(c_g - C\varepsilon\tau)\|v\|^2 + C\varepsilon\Psi,
\end{aligned}
$$

again from inequality (2.8). Finally, for ε_0 sufficiently small, we obtain

$$
\frac{d\mathcal{E}^{\boldsymbol{h}}}{dt} \leq -\eta\varepsilon\mathcal{E}^{\boldsymbol{h}} - \eta c_g\|v\|^2 + C\varepsilon\Psi. \tag{3.12}
$$

Direct differentiation gives

$$
\frac{d\Psi}{dt} = 2\sum_{j=1}^{N}\langle\mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}}\rangle\Big\{\langle\mathcal{L}(u^{\boldsymbol{h}}), \nabla_{\boldsymbol{h}}k_j^{\boldsymbol{h}} \cdot \boldsymbol{h}'\rangle - \langle L^{\boldsymbol{h}}\nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}', k_j^{\boldsymbol{h}}\rangle\Big\}.
$$

Using the estimates provided by Proposition 2.1 and by inequalities (2.12) and (3.7), we deduce

$$
\big|\langle\mathcal{L}(u^{\boldsymbol{h}}), \nabla_{\boldsymbol{h}}k_j^{\boldsymbol{h}} \cdot \boldsymbol{h}'\rangle\big| \leq |\boldsymbol{h}'|_\infty\|\mathcal{L}(u^{\boldsymbol{h}})\|\sum_{i=1}^{N}\|k_{ji}^{\boldsymbol{h}}\| \leq C\varepsilon^{-1}\|\mathcal{L}(u^{\boldsymbol{h}})\|\|v\|,
$$

$$
\big|\langle L^{\boldsymbol{h}}\nabla_{\boldsymbol{h}}u^{\boldsymbol{h}} \cdot \boldsymbol{h}', k_j^{\boldsymbol{h}}\rangle\big| \leq |\boldsymbol{h}'|_\infty\|k_j^{\boldsymbol{h}}\|\sum_{i=1}^{N}\|L^{\boldsymbol{h}}u_i^{\boldsymbol{h}}\| \leq C\exp(-c/\varepsilon)\|v\|,
$$

thus, observing that $|\langle\mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}}\rangle| \leq C\varepsilon^{-1/2}\|\mathcal{L}(u^{\boldsymbol{h}})\|$, we infer the bound

$$
\left|\frac{d\Psi}{dt}\right| \leq C\varepsilon^{-1/2}\{\varepsilon^{-1}\|\mathcal{L}(u^{\boldsymbol{h}})\| + \exp(-c/\varepsilon)\}\|\mathcal{L}(u^{\boldsymbol{h}})\|\|v\|.
$$

Using the inequality (3.11), we obtain

$$\left|\Gamma\frac{d\Psi}{dt}\right| \leq C\Gamma\varepsilon^{-1/2}\left\{\Psi^{1/2}+\exp(-c/\varepsilon)\right\}\|v\|\Psi^{1/2}$$

$$\leq \eta\|v\|^2 + C\Gamma^2\varepsilon^{-1}\left\{\Psi^{1/2}+\exp(-c/\varepsilon)\right\}^2\Psi.$$

Hence, observing that $\Psi \leq C\exp(-c/\varepsilon)$, we end up with

$$\left|\Gamma\frac{d\Psi}{dt}\right| \leq \eta\|v\|^2 + C\Gamma^2\exp(-c/\varepsilon)\Psi. \tag{3.13}$$

Combining inequalities (3.12) and (3.13), we obtain that if $(\boldsymbol{h},w,v) \in \hat{\mathcal{Z}}_{\Gamma,\rho}$ is a solution of system (3.6), then

$$\frac{d}{dt}\left\{\mathcal{E}^{\boldsymbol{h}}[w,v] - \Gamma\Psi(\boldsymbol{h})\right\} \leq -\eta\varepsilon\mathcal{E}^{\boldsymbol{h}} + C\left(\varepsilon + \Gamma^2\exp(-c/\varepsilon)\right)\Psi,$$

for some $\eta \in (0,1)$. Therefore the estimate (3.10) follows from

$$C\exp(-c/\varepsilon)\Gamma^2 - \eta\varepsilon\Gamma + C\varepsilon \leq 0,$$

and the latter is verified for $\Gamma \in [\Gamma_1,\Gamma_2]$, provided $\varepsilon \in (0,\varepsilon_0)$ with ε_0 sufficiently small so that $\eta^2\varepsilon - 4C^2\exp(-c/\varepsilon) > 0$.   □

Now, we have all the tools needed to prove Theorem 1.1.

Proof. (**Proof of Theorem 1.1.**) Let $(u_0,v_0) \in \mathring{\mathcal{Z}}_{\Gamma,\rho}$ and let (u,v) be the solution of problem (1.8)-(1.9)-(1.10). Assume that $(u,v) \in \mathcal{Z}_{\Gamma,\rho}$ for $t \in [0,T_\varepsilon]$, where T_ε is maximal. Then, $u = u^{\boldsymbol{h}} + w$ and $(\boldsymbol{h},w,v) \in \hat{\mathcal{Z}}_{\Gamma,\rho}$ solves the system (3.6) for $t \in [0,T_\varepsilon]$. Let us apply Proposition 3.2; from estimate (3.10), it follows that

$$\frac{d}{dt}\left\{\exp(\eta\varepsilon t)(\mathcal{E}^{\boldsymbol{h}}[w,v] - \Gamma\Psi(\boldsymbol{h}))\right\} \leq 0, \qquad t \in [0,T_\varepsilon]$$

and so,

$$\exp(\eta\varepsilon t)\{\mathcal{E}^{\boldsymbol{h}}[w,v] - \Gamma\Psi(\boldsymbol{h})\}(t) \leq \{\mathcal{E}^{\boldsymbol{h}}[w,v] - \Gamma\Psi(\boldsymbol{h})\}(0) < 0, \qquad t \in [0,T_\varepsilon].$$

Therefore, the solution (u,v) remains in the channel $\mathcal{Z}_{\Gamma,\rho}$ while $\boldsymbol{h} \in \Omega_\rho$ and if $T_\varepsilon < +\infty$ is maximal, then $\boldsymbol{h}(T_\varepsilon) \in \partial\Omega_\rho$, that is

$$h_j(T_\varepsilon) - h_{j-1}(T_\varepsilon) = \varepsilon/\rho \qquad \text{for some } j. \tag{3.14}$$

For Proposition 3.1, in the channel the solution satisfies inequalities (1.16) and (1.17). In particular, the transition points move with exponentially small velocity. This implies that (u,v) remains in the channel for an exponentially long time. Indeed, from inequality (1.17) it follows that for all $t \in [0,T_\varepsilon]$, one has

$$|h_j(t) - h_j(0)| \leq C(\varepsilon/\tau)^{1/2}\exp(-A\ell^{\boldsymbol{h}(t)}/\varepsilon)t \qquad \text{for any } j = 1,\ldots,N, \tag{3.15}$$

where $\ell^{\boldsymbol{h}(t)}$ is the minimum distance between layers at the time t. Combining the results (3.14) and (3.15), we obtain

$$\varepsilon/\rho \geq \ell^{\boldsymbol{h}(0)} - 2C(\varepsilon/\tau)^{1/2}\exp(-A/\rho)T_\varepsilon.$$

Hence, using conditions (1.12) we have

$$T_\varepsilon \geq C\left(\ell^{\boldsymbol{h}(0)} - \varepsilon/\rho\right)(\varepsilon/\tau)^{-1/2}\exp(A/\rho) \geq C\left(\ell^{\boldsymbol{h}(0)} - \varepsilon/\rho\right)(\varepsilon/\tau)^{-1/2}\exp(A\delta/\varepsilon),$$

and the proof is complete.   □

4. Reduced dynamics on the base manifold

In the previous section, we derived the equation (3.3) for the motion of the transition points and, by studying the ODE-PDE coupled system (3.6), we obtained the estimate (3.9) for the velocity of the transitions. In this section, we derive an ordinary differential equation approximating the equation for $\boldsymbol{h}$ to obtain further information on the motion of the transition points and analyze the differences with the parabolic case (1.7).

