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LONG-TIME BEHAVIOR OF SOLUTIONS TO THE NON-ISENTROPIC EULER-POISSON SYSTEM IN $\mathbb{R}^3$*

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Abstract. We study the global existence and asymptotic behavior of smooth solutions near a non-flat steady state to the compressible non-isentropic Euler–Poisson system in $\mathbb{R}^3$. Using some concise energy estimates and an interpolation trick, we show that the solution converges to the stationary solution exponentially fast. Here our results can follow from that the H^3 norms of the initial density, velocity and temperature are small. In this sense, we reduce the regularity of the initial temperature in [Y.P. Li, J. Differential Equations, 225:134–167, 2006].

Keywords. Non-isentropic Euler–Poisson system; long-time behavior; energy method; interpolation

AMS subject classifications. 35M10; 35Q60; 76N10; 35Q35; 35B40

1. Introduction

In the present paper, we consider the Cauchy problem of the compressible non-isentropic Euler–Poisson system of one carrier type

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t u + u\cdot\nabla u + \dfrac{1}{\rho}\nabla(\rho\Theta) = \nabla\phi - \dfrac{u}{\tau_1}, \\ \partial_t \Theta + u\cdot\nabla\Theta + \dfrac{2}{3}\Theta\,\mathrm{div}\,u - \dfrac{2}{3\rho}\mathrm{div}(\kappa\nabla\Theta) = \dfrac{2\tau_2-\tau_1}{3\tau_1\tau_2}|u|^2 - \dfrac{\Theta - T}{\tau_2}, \\ \Delta\phi = \rho - b, \\ (\rho,u,\Theta)(x,t)|_{t=0} = (\rho_0,u_0,\Theta_0)(x), \quad (x,t)\in\mathbb{R}^3\times[0,+\infty). \end{cases} \tag{1.1}$$

The unknown functions ρ, u, Θ, ϕ represent the electron density, the electron velocity, the absolutely temperature and the electrostatic potential, respectively. The coefficients κ, τ_1 and τ_2 denote the thermal conductivity, the velocity relaxation time and the temperature relaxation time, respectively. The functions $T = T(x)$ and $b = b(x)$ are the ambient device temperature and the doping profile, respectively.

The Euler–Poisson system as a hydrodynamic model is usually used to describe the transmission of charged fluid particles in semiconductor devices [28, 40] or in plasma

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physics [4, 50]. Due to its physical importance, the Euler–Poisson system has attracted considerable attention. About the stationary solution: Degond and Markowich [6] proved the existence and uniqueness of subsonic solutions in one-dimensional (1-D) interval under a smallness assumption on the current, and later Gamba [10] removed the smallness and constructed a transonic weak solution; Degond and Markowich also extended their 1-D results to the three-dimensional (3-D) irrotational case in [7]; the readers can also refer to [9, 38, 51] and the references therein. About the stability of stationary solutions: Luo, Natalini and Xin [36] showed the stability of stationary solutions with exponential decay rates for the Cauchy problem of 1-D isentropic system under zero steady current density; [36]'s work was extended by Hsiao et al. [17, 18] to 1-D initial-boundary value problem, by Hsiao, Ju and Wang [16] and Wang and Tan [55] to N-D ($N=2,3$) isentropic cases, by Alì et al. [1, 2] to N-D ($N\geqslant 1$) non-isentropic cases with zero-thermoconductivity and by Huang et al. [25, 26] to the 1-D and 3-D whole space cases with nonzero steady current density. For 1-D initial-boundary value problem with nonzero steady current density, Li, Markowich and Mei [30] showed the stability of smooth subsonic steady-state solution with exponential decay rates under flatness assumption on the doping profile, and Nishibata and Suzuki showed the similar results by removing the flatness assumption in [45] and extended them to non-isentropic case in [46]; maybe earlier, Guo and Strauss [13] extended [30]'s work to the 3-D case without flatness assumption on the doping profile. Regarding other topics, including entropy weak solutions, shock schemes and relaxation limits, and so on, the readers can refer to [5, 12, 20, 21, 27, 29, 31, 33, 39, 48, 49, 53, 54, 56, 57] and the references therein.

In this paper, our main purpose is to relax the regularity of the initial temperature in [32], where Li extended Alì's work [1] to the case with thermoconductivity. Since people are more and more interested in the bipolar Euler–Poisson system, we also review some research results about it in the following.

In the 1-D case, Gasser, Hsiao and Li [11] and Huang and Li [22] studied the asymptotic behavior of both small smooth and weak solutions, respectively. Natalini [43] and Hsiao and Zhang [19] constructed the global entropy weak solutions on the whole real line and some bounded domain by the method of compensated compactness, respectively. Zhu and Hattori [58] showed the stability of steady-state solutions for a recombined bipolar Euler–Poisson system. Huang et al. [24] proved the stability of the stationary solutions with Dirichlet or Neumann boundary conditions. Different from the previous series of studies for the case with two identical pressure functions and zero doping profile, Donatelli et al. [8] studied the Cauchy problem with two different pressure functions and a non-flat doping profile. In multi-dimensional case, Huang, Mei and Wang [23] showed the stability of planar diffusion waves by ingeniously constructing some new correct functions to delete the gaps between the original solutions and the diffusion waves in L^2 space. In addition, the readers can refer to [3, 15, 34, 35, 42, 47, 52] and the references therein.

Notation. In this paper, we use $H^k(\mathbb{R}^3)$, $k\in\mathbb{N}$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^k}$ and $L^p(\mathbb{R}^3)$, $1\leqslant p\leqslant+\infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. For $p=2$, we simply denote $\|\cdot\|$. Throughout this paper, we let C denote some positive universal constants. We will use $f\lesssim g$ if $f\leqslant Cg$ and $f\gtrsim g$ if $f\geqslant Cg$. And $f\sim g$ means that $f\lesssim g$ and $g\lesssim f$. For simplicity, we write $\|(A,B)\|_X:=\|A\|_X+\|B\|_X$ and $\int f:=\int_{\mathbb{R}^3} f\,dx$.

Without loss of generality, we set $\kappa=\tau_1=\tau_2=1$ in system (1.1). We assume that

$b(x)$ satisfies

$$\lim_{|x|\to+\infty} b(x)=\bar{b}>0,\ b(x)>0,\ b(x)\in C^{k+3}(\mathbb{R}^3) \text{ and } \nabla b(x)\in H^{k+2}(\mathbb{R}^3),\ k\geqslant 3, \tag{1.2}$$

and $T(x)$ satisfies

$$\lim_{|x|\to+\infty} T(x)=\bar{T}>0,\ T(x)>0,\ T(x)\in C^{k+3}(\mathbb{R}^3) \text{ and } \nabla T(x)\in H^{k+2}(\mathbb{R}^3),\ k\geqslant 3. \tag{1.3}$$

Now we consider the steady-state system when the velocity $u\equiv 0$ (the thermal equilibrium state), namely, we investigate the stationary solution (ρ_s,Θ_s,ϕ_s) of the system

$$\begin{cases} \nabla(\rho_s\Theta_s)=\rho_s\nabla\phi_s,\\ \frac{2}{3\rho_s}\Delta\Theta_s=\Theta_s-T,\\ \Delta\phi_s=\rho_s-b, \end{cases} \tag{1.4}$$

under the assumptions of

$$\rho_s-b\in H^{k+2}(\mathbb{R}^3), \quad \Theta_s-T\in H^{k+2}(\mathbb{R}^3). \tag{1.5}$$

We record the following proposition about the existence and uniqueness of the stationary solution.

PROPOSITION 1.1. *Let b and T satisfy conditions (1.2) and (1.3) respectively. Moreover, we assume that $\|(\nabla b,\nabla T)\|_{H^5}$ is sufficiently small. Then the system (1.4)–(1.5) has a unique classical solution (ρ_s,Θ_s,ϕ_s) such that*

$$0<\inf_{x\in\mathbb{R}^3} b(x)\leqslant\rho_s(x)\leqslant\sup_{x\in\mathbb{R}^3} b(x),\ 0<\inf_{x\in\mathbb{R}^3} T(x)\leqslant\Theta_s(x)\leqslant\sup_{x\in\mathbb{R}^3} T(x), \tag{1.6}$$

and

$$\|(\nabla\rho_s,\nabla\Theta_s)\|_{H^{k+2}}\lesssim\|(\nabla b,\nabla T)\|_{H^{k+2}}. \tag{1.7}$$

Proof. We can refer to [32] for $k=4$, but the case $k>4$ can be handled in the same fashion and so we omit the details. In fact, the smallness of $\|(\nabla b,\nabla T)\|_{H^5}$ could be deleted by referring to [16] when we consider the existence of the stationary solution. However, the smallness of $\|(\nabla b,\nabla T)\|_{H^5}$ is necessary for the uniqueness of the stationary solution from the proof of [32]. ∎

We define the perturbation

$$n=\rho-\rho_s,\ u=u,\ \theta=\Theta-\Theta_s,\ \Phi=\phi-\phi_s.$$

Then the Cauchy problem (1.1) is reformulated equivalently as

$$\begin{cases} \partial_t n=-u\cdot\nabla(n+\rho_s)-(n+\rho_s)\operatorname{div}u,\\ \partial_t u+u-\nabla\Phi+\nabla\theta=-u\cdot\nabla u-\dfrac{\theta+\Theta_s}{n+\rho_s}\nabla n-\dfrac{\theta\rho_s-n\Theta_s}{\rho_s(n+\rho_s)}\nabla\rho_s,\\ \partial_t\theta+\theta-\dfrac{2}{3(n+\rho_s)}\Delta\theta=-u\cdot\nabla(\theta+\Theta_s)-\dfrac{2}{3}(\theta+\Theta_s)\operatorname{div}u-\dfrac{2n\Delta\Theta_s}{3\rho_s(n+\rho_s)}+\dfrac{1}{3}|u|^2,\\ \Delta\Phi=n,\\ (n,u,\theta)(x,t)|_{t=0}=(n_0,u_0,\theta_0)(x), \quad (x,t)\in\mathbb{R}^3\times[0,+\infty). \end{cases} \tag{1.8}$$

