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INFINITE-DIMENSIONAL HILBERT TENSORS ON SPACES OF ANALYTIC FUNCTIONS*

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Abstract. In this paper, the mth order infinite dimensional Hilbert tensor (hypermatrix) is introduced to define an $(m-1)$-homogeneous operator on the spaces of analytic functions, which is called the Hilbert tensor operator. The boundedness of the Hilbert tensor operator is presented on Bergman spaces A^p $(p>2(m-1))$. On the base of the boundedness, two positively homogeneous operators are introduced to the spaces of analytic functions, and hence the upper bounds of norm of the two operators are found on Bergman spaces A^p $(p>2(m-1))$. In particular, the norms of such two operators on Bergman spaces $A^{4(m-1)}$ are smaller than or equal to π and $\pi^{\frac{1}{m-1}}$, respectively.

Keywords. Hilbert tensor; analytic function; upper bound; Bergman space; gamma function.

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1. Introduction

The Hilbert matrix H is a matrix with entries H_{ij} being the unit fractions for nonnegative integers i,j, i.e.,

$$H_{ij}=\frac{1}{i+j+1},\ i,j=0,1,2,\cdots$$

which was introduced by Hilbert [1]. Let $i,j=0,1,2,\cdots,n$. Then such an $(n+1)$-dimensional Hilbert matrix is a compact linear operator on finite-dimensional space $\mathbb{R}^n$. The properties of the n-dimensional Hilbert matrix have been studied by Frazer [2] and Taussky [3]. An infinite-dimensional Hilbert matrix H may be regarded as a bounded linear operator from the sequence space l^2 into itself, but not compact operator (Choi [4] and Ingham [5]). Magnus [6] and Kato [7] showed the spectral properties of such a class of matrices. The infinite-dimensional Hilbert matrix H induces an operator defined on the sequence space l^p $(1\le p)$, for $x=(x_k)_{k=0}^{\infty}\in l^p$,

$$H(x)=\left(\sum_{j=0}^{+\infty}\frac{x_j}{i+j+1}\right)_{i=0}^{\infty}. \tag{1.1}$$

If $1<p<+\infty$, the well-known Hilbert inequality ([37]) implies that H is an operator on l^p and its operator norm $\|H\|=\sup\limits_{\|x\|_{l_p}=1}\|H(x)\|_{l_p}$ is the following:

$$\|H\|=\frac{\pi}{\sin(\frac{\pi}{p})},\ 1<p<+\infty. \tag{1.2}$$

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On the other hand, the infinite-dimensional Hilbert matrix H also induces a bounded operator on the spaces of analytic functions defined by

$$H(f)(z)=\sum_{i=0}^{+\infty}\left(\sum_{j=0}^{+\infty}\frac{a_j}{i+j+1}\right)z^i, \tag{1.3}$$

for all analytic functions $f(z)=\sum_{k=0}^{+\infty}a_kz^k$ with the convergent coefficients $\sum_{j=0}^{+\infty}\frac{a_j}{i+j+1}$ for each i. In Hardy spaces H^p, Diamantopoulos and Siskakis [8] proved that H is bounded for $p>1$ and found an upper bound of its operator norm. In 2004, Diamantopoulos [9] showed that H is bounded on Bergman spaces for $p>2$ and obtained the upper bound of its operator norm. Aleman, Montes-Rodrguez, Sarafoleanu [10] studied the eigenfunctions of the Hilbert matrix operator on Hardy spaces H^p $(p>1)$.

As a natural extension of a Hilbert matrix, the entries of an mth order infinite-dimensional Hilbert tensor (hypermatrix) $\mathcal{H}=(\mathcal{H}_{i_1i_2\cdots i_m})$ are defined by

$$\mathcal{H}_{i_1i_2\cdots i_m}=\frac{1}{i_1+i_2+\cdots+i_m+1},\ i_k=0,1,2,\cdots,\ k=1,2,\cdots,m.$$

Each entry of $\mathcal{H}$ is derived from the integral

$$\mathcal{H}_{i_1i_2\cdots i_m}=\int_0^1 t^{i_1+i_2+\cdots+i_m}dt. \tag{1.4}$$

Clearly, $\mathcal{H}$ is positive ($\mathcal{H}_{i_1i_2\cdots i_m}>0$) and symmetric ($\mathcal{H}_{i_1i_2\cdots i_m}$ are invariant for any permutation of the indices), and $\mathcal{H}$ is a Hankel tensor with $v=(1,\frac{1}{2},\frac{1}{3},\cdots,\frac{1}{n},\cdots)$ (Qi [11]). Song and Qi [12] studied infinite- and finite-dimensional Hilbert tensors, and showed that $\mathcal{H}$ defines a bounded and positively $(m-1)$-homogeneous operator from l^1 into l^p $(1<p<\infty)$, and found the upper bound of the corresponding positively homogeneous operator norm.

A real mth order n-dimensional tensor (hypermatrix) $\mathcal{A}=(a_{i_1\cdots i_m})$ is a multi-array of real entries $a_{i_1\cdots i_m}$, where $i_j\in\{1,2,\cdots,n\}$ for $j\in\{1,2,\cdots,m\}$. Denote the set of all real mth order n-dimensional tensors by $T_{m,n}$. Then $T_{m,n}$ is a linear space of dimension n^m. Let $\mathcal{A}=(a_{i_1\cdots i_m})\in T_{m,n}$. If the entries $a_{i_1\cdots i_m}$ are invariant under any permutation of their indices, then $\mathcal{A}$ is called a symmetric tensor. Let $\mathcal{A}=(a_{i_1\cdots i_m})\in T_{m,n}$ and $\mathbf{x}\in\mathbb{R}^n$. Then $\mathcal{A}\mathbf{x}^{m-1}$ is a vector in $\mathbb{R}^n$ with its ith component as

