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ERRATUM TO "GLOBAL CLASSICAL SOLUTIONS OF THE 'ONE AND ONE-HALF' DIMENSIONAL VLASOV-MAXWELL-FOKKER-PLANCK SYSTEM"

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Abstract. This note is an erratum to [S. Pankavich and J. Schaeffer, Comm. Math. Sci., 14(1):209–232, 2016] and corrects an L^2 estimate concerning derivatives of a Green's function for the linear Vlasov–Fokker–Planck operator. Here, the proof that relies on this estimate is corrected using alternative means.

Key words. Kinetic Theory, Vlasov, Fokker–Planck equation, global existence.

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1. Introduction

The purpose of this article is to correct an error in [1]. The notation and assumptions used in [1] will be used here, and it is assumed that the reader is familiar with [1]. In particular, the estimate

$$\left(\iint |\nabla_w \mathcal{G}(t,x,v,\tau,y,w)|^2 dwdy\right)^{\frac{1}{2}} \leq C(t-\tau)^{-1/2}, \tag{1.1}$$

which appears just above line (4.28) of [1], is incorrect. What follows completes the proof of Theorem 1.1 of [1] without using the estimate (1.1) by proving that the solution possesses the stated regularity.

2. Correction to Proof of Theorem 1.1

Proceeding as in (4.16) of [1] we have

$$\begin{aligned}
&\frac{d}{dt}\iint v_0^a \left(f^{n+1}-f^{\ell+1}\right)^2 dvdx \\
&= -2\iint v_0^a \left|\nabla_v \left(f^{n+1}-f^{l+1}\right)\right|^2 dvdx \\
&+ \iint \left(f^{n+1}-f^{\ell+1}\right)^2 (\Delta_v v_0^a + E^n \cdot \nabla_v v_0^a) dvdx \\
&+2\iint f^{\ell+1}\left(K^n-K^\ell\right)\cdot\left(v_0^a \nabla_v\left(f^{n+1}-f^{\ell+1}\right)\left(f^{n+1}-f^{\ell+1}\right)\nabla_v v_0^a\right) dvdx.
\end{aligned} \tag{2.1}$$

We will use the notation

$$\sigma^{n,l} \leq o^{n,l}$$

to mean $\forall \epsilon > 0\ \exists N$ such that $n,\ell > N \Rightarrow \sigma^{n,l} \leq \epsilon$. Recall that

$$\sup_{t\in[0,T]} \left|\left|(E^n B^n) - \left(E^\ell, B^\ell\right)\right|\right|_{H^1(\mathbb{R})} + \sup_{t\in[0,T]} \iint v_0^a \left(f^{n+1}-f^{\ell+1}\right)^2 dvdx \leq o^{n,\ell}.$$

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By (4.14) of [1] we have

$$\begin{aligned}\left|K^n-K^\ell\right| &\le v_0\left|(E^n,B^n)-\left(E^\ell,B^\ell\right)\right|+v_0^{1+\epsilon/2}\left|B^\ell\right|2^{-n\epsilon/2}\\ &\le v_0^{1+\epsilon/2}o^{n,\ell}\end{aligned}$$

where we assume $\ell>n$.

Recall that $\alpha=a+2+\epsilon$ and note that

$$\begin{aligned}&\iint\left|f^{\ell+1}\left(K^n-K^\ell\right)\cdot v_0^a\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|dvdx\\ &\le o^{n,\ell}\iint\left|f^{\ell+1}\right|v_0^{a+1+\epsilon/2}\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|dvdx\\ &\le o^{n,\ell}\sqrt{\iint v_o^\alpha\left(f^{\ell+1}\right)^2dvdx}\sqrt{\iint v_0^a\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|^2dvdx}\\ &\le o^{n,\ell}\sqrt{\iint v_0^a\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|^2dvdx}.\end{aligned}$$

Similarly,

$$\begin{aligned}&\iint\left|f^{\ell+1}\left(K^n-K^\ell\right)\cdot\left(f^{n+1}-f^{\ell+1}\right)\nabla_v v_0^a\right|dvdx\\ &\le o^{n,\ell}\sqrt{\iint v_0^a\left(f^{n+1}-f^{\ell+1}\right)^2dvdx}\le o^{n,\ell}.\end{aligned}$$