Our main result is stated in the following theorem:

THEOREM 1.1. *Let $k \geqslant 3$. Assume that $n_0, u_0, \theta_0 \in H^k$, $\nabla b, \nabla T \in H^{k+2}$ and $\nabla \Phi_0 \in L^2$. If $\|(n_0,u_0,\theta_0)\|_{H^k} + \|\nabla\Phi_0\| + \|(\nabla b, \nabla T)\|_{H^5}$ is sufficiently small, then there exists a unique global solution $(n,u,\theta,\nabla\Phi)$ to the Cauchy problem (1.8) such that for some constant $\beta > 0$,*

$$\|(n,u,\theta,\nabla\Phi)(t)\|_{H^k}^2 \lesssim \|(n_0,u_0,\theta_0,\nabla\Phi_0)\|_{H^k}^2 e^{-\beta t}. \tag{1.9}$$

In the following, we give some remarks.

REMARK 1.1. In [32], the initial density and velocity belong to H^3, however, the initial temperature has to lie in H^4. Here, we manage to relax the initial temperature to H^3 by using some detailed estimates in Section 2. In fact, we need to carefully deal with the difficulty resulted by the terms involved with $\nabla^2\theta$. We can overcome this difficulty by using an interpolation trick, like estimate (2.17).

REMARK 1.2. In this paper, we solve the Cauchy problem (1.8) with the hyperbolic-parabolic-elliptic structure. So, our methods could be applied to the other equations with the similar structure, for instance, non-isentropic bipolar Euler–Poisson system, MHD equations with the Coulomb force, etc.

REMARK 1.3. Note that the smallness of $\|(\nabla b, \nabla T)\|_{H^5}$ in Theorem 1.1 may be too rigid. It is possible to relax it and this work will be left in the future.

REMARK 1.4. Similar result for isentropic Euler–Poisson system was obtained in [55]. Here, for more complicated non-isentropic Euler–Poisson system, we need to do more detailed estimates to control the temperature.

The present paper is structured as follows. In Section 2, we will establish the refined energy estimates for the solution to the Euler–Poisson system (1.8). We will prove the Theorem 1.1 in Section 3.

2. Nonlinear energy estimates

In this section, we will do the a priori estimates by assuming that for $k \geqslant 3$,

$$\|(n,u,\theta)(t)\|_{H^k} + \|\nabla\Phi(t)\| + \|(\nabla b, \nabla T)\|_{H^5} \leqslant \delta \ll 1. \tag{2.1}$$

Then by Sobolev's inequality, we have for some $b_0 > 0$ and some $T_0 > 0$,

$$\frac{b_0}{2} \leqslant n + \rho_s \leqslant 2b_0, \quad \frac{T_0}{2} \leqslant \theta + \Theta_s \leqslant 2T_0. \tag{2.2}$$

2.1. Preliminary.

In this subsection, we collect the analytic tools used later in the paper. We first recall the Sobolev interpolation of the Gagliardo–Nirenberg inequality.

LEMMA 2.1. *Let $2 \leqslant p \leqslant +\infty$ and $\alpha,\beta,\gamma \geqslant 0$. Then we have*

$$\|\nabla^\alpha f\|_{L^p} \lesssim \left\|\nabla^\beta f\right\|^{1-\vartheta} \|\nabla^\gamma f\|^{\vartheta}. \tag{2.3}$$

Here $0 \leqslant \vartheta \leqslant 1$ (if $p = +\infty$, then we require that $0 < \vartheta < 1$) and α satisfy

$$\alpha + 3\left(\frac{1}{2} - \frac{1}{p}\right) = \beta(1-\vartheta) + \gamma\vartheta.$$

Proof. We can refer to Theorem (p. 125) in [44] or Lemma A.1 in [14]. □

We recall the following commutator and product estimates:

LEMMA 2.2. *Let $l\geqslant 1$ be an integer and define the commutator*

$$[\nabla^l,g]\,h=\nabla^l(gh)-g\nabla^l h. \tag{2.4}$$

Then we have

$$\left\|[\nabla^l,g]\,h\right\|_{L^{p_0}}\lesssim\|\nabla g\|_{L^{p_1}}\left\|\nabla^{l-1}h\right\|_{L^{p_2}}+\left\|\nabla^l g\right\|_{L^{p_3}}\|h\|_{L^{p_4}}. \tag{2.5}$$

In addition, we have that for $l\geqslant 0$,

$$\left\|\nabla^l(gh)\right\|_{L^{p_0}}\lesssim\|g\|_{L^{p_1}}\left\|\nabla^l h\right\|_{L^{p_2}}+\left\|\nabla^l g\right\|_{L^{p_3}}\|h\|_{L^{p_4}}. \tag{2.6}$$

In the above, $p_0,p_1,p_2,p_3,p_4\in[1,+\infty]$ such that

$$\frac{1}{p_0}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Proof. Referring to Lemma 3.4 (p. 129) in [37], we give a complete and simple proof in the following. We first prove estimate (2.6). Let $p_0,p_1,p_2,p_3,p_4\in[1,+\infty]$ such that

$$\frac{1}{p_0}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Assume $\ell=0,1,\dots,l$. We choose q_1,q_2 by

$$\frac{1}{q_1}=\frac{1}{p_1}\left(1-\frac{\ell}{l}\right)+\frac{1}{p_3}\frac{\ell}{l},\quad \frac{1}{q_2}=\frac{1}{p_2}\left(1-\frac{\ell}{l}\right)+\frac{1}{p_4}\frac{\ell}{l}.$$

Thus, we have

$$\frac{1}{p_0}=\frac{1}{q_1}+\frac{1}{q_2}.$$

By Hölder's, Gagliardo–Nirenberg's and Young's inequalities, we have for $l\geqslant 0$,

$$\begin{aligned}
\left\|\nabla^l(gh)\right\|_{L^{p_0}}&=\left\|\sum_{\ell=0}^{l}\nabla^\ell g\nabla^{l-\ell}h\right\|_{L^{p_0}}\\
&\lesssim\sum_{\ell=0}^{l}\left\|\nabla^\ell g\right\|_{L^{q_1}}\left\|\nabla^{l-\ell}h\right\|_{L^{q_2}}\\
&\lesssim\|g\|_{L^{p_1}}^{1-\frac{\ell}{l}}\left\|\nabla^l g\right\|_{L^{p_3}}^{\frac{\ell}{l}}\|h\|_{L^{p_4}}^{\frac{\ell}{l}}\left\|\nabla^l h\right\|_{L^{p_2}}^{1-\frac{\ell}{l}}\\
&=\left(\|g\|_{L^{p_1}}\left\|\nabla^l h\right\|_{L^{p_2}}\right)^{1-\frac{\ell}{l}}\left(\left\|\nabla^l g\right\|_{L^{p_3}}\|h\|_{L^{p_4}}\right)^{\frac{\ell}{l}}\\
&\lesssim\|g\|_{L^{p_1}}\left\|\nabla^l h\right\|_{L^{p_2}}+\left\|\nabla^l g\right\|_{L^{p_3}}\|h\|_{L^{p_4}}.
\end{aligned} \tag{2.7}$$

Note that for $l\geqslant 1$,

$$[\nabla^l,g]\,h=\sum_{\ell=1}^{l}\nabla^\ell g\nabla^{l-\ell}h. \tag{2.8}$$

We can prove estimate (2.5) in the same way as in (2.7). □

2.2. Energy estimates. In this subsection, we will derive the basic energy estimates for the solution to the Euler–Poisson system (1.8). We begin with the zero-order energy estimates.