$$\left(\mathcal{A}\mathbf{x}^{m-1}\right)_i:=\sum_{i_2,\cdots,i_m=1}^{n}a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}$$

for $i\in\{1,2,\cdots,n\}$ ([13]). For mth order finite-dimensional tensors, various structured tensors were studied well. For more details, M-tensors see Zhang, Qi and Zhou [14] and Ding, Qi and Wei [15]; P-(B-)tensors see Song and Qi [16], Qi and Song [17]; copositive tensors see Song and Qi [18]; Cauchy tensor see Chen and Qi [19]; the applications in nonlinear complementarity problem, tensor complementarity problem see Song and Qi [20], Che, Qi, Wei [21], Song and Yu [22], Luo, Qi and Xiu [23], Gowda, Luo, Qi and Xiu [24], Bai, Huang and Wang [25], Wang, Huang and Bai [26], Ding, Luo and Qi [27], Suo and Wang [28], Song and Qi [29], Ling, He, Qi [30, 31], Chen, Yang, Ye [32].

However, for the infinite-dimensional tensor (hypermatrix), the corresponding results are fewer. Clearly, this class of tensors may be referred to as a class of nonlinear operators with special structure on some infinite-dimensional space. Then by means of its own specific structure, to study the properties of the infinite-dimensional tensor on various infinite-dimensional spaces will be very interesting, which may help to interpret the structure and properties of some infinite-dimensional spaces.

In this paper, we show that an mth order infinite dimensional Hilbert tensor defines an $(m-1)$-homogeneous operator on the spaces of analytic functions (Hardy spaces H^p $(p>m-1)$ and Bergman spaces A^p $(p>2(m-1))$),

$$\mathcal{H}(f)(z)=\sum_{k=0}^{+\infty}\left(\sum_{i_2,i_3,\cdots,i_m=0}^{+\infty}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{k+i_2+i_3+\cdots+i_m+1}\right)z^k \tag{1.5}$$

for all analytic functions $f(z)=\sum\limits_{k=0}^{+\infty}a_kz^k$. The upper bound of the Hilbert tensor operator $\mathcal{H}(f)$ is found on Bergman spaces A^p $(p>2(m-1))$ with the help of the proof technique of Diamantopoulos [9]. So two positively homogeneous operators may be defined on Bergman spaces A^p by the formula

$$T_{\mathcal{H}}(f)(z):=\begin{cases}\|f\|_{A^{p(m-1)}}^{2-m}\mathcal{H}(f)(z), & f\neq 0\\ 0, & f=0\end{cases} \quad \text{and } F_{\mathcal{H}}(f)(z):=(\mathcal{H}(f)(z))^{\frac{1}{m-1}}\ (m \text{ is even}), \tag{1.6}$$

where $T_{\mathcal{H}}:A^{p(m-1)}\to A^p$ and $F_{\mathcal{H}}:A^p\to A^p$. We obtain the upper bounds of the operator norms $\|T_{\mathcal{H}}\|$ and $\|F_{\mathcal{H}}\|$. In particular, when $p=4(m-1)$,

$$\|T_{\mathcal{H}}\|\leq\pi \text{ and } \|F_{\mathcal{H}}\|\leq\pi^{\frac{1}{m-1}}. \tag{1.7}$$

The paper is organized as follows: In Section 2, we will give some basic definitions and facts, which will be used to the proof of main results. In Section 3, we first study the definition of a Hilbert tensor operator and give the corresponding proof to show that such an operator is well-defined. We prove the integral form of the Hilbert tensor operator. Secondly, the boundedness of the Hilbert tensor operator is proved on Bergman spaces A^p $(p>2(m-1))$ by means of its integral form. Finally, we define two positively homogeneous operators induced by the mth order infinite-dimensional Hilbert tensor and prove the upper boundedness of their operator norms.

2. Preliminaries and basic facts

In this section, we will collect some basic definitions and facts, which will be used later on. Throughout this paper, let $\mathbb{C}$ be the complex plane, and let

$$\mathbb{B}:=\{z\in\mathbb{C}:\|z\|<1\}$$

be the open unit disk in $\mathbb{C}$. Likewise, we write $\mathbb{R}$ for the real line. The normalized Lebesgue measure on $\mathbb{B}$ will be denoted by $d\mu$. Obviously,

$$d\mu(z)=\frac{1}{\pi}dxdy=\frac{1}{\pi}rdrd\theta$$

for $z=x+yi=re^{i\theta}$. For $0<p<+\infty$, the Bergman space A^p is the space of all analytic functions f in $\mathbb{B}$ with

$$\|f\|_{A^p}=\left(\int_{\mathbb{B}}|f(z)|^pd\mu(z)\right)^{\frac{1}{p}}<+\infty. \tag{2.1}$$

The Hardy space H^p is the space of all analytic functions f in $\mathbb{B}$ with

$$\|f\|_{H^p} = \sup_{r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < +\infty. \tag{2.2}$$

It is well known that both the Hardy space H^p and the Bergman space A^p are Banach spaces for $1 \le p$, that $H^p \subset A^p$, that both H^p and A^p are embedded as closed subspaces in Lebesgue space $L^p(\mathbb{B})$, and that $H^q \subset H^p$ and $A^q \subset A^p$ for $q \le p$ (for more details, see [33, 34]).

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces, let $T : K \subset X \to Y$ be an operator, and let $r \in \mathbb{R}$. T is called

(i) *r-homogeneous* if $T(tx) = t^r Tx$ for each $t \in \mathbb{C}$ and all $x \in K$;

(ii) *positively homogeneous* if $T(tx) = tTx$ for each $t > 0$ and all $x \in K$;

(iii) *bounded* if there is a real number $M > 0$ such that

$$\|Tx\|_Y \le M \|x\|_X, \text{ for all } x \in K.$$

The gamma function $\Gamma(z)$ is defined by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \tag{2.3}$$

whenever the complex variable z has a positive real part, i.e., $\Re(z) > 0$. The beta function $\beta(u,v)$ is defined by the formula

$$\beta(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1} dt, \ \Re(u) > 0, \ \Re(v) > 0. \tag{2.4}$$

The formula relating the beta function to the gamma function is the following:

$$\beta(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{2.5}$$

Furthermore, the gamma function has the following properties ([35]):

(i) $\Gamma(1) = 1$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$;

(ii) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ for non-integral complex numbers z.