4.1. Derivation of the reduced system. Since w is very small, we use the approximation $w = 0$ in equation (3.2) and then

$$\sum_{i=1}^{N} \langle u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle \boldsymbol{h}_i' = \langle v, k_j^{\boldsymbol{h}} \rangle, \qquad j = 1, \dots, N. \tag{4.1}$$

In order to eliminate v, let us differentiate and multiply by τ equation (4.1). We have

$$\tau \sum_{i,l=1}^{N} \big(\langle u_{il}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle + \langle u_i^{\boldsymbol{h}}, k_{jl}^{\boldsymbol{h}} \rangle \big) \boldsymbol{h}_l' \boldsymbol{h}_i' + \tau \sum_{i=1}^{N} \langle u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle \boldsymbol{h}_i''$$

$$= -\langle \mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}} \rangle - \langle g(u^{\boldsymbol{h}}, \tau) v, k_j^{\boldsymbol{h}} \rangle + \tau \sum_{l=1}^{N} \langle v, k_{jl}^{\boldsymbol{h}} \rangle \boldsymbol{h}_l', \qquad j = 1, \dots, N.$$

Using the approximation $v = \nabla_{\boldsymbol{h}} u^{\boldsymbol{h}} \cdot \boldsymbol{h}$, we obtain

$$\tau \sum_{i,l=1}^{N} \big(\langle u_{il}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle + \langle u_i^{\boldsymbol{h}}, k_{jl}^{\boldsymbol{h}} \rangle \big) \boldsymbol{h}_l' \boldsymbol{h}_i' + \tau \sum_{i=1}^{N} \langle u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle \boldsymbol{h}_i''$$

$$= -\langle \mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}} \rangle - \sum_{i=1}^{N} \langle g(u^{\boldsymbol{h}}, \tau) u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle \boldsymbol{h}_i' + \tau \sum_{i,l=1}^{N} \langle u_i^{\boldsymbol{h}}, k_{jl}^{\boldsymbol{h}} \rangle \boldsymbol{h}_i' \boldsymbol{h}_l', \qquad j = 1, \dots, N.$$

Let us denote by $\nabla_{\boldsymbol{h}}^2 u^{\boldsymbol{h}}$ the Hessian of $u^{\boldsymbol{h}}$ with respect to $\boldsymbol{h}$ and by $q(\boldsymbol{\xi}) := \sum_{i,l=1}^{N} u_{il}^{\boldsymbol{h}} \boldsymbol{\xi}_l \boldsymbol{\xi}_i$ the quadratic form associated to $\nabla_{\boldsymbol{h}}^2 u^{\boldsymbol{h}}$. Simplifying, we get

$$\tau \sum_{i=1}^{N} \langle u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle \boldsymbol{h}_i'' + \sum_{i=1}^{N} \langle g(u^{\boldsymbol{h}}, \tau) u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle \boldsymbol{h}_i' + \tau \langle q(\boldsymbol{h}'), k_j^{\boldsymbol{h}} \rangle = -\langle \mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}} \rangle, \tag{4.2}$$

for $j = 1, \dots, N$. If $u^{\boldsymbol{h}(t)}(x)$ is a solution of the hyperbolic Allen–Cahn Equation (1.6), $\boldsymbol{h}(t)$ satisfies equation (4.2). Observe that with respect to the parabolic case, besides the coefficient $g(u^{\boldsymbol{h}}, \tau)$ which is in general different from 1, there are two new terms: the term involving $\boldsymbol{h}_i''$ and the one involving the quadratic form associated to the Hessian of $u^{\boldsymbol{h}}$. By inverting the matrix $D_{ij}(\boldsymbol{h}) = \langle u_j^{\boldsymbol{h}}, k_i^{\boldsymbol{h}} \rangle$, introduced in Section 3, we rewrite equation (4.2) as follows:

$$\tau \boldsymbol{h}'' + \mathcal{G}(\boldsymbol{h}) \boldsymbol{h}' + \tau \mathcal{Q}(\boldsymbol{h}, \boldsymbol{h}') = \mathcal{P}(\boldsymbol{h}), \tag{4.3}$$

where

$$\mathcal{G}_{ij}(\boldsymbol{h}) := \sum_{l=1}^{N} D_{il}^{-1}(\boldsymbol{h}) \langle g(u^{\boldsymbol{h}}, \tau) u_j^{\boldsymbol{h}}, k_l^{\boldsymbol{h}} \rangle, \qquad \mathcal{Q}_i(\boldsymbol{h}, \boldsymbol{h}') := \sum_{j=1}^{N} D_{ij}^{-1}(\boldsymbol{h}) \langle q(\boldsymbol{h}'), k_j^{\boldsymbol{h}} \rangle,$$

and

$$\mathcal{P}_i(\boldsymbol{h}) := -\sum_{j=1}^N D_{ij}^{-1}(\boldsymbol{h})\langle \mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}}\rangle.$$

Now we want to identify the leading terms in equation (4.3), having in mind the estimates for $k_j^{\boldsymbol{h}}$, $u^{\boldsymbol{h}}$ and their derivatives; namely we shall rewrite $\mathcal{G}$, $\mathcal{Q}$ and $\mathcal{P}$ by neglecting the exponentially small remainders in the asymptotic expansion for $\varepsilon \to 0$.

As proven by Carr and Pego [10, Corollary 3.6], defining

$$D_\infty := \int_{-1}^1 \sqrt{2F(s)}\,ds \quad \text{and} \quad \mathcal{P}_j^*(\boldsymbol{h}) := -\varepsilon D_\infty^{-1}\langle \mathcal{L}(u^{\boldsymbol{h}}), k_j^{\boldsymbol{h}}\rangle = \varepsilon D_\infty^{-1}(\alpha^{j+1/2} - \alpha^{j-1/2}),$$

there exists $C > 0$ such that if ρ is sufficiently small and $\boldsymbol{h} \in \Omega_\rho$, we have

$$|\mathcal{P}(\boldsymbol{h}) - \mathcal{P}^*(\boldsymbol{h})|_\infty \le C|\mathcal{P}^*(\boldsymbol{h})|_\infty \exp(-A\ell^{\boldsymbol{h}}/2\varepsilon), \tag{4.4}$$

where $|\mathcal{P}(\boldsymbol{h})|_\infty = \max|\mathcal{P}_j(\boldsymbol{h})|$. Since $\alpha^{j-1/2} = F(\phi^j) - \frac{1}{2}\varepsilon^2\left(\phi_x^j\right)^2 = F(\phi^j(h_{j-1/2}))$, it follows that $\mathcal{P}_j(\boldsymbol{h})$ depends essentially on the differences between values of the potential $F(\phi)$.