LEMMA 2.3. *It holds that*

$$\frac{d}{dt}\|(n,u,\theta,\nabla\Phi)\|^2+C\|(u,\theta,\nabla\theta)\|^2\lesssim\delta\|n\|^2. \tag{2.9}$$

Proof. Multiplying the first three equations in system (1.8) by $\frac{\Theta_s}{\rho_s}n$, $\rho_s u$, $\frac{3\rho_s}{2\Theta_s}\theta$, respectively, summing them up and then integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\left(\frac{\Theta_s}{\rho_s}n^2+\rho_s|u|^2+\frac{3\rho_s}{2\Theta_s}\theta^2\right)+\int\left(\rho_s|u|^2+\frac{3\rho_s}{2\Theta_s}\theta^2\right)-\int\frac{\rho_s}{\Theta_s(n+\rho_s)}\Delta\theta\theta\\
&=\int\rho_s u\cdot\nabla\Phi-\int(\Theta_s n\operatorname{div}u+\Theta_s\nabla n\cdot u+\rho_s\theta\operatorname{div}u+\rho_s\nabla\theta\cdot u)\\
&\quad-\int(n\operatorname{div}u+u\cdot\nabla n+u\cdot\nabla\rho_s)\frac{\Theta_s}{\rho_s}n\\
&\quad-\int\left(u\cdot\nabla u+\frac{\theta}{n+\rho_s}\nabla n-\frac{\Theta_s}{\rho_s(n+\rho_s)}n\nabla n+\frac{\theta\rho_s-n\Theta_s}{\rho_s(n+\rho_s)}\nabla\rho_s\right)\cdot\rho_s u\\
&\quad-\int\left(u\cdot\nabla\theta+u\cdot\nabla\Theta_s+\frac{2}{3}\theta\operatorname{div}u+\frac{2\Delta\Theta_s n}{3\rho_s(n+\rho_s)}-\frac{|u|^2}{3}\right)\frac{3\rho_s}{2\Theta_s}\theta.
\end{aligned}\tag{2.10}$$

By the integration by parts, we obtain

$$\begin{aligned}
-\int\frac{\rho_s}{\Theta_s(n+\rho_s)}\Delta\theta\theta&=\int\frac{\rho_s}{\Theta_s(n+\rho_s)}|\nabla\theta|^2+\int\nabla\left(\frac{\rho_s}{\Theta_s(n+\rho_s)}\right)\cdot\nabla\theta\theta\\
&\geqslant\int\frac{\rho_s}{\Theta_s(n+\rho_s)}|\nabla\theta|^2-C\delta\|(\theta,\nabla\theta)\|^2.
\end{aligned}\tag{2.11}$$

By equation $(1.8)_1$, we obtain

$$-\operatorname{div}(\rho_s u)=\partial_t n+\operatorname{div}(nu).$$

Then integrating by parts and using equation $(1.8)_4$, we obtain

$$\begin{aligned}
&\int\rho_s u\cdot\nabla\Phi=-\int\Phi\operatorname{div}(\rho_s u)=\int\Phi\partial_t n+\int\Phi\operatorname{div}(nu)\\
&=\int\Phi\partial_t\Delta\Phi-\int nu\cdot\nabla\Phi=-\frac{1}{2}\frac{d}{dt}\int|\nabla\Phi|^2-\int nu\cdot\nabla\Phi\\
&\leqslant-\frac{1}{2}\frac{d}{dt}\int|\nabla\Phi|^2+C\|n\|\|u\|_{L^3}\|\nabla\Phi\|_{L^6}\\
&\leqslant-\frac{1}{2}\frac{d}{dt}\int|\nabla\Phi|^2+C\delta\|n\|^2,
\end{aligned}\tag{2.12}$$

where we have used the standard elliptic estimates $\|\nabla^l\Phi\|\lesssim\|\nabla^{l-2}n\|$ for $l\geqslant 2$. By the integration by parts again, we obtain

$$-\int(\Theta_s n\operatorname{div}u+\Theta_s\nabla n\cdot u+\rho_s\theta\operatorname{div}u+\rho_s\nabla\theta\cdot u)=\int(n\nabla\Theta_s\cdot u+\theta\nabla\rho_s\cdot u)$$

$$\lesssim \|n\| \|\nabla\Theta_s\|_{L^\infty} \|u\| + \|\theta\| \|\nabla\rho_s\|_{L^\infty} \|u\| \lesssim \delta \|(n,u,\theta)\|^2. \tag{2.13}$$

For the remaining terms on the right-hand side of equation (2.10), by Hölder's, Sobolev's and Cauchy's inequalities, we can bound them by

$$\delta \|(n,u,\theta)\|^2. \tag{2.14}$$

Plugging the estimates (2.11)–(2.14) into equation (2.10), since δ is small, we deduce the estimate (2.9) from inequalities (1.6). ☐

We then derive the energy estimates for the higher derivatives.

LEMMA 2.4. *Let $k\geqslant 3$. For $l=1,\dots,k$, we have that for any $\varepsilon>0$,*

$$\begin{aligned}&\frac{d}{dt}\left\|\nabla^l(n,u,\theta,\nabla\Phi)\right\|^2 + C\left(\left\|\nabla^l(u,\theta)\right\|^2 + \left\|\nabla^{l+1}\theta\right\|^2\right)\\ &\lesssim (\delta+\varepsilon)\left\|\nabla^l n\right\|^2 + C_\varepsilon\left(\|(n,u)\|_{H^2}^2 + \|\theta\|_{H^3}^2\right) + \delta\left\|\nabla^{k+1}\theta\right\|^2.\end{aligned} \tag{2.15}$$

Proof. Applying ∇^l to the first three equations in system (1.8) and then multiplying the resulting identities by $\frac{\theta+\Theta_s}{n+\rho_s}\nabla^l n$, $(n+\rho_s)\nabla^l u$, $\frac{3}{2}\frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\theta$, respectively, summing them up and then integrating over $\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int \frac{\theta+\Theta_s}{n+\rho_s}|\nabla^l n|^2 + (n+\rho_s)|\nabla^l u|^2 + \frac{3}{2}\frac{n+\rho_s}{\theta+\Theta_s}|\nabla^l\theta|^2\\
&\quad + 2\int (n+\rho_s)|\nabla^l u|^2 + \int \frac{3(n+\rho_s)}{\theta+\Theta_s}|\nabla^l\theta|^2\\
&= \int \left(\frac{\partial_t\theta}{n+\rho_s} - \frac{\theta+\Theta_s}{(n+\rho_s)^2}\partial_t n\right)|\nabla^l n|^2 + \partial_t n|\nabla^l u|^2 + \frac{3}{2}\left(\frac{\partial_t n}{\theta+\Theta_s} - \frac{n+\rho_s}{(\theta+\Theta_s)^2}\partial_t\theta\right)|\nabla^l\theta|^2\\
&\quad + 2\int (n+\rho_s)\nabla^l\nabla\Phi\cdot\nabla^l u - 2\int (n+\rho_s)\nabla^l\left(\frac{\theta\rho_s - n\Theta_s}{\rho_s(n+\rho_s)}\nabla\rho_s\right)\cdot\nabla^l u\\
&\quad - 2\int \frac{\theta+\Theta_s}{n+\rho_s}\nabla^l(u\cdot\nabla n + u\cdot\nabla\rho_s)\nabla^l n + (n+\rho_s)\nabla^l(u\cdot\nabla u)\cdot\nabla^l u\\
&\quad - 3\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l(u\cdot\nabla\theta + u\cdot\nabla\Theta_s)\nabla^l\theta\\
&\quad + 2\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(\frac{\Delta\theta}{n+\rho_s} - \frac{n\Delta\Theta_s}{\rho_s(n+\rho_s)} + \frac{|u|^2}{2}\right)\nabla^l\theta\\
&\quad - 2\int \frac{\theta+\Theta_s}{n+\rho_s}\nabla^l((n+\rho_s)\operatorname{div}u)\nabla^l n + (n+\rho_s)\nabla^l\left(\frac{\theta+\Theta_s}{n+\rho_s}\nabla n\right)\cdot\nabla^l u\\
&\quad - 2\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l((\theta+\Theta_s)\operatorname{div}u)\nabla^l\theta + (n+\rho_s)\nabla^l\nabla\theta\cdot\nabla^l u\\
&:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned} \tag{2.16}$$

First, we estimate the term I_1. It follows from equation $(1.8)_1$ and equation $(1.8)_3$ that

$$\begin{aligned}
&\partial_t n = -n\operatorname{div}u - \rho_s\operatorname{div}u - u\cdot\nabla n - u\cdot\nabla\rho_s,\\
&\partial_t\theta = -u\cdot\nabla\theta - u\cdot\nabla\Theta_s - \frac{2}{3}(\theta+\Theta_s)\operatorname{div}u + \frac{2}{3(n+\rho_s)}\Delta\theta - \frac{2\Delta\Theta_s}{3\rho_s(n+\rho_s)}n + \frac{1}{3}|u|^2 - \theta.
\end{aligned}$$

In the following, we must carefully deal with the term involved with $\nabla^2\theta$ since $\left\|\nabla^2\theta\right\|_{L^\infty}$ is not small if $k=3$. By estimates (1.7), (2.1)–(2.2) and Hölder's, Sobolev's and Cauchy's inequalities, we obtain

$$
\begin{aligned}
&\int\left(\frac{\partial_t\theta}{n+\rho_s}-\frac{\theta+\Theta_s}{(n+\rho_s)^2}\partial_t n\right)|\nabla^l n|^2 \lesssim (\|\partial_t n\|_{L^\infty}+\|\partial_t\theta\|_{L^\infty})\left\|\nabla^l n\right\|^2\\
&\lesssim (\|(n,\rho_s)\|_{L^\infty}\|\nabla u\|_{L^\infty}+\|u\|_{L^\infty}\|(\nabla n,\nabla\rho_s)\|_{L^\infty})\left\|\nabla^l n\right\|^2\\
&\quad+(\|u\|_{L^\infty}\|(\nabla\theta,\nabla\Theta_s)\|_{L^\infty}+\|(\theta,\Theta_s)\|_{L^\infty}\|\nabla u\|_{L^\infty})\left\|\nabla^l n\right\|^2\\
&\quad+\left(\left\|\nabla^2\theta\right\|_{L^\infty}+\|n\|_{L^\infty}\left\|\nabla^2\Theta_s\right\|_{L^\infty}+\|u\|_{L^\infty}^2+\|\theta\|_{L^\infty}\right)\left\|\nabla^l n\right\|^2\\
&\lesssim\delta\left\|\nabla^l n\right\|^2+\left\|\nabla^2\theta\right\|_{L^\infty}\left\|\nabla^l n\right\|^2\lesssim\delta\left(\left\|\nabla^l n\right\|^2+\left\|\nabla^2\theta\right\|_{L^\infty}^2\right)\\
&\lesssim\delta\left(\left\|\nabla^l n\right\|^2+\|\theta\|^2+\left\|\nabla^{k+1}\theta\right\|^2\right),
\end{aligned}
$$

where we have used the interpolation estimates

$$
\left\|\nabla^2\theta\right\|_{L^\infty}\lesssim\|\theta\|^{\frac{2k-5}{2k+2}}\left\|\nabla^{k+1}\theta\right\|^{\frac{7}{2k+2}} \text{ for } k\geqslant 3. \tag{2.17}
$$