(iii) The duplication formula: $\Gamma(z)\Gamma(z+\frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$, i.e.,

$$\frac{\Gamma(z)\Gamma(z)}{\Gamma(2z)} = 2^{1-2z} \frac{\Gamma(z)\Gamma(\frac{1}{2})}{\Gamma(z+\frac{1}{2})}. \tag{2.6}$$

LEMMA 2.1. ([33, Page 36, Lemma]) *If $f \in H^p$ and $0 < p < +\infty$, then*

$$|f(z)| \le \left(\frac{2}{1-|z|} \right)^{\frac{1}{p}} \|f\|_{H^p}. \tag{2.7}$$

LEMMA 2.2. ([36, Page 755, Corollary]) *If $f \in A^p$ and $0 < p < +\infty$, then*

$$|f(z)| \le \left(\frac{1}{1-|z|^2} \right)^{\frac{2}{p}} \|f\|_{A^p}. \tag{2.8}$$

3. Hilbert tensor operators

3.1. Intergral form of the Hilbert tensor operator.

LEMMA 3.1. *Let $\mathcal{H}$ be an mth order infinite-dimensional Hilbert tensor, and let $f(z)=\sum\limits_{k=0}^{+\infty} a_k z^k \in L^{m-1}(\mathbb{B})$. Then*

$$\mathcal{H}(f)(z)=\sum_{k=0}^{+\infty}\left(\sum_{i_2,i_3,\cdots,i_m=0}^{+\infty}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{k+i_2+i_3+\cdots+i_m+1}\right)z^k \tag{3.1}$$

is a well-defined analytic function on the unit disc $\mathbb{B}$. Furthermore, $\mathcal{H}(f)(z)$ is well-defined on the Hardy space H^p or on the Bergman space A^p $(m-1<p<+\infty)$.

Proof. Let $f_l(z)=\sum\limits_{k=0}^{l} a_k z^k$ for all positive integers l. Obviously, $\lim\limits_{l\to\infty} f_l(z)=f(z)$, and so, $\lim\limits_{l\to\infty}(f_l(z))^{m-1}=(f(z))^{m-1}$. Thus for each $z\in\mathbb{B}$, there is a positive integer N such that $|f_l(z)|^{m-1}\le|f(z)|^{m-1}+1$ for all positive integers $l>N$. So for all positive integers $l>N$, we have

$$\begin{aligned}\left|\sum_{i_2,i_3,\cdots,i_m=0}^{l}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{i_1+i_2+\cdots+i_m+1}\right| &=\left|\sum_{i_2,i_3,\cdots,i_m=0}^{l} a_{i_2}a_{i_3}\cdots a_{i_m}\int_0^1 s^{i_1+\cdots+i_m}ds\right|\\ &=\left|\int_0^1\left(\sum_{i_2,i_3,\cdots,i_m=0}^{l} a_{i_2}a_{i_3}\cdots a_{i_m}s^{i_2+\cdots+i_m}\right)s^{i_1}ds\right|\\ &=\left|\int_0^1\left(\sum_{i=0}^{l}a_i s^i\right)^{m-1}s^{i_1}ds\right|\\ &=\left|\int_0^1 (f_l(s))^{m-1}s^{i_1}ds\right|\\ &\le\int_0^1|f_l(s)|^{m-1}|s|^{i_1}ds\le\int_0^1|f_l(s)|^{m-1}ds\\ &\le\int_0^1(|f(s)|^{m-1}+1)ds<+\infty\quad(\text{since } f\in L^{m-1}(\mathbb{B})).\end{aligned}$$

Then,

$$\left|\sum_{i_2,i_3,\cdots,i_m=0}^{+\infty}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{i_1+i_2+\cdots+i_m+1}\right|<+\infty,$$

and hence, the coefficient $\sum\limits_{i_2,i_3,\cdots,i_m=0}^{+\infty}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{k+i_2+\cdots+i_m+1}$ of the power series

$$\sum_{k=0}^{+\infty}\left(\sum_{i_2,i_3,\cdots,i_m=0}^{+\infty}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{k+i_2+i_3+\cdots+i_m+1}\right)z^k$$

is bounded. So for $|z|<1$, the above power series is absolutely convergent, denoted by $\mathcal{H}(f)(z)$. That is,

$$\mathcal{H}(f)(z)=\sum_{k=0}^{+\infty}\left(\sum_{i_2,i_3,\cdots,i_m=0}^{+\infty}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{k+i_2+i_3+\cdots+i_m+1}\right)z^k,$$

and then, the convergence radius of the power series $\mathcal{H}(f)(z)$ is greater than or equal to 1. Thus $\mathcal{H}(f)(z)$ is an analytic function on the unit disc $\mathbb{B}$. The desired conclusions follow. □

LEMMA 3.2. *Let*

$$\mathcal{G}(f)(z)=\int_0^1\frac{(f(s))^{m-1}}{1-zs}ds\ (m\geq 2) \tag{3.2}$$

for $z\in\mathbb{B}$. The operator $\mathcal{G}(f)(z)$ is well-defined on the Hardy space H^p $(m-1<p<+\infty)$ or on the Bergman space A^p $(2(m-1)<p<+\infty)$.