Similar result holds for the matrix $\mathcal{G}(\boldsymbol{h})$, namely for the scalar products $\langle g(u^{\boldsymbol{h}}, \tau)u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}}\rangle$, thus generalizing the aforementioned result to the case $g \not\equiv 1$. To this end, we recall the following result (see [10, Lemmas 7.8-7.9-8.1]).

LEMMA 4.1 (Carr–Pego [10]).   *The interval $[h_{j-1} - \varepsilon, h_{j+1} + \varepsilon]$ contains the support of $u_j^{\boldsymbol{h}}$ and*

$$u_j^{\boldsymbol{h}} = \begin{cases} \chi^{j-1}\nu^j & x \in I_{j-1}, \\ (1-\chi^j)(-\phi_x^j + \nu^j) + \chi^j(-\phi_x^{j+1} - \nu^{j+1}) & \\ \quad + \chi_x^j(\phi^j - \phi^{j+1}) & x \in I_j, \\ -(1-\chi^{j+1})\nu^{j+1} & x \in I_{j+1}, \end{cases}$$

where $\nu^j(x) := \nu(x - h_{j-1/2}, h_j - h_{j-1}, (-1)^j)$ for $x \in I_j$ and there exists $r_0 > 0$ such that, for $0 < r < r_0$,

$$|\nu(x, \ell, \pm 1)| \le C\varepsilon^{-1}\beta_\pm(r), \qquad \text{for } x \in \left[-\tfrac{\ell}{2} - \varepsilon, \tfrac{\ell}{2} + \varepsilon\right]. \tag{4.5}$$

In order to compute $u_j^{\boldsymbol{h}} = \partial_{h_j}u^{\boldsymbol{h}}$, one needs to obtain an expression for ϕ_ℓ. Since ϕ solution of equation (2.4) depends on ℓ through its boundary value, the latter can be obtained by differentiating the integrated version of that equation with respect to ℓ, that is, the ε–rescaled version of equation (2.2). Finally, for $x \in [-\ell, \ell]$, we end up with

$$\phi_\ell(x, \ell, \pm 1) = \nu(x, \ell, \pm 1) - \tfrac{1}{2}(\operatorname{sgn} x)\phi_x(x, \ell, \pm 1),$$

and ν is an even function of x satisfying

$$\varepsilon^2\nu_{xx} = f'(\phi)\nu, \qquad \text{for } x \in [0, \ell];$$

see [10, Lemma 7.8] for details.

From inequality (4.5), it follows that

$$|\nu^j(x)| \le C\varepsilon^{-1}\beta^{j-1/2}, \qquad \text{for } x \in [h_{j-1} - \varepsilon, h_j + \varepsilon],$$

and so, for $x \in [h_{j-1} - \varepsilon, h_{j+1} + \varepsilon]$,

$$|(1 - \chi^j)\nu^j| + |\chi^j \nu^{j+1}| \leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\} \leq C\varepsilon^{-1} \exp(-A\ell^{\boldsymbol{h}}/2\varepsilon). \qquad (4.6)$$

Note that, for $x \in I_j$, one has

$$u_x^{\boldsymbol{h}} = (1 - \chi^j)\phi_x^j + \chi^j \phi_x^{j+1} + \chi_x^j(\phi^{j+1} - \phi^j) \text{ and } u_j^{\boldsymbol{h}} = -u_x^{\boldsymbol{h}} + (1 - \chi^j)\nu^j - \chi^j \nu^{j+1}. \quad (4.7)$$

Thanks to Lemma 4.1, we can prove the following proposition.

PROPOSITION 4.1.   *Let $F \in C^3$ be such that conditions* (1.3) *hold and $g \in C^1(\mathbb{R})$. Set*

$$C_{F,g} := \int_{-1}^{1} \sqrt{2F(s)} g(s) ds.$$

If ρ is sufficiently small and $\boldsymbol{h} \in \Omega_\rho$, then there exists $C > 0$ such that, for $j = 1, \ldots, N$,

$$\left|\langle g(u^{\boldsymbol{h}}) u_j^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle - \varepsilon^{-1} C_{F,g}\right| \leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\} \leq C\varepsilon^{-1} \exp(-A\ell^{\boldsymbol{h}}/2\varepsilon), \quad (4.8)$$

$$\left|\langle g(u^{\boldsymbol{h}}) u_j^{\boldsymbol{h}}, k_{j+1}^{\boldsymbol{h}} \rangle\right| + \left|\langle g(u^{\boldsymbol{h}}) u_{j+1}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle\right| \leq C\varepsilon^{-1} \beta^{j+1/2} \leq C\varepsilon^{-1} \exp(-A\ell^{\boldsymbol{h}}/2\varepsilon), \qquad (4.9)$$

$$\langle g(u^{\boldsymbol{h}}) u_j^{\boldsymbol{h}}, k_i^{\boldsymbol{h}} \rangle = 0 \qquad if \; |j - i| > 1. \qquad (4.10)$$

Proof.   Firstly, we recall that $k_j^{\boldsymbol{h}}$ is supported in I_j. Since the support of $u_j^{\boldsymbol{h}}$ is contained in $[h_{j-1} - \varepsilon, h_{j+1} + \varepsilon]$, we have equation (4.10). From Lemma 4.1, it follows that $|u_{j+1}^{\boldsymbol{h}}(x)| \leq C\varepsilon^{-1} \beta^{j+1/2}$, for $x \in I_j$ and $|u_j^{\boldsymbol{h}}(x)| \leq C\varepsilon^{-1} \beta^{j+1/2}$ for $x \in I_{j+1}$. Then,

$$\left|\langle g(u^{\boldsymbol{h}}) u_j^{\boldsymbol{h}}, k_{j+1}^{\boldsymbol{h}} \rangle\right| + \left|\langle g(u^{\boldsymbol{h}}) u_{j+1}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle\right| \leq C\varepsilon^{-1} \beta^{j+1/2} \left(\int_{I_j} |k_j^{\boldsymbol{h}}| + \int_{I_{j+1}} |k_{j+1}^{\boldsymbol{h}}|\right).$$

However, $u_x^{\boldsymbol{h}}$ is of one sign in I_j, thus $\int_{I_j} |k_j^{\boldsymbol{h}}| \leq C$ and we obtain inequality (4.9). It remains to prove inequality (4.8). To do this, for $x \in I_j$, we write $u_j^{\boldsymbol{h}} = y_1 + y_2$ and $k_j^{\boldsymbol{h}} = y_1 + y_3$, where

$$\begin{aligned}
y_1 &= -(1 - \chi^j)\phi_x^j - \chi^j \phi_x^{j+1}, \\
y_2 &= -\chi_x^j(\phi^{j+1} - \phi^j) + (1 - \chi^j)\nu^j - \chi^j \nu^{j+1}, \\
y_3 &= (1 - \gamma^j) u_x^{\boldsymbol{h}} - \chi_x^j(\phi^{j+1} - \phi^j).
\end{aligned}$$