Similarly, we can also obtain

$$
\begin{aligned}
&\int\left(\partial_t n|\nabla^l u|^2+\frac{3}{2}\left(\frac{\partial_t n}{\theta+\Theta_s}-\frac{n+\rho_s}{(\theta+\Theta_s)^2}\partial_t\theta\right)|\nabla^l\theta|^2\right)\\
&\lesssim\delta\left\|\nabla^l(u,\theta)\right\|^2+\delta\left(\|\theta\|^2+\left\|\nabla^{k+1}\theta\right\|^2\right).
\end{aligned}
$$

Thus, we have

$$
I_1\lesssim\delta\left\|\nabla^l(n,u,\theta)\right\|^2+\delta\left(\|\theta\|^2+\left\|\nabla^{k+1}\theta\right\|^2\right).
$$

For the term I_2, by integration by parts and equation $(1.8)_1$, we can rewrite I_2 as

$$
\begin{aligned}
I_2&=2\int(n+\rho_s)\nabla^l\nabla\Phi\cdot\nabla^l u\\
&=-2\int(\nabla n+\nabla\rho_s)\cdot\nabla^l u\nabla^l\Phi-2\int(n+\rho_s)\nabla^l\Phi\nabla^l\operatorname{div}u\\
&=-2\int(\nabla n+\nabla\rho_s)\cdot\nabla^l u\nabla^l\Phi+2\int(n+\rho_s)\nabla^l\Phi\nabla^l\left(\frac{1}{n+\rho_s}(\partial_t n+u\cdot\nabla n+u\cdot\nabla\rho_s)\right)\\
&:=I_{21}+I_{22}.
\end{aligned}
$$

By Hölder's, Sobolev's and Cauchy's inequalities, equation $(1.8)_1$, equation $(1.8)_3$ and Lemma 2.2, we obtain

$$
\begin{aligned}
I_{21}&=-2\int(\nabla n+\nabla\rho_s)\cdot\nabla^l u\nabla^l\Phi\\
&\lesssim\|(\nabla n+\nabla\rho_s)\|_{L^3}\left\|\nabla^l u\right\|\left\|\nabla^l\Phi\right\|_{L^6}\\
&\lesssim\|(n,\nabla\rho_s)\|_{H^3}\left\|\nabla^l u\right\|\left\|\nabla^l\nabla\Phi\right\|\\
&\lesssim\|(n,\nabla\rho_s)\|_{H^3}\left\|\nabla^l u\right\|\left\|\nabla\Delta^{-1}\nabla^l n\right\|\\
&\lesssim\delta\left(\left\|\nabla^{l-1}n\right\|^2+\left\|\nabla^l u\right\|^2\right)
\end{aligned}
$$

and

$$
\begin{aligned}
I_{22}=&2\int(n+\rho_s)\nabla^l\Phi\nabla^l\left(\frac{1}{n+\rho_s}(\partial_t n+u\cdot\nabla n+u\cdot\nabla\rho_s)\right)\\
=&2\int(n+\rho_s)\nabla^l\Phi\nabla^l\left(\frac{1}{n+\rho_s}\partial_t n\right)+2\int(n+\rho_s)\nabla^l\Phi\nabla^l\left(\frac{1}{n+\rho_s}(u\cdot\nabla n+u\cdot\nabla\rho_s)\right)\\
=&2\int(n+\rho_s)\nabla^l\Phi\left(\frac{1}{n+\rho_s}\nabla^l\partial_t n+\left[\nabla^l,\frac{1}{n+\rho_s}\right]\partial_t n\right)\\
&+2\int(n+\rho_s)\nabla^l\Phi\left(\frac{1}{n+\rho_s}\nabla^l(u\cdot\nabla n+u\cdot\nabla\rho_s)+\left[\nabla^l,\frac{1}{n+\rho_s}\right](u\cdot\nabla n+u\cdot\nabla\rho_s)\right)\\
=&2\int\nabla^l\Phi\nabla^l\partial_t n+2\int(n+\rho_s)\nabla^l\Phi\left[\nabla^l,\frac{1}{n+\rho_s}\right]\partial_t n\\
&-2\int\nabla^l\nabla\Phi\nabla^{l-1}(u\cdot\nabla n+u\cdot\nabla\rho_s)+2\int(n+\rho_s)\nabla^l\Phi\left[\nabla^l,\frac{1}{n+\rho_s}\right](u\cdot\nabla n+u\cdot\nabla\rho_s)\\
\lesssim&-\frac{d}{dt}\int|\nabla^l\nabla\Phi|^2+\left\|\nabla^l\Phi\right\|_{L^6}\left\|\left[\nabla^l,\frac{1}{n+\rho_s}\right]\partial_t n\right\|_{L^{6/5}}\\
&+\left\|\nabla^l\nabla\Phi\right\|\left\|\nabla^{l-1}(u\cdot\nabla n+u\cdot\nabla\rho_s)\right\|\\
&+\left\|\nabla^l\Phi\right\|_{L^6}\left\|\left[\nabla^l,\frac{1}{n+\rho_s}\right](u\cdot\nabla n+u\cdot\nabla\rho_s)\right\|_{L^{6/5}}.
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\left\|\nabla^{l-1}\partial_t n\right\|=&\left\|\nabla^{l-1}(n\operatorname{div}u+\rho_s\operatorname{div}u+u\cdot\nabla n+u\cdot\nabla\rho_s)\right\|\\
\lesssim&\,\delta\left\|\nabla^l(n,u)\right\|+\left\|\nabla^l u\right\|+\|u\|_{H^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\partial_t n\|_{L^3}=&\|n\operatorname{div}u+\rho_s\operatorname{div}u+u\cdot\nabla n+u\cdot\nabla\rho_s\|_{L^3}\\
\lesssim&\,\delta\|n\|_{H^2}+\|u\|_{H^2}.
\end{aligned}
$$

By Lemma 2.2, we can estimate

$$
\begin{aligned}
&\left\|\nabla^l\Phi\right\|_{L^6}\left\|\left[\nabla^l,\frac{1}{n+\rho_s}\right]\partial_t n\right\|_{L^{6/5}}\\
\lesssim&\left\|\nabla^l\nabla\Phi\right\|\left(\left\|\nabla\left(\frac{1}{n+\rho_s}\right)\right\|_{L^3}\left\|\nabla^{l-1}\partial_t n\right\|+\left\|\nabla^l\left(\frac{1}{n+\rho_s}\right)\right\|\|\partial_t n\|_{L^3}\right)\\
\lesssim&\left\|\nabla^{l-1}n\right\|\left((1+\|\nabla n\|_{L^3})\left\|\nabla^{l-1}\partial_t n\right\|+(1+\left\|\nabla^l n\right\|)\|\partial_t n\|_{L^3}\right)\\
\lesssim&\,\delta\left(\left\|\nabla^{l-1}n\right\|^2+\left\|\nabla^l(n,u)\right\|^2+\|(n,u)\|_{H^2}^2\right)+\left\|\nabla^{l-1}n\right\|\left\|\nabla^l u\right\|+\left\|\nabla^{l-1}n\right\|\|u\|_{H^2}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
I_2\lesssim&-\frac{d}{dt}\int|\nabla^l\nabla\Phi|^2+\delta\left(\left\|\nabla^{l-1}n\right\|^2+\left\|\nabla^l(n,u)\right\|^2+\|(n,u)\|_{H^2}^2\right)\\
&+\left\|\nabla^{l-1}n\right\|\left\|\nabla^l u\right\|+\left\|\nabla^{l-1}n\right\|\|u\|_{H^2}.
\end{aligned}
$$

Next, we estimate the term I_3. By estimates (1.7), (2.1)–(2.2), the product estimates (2.6), and Hölder's, Sobolev's and Cauchy's inequalities, we obtain

$$
I_3=-2\int(n+\rho_s)\nabla^l\left(\frac{\theta\rho_s-n\Theta_s}{n+\rho_s}\cdot\frac{\nabla\rho_s}{\rho_s}\right)\cdot\nabla^l u
$$

$$
\begin{aligned}
&\lesssim \left(\left\| \frac{\theta\rho_s - n\Theta_s}{n+\rho_s} \right\|_{L^\infty} \left\| \nabla^l \nabla \ln \rho_s \right\| + \left\| \nabla^l \left(\frac{\theta\rho_s - n\Theta_s}{n+\rho_s} \right) \right\| \left\| \nabla \ln \rho_s \right\|_{L^\infty} \right) \left\| \nabla^l u \right\| \\
&\lesssim \left\| \nabla^l \nabla \ln \rho_s \right\| (\|\theta\|_{L^\infty} + \|n\|_{L^\infty}) \left\| \nabla^l u \right\| \\
&\quad + \|\nabla \ln \rho_s\|_{L^\infty} \left(\left\| \nabla^l (\theta\rho_s) \right\| + \left\| \nabla^l (n\Theta_s) \right\| \right) \left\| \nabla^l u \right\| \\
&\lesssim (\|\theta\|_{H^2} + \|n\|_{H^2}) \left\| \nabla^l u \right\| \\
&\quad + \delta \left(\|\theta\|_{L^\infty} \left\| \nabla^l \rho_s \right\| + \|\rho_s\|_{L^\infty} \left\| \nabla^l \theta \right\| + \|n\|_{L^\infty} \left\| \nabla^l \Theta_s \right\| + \|\Theta_s\|_{L^\infty} \left\| \nabla^l n \right\| \right) \left\| \nabla^l u \right\| \\
&\lesssim \delta \left(\left\| \nabla^l (n,u,\theta) \right\|^2 + \|(n,\theta)\|_{H^2}^2 \right) + \|(n,\theta)\|_{H^2} \left\| \nabla^l u \right\|. \qquad (2.18)
\end{aligned}
$$