Proof.
(1) For $f\in H^p$, from Lemma 2.1 and the fact that

$$\frac{1}{|1-zs|}\leq\frac{1}{1-|z||s|}\leq\frac{1}{1-|z|},$$

it follows that

$$\begin{aligned}|\mathcal{G}(f)(z)|&\leq\int_0^1\frac{(|f(s)|)^{m-1}}{|1-zs|}ds\\&\leq\int_0^1\frac{\left(\left(\frac{2}{1-s}\right)^{\frac{1}{p}}\|f\|_{H^p}\right)^{m-1}}{1-|z|}ds\\&=\frac{2^{\frac{m-1}{p}}\|f\|_{H^p}^{m-1}}{1-|z|}\int_0^1\frac{1}{(1-s)^{\frac{m-1}{p}}}ds<+\infty\ (\frac{m-1}{p}<1)\end{aligned}$$

since the integral $\int_0^1\frac{1}{(1-s)^r}ds$ converges for $r<1$.

(2) For $f\in A^p$, it follows from Lemma 2.2 that

$$\begin{aligned}|\mathcal{G}(f)(z)|&\leq\int_0^1\frac{\left(\left(\frac{1}{1-s}\right)^{\frac{2}{p}}\|f\|_{A^p}\right)^{m-1}}{1-|z|}ds\\&=\frac{\|f\|_{A^p}^{m-1}}{1-|z|}\int_0^1\frac{1}{(1-s)^{\frac{2(m-1)}{p}}}ds<+\infty\ (\frac{2(m-1)}{p}<1).\end{aligned}$$

The desired conclusions follow. □

LEMMA 3.3. *Let $\mathcal{H}$ be an mth order infinite-dimensional Hilbert tensor, and let $f\in H^p$ $(m-1<p<+\infty)$ or $f\in A^p$ $(2(m-1)<p<+\infty)$. Then for each $z\in\mathbb{B}$,*

(i) $\mathcal{H}(f)(z)=\mathcal{G}(f)(z)=\int_0^1\frac{(f(s))^{m-1}}{1-zs}ds;$

(ii) $\mathcal{H}(f)(z)=\mathcal{G}(f)(z)=\int_0^1\left(f(\frac{s}{(s-1)z+1})\right)^{m-1}\frac{1}{(s-1)z+1}ds.$

Proof. (i) Let $f_l(z)=\sum\limits_{k=0}^{l} a_k z^k$. Obviously, $\lim\limits_{l\to\infty} f_l(z)=f(z)$, and hence, $\lim\limits_{l\to\infty}(f_l(z))^{m-1}=(f(z))^{m-1}$. Now we may define a power series

$$\mathcal{H}(f_l)(z)=\sum_{k=0}^{+\infty}\left(\sum_{i_2,i_3,\cdots,i_m=0}^{l}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{k+i_2+i_3+\cdots+i_m+1}\right)z^k.$$

Then we have

$$\begin{aligned}
\mathcal{H}(f_l)(z)&=\sum_{k=0}^{+\infty}z^k\sum_{i_2,i_3,\cdots,i_m=0}^{l}a_{i_2}a_{i_3}\cdots a_{i_m}\int_0^1 s^{k+i_2+\cdots+i_m}ds\\
&=\sum_{k=0}^{+\infty}z^k\int_0^1\left(\sum_{i_2,i_3,\cdots,i_m=0}^{l}a_{i_2}a_{i_3}\cdots a_{i_m}s^{i_2+\cdots+i_m}\right)s^k ds\\
&=\sum_{k=0}^{+\infty}z^k\int_0^1\left(\sum_{i=0}^{l}a_i s^i\right)^{m-1}s^k ds\\
&=\sum_{k=0}^{+\infty}\int_0^1 (f_l(s))^{m-1}(zs)^k ds\\
&=\int_0^1 (f_l(s))^{m-1}\left(\sum_{k=0}^{+\infty}(zs)^k\right)ds\\
&=\int_0^1\frac{(f_l(s))^{m-1}}{1-zs}ds.
\end{aligned}$$

For $z\in\mathbb{B}$ and $p>m-1$, it is obvious that the fact that $f(z)\in H^p$ implies that $(f(z))^{m-1}\in H^{\frac{p}{m-1}}$, and hence, $(f(z))^{m-1}-(f_l(z))^{m-1}\in H^{\frac{p}{m-1}}$. Furthermore, from Lemma 2.1, it follows that

$$\begin{aligned}
|\mathcal{H}(f_l)(z)-\mathcal{G}(f)(z)|&=\left|\int_0^r\frac{(f_l(s))^{m-1}-(f(s))^{m-1}}{1-zs}ds\right|\\
&\leq\int_0^1\frac{|(f_l(s))^{m-1}-(f(s))^{m-1}|}{|1-zs|}ds\\
&\leq\int_0^1\frac{\left(\frac{2}{1-s}\right)^{\frac{m-1}{p}}\|f_l^{m-1}-f^{m-1}\|_{H^{\frac{p}{m-1}}}}{1-|z|}ds\\
&=\left(\frac{2^{\frac{m-1}{p}}}{1-|z|}\int_0^1\frac{1}{(1-s)^{\frac{m-1}{p}}}ds\right)\|f_l^{m-1}-f^{m-1}\|_{H^{\frac{p}{m-1}}}.
\end{aligned}$$

Therefore, for each $z\in\mathbb{B}$,

$$\lim_{l\to\infty}\mathcal{H}(f_l)(z)=\sum_{k=0}^{+\infty}\left(\sum_{i_2,i_3,\cdots,i_m=0}^{+\infty}\frac{a_{i_2}a_{i_3}\cdots a_{i_m}}{k+i_2+i_3+\cdots+i_m+1}\right)z^k=\mathcal{G}(f)(z).$$

Then $\mathcal{G}(f)(z)$ defines an analytic function $\mathcal{H}(f)(z)=\lim_{l\to\infty}\mathcal{H}(f_l)(z)$. That is,

$$\mathcal{H}(f)(z)=\mathcal{G}(f)(z)=\int_0^1\frac{(f(s))^{m-1}}{1-zs}ds$$

for each $f\in H^p$ $(m-1<p<+\infty)$.