Then,

$$\langle g(u^{\boldsymbol{h}}) u_j^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle = \int_{h_{j-1/2}}^{h_{j+1/2}} g(u^{\boldsymbol{h}}(x))\{y_1(x)^2 + y_1(x)(y_2(x) + y_3(x)) + y_2(x)y_3(x)\} dx.$$

From definition (2.5) and the definition of χ, it follows that

$$\int_{h_{j-1/2}}^{h_{j+1/2}} g(u^{\boldsymbol{h}}) y_1^2 dx = \int_{h_{j-1/2}}^{h_j} g(\phi^j)\left(\phi_x^j\right)^2 dx + \int_{h_j}^{h_{j+1/2}} g(\phi^{j+1})\left(\phi_x^{j+1}\right)^2 dx + E, \quad (4.11)$$

where

$$E = \int_{h_j - \varepsilon}^{h_j + \varepsilon} g(u^{\boldsymbol{h}}) y_1^2 dx - \int_{h_j - \varepsilon}^{h_j} g(\phi^j)(\phi_x^j)^2 dx - \int_{h_j}^{h_j + \varepsilon} g(\phi^{j+1})(\phi_x^{j+1})^2 dx.$$

By writing $y_1 = \chi^j(\phi_x^j - \phi_x^{j+1}) - \phi_x^j = (1-\chi^j)(\phi_x^{j+1} - \phi_x^j) - \phi_x^{j+1}$, we get

$$E = \int_{h_j-\varepsilon}^{h_j} \{g(u^{\boldsymbol{h}}) - g(\phi^j)\}(\phi_x^j)^2 dx + \int_{h_j}^{h_j+\varepsilon} \{g(u^{\boldsymbol{h}}) - g(\phi^{j+1})\}(\phi_x^{j+1})^2 dx + R.$$

Using inequality (2.14) and the estimate $|\phi_x^j| \le C\varepsilon^{-1}$, we deduce that R satisfies

$$|R| \le C\left(\int_{h_j-\varepsilon}^{h_j} g(u^{\boldsymbol{h}})|\phi_x^j||\phi_x^{j+1} - \phi_x^j| + \int_{h_j}^{h_j+\varepsilon} g(u^{\boldsymbol{h}})|\phi_x^{j+1}||\phi_x^{j+1} - \phi_x^j| \right.$$
$$\left. + \int_{h_j-\varepsilon}^{h_j+\varepsilon} g(u^{\boldsymbol{h}})|\phi_x^{j+1} - \phi_x^j|^2 dx \right) \le C\varepsilon^{-1} \max\{\alpha^{j-1/2}, \alpha^{j+1/2}\}.$$

Moreover, for inequality (2.14) we have

$$\left| \int_{h_j-\varepsilon}^{h_j} \{g(u^{\boldsymbol{h}}) - g(\phi^j)\}(\phi_x^j)^2 dx \right| \le C \int_{h_j-\varepsilon}^{h_j} |u^{\boldsymbol{h}} - \phi^j|(\phi_x^j)^2 dx$$
$$\le C\varepsilon^{-2} \int_{h_j-\varepsilon}^{h_j} \chi^j |\phi^{j+1} - \phi^j| \le C\varepsilon^{-1}|\alpha^{j-1/2} - \alpha^{j+1/2}|.$$

Similarly, we can estimate the other term and obtain $|E| \le C\varepsilon^{-1} \max\{\alpha^{j-1/2}, \alpha^{j+1/2}\}$.

Let us now compute

$$\int_{h_{j-1/2}}^{h_j} g(\phi^j)(\phi_x^j)^2 dx;$$

the other remaining term in equatio (4.11) is evaluated similarly. To do this, we observe that since $\phi^j(x) = \phi(x - h_{j-1/2}, h_j - h_{j-1}, (-1)^j)$ and $\phi(x, \ell, \pm 1)$ is solution of equation (2.4), positive or negative respectively, we have

$$\varepsilon^2(\phi_x^j)^2 = 2(F(\phi^j) - \alpha^{j-1/2}), \tag{4.12}$$

where $\alpha^{j-1/2} = F(\phi^j(h_{j-1/2}))$. In what follows, we are considering the case $\phi^j(x) < 0$ in $[h_{j-1/2}, h_j]$ (i.e. j odd); treatment of $\phi(x - h_{j-1/2}, h_j - h_{j-1}, +1)$ is similar. By using equation (4.12) and changing variable, we obtain

$$\varepsilon \int_{h_{j-1/2}}^{h_j} g(\phi^j)(\phi_x^j)^2 dx = \int_{\phi^j(h_{j-1/2})}^{0} g(s)\sqrt{2(F(s) - \alpha^{j-1/2})}\, ds$$
$$- \int_{\phi^j(h_{j-1/2})}^{0} g(s)\sqrt{2F(s)}\, ds + \int_{\phi^j(h_{j-1/2})}^{0} g(s)\sqrt{2F(s)}\, ds$$
$$= \int_{-1}^{0} g(s)\sqrt{2F(s)}\, ds - \int_{-1}^{\phi^j(h_{j-1/2})} g(s)\sqrt{2F(s)}\, ds$$
$$- \sqrt{2} \int_{\phi^j(h_{j-1/2})}^{0} \frac{\alpha^{j-1/2} g(s)}{\sqrt{F(s) - \alpha^{j-1/2}} + \sqrt{F(s)}}\, ds.$$

Since $F(s) \ge \alpha^{j-1/2}$ for $\phi^j(h_{j-1/2}) \le s \le 0$ and $F(s) \le \alpha^{j-1/2}$ for $-1 \le s \le \phi^j(h_{j-1/2})$, we have

$$\int_{-1}^{\phi^j(h_{j-1/2})} g(s)\sqrt{2F(s)}\, ds + \sqrt{2} \int_{\phi^j(h_{j-1/2})}^{0} \frac{\alpha^{j-1/2} g(s)}{\sqrt{F(s) - \alpha^{j-1/2}} + \sqrt{F(s)}}\, ds$$

$$\leq C\sqrt{\alpha^{j-1/2}} \leq C\beta^{j-1/2}.$$

Then, we can conclude that

$$\left| \int_{h_{j-1/2}}^{h_{j+1/2}} g(u^{\boldsymbol{h}}) y_1^2 \, dx - \varepsilon^{-1} C_{F,g} \right| \leq C\varepsilon^{-1} \max\left\{ \beta^{j-1/2}, \beta^{j+1/2} \right\}.$$

Moreover, from inequalities (2.14) and (4.6), it follows that for $x \in [h_{j-1/2}, h_{j+1/2}]$ we have

$$|y_2| \leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\}, \qquad |y_3| \leq |(1-\gamma^j) u_x^{\boldsymbol{h}}| + C\varepsilon^{-1} |\alpha^{j-1/2} - \alpha^{j+1/2}|.$$

We claim that

$$|(1-\gamma^j) u_x^{\boldsymbol{h}}| \leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\}. \tag{4.13}$$

Indeed, $(1-\gamma^j) u_x^{\boldsymbol{h}} = 0$ in $[h_{j-1/2} + 2\varepsilon, h_{j+1/2} - 2\varepsilon]$ and

$$u_x^{\boldsymbol{h}}(x) = \begin{cases} \phi_x^j(x) & x \in [h_{j-1/2}, h_{j-1/2} + 2\varepsilon], \\ \phi_x^{j+1}(x) & x \in [h_{j+1/2} - 2\varepsilon, h_{j+1/2}]. \end{cases}$$