Next, we estimate the term I_4. We rewrite I_4 as

$$
\begin{aligned}
I_4 &= -2\int \frac{\theta+\Theta_s}{n+\rho_s} \nabla^l (u\cdot\nabla n + u\cdot\nabla\rho_s) \nabla^l n + (n+\rho_s) \nabla^l (u\cdot\nabla u)\cdot\nabla^l u \\
&= -2\int \frac{\theta+\Theta_s}{n+\rho_s} \nabla^l (u\cdot\nabla n) \nabla^l n - 2\int (n+\rho_s) \nabla^l (u\cdot\nabla u)\cdot\nabla^l u \\
&\quad - 2\int \frac{\theta+\Theta_s}{n+\rho_s} \nabla^l (u\cdot\nabla\rho_s) \nabla^l n := I_{41} + I_{42} + I_{43}.
\end{aligned}
$$

First, we estimate I_{41}. By the commutator notation (2.4), we have

$$
\begin{aligned}
I_{41} &= -2\int \frac{\theta+\Theta_s}{n+\rho_s} \nabla^l (u\cdot\nabla n) \nabla^l n = -2\int \frac{\theta+\Theta_s}{n+\rho_s} \left(u\cdot\nabla\nabla^l n + \left[\nabla^l, u\right]\cdot\nabla n \right) \nabla^l n \\
&\lesssim \left| \int \frac{\theta+\Theta_s}{n+\rho_s} u\cdot\nabla(\nabla^l n \nabla^l n) + \frac{\theta+\Theta_s}{n+\rho_s} \left[\nabla^l, u\right]\cdot\nabla n \nabla^l n \right|.
\end{aligned}
$$

By integrating by parts and estimate (2.2), we have

$$
\begin{aligned}
&\left| \int \frac{\theta+\Theta_s}{n+\rho_s} u\cdot\nabla(\nabla^l n \nabla^l n) \right| = \left| \int \frac{\theta+\Theta_s}{n+\rho_s} \operatorname{div} u |\nabla^l n|^2 + \int \nabla \left(\frac{\theta+\Theta_s}{n+\rho_s} \right) \cdot u |\nabla^l n|^2 \right| \\
&\lesssim (\|\operatorname{div} u\|_{L^\infty} + \|(\nabla n, \nabla\theta, \nabla\rho_s, \nabla\Theta_s)\|_{L^\infty} \|u\|_{L^\infty}) \left\| \nabla^l n \right\|^2 \lesssim \delta \left\| \nabla^l n \right\|^2.
\end{aligned}
$$

On the other hand, by estimate (2.2) and the commutator estimates (2.5), we have

$$
\begin{aligned}
&\left| \int \frac{\theta+\Theta_s}{n+\rho_s} \left[\nabla^l, u\right]\cdot\nabla n \nabla^l n \right| \\
&\lesssim \left(\|\nabla u\|_{L^\infty} \left\| \nabla^{l-1} \nabla n \right\| + \left\| \nabla^l u \right\| \|\nabla n\|_{L^\infty} \right) \left\| \nabla^l n \right\| \lesssim \delta \left\| \nabla^l (n,u) \right\|^2.
\end{aligned}
$$

Then applying the similar arguments to I_{42}, we obtain

$$
I_{41} + I_{42} \lesssim \delta \left\| \nabla^l (n,u) \right\|^2.
$$

For the term I_{43}, by estimate (2.2) and the product estimates (2.6), we have

$$
\begin{aligned}
I_{43} &\lesssim \left(\|u\|_{L^\infty} \left\| \nabla^l \nabla\rho_s \right\| + \|\nabla\rho_s\|_{L^\infty} \left\| \nabla^l u \right\| \right) \left\| \nabla^l n \right\| \\
&\lesssim \left(\|u\|_{H^2} \left\| \nabla^l \nabla\rho_s \right\| + \|\nabla\rho_s\|_{L^\infty} \left\| \nabla^l u \right\| \right) \left\| \nabla^l n \right\| \\
&\lesssim \delta \left\| \nabla^l (n,u) \right\|^2 + \|u\|_{H^2} \left\| \nabla^l n \right\|.
\end{aligned}
$$

Like I_4, we can similarly estimate I_5. Hence, we obtain

$$
I_4 + I_5 \lesssim \delta \left\| \nabla^l (n,u,\theta) \right\|^2 + \|u\|_{H^2} \left\| \nabla^l n \right\| + \|u\|_{H^2} \left\| \nabla^l \theta \right\|.
$$

Now we estimate the term I_6. We rewrite I_6 as

$$\begin{aligned}
I_6 &= 2\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(\frac{\Delta\theta}{n+\rho_s}-\frac{n\Delta\Theta_s}{\rho_s(n+\rho_s)}+\frac{|u|^2}{2}\right)\nabla^l\theta \\
&= 2\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(\frac{\Delta\theta}{n+\rho_s}\right)\nabla^l\theta - 2\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(\frac{n\Delta\Theta_s}{\rho_s(n+\rho_s)}\right)\nabla^l\theta \\
&\quad + \int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(|u|^2\right)\nabla^l\theta \\
&:= I_{61}+I_{62}+I_{63}.
\end{aligned}$$

By estimates (1.7), (2.1)–(2.2), integration by parts and the commutator estimates (2.5), we obtain

$$\begin{aligned}
I_{61} &= 2\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(\frac{\Delta\theta}{n+\rho_s}\right)\nabla^l\theta \\
&= \int \frac{2}{\theta+\Theta_s}\nabla^l\Delta\theta\nabla^l\theta + 2\int\frac{n+\rho_s}{\theta+\Theta_s}\left[\nabla^l,\frac{1}{n+\rho_s}\right]\Delta\theta\nabla^l\theta \\
&= -\int \frac{2}{\theta+\Theta_s}\left|\nabla^{l+1}\theta\right|^2 - 2\int\nabla\left(\frac{1}{\theta+\Theta_s}\right)\cdot\nabla^{l+1}\theta\nabla^l\theta + 2\int\frac{n+\rho_s}{\theta+\Theta_s}\left[\nabla^l,\frac{1}{n+\rho_s}\right]\Delta\theta\nabla^l\theta \\
&\lesssim -\left\|\nabla^{l+1}\theta\right\|^2 + (\|\nabla\theta\|_{L^\infty}+\|\nabla\Theta_s\|_{L^\infty})\left\|\nabla^{l+1}\theta\right\|\left\|\nabla^l\theta\right\| \\
&\quad + \left\|\left[\nabla^l,\frac{1}{n+\rho_s}\right]\Delta\theta\right\|_{L^{6/5}}\left\|\nabla^l\theta\right\|_{L^6} \\
&\lesssim -\left\|\nabla^{l+1}\theta\right\|^2 + (\|\nabla\theta\|_{L^\infty}+\|\nabla\Theta_s\|_{L^\infty})\left\|\nabla^{l+1}\theta\right\|\left\|\nabla^l\theta\right\| \\
&\quad + \left\|\nabla\left(\frac{1}{n+\rho_s}\right)\right\|_{L^3}\left\|\nabla^{l+1}\theta\right\|\left\|\nabla^l\theta\right\|_{L^6} + \left(\left\|\nabla^l n\right\|\|\Delta\theta\|_{L^3}+\left\|\nabla^l\rho_s\right\|\|\Delta\theta\|_{L^3}\right)\left\|\nabla^l\theta\right\|_{L^6} \\
&\lesssim -\left\|\nabla^{l+1}\theta\right\|^2 + \delta\left(\left\|\nabla^l(n,\theta)\right\|^2+\left\|\nabla^{l+1}\theta\right\|^2\right) + \|\theta\|_{H^3}\left\|\nabla^{l+1}\theta\right\|.
\end{aligned}$$