So, by Equation (2.1) we find

$$\begin{aligned}&\frac{d}{dt}\iint v_0^a\left(f^{n+1}-f^{\ell+1}\right)^2dvdx\\ &\le-\iint v_0^a\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|^2dvdx+o^{n,\ell}\\ &+\left[o^{n,\ell}\sqrt{\iint v_0^a\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|^2dvdx}-\iint v_0^a\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|^2dvdx\right]\\ &\le o^{n,\ell}-\iint v_0^a\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|^2dvdx.\end{aligned}$$

It follows that

$$\int_0^T\iint v_0^a\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|^2dvdxd\tau\le o^{n,\ell}. \tag{2.2}$$

In a similar manner we may show that

$$\int_0^T\iint v_0^\alpha\left|\nabla_v f^{n+1}\right|^2dvdxd\tau\le C. \tag{2.3}$$

Note that the exponent of v_0 in the inequality (2.2) is a, but in the inequality (2.3) it is α.

Next, we derive L^p bounds on $\mathcal{G}$. Considering $0\le\tau\le t\le T$, letting $p\ge 1,\ b,\theta\ge 0$, and using the change of variables

$$u=\frac{v-w}{\sqrt{t-\tau}},\qquad z=\frac{x-y-\frac{t-\tau}{2}(v_1+w_1)}{(t-\tau)^{3/2}}$$

we have

$$\begin{aligned}
&\iint w_0^{-b\theta}\mathcal{G}^p dwdy\\
&=C\iint\left(\sqrt{1+|v-\sqrt{t-\tau}u|^2}\right)^{-b\theta}\left[(t-\tau)^{-5/2}e^{-|u|^2/4}e^{-3z^2}\right]^p(t-\tau)^{5/2}dzdu\\
&=C(t-\tau)^{\frac{5}{2}(1-p)}\left(\int_{|u|<\frac{1}{2}(t-\tau)^{-1/2}|v|}\left(\sqrt{1+|v-\sqrt{t-\tau}u|^2}\right)^{-b\theta}e^{-p|u|^2/4}du\right.\\
&\left.+\int_{|u|>\frac{1}{2}(t-\tau)^{-1/2}|v|}\left(\sqrt{1+|v-\sqrt{t-\tau}u|^2}\right)^{-b\theta}e^{-p|u|^2/4}du\right)\\
&\le C(t-\tau)^{\frac{5}{2}(1-p)}\left[\left(\sqrt{1+\left(\frac{1}{2}|v|\right)^2}\right)^{-b\theta}\int e^{-|u|^2/4}du+e^{-\frac{1}{8}\left(\frac{|v|}{2\sqrt{t-\tau}}\right)^2}\int e^{-|u|^2/8}du\right]\\
&\le C(t-\tau)^{\frac{5}{2}(1-p)}\left[\left(\sqrt{1+|v|^2}\right)^{-b\theta}+e^{-C|v|^2}\right]\\
&\le C(t-\tau)^{\frac{5}{2}(1-p)}v_0^{-b\theta}.
\end{aligned}$$

So for $1\le p<7/5$ we have

$$\int_0^t\iint w_0^{-b\theta}\mathcal{G}^p dwdyd\tau\le Cv_0^{-b\theta}. \tag{2.4}$$

Here, constants may depend on p,b, and θ. Later, specific choices of p,b, and θ are used and this dependence is removed.

Next, we bound $|\nabla_w\mathcal{G}|$ and $|\nabla_v\mathcal{G}|$. Note that

$$\begin{aligned}
&\iint w_0^{-b\theta}\left(\frac{|v-w|}{t-\tau}\mathcal{G}\right)^p dwdy\\
&=C\iint\left(\sqrt{1+|v-\sqrt{t-\tau}u|^2}\right)^{-b\theta}\left[\frac{|u|}{\sqrt{t-\tau}}(t-\tau)^{-5/2}e^{-|u|^2/4}e^{-3z^2}\right]^p(t-\tau)^{5/2}dzdu\\
&=C(t-\tau)^{\frac{5}{2}-3p}\int\left(\sqrt{1+|v-\sqrt{t-\tau}u|^2}\right)^{-b\theta}\left[|u|e^{-|u|^2/4}\right]^p du\\
&\le C(t-\tau)^{\frac{5}{2}-3p}v_0^{-b\theta}
\end{aligned}$$

and similarly,

$$\iint w_0^{-b\theta}\left(\frac{\left|x-y-\frac{t-\tau}{2}(v_1+w_1)\right|}{(t-\tau)^2}\mathcal{G}\right)^p dwdy\le C(t-\tau)^{\frac{5}{2}-3p}v_0^{-b\theta}.$$