Using the fact that

$$|\phi_x(x, \ell, \pm 1)| \leq C\varepsilon^{-1} \sqrt{F(\phi(x, \ell, \pm 1))} \leq C\varepsilon^{-1}(1 \mp \phi(0, \ell, \pm 1)) = C\varepsilon^{-1}\beta_\pm(r),$$

for $|x| \leq 2\varepsilon$, we obtain inequality (4.13) and so, $|y_3| \leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\}$. Therefore,

$$\left| \int_{h_{j-1/2}}^{h_{j+1/2}} g(u^{\boldsymbol{h}}) y_1 (y_2 + y_3) \right| \leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\} \int_{h_{j-1/2}}^{h_{j+1/2}} |y_1|$$

$$\leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\}.$$

Also,

$$\left| \int_{h_{j-1/2}}^{h_{j+1/2}} g(u^{\boldsymbol{h}}) y_2 y_3 \right| \leq C\varepsilon^{-2} \max\{\beta^{j-1/2}, \beta^{j+1/2}\} \varepsilon \leq C\varepsilon^{-1} \max\{\beta^{j-1/2}, \beta^{j+1/2}\},$$

because y_3 is supported on a set of measure proportional to ε. $\qquad\square$

We are ready to analyze the term $\mathcal{G}(\boldsymbol{h})$. To this aim, let us introduce the constant

$$\gamma_\tau := \frac{\sqrt{2}}{D_\infty} \int_{-1}^{1} \sqrt{F(s)} g(s, \tau) \, ds = \frac{C_{F,g}}{D_\infty}.$$

Then, in view of the results (4.8)–(4.9)–(4.10), we obtain

$$|\langle g(u^{\boldsymbol{h}}, \tau) u_i^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle| \leq C\varepsilon^{-1}, \text{ for any } i, j,$$

and, being

$$D^{-1}(\boldsymbol{h}) = \varepsilon D_\infty^{-1} \left\{ \mathbb{I}_N - \left(\mathbb{I}_N - \varepsilon D_\infty^{-1} D(\boldsymbol{h}) \right) \right\}^{-1} = \varepsilon D_\infty^{-1} \sum_{k=0}^{\infty} \left\{ \varepsilon D_\infty^{-1} \left(\varepsilon^{-1} D_\infty \mathbb{I}_N - D(\boldsymbol{h}) \right) \right\}^k,$$

one has

$$\|D^{-1}(\boldsymbol{h}) - \varepsilon D_\infty^{-1}\mathbb{I}_N\|_\infty \le \varepsilon C \exp(-A\ell^{\boldsymbol{h}}/2\varepsilon).$$

Hence, for $g = g(u^{\boldsymbol{h}}, \tau)$,

$$|\mathcal{G}_{ij}(\boldsymbol{h}) - \gamma_\tau \delta_{ij}| \le \left| \sum_{l=1}^N (D_{il}^{-1}(\boldsymbol{h}) - \varepsilon D_\infty^{-1}\delta_{il})\langle g\, u_j^{\boldsymbol{h}}, k_l^{\boldsymbol{h}} \rangle \right| + \left| \varepsilon D_\infty^{-1} \langle g\, u_j^{\boldsymbol{h}}, k_i^{\boldsymbol{h}} \rangle - \frac{C_{F,g}}{D_\infty}\delta_{ij} \right|$$

$$\le C \exp(-A\ell^{\boldsymbol{h}}/2\varepsilon). \tag{4.14}$$

Therefore, in equation (4.3) we substitute the matrix $\mathcal{G}(\boldsymbol{h})$ with $\gamma_\tau \mathbb{I}_N$.

Let us now focus our attention on the term $\tau\mathcal{Q}(\boldsymbol{h}, \boldsymbol{h}')$; analogously to the previous terms we have

$$|\mathcal{Q}(\boldsymbol{h}, \boldsymbol{h}') - \mathcal{Q}^*(\boldsymbol{h}, \boldsymbol{h}')|_\infty \le C|\mathcal{Q}^*(\boldsymbol{h}, \boldsymbol{h}')|_\infty \exp(-A\ell^{\boldsymbol{h}}/2\varepsilon),$$

where

$$\mathcal{Q}_j^*(\boldsymbol{h}, \boldsymbol{h}') := \varepsilon D_\infty^{-1} \sum_{i,l=1}^N \langle u_{il}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle h_l' h_i'.$$

Then, let us study the elements $\langle u_{il}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle$, which shall be treated in a similar way of Proposition 4.1. Since $k_j^{\boldsymbol{h}}$ is supported in I_j, it follows that $\langle u_{il}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle = 0$ if either $|i - j| > 1$ or $|l - j| > 1$. For all the remaining terms, from the expression of $u_j^{\boldsymbol{h}}$ in Lemma 4.1, and using the bounds in [10, 11], the only one which may not be exponentially small is for $i = l = j$. Therefore, here we omit the tedious, but straightforward control of such terms and we discuss only $\langle u_{jj}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle$. To this end, observe that in the interval I_j, by differentiating equations (4.7) with respect to h_j, we have

$$u_{xj}^{\boldsymbol{h}} = \phi_{jx}^j + \chi^j(\phi_{jx}^{j+1} - \phi_{jx}^j) + \chi_{xx}^j(\phi^j - \phi^{j+1}) + \chi_x^j[(\phi_j^{j+1} - \phi_j^j) - (\phi_x^{j+1} - \phi_x^j)],$$
$$u_{jj}^{\boldsymbol{h}} = -u_{xj}^{\boldsymbol{h}} + \nu_j^j - \chi^j(\nu_j^j + \nu_j^{j+1}) - \chi_j^j(\nu^j + \nu^{j+1}).$$

Using

$$\phi_j^j = -\phi_x^j + \nu^j, \quad \phi_j^{j+1} = -\phi_x^{j+1} - \nu^{j+1}, \quad \phi_{jx}^j = -\phi_{xx}^j + \nu_x^j, \quad \phi_{jx}^{j+1} = -\phi_{xx}^{j+1} - \nu_x^{j+1},$$

and from the expression of $u_{xx}^{\boldsymbol{h}}$ obtained again from equations (4.7), in $x \in I_j$ we infer

$$u_{xj}^{\boldsymbol{h}} = -u_{xx}^{\boldsymbol{h}} + \nu_x^j - \chi^j(\nu_x^{j+1} + \nu_x^j) - \chi_x^j(\nu^{j+1} + \nu^j).$$

Hence, we can conclude that $u_{jj}^{\boldsymbol{h}} = u_{xx}^{\boldsymbol{h}} + R^j$, and so

$$\langle u_{jj}^{\boldsymbol{h}}, k_j^{\boldsymbol{h}} \rangle = -\int_{I_j} u_{xx}^{\boldsymbol{h}} u_x^{\boldsymbol{h}}\, dx + \int_{I_j} u_{xx}^{\boldsymbol{h}}(1 - \gamma^j) u_x^{\boldsymbol{h}}\, dx + \int_{I_j} R^j k_j^{\boldsymbol{h}}\, dx.$$