By estimates (1.7), (2.1)–(2.2) and the product estimates (2.6), we obtain

$$\begin{aligned}
I_{62} &= -2\int\frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(\frac{n\Delta\Theta_s}{\rho_s(n+\rho_s)}\right)\nabla^l\theta \lesssim \left\|\nabla^l\left(\frac{n\Delta\Theta_s}{\rho_s(n+\rho_s)}\right)\right\|\left\|\nabla^l\theta\right\| \\
&\lesssim \left(\left\|\frac{n}{\rho_s(n+\rho_s)}\right\|_{L^\infty}\left\|\nabla^l\Delta\Theta_s\right\| + \left\|\nabla^l\left(\frac{n}{\rho_s(n+\rho_s)}\right)\right\|\|\Delta\Theta_s\|_{L^\infty}\right)\left\|\nabla^l\theta\right\| \\
&\lesssim \left(\|n\|_{L^\infty}\left\|\nabla^l\Delta\Theta_s\right\| + \left\|\nabla^l\left(\frac{n}{\rho_s(n+\rho_s)}\right)\right\|\|\Delta\Theta_s\|_{L^\infty}\right)\left\|\nabla^l\theta\right\| \\
&\lesssim \|n\|_{H^2}\left\|\nabla^l\theta\right\| + \left\|\frac{1}{\rho_s(n+\rho_s)}\right\|_{L^\infty}\left\|\nabla^l n\right\|\|\nabla\Theta_s\|_{H^3}\left\|\nabla^l\theta\right\| \\
&\quad + \left\|\nabla^l\left(\frac{1}{\rho_s(n+\rho_s)}\right)\right\|\|n\|_{L^\infty}\|\nabla\Theta_s\|_{H^3}\left\|\nabla^l\theta\right\| \\
&\lesssim \|n\|_{H^2}\left\|\nabla^l\theta\right\| + \|\nabla\Theta_s\|_{H^3}\left\|\nabla^l n\right\|\left\|\nabla^l\theta\right\| \\
&\quad + \left\|\nabla^l\left(\frac{1}{\rho_s(n+\rho_s)}\right)\right\|\|n\|_{H^2}\|\nabla\Theta_s\|_{H^3}\left\|\nabla^l\theta\right\| \\
&\lesssim \|n\|_{H^2}\left\|\nabla^l\theta\right\| + \|\nabla\Theta_s\|_{H^3}\left\|\nabla^l n\right\|\left\|\nabla^l\theta\right\| \\
&\quad + \left(\left\|\frac{1}{\rho_s}\right\|_{L^\infty}\left\|\nabla^l\left(\frac{1}{n+\rho_s}\right)\right\| + \left\|\nabla^l\left(\frac{1}{\rho_s}\right)\right\|\left\|\frac{1}{n+\rho_s}\right\|_{L^\infty}\right)\|n\|_{H^2}\|\nabla\Theta_s\|_{H^3}\left\|\nabla^l\theta\right\|
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|n\|_{H^2}\left\|\nabla^l\theta\right\| + \|\nabla\Theta_s\|_{H^3}\left\|\nabla^l n\right\|\left\|\nabla^l\theta\right\| + \left\|\nabla^l(n,\rho_s)\right\|\|n\|_{H^2}\|\nabla\Theta_s\|_{H^3}\left\|\nabla^l\theta\right\| \\
&\lesssim \delta\left\|\nabla^l(n,\theta)\right\|^2 + \|n\|_{H^2}\left\|\nabla^l\theta\right\|
\end{aligned}
$$

and

$$
I_{63} = \int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l\left(|u|^2\right)\nabla^l\theta \lesssim \left\|\nabla^l\left(|u|^2\right)\right\|\left\|\nabla^l\theta\right\| \lesssim \|u\|_{L^\infty}\left\|\nabla^l u\right\|\left\|\nabla^l\theta\right\| \lesssim \delta\left\|\nabla^l(u,\theta)\right\|^2.
$$

Hence, we obtain

$$
I_6 \lesssim -\left\|\nabla^{l+1}\theta\right\|^2 + \delta\left(\left\|\nabla^l(n,u,\theta)\right\|^2 + \left\|\nabla^{l+1}\theta\right\|^2\right) + \|\theta\|_{H^3}\left\|\nabla^{l+1}\theta\right\| + \|n\|_{H^2}\left\|\nabla^l\theta\right\|.
$$

We now estimate the term I_7. By the commutator notation (2.4), we can rewrite I_7 as

$$
\begin{aligned}
I_7 &= -2\int \frac{\theta+\Theta_s}{n+\rho_s}\nabla^l((n+\rho_s)\operatorname{div}u)\nabla^l n + (n+\rho_s)\nabla^l\left(\frac{\theta+\Theta_s}{n+\rho_s}\nabla n\right)\cdot\nabla^l u \\
&= -2\int(\theta+\Theta_s)(\nabla^l\operatorname{div}u\nabla^l n + \nabla^l\nabla n\cdot\nabla^l u) \\
&\quad -2\int \frac{\theta+\Theta_s}{n+\rho_s}\left[\nabla^l, n+\rho_s\right]\operatorname{div}u\nabla^l n + (n+\rho_s)\left[\nabla^l, \frac{\theta+\Theta_s}{n+\rho_s}\right]\nabla n\cdot\nabla^l u.
\end{aligned}
$$

By integration by parts, estimates (1.7), (2.1)–(2.2) and Hölder's, Sobolev's and Cauchy's inequalities, we obtain

$$
\begin{aligned}
&-2\int(\theta+\Theta_s)(\nabla^l\operatorname{div}u\nabla^l n + \nabla^l\nabla n\cdot\nabla^l u) = -2\int(\theta+\Theta_s)\operatorname{div}(\nabla^l n\nabla^l u) \\
&= 2\int(\nabla\theta+\nabla\Theta_s)\cdot\nabla^l u\nabla^l n \lesssim (\|\nabla\theta\|_{L^\infty} + \|\nabla\Theta_s\|_{L^\infty})\left\|\nabla^l u\right\|\left\|\nabla^l n\right\| \lesssim \delta\left\|\nabla^l(n,u)\right\|^2.
\end{aligned}
$$

By estimates (1.7), (2.1)–(2.2) and Lemma 2.2, we obtain

$$
\begin{aligned}
&-2\int \frac{\theta+\Theta_s}{n+\rho_s}\left[\nabla^l, n+\rho_s\right]\operatorname{div}u\nabla^l n + (n+\rho_s)\left[\nabla^l, \frac{\theta+\Theta_s}{n+\rho_s}\right]\nabla n\cdot\nabla^l u \\
&\lesssim \left(\|\nabla(n,\rho_s)\|_{L^\infty}\left\|\nabla^{l-1}\operatorname{div}u\right\| + \|\operatorname{div}u\|_{L^\infty}\left\|\nabla^l n\right\| + \|\operatorname{div}u\|_{L^3}\left\|\nabla^l\rho_s\right\|_{L^6}\right)\left\|\nabla^l n\right\| \\
&\quad + \left(\|\nabla(n,\theta,\rho_s,\Theta_s)\|_{L^\infty}\left\|\nabla^l n\right\| + \|\nabla n\|_{L^\infty}\left\|\nabla^l(n,\theta)\right\|\right)\left\|\nabla^l u\right\| \\
&\quad + \|\nabla n\|_{L^3}\left\|\nabla^l(\rho_s,\Theta_s)\right\|_{L^6}\left\|\nabla^l u\right\| \\
&\lesssim \left(\|\nabla(n,\rho_s)\|_{L^\infty}\left\|\nabla^l u\right\| + \|\nabla u\|_{L^\infty}\left\|\nabla^l n\right\|\right)\left\|\nabla^l n\right\| \\
&\quad + \left(\|\nabla(n,\theta,\rho_s,\Theta_s)\|_{L^\infty}\left\|\nabla^l n\right\| + \|\nabla n\|_{L^\infty}\left\|\nabla^l(n,\theta)\right\|\right)\left\|\nabla^l u\right\| \\
&\quad + \|u\|_{H^2}\left\|\nabla^{l+1}\rho_s\right\|\left\|\nabla^l n\right\| + \|n\|_{H^2}\left\|\nabla^{l+1}(\rho_s,\Theta_s)\right\|\left\|\nabla^l u\right\| \\
&\lesssim \delta\left\|\nabla^l(n,u,\theta)\right\|^2 + \|u\|_{H^2}\left\|\nabla^l n\right\| + \|n\|_{H^2}\left\|\nabla^l u\right\|.
\end{aligned}
\tag{2.19}
$$

Hence, we obtain

$$
I_7 \lesssim \delta\left\|\nabla^l(n,u,\theta)\right\|^2 + \|u\|_{H^2}\left\|\nabla^l n\right\| + \|n\|_{H^2}\left\|\nabla^l u\right\|.
$$

For the last term I_8, by estimates (1.7), (2.1)–(2.2), integration by parts and the commutator estimates (2.5), we obtain

$$
I_8 = -2\int \frac{n+\rho_s}{\theta+\Theta_s}\nabla^l((\theta+\Theta_s)\operatorname{div}u)\nabla^l\theta + (n+\rho_s)\nabla^l\nabla\theta\cdot\nabla^l u
$$

$$
\begin{aligned}
&= -2\int (n+\rho_s)\operatorname{div}(\nabla^l u \nabla^l \theta) - 2\int \frac{n+\rho_s}{\theta+\Theta_s}\left[\nabla^l, \theta+\Theta_s\right]\operatorname{div} u \nabla^l\theta \\
&= 2\int (\nabla n + \nabla\rho_s)\cdot \nabla^l u \nabla^l\theta - 2\int \frac{n+\rho_s}{\theta+\Theta_s}\left[\nabla^l, \theta+\Theta_s\right]\operatorname{div} u \nabla^l\theta \\
&\lesssim \|(\nabla n, \nabla\rho_s)\|_{L^\infty}\left\|\nabla^l u\right\|\left\|\nabla^l\theta\right\| + \left\|\left[\nabla^l, \theta+\Theta_s\right]\operatorname{div} u\right\|\left\|\nabla^l\theta\right\| \\
&\lesssim \|(\nabla n, \nabla\rho_s)\|_{L^\infty}\left\|\nabla^l u\right\|\left\|\nabla^l\theta\right\| + \|(\nabla\theta, \nabla\Theta_s)\|_{L^\infty}\left\|\nabla^{l-1}\operatorname{div} u\right\|\left\|\nabla^l\theta\right\| \\
&\quad + \left\|\nabla^l\theta\right\|\|\operatorname{div} u\|_{L^\infty}\left\|\nabla^l\theta\right\| + \left\|\nabla^l\Theta_s\right\|_{L^6}\|\operatorname{div} u\|_{L^3}\left\|\nabla^l\theta\right\| \\
&\lesssim \delta\left\|\nabla^l(u,\theta)\right\|^2 + \|u\|_{H^2}\left\|\nabla^l\theta\right\|.
\end{aligned}
$$