Hence,

$$\iint w_0^{-b\theta}\left(|\nabla_w\mathcal{G}|^p+|\nabla_v\mathcal{G}|^p\right)dwdy\le C(t-\tau)^{\frac{5}{2}-3p}v_0^{-b\theta}$$

and for $1\le p<7/6$

$$\int_0^t\iint w_0^{-b\theta}\left(|\nabla_w\mathcal{G}|^p+|\nabla_v\mathcal{G}|^p\right)dwdyd\tau\le Cv_0^{-b\theta}. \tag{2.5}$$

In a very similar manner it may be shown that

$$\int_0^T\iint v_0^{-b\theta}\mathcal{G}^p dvdxdt\le Cw_0^{-b\theta} \tag{2.6}$$

for $p<7/5$ and

$$\int_0^T\iint v_0^{-b\theta}\left(|\nabla_v\mathcal{G}|^p+|\nabla_w\mathcal{G}|^p\right)dvdxdt\le Cw_0^{-b\theta}$$

for $p<7/6$.

Next we derive two inequalities that will be used repeatedly. Let $p,q\in[1,\infty)$ and $r\in[1,\infty]$ satisfy

$$\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}.$$

We will first consider the case $r\neq\infty$, but what follows may be easily adapted to the case $r=\infty$. Let $\theta\ge 0$ and define

$$b=\left(\frac{1}{p}-\frac{1}{r}\right)^{-1}\qquad c=\left(\frac{1}{q}-\frac{1}{r}\right)^{-1}$$

and note that

$$\frac{1}{r}+\frac{1}{b}+\frac{1}{c}=1.$$

By Hölder's inequality and using the inequality (2.4), we have for $p<7/5$ and any $h(\tau,y,w)\ge 0$

$$\begin{aligned}
&\int_0^t\iint\mathcal{G}h\,dwdyd\tau=\int_0^t\iint\left[\mathcal{G}^{p/r}(w_0^\theta h)^{q/r}\right]\left[\mathcal{G}^{p\left(\frac{1}{p}-\frac{1}{r}\right)}w_0^{-\theta}\right]\left[w_0^\theta h\right]^{q\left(\frac{1}{q}-\frac{1}{r}\right)}dwdyd\tau\\
&\le\left(\int_0^t\iint\mathcal{G}^p\left(w_0^\theta h\right)^q dwdyd\tau\right)^{\frac{1}{r}}\left(\int_0^t\iint\mathcal{G}^p w_0^{-b\theta}dwdyd\tau\right)^{\frac{1}{b}}\left(\int_0^t\iint\left(w_0^\theta h\right)^q dwdyd\tau\right)^{\frac{1}{c}}\\
&\le Cv_0^{-\theta}\left(\int_0^t\iint\mathcal{G}^p\left(w_0^\theta h\right)^q dwdyd\tau\right)^{1/r}\left(\int_0^t\iint\left(w_0^\theta h\right)^q dwdyd\tau\right)^{1/c}.
\end{aligned}$$

Hence, using the inequality (2.6) we have, for $p<7/5$

$$\begin{aligned}
&\left[\int_0^T\iint\left(v_0^\theta\int_0^t\iint\mathcal{G}h\,dwdyd\tau\right)^r dvdxdt\right]^{1/r}\\
&\leq C\left[\int_0^T\iint\left(\int_0^t\iint\mathcal{G}^p\left(w_0^\theta h\right)^q dwdyd\tau\right)\right.\\
&\qquad\left.\left(\int_0^t\iint\left(w_0^\theta h\right)^q dwdyd\tau\right)^{r/c} dvdxdt\right]^{1/r}\\
&\leq C\left[\left(\int_0^T\iint\left(w_0^\theta h\right)^q dwdyd\tau\right)^{r/c}\right.\\
&\qquad\left.\int_0^T\iint\left(\int_0^T\iint\mathcal{G}^p dvdxdt\right)\left(w_0^\theta h\right)^q dwdyd\tau\right]^{1/r}\\
&\leq C\left[\left(\int_0^T\iint\left(w_0^\theta h\right)^q dwdyd\tau\right)^{1+r/c}\right]^{1/r}\\
&=C\left[\int_0^T\iint\left(w_0^\theta h\right)^q dwdyd\tau\right]^{1/q}.
\end{aligned}\tag{2.7}$$