Reasoning as in the proof of the Proposition 4.1, and taking into account the needed bounds for the higher involved derivatives [10, 11], one can prove that the last two integrals are exponentially small, whereas for the first integral we obtain

$$\int_{h_{j-1/2}}^{h_{j+1/2}} \left(-u_{xx}^{\boldsymbol{h}}(x) u_x^{\boldsymbol{h}}(x) \right) dx = \frac{1}{2}\left(u_x^{\boldsymbol{h}}(h_{j-1/2})^2 - u_x^{\boldsymbol{h}}(h_{j+1/2})^2 \right) = 0,$$

because $u_x^{\boldsymbol{h}}(h_{j-1/2})=0$ for all $j=1,\ldots,N$. Therefore, we obtain that there exists $c>0$ such that

$$|\mathcal{Q}(\boldsymbol{h},\boldsymbol{h}')|_\infty \leq C\exp(-c/\varepsilon)|\boldsymbol{h}'|_\infty^2. \tag{4.15}$$

In conclusion, using the estimates (4.4), (4.14) and (4.15), and neglecting all the exponentially small terms in equation (4.3), we end up with the reduced system

$$\tau\boldsymbol{h}''+\gamma_\tau\boldsymbol{h}'=\mathcal{P}^*(\boldsymbol{h}). \tag{4.16}$$

In the case of the damped wave equation with bistable nonlinearity, $g(u,\tau)\equiv 1$, and therefore $\gamma_\tau=1$. Moreover, for the Allen–Cahn equation with relaxation, $g(u,\tau)=1+\tau f'(u)$ and

$$\gamma_\tau=1+\frac{\sqrt{2}\tau}{D_\infty}\int_{-1}^1\sqrt{F(s)}F''(s)\,ds=1-\frac{\tau}{D_\infty}\int_{-1}^1\frac{(F'(s))^2}{\sqrt{2F(s)}}\,ds<1.$$

Hence, in the latter case, the effect of the parameter $\tau>0$ is present also in the friction term $\gamma_\tau\boldsymbol{h}'$, and in particular it speeds up the dynamics with respect to the simpler nonlinear damped wave equation, being the coefficient smaller. This richer effect on the dynamics in the present analysis confirms what has been already observed in the study of traveling waves in [33], where again the relaxation parameter τ in the case of the Allen–Cahn equation with relaxation affects the speed of the wave also though a modification of the friction effects.

4.2. Comparison with the parabolic case.   Now, if $\gamma_\tau\to 1$ as $\tau\to 0$, taking formally the limit in equation (4.16) we obtain the system $\boldsymbol{h}'=\mathcal{P}^*(\boldsymbol{h})$. The structure of solutions of this system of ordinary differential equations is studied in [10] to describe the evolution of the layer positions in the parabolic case (1.7). We can write

$$\mathcal{P}^*(\boldsymbol{h})=-\nabla W(\boldsymbol{h}),$$

where W is defined in the following way. For $s>\rho^{-1}$, define $W_\pm$ by $W_\pm'(s)=D_\infty^{-1}\alpha_\pm(s^{-1})$ and set $W_j=W_+$ for j even, $W_j=W_-$ for j odd. For $\boldsymbol{h}\in\Omega_\rho$ let

$$W(\boldsymbol{h}):=\varepsilon^2\left[\tfrac{1}{2}W_1((h_1-h_0)/\varepsilon)+\sum_{j=2}^N W_j((h_j-h_{j-1})/\varepsilon)+\tfrac{1}{2}W_{N+1}((h_{N+1}-h_N)/\varepsilon)\right].$$

Since $h_0=-h_1$ and $h_{N+1}=2-h_N$, we have

$$\frac{\partial W(\boldsymbol{h})}{\partial h_j}=\varepsilon\left[W_j'((h_j-h_{j-1})/\varepsilon)-W_{j+1}'((h_{j+1}-h_j)/\varepsilon)\right]$$

$$=-\varepsilon D_\infty^{-1}(\alpha^{j+1/2}-\alpha^{j-1/2}).$$

Then, we can write equation (4.16) as $\tau\boldsymbol{h}''+\boldsymbol{h}'=-\nabla W(\boldsymbol{h})$. For a solution $\boldsymbol{h}$ with values in Ω_ρ, the energy $\mathsf{E}_\tau=\tfrac{1}{2}\tau|\boldsymbol{h}'|^2+W(\boldsymbol{h})$ is nonincreasing and

$$\frac{d\mathsf{E}_\tau}{dt}=\tau\boldsymbol{h}'\cdot\boldsymbol{h}''+\nabla W(\boldsymbol{h})\cdot\boldsymbol{h}'=-\gamma_\tau|\boldsymbol{h}'|^2\leq 0.$$

Note that γ_τ large implies a greater dissipation of energy.

PROPOSITION 4.2 (Carr–Pego [10]).   *If ρ is sufficiently small, then the function W has a unique critical point $\boldsymbol{h}^e$, which is a strict local maximum.*

Proof.   If $\boldsymbol{h}^e$ is a critical point of W, then $\alpha^{j-1/2} = \alpha^{j+1/2}$ for $j = 1,\dots,N$. Define $\ell_j := h_j - h_{j-1}$. From Proposition 2.3, it follows that for r sufficiently small $\alpha_{\pm}(r)$ are monotone and so, $\ell_j = \ell_{j+2}$ for $j = 1,\dots,N-1$. Let $\ell_- = \ell_1$, $\ell_+ = \ell_2$. Since $\boldsymbol{h} \in \Omega_\rho$, we have that $\frac{1}{2}\ell_1 + \ell_2 + \cdots + \ell_N + \frac{1}{2}\ell_{+1} = 1$ and then $\ell_- + \ell_+ = 2/N$. The condition $\alpha^{j-1/2} = \alpha^{j+1/2}$, also, gives $\alpha_+(\varepsilon/\ell_+) = \alpha_-(\varepsilon/\ell_-)$. Using Proposition 2.3, we have that critical points correspond to zeros of

$$\gamma = \varepsilon^{-1}(A_-\ell_- - A_+\ell_+) + 2\ln(K_+A_+/K_-A_-) + O\big(\rho^{-1}\exp(-A/2\rho)\big),$$

for $\ell_- \in [\varepsilon\rho^{-1}, 2N^{-1} - \varepsilon\rho^{-1}]$ with $\ell_+ = 2N^{-1} - \ell_-$. For ρ sufficiently small, $\gamma > 0$ when $\ell_- = \varepsilon\rho^{-1}$ and $\gamma < 0$ when $\ell_- = 2N^{-1} - \varepsilon\rho^{-1}$; hence a critical point must exist. It is unique, because α_+ and α_- are monotone.