Plugging the estimates for $I_1 - I_8$ into equation (2.16), it follows from estimate (2.2) that

$$
\begin{aligned}
&\frac{d}{dt}\left\|\nabla^l(n,u,\theta,\nabla\Phi)\right\|^2 + C\left(\left\|\nabla^l(u,\theta)\right\|^2 + \left\|\nabla^{l+1}\theta\right\|^2\right) \\
&\lesssim \delta\left(\left\|\nabla^{k+1}\theta\right\|^2 + \left\|\nabla^{l+1}\theta\right\|^2 + \left\|\nabla^l(n,u,\theta)\right\|^2 + \left\|\nabla^{l-1}n\right\|^2 + \|(n,u,\theta)\|_{H^2}^2\right) \\
&\quad + \|\theta\|_{H^3}\left\|\nabla^{l+1}\theta\right\| + \|n\|_{H^2}\left\|\nabla^l\theta\right\| + \|u\|_{H^2}\left\|\nabla^l(n,\theta)\right\| \\
&\quad + \left\|\nabla^{l-1}n\right\|\left\|\nabla^l u\right\| + \left\|\nabla^{l-1}n\right\|\|u\|_{H^2} + \|(n,\theta)\|_{H^2}\left\|\nabla^l u\right\|.
\end{aligned}
$$

By the interpolation inequality (2.3) and Young's inequality, since δ is small, we deduce estimate (2.15). ◻

Note that in Lemmas 2.3–2.4 we only derive the dissipation estimates of u and θ. We now recover the dissipation estimates of n and $\nabla\Phi$ by constructing some interactive energy functionals in the following lemmas. First, we will establish the dissipation estimates for lower-order n and $\nabla\Phi$.

Lemma 2.5. *It holds that for some small positive constant η,*

$$
\frac{d}{dt}\left(\int u\cdot\nabla n - \eta\int u\cdot\nabla\Phi\right) + C\left(\|n\|_{H^1}^2 + \|\nabla\Phi\|^2\right) \lesssim \|(u,\theta)\|_{H^1}^2. \tag{2.20}
$$

Proof. We divide the proof into three steps.

Step 1: Dissipation estimate of n.

Multiplying equation $(1.8)_2$ by ∇n and thus integrating over $\mathbb{R}^3$, by integration by parts, equation $(1.8)_4$, estimates (1.7) and (2.1)–(2.2) and Hölder's, Sobolev's and Cauchy's inequalities, we obtain for any $\varepsilon > 0$,

$$
\begin{aligned}
&\int \partial_t u\cdot\nabla n + \int \frac{\theta+\Theta_s}{n+\rho_s}|\nabla n|^2 \\
&\leqslant \int \nabla\Phi\cdot\nabla n + \|u\|\|\nabla n\| + \|\nabla\theta\|\|\nabla n\| + \|u\cdot\nabla u\|\|\nabla n\| + \left\|\frac{\theta\rho_s - n\Theta_s}{\rho_s(n+\rho_s)}\nabla\rho_s\right\|\|\nabla n\| \\
&\leqslant -\|n\|^2 + \varepsilon\|\nabla n\|^2 + C_\varepsilon\left(\|u\|^2 + \|\nabla\theta\|^2\right) + C\delta\|(n,u,\theta)\|^2.
\end{aligned} \tag{2.21}
$$

We now estimate the first term on the left-hand side of inequality (2.21). Making use of equation $(1.8)_1$, by integrating by parts for both t- and x-variables, we obtain

$$
\int \partial_t u\cdot\nabla n = \frac{d}{dt}\int u\cdot\nabla n - \int u\cdot\nabla\partial_t n = \frac{d}{dt}\int u\cdot\nabla n + \int \operatorname{div} u\,\partial_t n
$$

$$
\begin{aligned}
&= \frac{d}{dt}\int u\cdot\nabla n - \int \operatorname{div} u\,(u\cdot\nabla n + n\operatorname{div} u + \rho_s \operatorname{div} u + u\cdot\nabla\rho_s)\\
&\geqslant \frac{d}{dt}\int u\cdot\nabla n - \|\operatorname{div} u\|\,(\|u\|_{L^6}\|\nabla n\|_{L^3} + \|n\|_{L^\infty}\|\operatorname{div} u\|)\\
&\quad - \|\operatorname{div} u\|\,(\|\rho_s\|_{L^\infty}\|\operatorname{div} u\| + \|u\|_{L^6}\|\nabla\rho_s\|_{L^3})\\
&\geqslant \frac{d}{dt}\int u\cdot\nabla n - C\|\nabla u\|^2.
\end{aligned}\tag{2.22}
$$

Substituting inequality (2.22) into inequality (2.21) and taking ε sufficiently small, by estimate (2.2), we obtain

$$
\frac{d}{dt}\int u\cdot\nabla n + \|n\|_{H^1}^2 \lesssim \|u\|_{H^1}^2 + \|\nabla\theta\|^2 + \delta\|(n,u,\theta)\|^2. \tag{2.23}
$$

Step 2: Dissipation estimate of $\nabla\Phi$.

Multiplying equation $(1.8)_2$ by $-\nabla\Phi$ and thus integrating over $\mathbb{R}^3$, by estimate (2.2) and Hölder's, Sobolev's and Cauchy's inequalities, we obtain

$$
\begin{aligned}
-\int \partial_t u\cdot\nabla\Phi + \|\nabla\Phi\|^2 &\lesssim (\|u\| + \|u\cdot\nabla u\| + \|\nabla n + \theta\rho_s\nabla\rho_s + n\Theta_s\nabla\rho_s + \nabla\theta\|)\,\|\nabla\Phi\|\\
&\lesssim \|\nabla n\|^2 + \|\nabla\theta\|^2 + \|u\|^2 + \delta\|(n,u,\theta)\|^2.
\end{aligned}\tag{2.24}
$$

Next, we need to estimate the first term on the left-hand side of estimate (2.24). By equation $(1.8)_1$ and equation $(1.8)_4$, we obtain

$$
\begin{aligned}
-\int \partial_t u\cdot\nabla\Phi &= -\frac{d}{dt}\int u\cdot\nabla\Phi + \int u\cdot\nabla\Delta^{-1} n_t\\
&= -\frac{d}{dt}\int u\cdot\nabla\Phi - \int u\cdot\nabla\Delta^{-1}\operatorname{div}((n+\rho_s)u)\\
&\geqslant -\frac{d}{dt}\int u\cdot\nabla\Phi - C\|u\|\,\|(n+\rho_s)u\|\\
&\geqslant -\frac{d}{dt}\int u\cdot\nabla\Phi - C\|u\|^2.
\end{aligned}\tag{2.25}
$$

Plugging inequality (2.25) into estimate (2.24), we obtain

$$
-\frac{d}{dt}\int u\cdot\nabla\Phi + \|\nabla\Phi\|^2 \lesssim \|\nabla n\|^2 + \|\nabla\theta\|^2 + \|u\|^2 + \delta\|(n,u,\theta)\|^2. \tag{2.26}
$$

Step 3: Conclusion.

Multiplying estimate (2.26) by a small enough but fixed constant η and then adding it to estimate (2.23) so that the first term on the right-hand side of estimate (2.26) can be absorbed, since δ is small, we immediately obtain estimate (2.20). ∎

Next we will establish the dissipation estimates for higher-order n.

LEMMA 2.6. *Let $k\geqslant 3$. For $l=1,\ldots,k-1$, we have*

$$
\frac{d}{dt}\int \nabla^l u\cdot\nabla\nabla^l n + C\left(\|\nabla^{l+1} n\|^2 + \|\nabla^l n\|^2\right) \lesssim \|\nabla^{l+1}\theta\|^2 + \|\nabla^{l+1} u\|^2 + \|(n,u,\theta)\|^2. \tag{2.27}
$$

Proof. Applying ∇^l to equation $(1.8)_2$ and then taking the L^2 inner product with $\nabla\nabla^l n$, by the commutator notation (2.4), we obtain

$$\begin{aligned}&\int \nabla^l\partial_t u\cdot\nabla\nabla^l n+\int\frac{\theta+\Theta_s}{n+\rho_s}|\nabla\nabla^l n|^2\\ &\lesssim\int\nabla^l\nabla\Phi\cdot\nabla\nabla^l n+\|\nabla^l u\|\,\|\nabla^{l+1}n\|+\|\nabla^{l+1}\theta\|\,\|\nabla^{l+1}n\|+\|\nabla^l(u\cdot\nabla u)\|\,\|\nabla^{l+1}n\|\\ &\quad+\left\|\left[\nabla^l,\frac{\theta+\Theta_s}{n+\rho_s}\right]\nabla n\right\|\|\nabla^{l+1}n\|+\left\|\nabla^l\left(\frac{\theta\rho_s-n\Theta_s}{\rho_s(n+\rho_s)}\nabla\rho_s\right)\right\|\|\nabla^{l+1}n\|.\end{aligned}\tag{2.28}$$