Similarly, for $f\in A^p$, it follows from Lemma 2.2 that

$$\begin{aligned}|\mathcal{H}(f_l)(z)-\mathcal{G}(f)(z)|&\le\int_0^1\frac{|(f_l(s))^{m-1}-(f(s))^{m-1}|}{|1-zs|}ds\\&\le\int_0^1\frac{\left(\frac{1}{1-s}\right)^{\frac{2(m-1)}{p}}\|f_l^{m-1}-f^{m-1}\|_{A^{\frac{p}{m-1}}}}{1-|z|}ds\\&=\left(\frac{1}{1-|z|}\int_0^1\frac{1}{(1-s)^{\frac{2(m-1)}{p}}}ds\right)\|f_l^{m-1}-f^{m-1}\|_{A^{\frac{p}{m-1}}},\end{aligned}$$

and so, $\mathcal{H}(f)(z)=\mathcal{G}(f)(z)$ for every $f\in A^p$ $(2(m-1)<p<+\infty)$.

(ii) Given $f\in H^p$ $(m-1<p<+\infty)$ or $f\in A^p$ $(2(m-1)<p<+\infty)$, the integral $\mathcal{G}(f)(z)$ is independent of the path of integration. Then for $z\in\mathbb{B}$, we may choose the path of integration

$$s(t)=\frac{t}{(t-1)z+1},0\le t\le 1,$$

and hence

$$s'(t)=\frac{ds(t)}{dt}=\frac{((t-1)z+1)-tz}{((t-1)z+1)^2}=\frac{1-z}{((t-1)z+1)^2}.$$

So we have

$$\begin{aligned}\mathcal{G}(f)(z)&=\int_0^1\frac{\left(f(\frac{t}{(t-1)z+1})\right)^{m-1}}{1-z\frac{t}{(t-1)z+1}}s'(t)dt\\&=\int_0^1\left(f(\frac{t}{(t-1)z+1})\right)^{m-1}\frac{(t-1)z+1}{(t-1)z+1-zt}\frac{1-z}{((t-1)z+1)^2}dt\\&=\int_0^1\left(f(\frac{t}{(t-1)z+1})\right)^{m-1}\frac{1}{(t-1)z+1}dt.\end{aligned}$$

The desired conclusion follows. ∎

3.2. Boundedness of the Hilbert tensor operator.

THEOREM 3.1. *Let $\mathcal{H}$ be an mth order infinite-dimensional Hilbert tensor, and let $\mathcal{H}(f)$ be as in Lemma 3.1. Then $\mathcal{H}$ is bounded and $(m-1)$-homogeneous on the Bergman space $A^{p(m-1)}$ for $2<p<+\infty$, and satisfies the following:*

(i) If $4\le p<+\infty$ and $f\in A^{p(m-1)}$, then

$$\|\mathcal{H}(f)\|_{A^p}\le\frac{\pi}{\sin(\frac{2\pi}{p})}\|f\|^{m-1}_{A^{p(m-1)}}. \tag{3.3}$$

(ii) If $2<p\leq 4$ *and* $f\in A^{p(m-1)}$*, then*

$$\|\mathcal{H}(f)\|_{A^p}\leq M\|f\|^{m-1}_{A^{p(m-1)}}, \tag{3.4}$$

where $M=4^{\frac{4}{p}-1}\sqrt{\pi}\frac{\Gamma(1-\frac{2}{p})}{\Gamma(\frac{3}{2}-\frac{2}{p})}$.

Proof. Let

$$\varphi(t,z)=\frac{t}{(t-1)z+1} \text{ and } \psi(t,z)=\frac{1}{(t-1)z+1}$$

for all $z\in\mathbb{B}$ and all real number t with $0<t<1$. Then

$$\frac{\partial\varphi(t,z)}{\partial z}=\frac{-t(t-1)}{((t-1)z+1)^2}=t(1-t)(\psi(t,z))^2.$$

Let $\mathcal{T}_t(f)(z)=\psi(t,z)\left(f(\varphi(t,z))\right)^{m-1}$. Then for each $t\in(0,1)$, we have

$$\begin{aligned}
\|\mathcal{T}_t(f)\|^p_{A^p}&=\int_{\mathbb{B}}|\psi(t,z)|^p\left|(f(\varphi(t,z)))^{m-1}\right|^p d\mu(z)\\
&=\int_{\mathbb{B}}|\psi(t,z)|^{p-4}|f(\varphi(t,z))|^{p(m-1)}|\psi(t,z)|^4 d\mu(z)\\
&=\int_{\mathbb{B}}|\psi(t,z)|^{p-4}|f(\varphi(t,z))|^{p(m-1)}\frac{1}{t^2(1-t)^2}\left|\frac{\partial\varphi(t,z)}{\partial z}\right|^2 d\mu(z)\\
&=\frac{1}{t^2(1-t)^2}\int_{\mathbb{B}}|\psi(t,z)|^{p-4}|f(\varphi(t,z))|^{p(m-1)}\left|\frac{\partial\varphi(t,z)}{\partial z}\right|^2 d\mu(z).
\end{aligned}$$