Similarly, using the inequality (2.5) we find for $p<7/6$

$$\begin{aligned}
&\left[\int_0^T\iint\left(v_0^\theta\int_0^t\iint\left[|\nabla_w\mathcal{G}|+|\nabla_v\mathcal{G}|\right]h\,dwdyd\tau\right)^r dvdxdt\right]^{1/r}\\
&\leq C\left(\int_0^T\iint\left(w_0^\theta h\right)^q dwdyd\tau\right)^{1/q}.
\end{aligned}\tag{2.8}$$

Now we will show that f^n converges in L^∞. We have

$$f^{n+1}-f^{\ell+1}=-\int_0^t\iint\mathcal{G}\left(K^n\cdot\nabla_v f^{n+1}-K^\ell\cdot\nabla_v f^{\ell+1}\right)(\tau,y,w)\,dwdyd\tau. \tag{2.9}$$

Using (4.14) of [1] and taking $\ell>n$

$$\begin{aligned}
&\left|K^n\cdot\nabla_v f^{n+1}-K^\ell\cdot\nabla_v f^{\ell+1}\right|\\
&\leq\left|K^n-K^\ell\right|\left|\nabla_v f^{n+1}\right|+\left|K^\ell\right|\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|\\
&\leq\left|(E^n,B^n)-\left(E^\ell,B^\ell\right)\right|w_0\left|\nabla_v f^{n+1}\right|\\
&+Cw_0^{1+\epsilon/2}2^{-n\epsilon/2}\left|\nabla_v f^{n+1}\right|+Cw_0\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|\\
&\leq o^{n,\ell}w_0^{1+\epsilon/2}\left|\nabla_v f^{n+1}\right|+Cw_0\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|.
\end{aligned}\tag{2.10}$$

Next, we will apply the inequality (2.7) with $p<7/5$ and $q=2$. Note that by taking p close to $7/5$, we may make $r=\left(\frac{1}{p}+\frac{1}{q}-1\right)^{-1}$ close to

$$\left[\frac{5}{7}+\frac{1}{2}-1\right]^{-1}=14/3.$$

Thus, applying the inequality (2.7) with $p<7/5,\ q=2,\ \theta=\frac{a}{2}-1$, and

$$h=w_0\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|$$

while using the inequality (2.2) yields

$$\left[\int_0^T\iint\left(v_0^{\frac{a}{2}-1}\int_0^t\iint\mathcal{G}w_0\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|dwdyd\tau\right)^r dvdxdt\right]^{1/r}$$
$$\leq C\left[\int_0^T\iint\left(w_0^{\frac{a}{2}-1}w_0\left|\nabla_v\left(f^{n+1}-f^{\ell+1}\right)\right|\right)^2dwdyd\tau\right]^{\frac{1}{2}}\leq o^{n,\ell}$$

for $r<14/3$. Applying the inequality (2.7) with $p<7/5,\ q=2,\ \theta=\frac{\alpha}{2}-1-\epsilon/2$, and

$$h=w_0^{1+\epsilon/2}\left|\nabla_v f^{n+1}\right|$$

and then using the inequality (2.3) yields

$$\left[\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-1-\epsilon/2}\int_0^t\iint\mathcal{G}w_0^{1+\epsilon/2}\left|\nabla_v f^{n+1}\right|dwdyd\tau\right)^r dvdxdt\right]^{1/r}$$
$$\leq C\left[\int_0^T\iint\left(w_0^{\frac{\alpha}{2}-1-\epsilon/2}w_0^{1+\epsilon/2}\left|\nabla_v f^{n+1}\right|\right)^2dwdyd\tau\right]^{\frac{1}{2}}\leq C$$

for $r<14/3$. Since $\frac{\alpha}{2}-1-\frac{\epsilon}{2}>\frac{a}{2}-1$, Equation (2.9) and inequality (2.10) now yield for $r<14/3$

$$\int_0^T\iint\left(v_0^{\frac{a}{2}-1}\left|f^{n+1}-f^{\ell+1}\right|\right)^r dvdxdt\leq o^{n.\ell}. \tag{2.11}$$