We first estimate the first term on the left-hand side of estimate (2.28). Like inequality (2.22), we obtain

$$\begin{aligned}\int\nabla^l\partial_t u\cdot\nabla\nabla^l n&=\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l n-\int\nabla^l u\cdot\nabla\nabla^l\partial_t n\\ &=\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l n+\int\nabla^l\operatorname{div}u\nabla^l\partial_t n\\ &=\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l n-\int\nabla^l\operatorname{div}u\nabla^l(u\cdot\nabla n+n\operatorname{div}u+\rho_s\operatorname{div}u+u\cdot\nabla\rho_s)\\ &\geqslant\frac{d}{dt}\int\nabla^l u\cdot\nabla\nabla^l n-\|\nabla^{l+1}u\|\,\|\nabla^l(u\cdot\nabla n+n\operatorname{div}u+\rho_s\operatorname{div}u+u\cdot\nabla\rho_s)\|.\end{aligned}$$

Using the product estimates (2.6), we have

$$\|\nabla^l(u\cdot\nabla n)\|\lesssim\|u\|_{L^\infty}\|\nabla^{l+1}n\|+\|\nabla n\|_{L^\infty}\|\nabla^l u\|\lesssim\delta\left(\|\nabla^l u\|+\|\nabla^{l+1}n\|\right)\tag{2.29}$$

and

$$\|\nabla^l(\rho_s\operatorname{div}u)\|\lesssim\|\rho_s\|_{L^\infty}\|\nabla^{l+1}u\|+\|\nabla u\|_{L^3}\|\nabla^l\rho_s\|_{L^6}\lesssim\|\nabla^{l+1}u\|+\|u\|_{H^2}.$$

Similarly, we have

$$\|\nabla^l(n\operatorname{div}u)\|\lesssim\delta\left(\|\nabla^l n\|+\|\nabla^{l+1}u\|\right)$$

and

$$\|\nabla^l(u\cdot\nabla\rho_s)\|\lesssim\delta\|\nabla^l u\|+\|u\|_{H^2}.$$

Hence, by Cauchy's inequality, we obtain

$$\int\nabla^l\partial_t u\cdot\nabla\nabla^l n\geqslant\frac{d}{dt}\int\nabla^l u\cdot\nabla^l\nabla n-C\|\nabla^{l+1}u\|^2-C\delta\|\nabla^l(n,u)\|_{H^1}^2-C\|u\|_{H^2}^2.\tag{2.30}$$

Next, integrating by parts and using equation $(1.8)_4$, we have

$$\int\nabla^l\nabla\Phi\cdot\nabla\nabla^l n=-\int\nabla^l\Delta\Phi\nabla^l n=-\|\nabla^l n\|^2.\tag{2.31}$$

Like estimate (2.29), we have

$$\|\nabla^l(u\cdot\nabla u)\|\lesssim\delta\|\nabla^l u\|_{H^1}.\tag{2.32}$$

From estimates (2.18)–(2.19), we easily obtain

$$\left\|\left[\nabla^l,\frac{\theta+\Theta_s}{n+\rho_s}\right]\nabla n\right\|+\left\|\nabla^l\left(\frac{\theta\rho_s-n\Theta_s}{\rho_s(n+\rho_s)}\nabla\rho_s\right)\right\|\lesssim\delta\left\|\nabla^l(n,\theta)\right\|+\|(n,\theta)\|_{H^2}. \tag{2.33}$$

Plugging the estimates (2.30)–(2.33) into estimate (2.28), by Cauchy's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\int\nabla^l u\cdot\nabla^l\nabla n+\int\frac{\theta+\Theta_s}{n+\rho_s}|\nabla\nabla^l n|^2+\left\|\nabla^l n\right\|^2\\&\quad\lesssim\left\|\nabla^{l+1}(u,\theta)\right\|^2+\left\|\nabla^l u\right\|^2+\delta\left\|\nabla^l(n,u,\theta)\right\|_{H^1}^2+\|(n,u,\theta)\|_{H^2}^2.\end{aligned}$$

By the interpolation inequality (2.3) and Young's inequality, since δ is small, we then deduce estimate (2.27) from estimate (2.2). □

3. Proof of Theorem 1.1

In this section, we will make good use of Lemmas 2.3–2.6 obtained in Section 2 to prove Theorem 1.1.

Let $l=k-1$ in estimate (2.27) of Lemma 2.6. Then, we obtain

$$\frac{d}{dt}\int\nabla^{k-1}u\cdot\nabla^k n+C_1\left(\left\|\nabla^k n\right\|^2+\left\|\nabla^{k-1}n\right\|^2\right)\leqslant C_2\left\|\nabla^k(u,\theta)\right\|^2+C_3\|(n,u,\theta)\|^2. \tag{3.1}$$

Let $l=k$ in estimate (2.15) of Lemma 2.4. Then, by the interpolation inequality (2.3) and Young's inequality, since δ is small, we obtain for any $\varepsilon>0$,

$$\frac{d}{dt}\left\|\nabla^k(n,u,\theta,\nabla\Phi)\right\|^2+C_4\left\|\nabla^k(u,\theta)\right\|^2\leqslant C_5(\delta+\varepsilon)\left\|\nabla^k n\right\|^2+C_\varepsilon\|(n,u,\theta)\|^2. \tag{3.2}$$

Multiplying inequalities (3.1)–(3.2) by two small enough but fixed constants ϵ_1,ϵ_2, respectively, and then adding them to estimate (2.20) of Lemma 2.5, by the interpolation inequality (2.3) and Young's inequality, taking ε properly small and since δ is small, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\int u\cdot\nabla n-\eta\int u\cdot\nabla\Phi+\epsilon_1\int\nabla^{k-1}u\cdot\nabla^k n+\epsilon_2\left\|\nabla^k(n,u,\theta,\nabla\Phi)\right\|^2\right)\\&\quad+C_6\left(\left\|\nabla^k(n,u,\theta)\right\|^2+\|n\|^2+\|\nabla\Phi\|^2\right)\leqslant C_7\|(u,\theta)\|^2.\end{aligned} \tag{3.3}$$

Multiplying inequality (3.3) by a small enough but fixed constant ϵ_3 and then adding it to estimate (2.9) of Lemma 2.3, since δ is small, we obtain

$$\begin{aligned}&\frac{d}{dt}\Bigg(\|(n,u,\theta,\nabla\Phi)\|^2+\epsilon_3\bigg(\int u\cdot\nabla n-\eta\int u\cdot\nabla\Phi\\&\qquad+\epsilon_1\int\nabla^{k-1}u\cdot\nabla^k n+\epsilon_2\left\|\nabla^k(n,u,\theta,\nabla\Phi)\right\|^2\bigg)\Bigg)\\&\qquad+C_8\left(\left\|\nabla^k(n,u,\theta)\right\|^2+\|(n,u,\theta)\|^2+\|\nabla\Phi\|^2\right)\leqslant 0.\end{aligned} \tag{3.4}$$

If we take ϵ_1, ϵ_2, ϵ_3 and η properly small, then the expression under the time differentiation in inequality (3.4) is equivalent to $\left\|\nabla^k(n,u,\theta,\nabla\Phi)\right\|^2+\|(n,u,\theta,\nabla\Phi)\|^2$. Hence, we deduce for some constant $\beta'>0$,

$$\frac{d}{dt}\left(\left\|\nabla^k(n,u,\theta,\nabla\Phi)\right\|^2+\|(n,u,\theta,\nabla\Phi)\|^2\right)+\beta'\left(\left\|\nabla^k(n,u,\theta)\right\|^2+\|(n,u,\theta,\nabla\Phi)\|^2\right)\leqslant 0,$$

which implies for some constant $\beta > 0$,

$$\frac{d}{dt}\|(n,u,\theta,\nabla\Phi)\|_{H^k}^2 + \beta\|(n,u,\theta,\nabla\Phi)\|_{H^k}^2 \leqslant 0.$$

By Gronwall's inequality, we immediately obtain estimate (1.9). By a standard continuity argument, we then close the a priori estimates (2.1) if we assume at initial time that $\|(n_0,u_0,\theta_0)\|_{H^k} + \|\nabla\Phi_0\| + \|(\nabla b,\nabla T)\|_{H^5}$ is sufficiently small. The global solution in Theorem 1.1 then follows by a standard continuity argument under the help of the local existence of solutions and the a priori estimates. For completeness, we record the local existence of the solution and omit the details of proof since one can refer to [41].

Proposition 3.1. *Assume that the steady state $(\rho_s(x),\Theta_s(x),\phi_s(x))$ is shown by Proposition 1.1. If $(n_0,u_0,\theta_0) \in H^k$ with $k \geqslant 3$ and $\inf_{x\in\mathbb{R}^3}\{n(x,0)+\rho_s(x)\} > 0$, then there exists a positive T such that the Cauchy problem (1.8) has a unique solution*

$$\begin{cases} (n,u)(t) \in \mathcal{C}^0(0,T;H^k)\cap\mathcal{C}^1(0,T;H^{k-1}), \\ \theta(t) \in \mathcal{C}^0(0,T;H^k)\cap\mathcal{C}^1(0,T;H^{k-2}), \end{cases}$$

which satisfies for any $t \in [0,T]$,

$$\begin{cases} n(x,t)+\rho_s(x) \geqslant \inf_{x\in\mathbb{R}^3}\{n(x,0)+\rho_s(x)\}/2 > 0, \\ \|(n,u,\theta)(t)\|_{H^k} \leqslant C\|(n_0,u_0,\theta_0)\|_{H^k}. \end{cases}$$

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