To determine the nature of the critical point, consider the Hessian of W, $B_{ij} = \partial^2 W(\boldsymbol{h}^e)/\partial h_i \partial h_j$. The matrix B is symmetric and tri-diagonal with

$$\begin{aligned} B_{11} &= 2\omega_1 + \omega_2, & B_{jj} &= \omega_j + \omega_{j+1}, & &\text{for } j = 2,\dots,N-1, \\ B_{NN} &= \omega_N + 2\omega_{N+1}, & B_{j,j+1} &= -\omega_{j+1}, & &\text{for } j = 1,\dots,N-1, \end{aligned}$$

where $\omega_j = W_j''((h_j^e - h_{j-1}^e)/\varepsilon)$. From Proposition 2.3, $w_j < 0$ for $j = 1,\dots,N$. Then, for $y \in \mathbb{R}^N$,

$$\sum_{i,j=1}^N B_{ij} y_i y_j = 2\omega_1 y_1^2 + \sum_{j=2}^N \omega_j(y_{j-1} - y_j)^2 + 2\omega_{N+1} y_N^2,$$

so that B is negative definite and $\boldsymbol{h}^e$ is a local maximum.   □

If f is odd, all the $\pm$ subscripts can be ignored, e.g. $\alpha_+ = \alpha_-$, $\ell_+ = \ell_-$ and $h_j^e - h_{j-1}^e = 1/N$ for $j = 1,\dots,N$. In general, the steady state domain lengths satisfy

$$\begin{aligned} \ell_- &= 2\left(A_+/N + \varepsilon\ln(K_-A_-/K_+A_+)\right)/(A_+ + A_-) + O\big(\rho^{-1}\exp(-A/2\rho)\big), \\ \ell_+ &= 2\left(A_-/N + \varepsilon\ln(K_+A_+/K_-A_-)\right)/(A_+ + A_-) + O\big(\rho^{-1}\exp(-A/2\rho)\big). \end{aligned}$$

Thus, system (4.16) has a unique equilibrium point $(\boldsymbol{h}^e,0)$ with $\boldsymbol{h}^e \in \Omega_\rho$. From Proposition 2.4, $\mathcal{L}(u^{\boldsymbol{h}^e}) = 0$ and so $u^{\boldsymbol{h}^e}$ is a stationary solution of equation (1.6). In other words, there is a unique stationary solution u^e of equation (1.6) with N transition layers and $u(0) < 0$; moreover, $u^e \in \mathcal{M}$ with $u^e = u^{\boldsymbol{h}^e}$ where $\boldsymbol{h}^e = \mathcal{H}(u^e)$. Note, also, that by definition (2.10), $\Psi(\boldsymbol{h}) = 0$ if and only if $\boldsymbol{h} = \boldsymbol{h}^e$ so that the channel $\mathcal{Z}_{\Gamma,\rho}$ is "pinched" at $u = u^e$.

Now, let us study the stability of the equilibrium point $(\boldsymbol{h}^e,0)$ for system (4.16). To do this, rewrite it as the first order system

$$\begin{cases} \boldsymbol{h}' = \boldsymbol{\eta}, \\ \tau\boldsymbol{\eta}' = \mathcal{P}^*(\boldsymbol{h}) - \gamma_\tau\boldsymbol{\eta}. \end{cases} \tag{4.17}$$

PROPOSITION 4.3.   *System (4.17) has a unique equilibrium point $(\boldsymbol{h}^e,0)$, which is unstable. In particular, the Jacobian matrix evaluated at $(\boldsymbol{h}^e,0)$ has N negative eigenvalues and N positive eigenvalues.*

Proof.   From Lemma 4.2, it follows that the system (4.17) has a unique equilibrium point $(\boldsymbol{h}^e, 0)$. To determine the stability of this stationary point, we have to analyze the eigenvalues of the block matrix

$$J = \begin{pmatrix} 0_N & \mathbb{I}_N \\ -\frac{1}{\tau}B & -\frac{\gamma_\tau}{\tau}\mathbb{I}_N \end{pmatrix},$$

where, as above, B is the Hessian matrix of W evaluated at $\boldsymbol{h}^e$. To this end, we make use of the Schur complement, defined for a general block $n \times m$ matrix

$$M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

as follows: $M/A_1 = A_4 - A_3 A_1^{-1} A_2$, provided A_1 is an invertible square matrix. In this case, $\det(M) = \det(A_1) \det(M/A_1)$. For $M_\lambda = J - \lambda \mathbb{I}_{2N}$, we have

$$\det(M_\lambda) = \det \begin{pmatrix} -\lambda \mathbb{I}_N & \mathbb{I}_N \\ -\frac{1}{\tau}B & -\left(\frac{\gamma_\tau}{\tau} + \lambda\right)\mathbb{I}_N \end{pmatrix} = \det(-\lambda \mathbb{I}_N) \det(M_\lambda/(-\lambda \mathbb{I}_N)),$$

where

$$M_\lambda/(-\lambda \mathbb{I}_N) = -\left(\frac{\gamma_\tau}{\tau} + \lambda\right)\mathbb{I}_N - \frac{1}{\tau\lambda}B.$$

Then,

$$\det M_\lambda = (-\lambda)^N \det\left(-\frac{1}{\tau\lambda}B - \left(\frac{\gamma_\tau + \tau\lambda}{\tau}\right)\mathbb{I}_N\right) = \frac{1}{\tau^N}\det\left(B + (\gamma_\tau\lambda + \tau\lambda^2)\mathbb{I}_N\right).$$

It follows that λ is an eigenvalue of J if and only if $-\tau\lambda^2 - \gamma_\tau\lambda$ is an eigenvalue of B. As previously shown in Lemma 4.2, B is symmetric and negative definite, so all the eigenvalues of B are negative. Denote them by $-\mu_i^2$ for $i = 1, \ldots, N$. For each $-\mu_i^2$ there are two eigenvalues of J:

$$\lambda_i^+(\tau) = \frac{-\gamma_\tau + \sqrt{\gamma_\tau^2 + 4\tau\mu_i^2}}{2\tau} > 0, \qquad \lambda_i^-(\tau) = \frac{-\gamma_\tau - \sqrt{\gamma_\tau^2 + 4\tau\mu_i^2}}{2\tau} < 0.$$

In conclusion, the Jacobian matrix evaluated at $(\boldsymbol{h}^e, 0)$ has N positive eigenvalues $\lambda_i^+(\tau)$ and N negative eigenvalues $\lambda_i^-(\tau)$, and so $(\boldsymbol{h}^e, 0)$ is unstable. The eigenvalues satisfy

$$\lim_{\tau \to 0^+} \lambda_i^+(\tau) = \frac{\mu_i^2}{\gamma_0}, \qquad \lim_{\tau \to 0^+} \lambda_i^-(\tau) = -\infty,$$

if $\lim_{\tau \to 0^+} \gamma_\tau =: \gamma_0 > 0$. In particular, if $\gamma_0 = 1$, $\lambda_i^+(\tau)$ converge to the eigenvalues of the parabolic case, as expected.   □

To conclude this section, we use singular perturbation theory to compare, for τ small, the solutions of the system (4.17) and the ones of

$$\begin{cases} \boldsymbol{h}' = \boldsymbol{\eta}, \\ \boldsymbol{\eta} = \mathcal{P}^*(\boldsymbol{h}), \end{cases} \tag{4.18}$$

that is obtained by substituting $\tau = 0$ in system (4.17), assuming that $\gamma_\tau \to 1$ as $\tau \to 0$. Denote by $(\boldsymbol{h}_p, \boldsymbol{\eta}_p)$ the solutions of system (4.18); $\boldsymbol{h}_p$ is solution of the system $\boldsymbol{h}' = \mathcal{P}^*(\boldsymbol{h})$, that describes the evolution of layer positions in the parabolic case (1.7). Set

$$\mathrm{E}_\tau(t) := \gamma_\tau |\boldsymbol{h}(t) - \boldsymbol{h}_p(t)| + \tau |\boldsymbol{\eta}(t) - \boldsymbol{\eta}_p(t)|.$$

A general theorem of Tihonov on singular perturbations could be applied to systems (4.17)-(4.18). Specifically for the system (4.17) we have the following result.