Similarly, using the inequality (2.3) we may show that for $r<14/3$

$$\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-1}\left|f^{n+1}\right|\right)^r dvdxdt\leq C. \tag{2.12}$$

To use the inequality (2.11) we integrate by parts in Equation (2.9) to obtain

$$f^{n+1}-f^{\ell+1}=\int_0^t\iint\nabla_w\mathcal{G}\cdot\left(K^nf^{n+1}-K^\ell f^{\ell+1}\right)dwdyd\tau \tag{2.13}$$

and using (4.14) of [1]

$$\begin{aligned}&\left|K^nf^{n+1}-K^\ell f^{\ell+1}\right|\\&\leq\left|(E^n,B^n)-\left(E^\ell,B^\ell\right)\right|w_0\left|f^{n+1}\right|\\&+Cw_0^{1+\epsilon/2}2^{-n\epsilon/2}\left|f^{n+1}\right|+Cw_0\left|f^{n+1}-f^{\ell+1}\right|\\&\leq o^{n,\ell}w_0^{1+\epsilon/2}\left|f^{n+1}\right|+Cw_0\left|f^{n+1}-f^{\ell+1}\right|.\end{aligned} \tag{2.14}$$

Next, we will apply the inequality (2.8), but now with $p<7/6$ and $q<14/3$. Note that we may take p close to $7/6$ and q close to $14/3$ to make r close to

$$\left(\frac{6}{7}+\frac{3}{14}-1\right)^{-1}=14.$$

Thus, applying inequality (2.8) with $p<7/6$, $q<14/3$, $\theta=\frac{a}{2}-2$, and

$$h=w_0\left|f^{n+1}-f^{\ell+1}\right|$$

and then using the inequality (2.11) yields

$$\left[\int_0^T\iint\left(v_0^{\frac{a}{2}-2}\int_0^t\iint|\nabla_w\mathcal{G}|w_0\left|f^{n+1}-f^{\ell+1}\right|dwdyd\tau\right)^r dvdxdt\right]^{1/r}$$

$$\leq C\left[\int_0^T\iint\left(w_0^{\frac{a}{2}-2}w_0\left|f^{n+1}-f^{\ell+1}\right|\right)^q dwdyd\tau\right]^{1/q}\leq o^{n\ell}$$

for $r<14$. Applying inequality (2.8) again with $p<7/6$, $q<14/3$, $\theta=\frac{\alpha}{2}-2-\frac{\epsilon}{2}$ and

$$h=w_0^{1+\epsilon/2}\left|f^{n+1}\right|$$

yields

$$\left[\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-2-\epsilon/2}\int_0^t\iint|\nabla_w\mathcal{G}|w_0^{1+\epsilon/2}\left|f^{n+1}\right|dwdyd\tau\right)^r dvdxdt\right]^{1/r}$$

$$\leq\left[\int_0^T\iint\left(w_0^{\frac{\alpha}{2}-1}\left|f^{n+1}\right|\right)^q dwdyd\tau\right]^{1/q}\leq C$$

for $r<14$ by using the inequality (2.12). With these two estimates, (2.13) and (2.14) yield

$$\int_0^T\iint\left(v_0^{\frac{a}{2}-2}\left|f^{n+1}-f^{\ell+1}\right|\right)^r dvdxdt\leq o^{n,\ell}$$

for $r<14$. Similarly,

$$\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-2}\left|f^{n+1}\right|\right)^r dvdxdt\leq C.$$

Finally, we can apply estimate (2.8) with $p=\frac{7}{6.4}, q=\frac{14}{1.2}$, and $r=\infty$. Proceeding as above we obtain

$$\left|\left|v_0^{\frac{a}{2}-3}\left(f^{n+1}-f^{\ell+1}\right)\right|\right|_{L^\infty}\leq o^{n,\ell}.$$

Defining

$$f=\lim_{n\to\infty}f^n$$

it follows that

$$\left|\left|v_0^{\frac{a}{2}-3}(f^n-f)\right|\right|_{L^\infty}\to 0\quad\text{as}\quad n\to\infty.$$