(i) For $+\infty>p\geq 4$ and each $t\in(0,1)$, we have

$$\psi(t,z)=\frac{\varphi(t,z)}{t},$$

$$|\varphi(t,z)|=\frac{t}{|(t-1)z+1|}\leq\frac{t}{1-|t-1||z|}\leq\frac{t}{1-(1-t)}=1$$

and furthermore,

$$\begin{aligned}
\|\mathcal{T}_t(f)\|^p_{A^p}&=\frac{1}{t^2(1-t)^2}\int_{\mathbb{B}}\left|\frac{\varphi(t,z)}{t}\right|^{p-4}|f(\varphi(t,z))|^{p(m-1)}\left|\frac{\partial\varphi(t,z)}{\partial z}\right|^2 d\mu(z)\\
&=\frac{1}{t^{p-2}(1-t)^2}\int_{\mathbb{B}}|\varphi(t,z)|^{p-4}|f(\varphi(t,z))|^{p(m-1)}\left|\frac{\partial\varphi(t,z)}{\partial z}\right|^2 d\mu(z)\\
&\leq\frac{1}{t^{p-2}(1-t)^2}\int_{\mathbb{B}}|f(\varphi(t,z))|^{p(m-1)}\left|\frac{\partial\varphi(t,z)}{\partial z}\right|^2 d\mu(z)\\
&=\frac{1}{t^{p-2}(1-t)^2}\int_{\mathbb{D}}|f(y)|^{p(m-1)}d\mu(y),
\end{aligned}$$

where $y=\varphi(t,z)$, $\mathbb{D}=\{y=\varphi(t,z);z\in\mathbb{B}\}$ and $d\mu(y)=\left|\frac{\partial\varphi(t,z)}{\partial z}\right|^2 d\mu(z)$. Therefore, we have

$$\|\mathcal{T}_t(f)\|_{A^p}\leq\frac{1}{t^{1-\frac{2}{p}}(1-t)^{\frac{2}{p}}}\left(\int_{\mathbb{D}}|f(y)|^{p(m-1)}d\mu(y)\right)^{\frac{1}{p}}$$

$$
\begin{aligned}
&= \frac{1}{t^{1-\frac{2}{p}}(1-t)^{\frac{2}{p}}} \left(\left(\int_{\mathbb{D}} |f(y)|^{p(m-1)} d\mu(y) \right)^{\frac{1}{p(m-1)}} \right)^{m-1} \\
&= t^{\frac{2}{p}-1}(1-t)^{-\frac{2}{p}} \|f\|^{m-1}_{A^{p(m-1)}}. \qquad (3.5)
\end{aligned}
$$

From the equality

$$
\mathcal{H}(f)(z) = \int_0^1 \psi(t,z)\left(f(\varphi(t,z))\right)^{m-1} dt = \int_0^1 \mathcal{T}_t(f)(z) dt
$$

and Minkowski's integral inequality, it follows that

$$
\begin{aligned}
\|\mathcal{H}(f)\|_{A^p} &= \left(\int_{\mathbb{B}} |\mathcal{H}(f)(z)|^p d\mu(z) \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{B}} \left| \int_0^1 \mathcal{T}_t(f)(z) dt \right|^p d\mu(z) \right)^{\frac{1}{p}} \\
&\le \int_0^1 \left(\int_{\mathbb{B}} |\mathcal{T}_t(f)(z)|^p d\mu(z) \right)^{\frac{1}{p}} dt \\
&= \int_0^1 \|\mathcal{T}_t(f)\|_{A^p} dt, \qquad (3.6)
\end{aligned}
$$

and hence, using inequality (3.5), we have

$$
\begin{aligned}
\|\mathcal{H}(f)\|_{A^p} &\le \left(\int_0^1 t^{\frac{2}{p}-1}(1-t)^{(1-\frac{2}{p})-1} dt \right) \|f\|^{m-1}_{A^{p(m-1)}} \\
&= \beta\left(\frac{2}{p}, 1-\frac{2}{p}\right) \|f\|^{m-1}_{A^{p(m-1)}} \\
&= \frac{\Gamma(\frac{2}{p})\Gamma(1-\frac{2}{p})}{\Gamma(\frac{2}{p}+1-\frac{2}{p})} \|f\|^{m-1}_{A^{p(m-1)}} \\
&= \frac{\pi}{\sin(\frac{2\pi}{p})} \|f\|^{m-1}_{A^{p(m-1)}}.
\end{aligned}
$$

(ii) For $2 < p \le 4$ and each $t \in (0,1)$, we also have

$$
|(\psi(t,z))^{-1}| = |(t-1)z+1| \le 2,
$$

and so,

$$
\begin{aligned}
\|\mathcal{T}_t(f)\|^p_{A^p} &= \frac{1}{t^2(1-t)^2} \int_{\mathbb{B}} |\psi(t,z)|^{p-4} |f(\varphi(t,z))|^{p(m-1)} \left| \frac{\partial \varphi(t,z)}{\partial z} \right|^2 d\mu(z) \\
&= \frac{1}{t^2(1-t)^2} \int_{\mathbb{B}} \left|(\psi(t,z))^{-1}\right|^{4-p} |f(\varphi(t,z))|^{p(m-1)} \left| \frac{\partial \varphi(t,z)}{\partial z} \right|^2 d\mu(z) \\
&\le \frac{2^{4-p}}{t^2(1-t)^2} \int_{\mathbb{B}} |f(\varphi(t,z))|^{p(m-1)} \left| \frac{\partial \varphi(t,z)}{\partial z} \right|^2 d\mu(z) \\
&= \frac{2^{4-p}}{t^2(1-t)^2} \int_{\mathbb{D}} |f(y)|^{p(m-1)} d\mu(y),
\end{aligned}
$$

where $y=\varphi(t,z)$, $\mathbb{D}=\{y=\varphi(t,z); z\in\mathbb{B}\}$ and $d\mu(y)=\left|\frac{\partial\varphi(t,z)}{\partial z}\right|^2 d\mu(z)$. Therefore, we have

$$\begin{aligned}\|\mathcal{T}_t(f)\|_{A^p} \le& \frac{2^{\frac{4}{p}-1}}{t^{\frac{2}{p}}(1-t)^{\frac{2}{p}}}\left(\left(\int_{\mathbb{D}}|f(y)|^{p(m-1)}d\mu(y)\right)^{\frac{1}{p(m-1)}}\right)^{m-1}\\ =&2^{\frac{4}{p}-1}t^{-\frac{2}{p}}(1-t)^{-\frac{2}{p}}\|f\|^{m-1}_{A^{p(m-1)}}.\end{aligned} \tag{3.7}$$