THEOREM 4.1.   *Let $(\boldsymbol{h},\boldsymbol{\eta})$ be a solution of system (4.17) and $(\boldsymbol{h}_p,\boldsymbol{\eta}_p)$ a solution of system (4.18), with $\boldsymbol{h}(t),\boldsymbol{h}_p(t)\in\Omega_\rho$ for any $t\in[0,T]$. Then, there exists $C>0$ (independent of τ) such that*

$$\mathrm{E}_\tau(t)\leq C(\mathrm{E}_\tau(0)+1-\gamma_\tau+\tau), \qquad \text{for } t\in[0,T]. \tag{4.19}$$

Moreover,

$$\int_0^T|\boldsymbol{\eta}(t)-\boldsymbol{\eta}_p(t)|dt\leq\frac{C}{\gamma_\tau}(\mathrm{E}_\tau(0)+1-\gamma_\tau+\tau), \tag{4.20}$$

$$|\boldsymbol{\eta}(t)-\boldsymbol{\eta}_p(t)|\leq\frac{C}{\gamma_\tau}(\mathrm{E}_\tau(0)+1-\gamma_\tau+\tau), \qquad \text{for } t\in[t_1,T], \tag{4.21}$$

for all $t_1\in(0,T)$. In particular, from inequalities (4.19), (4.20) and (4.21), it follows that, if $\gamma_\tau\to 1$ and $\mathrm{E}_\tau(0)\to 0$ as $\tau\to 0$, then

$$\lim_{\tau\to 0}\sup_{t\in[0,T]}|\boldsymbol{h}(t)-\boldsymbol{h}_p(t)|=\lim_{\tau\to 0}\int_0^T|\boldsymbol{\eta}(t)-\boldsymbol{\eta}_p(t)|dt=\lim_{\tau\to 0}\sup_{t\in[t_1,T]}|\boldsymbol{\eta}(t)-\boldsymbol{\eta}_p(t)|=0,$$

for any $t_1\in(0,T)$.

   Proof.   For $t\in[0,T]$, define

$$\boldsymbol{\delta_h}(t):=\boldsymbol{h}(t)-\boldsymbol{h}_p(t), \qquad \boldsymbol{\delta_\eta}(t):=\boldsymbol{\eta}(t)-\boldsymbol{\eta}_p(t).$$

By hypotheses, $\boldsymbol{h}(t),\boldsymbol{h}_p(t)\in\Omega_\rho$ for $t\in[0,T]$, so $|\boldsymbol{h}(t)|$ and $|\boldsymbol{h}_p(t)|$ are uniformly bounded in $[0,T]$. Since $\mathcal{P}^*(\boldsymbol{h})$ is a regular function of $\boldsymbol{h}$, there exists $C>0$ such that

$$|\mathcal{P}^*(\boldsymbol{h}_p)|\leq C, \qquad |J\mathcal{P}^*(\boldsymbol{h}_p)|\leq C, \qquad |\mathcal{P}^*(\boldsymbol{h}_p+\boldsymbol{\delta_h})-\mathcal{P}^*(\boldsymbol{h}_p)|\leq C|\boldsymbol{\delta_h}|, \tag{4.22}$$

for all $t\in[0,T]$. Here and in what follows, C is a positive constant independent of τ whose value may change from line to line. We have

$$\boldsymbol{\delta_h}'=\boldsymbol{\eta}-\boldsymbol{\eta}_p, \qquad \tau\boldsymbol{\delta_\eta}'=\mathcal{P}^*(\boldsymbol{h}_p+\boldsymbol{\delta}_p)-\gamma_\tau\mathcal{P}^*(\boldsymbol{h}_p)-\gamma_\tau\boldsymbol{\delta_\eta}-\tau J\mathcal{P}^*(\boldsymbol{h}_p)\mathcal{P}^*(\boldsymbol{h}_p).$$

Since $\dfrac{d}{dt}|\boldsymbol{\delta}|=\dfrac{\boldsymbol{\delta}'\cdot\boldsymbol{\delta}}{|\boldsymbol{\delta}|}$ for any $\boldsymbol{\delta}(t)\in\mathbb{R}^N$, using inequalities (4.22) and the Cauchy–Schwarz inequality, we obtain

$$\frac{d}{dt}|\boldsymbol{\delta_h}|\leq|\boldsymbol{\delta_\eta}|, \qquad \tau\frac{d}{dt}|\boldsymbol{\delta_\eta}|\leq C\gamma_\tau|\boldsymbol{\delta_h}|+(1-\gamma_\tau)C-\gamma_\tau|\boldsymbol{\delta_\eta}|+\tau C.$$

Summing, one has

$$\frac{d}{dt}\left(\gamma_\tau|\boldsymbol{\delta_h}|+\tau|\boldsymbol{\delta_\eta}|\right)\leq C\gamma_\tau|\boldsymbol{\delta_h}|+C(1-\gamma_\tau+\tau),$$

and so

$$\frac{d}{dt}\mathrm{E}_\tau(t)\leq C(\mathrm{E}_\tau(t)+1-\gamma_\tau+\tau), \qquad \text{for } t\in[0,T]. \tag{4.23}$$

Integrating inequality (4.23) and applying Grönwall's Lemma, we obtain inequality (4.19). In particular, from inequality (4.19), it follows that

$$\gamma_\tau |\boldsymbol{\delta_h}(t)| \le C(\mathrm{E}_\tau(0) + 1 - \gamma_\tau + \tau), \qquad \text{for } t \in [0, T]. \qquad (4.24)$$

Substituting inequality (4.24) in the equation for $\boldsymbol{\delta_\eta}$, we obtain

$$\tau \frac{d}{dt}|\boldsymbol{\delta_\eta}| \le -\gamma_\tau |\boldsymbol{\delta_\eta}| + C(\mathrm{E}_\tau(0) + 1 - \gamma_\tau + \tau). \qquad (4.25)$$

Integrating inequality (4.25), we obtain inequality (4.20). Furthermore, for inequality (4.25), we have

$$\frac{d}{dt}\left(\tau e^{\gamma_\tau t/\tau}|\boldsymbol{\delta_\eta}(t)|\right) \le C(\mathrm{E}_\tau(0) + 1 - \gamma_\tau + \tau)e^{\gamma_\tau t/\tau},$$

and so

$$\begin{aligned}
|\boldsymbol{\delta_\eta}(t)| &\le \frac{C}{\gamma_\tau}(\mathrm{E}_\tau(0) + 1 - \gamma_\tau + \tau)\left(1 - e^{-\gamma_\tau t/\tau}\right) + |\boldsymbol{\delta_\eta}(0)|e^{-\gamma_\tau t/\tau} \\
&\le \frac{C}{\gamma_\tau}(\mathrm{E}_\tau(0) + 1 - \gamma_\tau + \tau) + \mathrm{E}_\tau(0)\frac{e^{-\gamma_\tau t/\tau}}{\tau},
\end{aligned}$$

for $t \in [0, T]$. Therefore, for any fixed $t_1 \in (0, T)$, we obtain inequality (4.21). $\qquad\square$

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