Next, we bound $\nabla_v f^n$ in L^∞. We have

$$f^{n+1}=H-\int_0^t\iint\mathcal{G}K^n\cdot\nabla_v f^{n+1}dwdyd\tau$$

so

$$\begin{aligned}\left|\nabla_v\left(f^{n+1}-H\right)\right|&=\left|\int_0^t\iint\nabla_v\mathcal{G}K^n\cdot\nabla_v f^{n+1}dwdyd\tau\right|\\&\le C\int_0^t\iint|\nabla_w\mathcal{G}|w_0\left|\nabla_v f^{n+1}\right|dwdyd\tau.\end{aligned}\tag{2.15}$$

Applying estimate (2.8) with $p<7/6$, $q=2$, $\theta=\frac{\alpha}{2}-1$, and $h=w_0\left|\nabla_v f^{n+1}\right|$, and then using (2.3) yields

$$\begin{aligned}&\left[\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-1}\int_0^t\iint|\nabla_w\mathcal{G}|w_0\left|\nabla_v f^{n+1}\right|dwdyd\tau\right)^r dvdxdt\right]^{1/r}\\&\le\left[\int_0^T\iint\left(w_0^{\frac{\alpha}{2}-1}w_0\left|\nabla_v f^{n+1}\right|\right)^2 dwdyd\tau\right]^{1/2}\le C.\end{aligned}$$

Note that taking p close to $7/6$ yields r close to

$$\left(\frac{6}{7}+\frac{1}{2}-1\right)^{-1}=14/5.$$

Hence, using estimate (2.15) we find for $r<14/5$

$$\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-1}\left|\nabla_v f^{n+1}\right|\right)^r dvdxdt\le C.$$

We then apply estimate (2.8) three more times. In each application we take $h=w_0\left|\nabla_v f^{n+1}\right|$. First, using $p<\frac{7}{6}$, $q<14/5$, and $\theta=\frac{\alpha}{2}-2$ yields

$$\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-2}\left|\nabla_v f^{n+1}\right|\right)^r dvdxdt\le C$$

for $r<14/3$. Using $p<7/6$, $q<14/3$, and $\theta=\frac{\alpha}{2}-3$ yields

$$\int_0^T\iint\left(v_0^{\frac{\alpha}{2}-3}\left|\nabla_v f^{n+1}\right|\right)^r dvdxdt\le C$$

for $r<14$. Using $p=\frac{7}{6.4}$, $q=\frac{14}{1.2}$, $r=\infty$, and $\theta=\frac{\alpha}{2}-4$ yields

$$\left\|v_0^{\frac{\alpha}{2}-4}\nabla_v f^{n+1}\right\|_{L^\infty}\le C.\tag{2.16}$$

Recall that $a>8$ so $\frac{\alpha}{2}-4=\frac{a+2+\epsilon}{2}-4>1$. Now $\forall h\in\mathbb{R}^2$

$$\left|f^{n+1}(t,x,v+h)-f^{n+1}(t,x,v)\right|\le C|h|,$$

and so

$$|f(t,x,v+h)-f(t,x,v)|\le C|h|.$$

Finally, we show that f is Hölder continuous in x. Let $h>0$ and

$$e(t,x,v)=f^{n+1}(t,x+h,v)-f^{n+1}(t,x,v),$$

then

$$\partial_t e + v_1 \partial_x e + K^n \cdot \nabla_v e - \Delta_v e = -(K^n(t,x+h,v) - K^n(t,x,v)) \cdot \nabla_v f^{n+1}(t,x+h,v).$$

Note that

$$\begin{aligned}|K^n(t,x+h,v) - K^n(t,x,v)| &\leq v_0 (|E^n(t,x+h) - E^n(t,x)| + |B^n(t,x+h) - B^n(t,x)|) \\ &\leq v_0 h^{1/2} \left(\sqrt{\int (\partial_x E)^2 \, dx} + \sqrt{\int (\partial_x B)^2 \, dx} \right) \\ &\leq C v_0 h^{1/2}.\end{aligned}$$

Thus, by the bound (2.16) we find

$$|\partial_t e + v_1 \partial_x e + K^n \cdot \nabla_v e - \Delta_v e| \leq C v_0 h^{1/2} C v_0^{4-\alpha/2} \leq C h^{1/2}.$$

By Lemma 2.1 of [1]

$$|e| \leq C h^{1/2}.$$

It follows that f is Hölder continuous with exponent $1/2$ in x.

Now, Theorem II.1 of [2] shows that f possesses the regularity stated in Theorem 1.1 of [1], and this completes the argument.

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