From inequalities (3.6) and (3.7) and the duplication formula (2.6) of the gamma function $\Gamma(\cdot)$, it follows that

$$\begin{aligned}\|\mathcal{H}(f)\|_{A^p} \le& \int_0^1 \|\mathcal{T}_t(z)\|_{A^p}dt\\ \le& 2^{\frac{4}{p}-1}\left(\int_0^1 t^{(1-\frac{2}{p})-1}(1-t)^{(1-\frac{2}{p})-1}dt\right)\|f\|^{m-1}_{A^{p(m-1)}}\\ =&2^{\frac{4}{p}-1}\beta(1-\frac{2}{p},1-\frac{2}{p})\|f\|^{m-1}_{A^{p(m-1)}}\\ =&2^{\frac{4}{p}-1}\frac{\Gamma(1-\frac{2}{p})\Gamma(1-\frac{2}{p})}{\Gamma(2-\frac{4}{p})}\|f\|^{m-1}_{A^{p(m-1)}}\\ =&2^{\frac{4}{p}-1}\left(2^{1-2(1-\frac{2}{p})}\sqrt{\pi}\frac{\Gamma(1-\frac{2}{p})}{\Gamma(1-\frac{2}{p}+\frac{1}{2})}\right)\|f\|^{m-1}_{A^{p(m-1)}}\\ =&4^{\frac{4}{p}-1}\sqrt{\pi}\frac{\Gamma(1-\frac{2}{p})}{\Gamma(\frac{3}{2}-\frac{2}{p})}\|f\|^{m-1}_{A^{p(m-1)}}.\end{aligned}$$

The desired conclusions follow. □

Define an operator $T_{\mathcal{H}}: A^{p(m-1)}\to A^p$ by the formula

$$T_{\mathcal{H}}(f)(z):=\begin{cases}\|f\|^{2-m}_{A^{p(m-1)}}\mathcal{H}(f)(z), & f\neq 0\\ 0, & f=0.\end{cases} \tag{3.8}$$

When m is even, define another operator $F_{\mathcal{H}}: A^p\to A^p$ by the formula

$$F_{\mathcal{H}}(f)(z):=(\mathcal{H}(f)(z))^{\frac{1}{m-1}}. \tag{3.9}$$

Clearly, both $F_{\mathcal{H}}$ and $T_{\mathcal{H}}$ are bounded and positively homogeneous by Theorem 3.1. So we may define the following operator norms ([38]):

$$\|T_{\mathcal{H}}\|=\sup_{\|f\|_{A^{p(m-1)}}=1}\|T_{\mathcal{H}}(f)\|_{A^p} \text{ and } \|F_{\mathcal{H}}\|=\sup_{\|f\|_{A^p}=1}\|F_{\mathcal{H}}(f)\|_{A^p}. \tag{3.10}$$

The following upper bounds and properities of the operator norm may be established.

THEOREM 3.2. *Let $\mathcal{H}$ be an mth order infinite-dimensional Hilbert tensor, and let $\mathcal{H}(f)$ be as in Lemma 3.1. Then $T_{\mathcal{H}}$ is a bounded and positively homogeneous operator from the Bergman space $A^{p(m-1)}$ to A^p for $2<p<+\infty$, and its norm satisfies the following:*

(i) If $4\le p<+\infty$, then

$$\|T_{\mathcal{H}}\|\le\frac{\pi}{\sin(\frac{2\pi}{p})}. \tag{3.11}$$

(ii) If $2<p\leq 4$, *then*

$$\|T_{\mathcal{H}}\|\leq 4^{\frac{4}{p}-1}\sqrt{\pi}\frac{\Gamma(1-\frac{2}{p})}{\Gamma(\frac{3}{2}-\frac{2}{p})}. \tag{3.12}$$

Proof. It follows from definition (3.8) of the operator $T_{\mathcal{H}}$ that

$$\|T_{\mathcal{H}}(f)\|_{A^p}=\|\|f\|^{2-m}_{A^{p(m-1)}}\mathcal{H}(f)\|_{A^p}=\|f\|^{2-m}_{A^{p(m-1)}}\|\mathcal{H}(f)\|_{A^p}.$$

Then an application of Theorem 3.1 yields the following:

(i) For $4\leq p<+\infty$,

$$\|T_{\mathcal{H}}(f)\|_{A^p}\leq \|f\|^{2-m}_{A^{p(m-1)}}\left(\frac{\pi}{\sin(\frac{2\pi}{p})}\|f\|^{m-1}_{A^{p(m-1)}}\right)=\frac{\pi}{\sin(\frac{2\pi}{p})}\|f\|_{A^{p(m-1)}}.$$

(ii) For $2<p\leq 4$,

$$\begin{aligned}\|T_{\mathcal{H}}(f)\|_{A^p}\leq&\|f\|^{2-m}_{A^{p(m-1)}}\left(4^{\frac{4}{p}-1}\sqrt{\pi}\frac{\Gamma(1-\frac{2}{p})}{\Gamma(\frac{3}{2}-\frac{2}{p})}\|f\|^{m-1}_{A^{p(m-1)}}\right)\\ =&4^{\frac{4}{p}-1}\sqrt{\pi}\frac{\Gamma(1-\frac{2}{p})}{\Gamma(\frac{3}{2}-\frac{2}{p})}\|f\|_{A^{p(m-1)}}.\end{aligned}$$

So the desired conclusions directly follow from the definition (3.10) of the operator norm. □

THEOREM 3.3. *Let* $\mathcal{H}$ *be an mth order infinite-dimensional Hilbert tensor, and let* $\mathcal{H}(f)$ *be as in Lemma 3.1. Then* $F_{\mathcal{H}}$ *is a bounded and positively homogeneous operator from the Bergman space* A^p *to* A^p *for* $2(m-1)<p<+\infty$ *if m is even, and its norm satisfies the following:*

(i) If $4(m-1)\leq p<+\infty$, *then*

$$\|F_{\mathcal{H}}\|\leq\left(\frac{\pi}{\sin(\frac{2(m-1)\pi}{p})}\right)^{\frac{1}{m-1}}. \tag{3.13}$$

(ii) If $2(m-1)<p\leq 4(m-1)$, *then*

$$\|F_{\mathcal{H}}\|\leq 4^{\frac{4}{p}}\left(\frac{\sqrt{\pi}\Gamma(1-\frac{2(m-1)}{p})}{4\Gamma(\frac{3}{2}-\frac{2(m-1)}{p})}\right)^{\frac{1}{m-1}}. \tag{3.14}$$

Proof. It follows from definition (3.9) of the operator $F_{\mathcal{H}}$ and Minkowski's integral inequality that

$$\begin{aligned}\|F_{\mathcal{H}}(f)\|_{A^p}&=\left(\int_{\mathbb{B}}\left|(\mathcal{H}(f)(z))^{\frac{1}{m-1}}\right|^p d\mu(z)\right)^{\frac{1}{p}}\\ &=\left(\left(\int_{\mathbb{B}}|\mathcal{H}(f)(z)|^{\frac{p}{m-1}}d\mu(z)\right)^{\frac{m-1}{p}}\right)^{\frac{1}{m-1}}\end{aligned}$$

$$
\begin{aligned}
&=\left(\left(\int_{\mathbb{B}}\left|\int_0^1 \mathcal{T}_t(f)(z)dt\right|^{\frac{p}{m-1}} d\mu(z)\right)^{\frac{m-1}{p}}\right)^{\frac{1}{m-1}}\\
&\le\left(\int_0^1\left(\int_{\mathbb{B}}|\mathcal{T}_t(f)(z)|^{\frac{p}{m-1}} p d\mu(z)\right)^{\frac{m-1}{p}} dt\right)^{\frac{1}{m-1}}\\
&=\left(\int_0^1 \|\mathcal{T}_t(f)\|_{A^{\frac{p}{m-1}}} dt\right)^{\frac{1}{m-1}}, \qquad (3.15)
\end{aligned}
$$

Using the proof technique of Theorem 3.1 (p is replaced by $\frac{p}{m-1}$), the following may be proved easily:

(i) For $4\le \frac{p}{m-1}<+\infty$,

$$
\|\mathcal{T}_t(f)\|_{A^{\frac{p}{m-1}}}\le t^{\frac{2(m-1)}{p}-1}(1-t)^{-\frac{2(m-1)}{p}}\|f\|_{A^p}^{m-1},
$$

and hence,

$$
\|F_{\mathcal{H}}(f)\|_{A^p}\le\left(\frac{\pi}{\sin(\frac{2(m-1)\pi}{p})}\|f\|_{A^p}^{m-1}\right)^{\frac{1}{m-1}}=\left(\frac{\pi}{\sin(\frac{2(m-1)\pi}{p})}\right)^{\frac{1}{m-1}}\|f\|_{A^p}.
$$

(ii) For $2<\frac{p}{m-1}\le 4$,

$$
\|\mathcal{T}_t(f)\|_{A^{\frac{p}{m-1}}}\le 2^{\frac{4(m-1)}{p}-1}t^{-\frac{2(m-1)}{p}}(1-t)^{-\frac{2(m-1)}{p}}\|f\|_{A^p}^{m-1},
$$

and hence,

$$
\begin{aligned}
\|F_{\mathcal{H}}(f)\|_{A^p}&\le\left(4^{\frac{4(m-1)}{p}-1}\sqrt{\pi}\frac{\Gamma(1-\frac{2(m-1)}{p})}{\Gamma(\frac{3}{2}-\frac{2(m-1)}{p})}\|f\|_{A^p}^{m-1}\right)^{\frac{1}{m-1}}\\
&=4^{\frac{4}{p}}\left(\frac{\sqrt{\pi}\Gamma(1-\frac{2(m-1)}{p})}{4\Gamma(\frac{3}{2}-\frac{2(m-1)}{p})}\right)^{\frac{1}{m-1}}\|f\|_{A^p}.
\end{aligned}
$$

So the desired conclusions directly follow from the definition (3.10) of the operator norm. □

Let $p=4$ in Theorem 3.2 ((i) or (ii)) and $p=4(m-1)$ in Theorem 3.3((i) or (ii)), respectively. Then the following conclusions are easily obtained.

Corollary 3.1. *Let $\mathcal{H}$ be an mth order infinite-dimensional Hilbert tensor, and let $\mathcal{H}(f)$ be as in Lemma 3.1. Then*

(i) $T_{\mathcal{H}}: A^{4(m-1)}\to A^4$ is a bounded and positively homogeneous operator and

$$
\|T_{\mathcal{H}}\|\le\pi;
$$

(ii) $F_{\mathcal{H}}: A^{4(m-1)}\to A^{4(m-1)}$ is a bounded and positively homogeneous operator if m is even and

$$
\|F_{\mathcal{H}}\|\le\pi^{\frac{1}{m-1}}.
$$

REMARK 3.1.

(i) In this paper, the boundedness of the Hilbert tensor operator is obtained on A^p for $p>2(m-1)$. For $0<p\leq 2(m-1)$, it is not clear that whether or not the Hilbert tensor operator is bounded on the Bergman space A^p or the Hardy space H^p or other spaces of analytic functions.

(ii) Are the upper bounds of the norm of the Hilbert tensor operator the best in this paper?

(iii) May the operator norms of $T_{\mathcal{H}}$ and $F_{\mathcal{H}}$ be given the exact value?

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