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ENFORCE THE DIRICHLET BOUNDARY CONDITION BY VOLUME CONSTRAINT IN POINT INTEGRAL METHOD*

ZUOQIANG SHI†

Abstract. Poisson equation on point cloud with Dirichlet boundary condition plays important role in many problems. In this paper, we use the volume constraint proposed by Du et.al to handle the Dirichlet boundary condition in the point integral method for Poisson equation on point cloud. We prove that the solution given by volume constraint converges to the true solution as the point cloud converges to the underlying smooth manifold.

Keywords. Point Integral Method; volume constraint; Dirichlet boundary; Laplace-Beltrami operator; Poisson equation.

AMS subject classifications. 65N12.

1. Introduction

Partial differential equations on manifolds arise in a wide variety of applications, including material science [8,15], fluid flow [17,19], biology and biophysics [1,2,16,29]. In the past several years, manifold model attracts more and more attentions in data analysis and image processing [3,10,30,31]. In many problems, data can be represented as a set of points in high dimensional Euclidean space, which is usually referred as point cloud. One fundamental problem is to infer the value of a function on the whole point cloud from the value on a subset of the point cloud. Harmonic function provides an efficient way to extend the function to the whole point cloud. In harmonic extension, one need to find a harmonic function such that it coincides with the given value in the subset of the point cloud. This harmonic function can be obtained by solving a Laplace equation with Dirichlet type boundary condition.

To solve PDEs on manifold, people have developed many numerical methods, such as surface finite element method [14], level set method [6,38], grid based particle method [22,23] and closest point method [28,32]. These methods are very powerful especially on 2D surfaces. However, it is difficult to generalize these methods to solve PDEs on point cloud in high dimensional space. Liang et al. proposed to discretize the differential operators on point cloud by local least square approximations of the manifold [26]. Later, Lai et al. proposed local mesh method to approximate the differential operators on point cloud [21]. The main idea is to approximate the manifold locally by polynomials or mesh. Once the local approximation is obtained, it is easy to discretize the differential operators. When the dimension of the manifold is high, even the local approximation is not easy to construct.

In [25], Li et.al. proposed a novel numerical method, point integral method (PIM), to solve the Poisson equation on point cloud. The main idea of the point integral method is to approximate the Poisson equation by an integral equation:

$$-\int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}_t(\mathbf{x},\mathbf{y}) \mathrm{d}\mathbf{y} \approx \frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y} - 2\int_{\partial\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\mathrm{d}\tau_{\mathbf{y}}, \tag{1.1}$$

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†Yau Mathematical Sciences Center, Tsinghua University, Beijing, 100084, P.R. China (zqshi@mail.tsinghua.edu.cn).

where $\mathbf{n}$ is the out normal of $\mathcal{M}$, $\mathcal{M}$ is a smooth k-dimensional manifold embedded in $\mathbb{R}^d$ and $\partial\mathcal{M}$ is the boundary of $\mathcal{M}$. $R_t(\mathbf{x},\mathbf{y})$ and $\bar{R}_t(\mathbf{x},\mathbf{y})$ are kernel functions given as follows

$$R_t(\mathbf{x},\mathbf{y}) = C_t R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right), \quad \bar{R}_t(\mathbf{x},\mathbf{y}) = C_t \bar{R}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) \tag{1.2}$$

where $C_t = \frac{1}{(4\pi t)^{k/2}}$ is the normalizing factor. $R \in C^2(\mathbb{R}^+)$ be a positive function which is integrable over $[0,+\infty)$,

$$\bar{R}(r) = \int_r^{+\infty} R(s)\mathrm{d}s.$$

$\Delta_{\mathcal{M}} = \mathrm{div}(\nabla)$ is the Laplace-Beltrami operator on $\mathcal{M}$. Let $\Phi : \Omega \subset \mathbb{R}^k \to \mathcal{M} \subset \mathbb{R}^d$ be a local parametrization of $\mathcal{M}$ and $\theta \in \Omega$. For any differentiable function $f : \mathcal{M} \to \mathbb{R}$, define the gradient on the manifold

$$\nabla f(\Phi(\theta)) = \sum_{i,j=1}^{k} g^{ij}(\theta)\frac{\partial \Phi}{\partial \theta_i}(\theta)\frac{\partial f(\Phi(\theta))}{\partial \theta_j}(\theta),$$

and for vector field $F : \mathcal{M} \to T_{\mathbf{x}}\mathcal{M}$ on $\mathcal{M}$, where $T_{\mathbf{x}}\mathcal{M}$ is the tangent space of $\mathcal{M}$ at $\mathbf{x} \in \mathcal{M}$, the divergence is defined as

$$\mathrm{div}(F) = \frac{1}{\sqrt{\det G}} \sum_{m=1}^{d} \sum_{i,j=1}^{k} \frac{\partial}{\partial \theta_i}\left(\sqrt{\det G}\, g^{ij} F^m(\Phi(\theta))\frac{\partial \Phi^m}{\partial \theta_j}\right),$$

where $(g^{ij})_{i,j=1,\cdots,k} = G^{-1}$, $\det G$ is the determinant of matrix G and $G(\theta) = (g_{ij})_{i,j=1,\cdots,k}$ is the first fundamental form which is defined by

$$g_{ij}(\theta) = \sum_{m=1}^{d} \frac{\partial \Phi_m}{\partial \theta_i}(\theta)\frac{\partial \Phi_m}{\partial \theta_j}(\theta), \quad i,j = 1,\cdots,k.$$

and $(F^1(\mathbf{x}),\cdots,F^d(\mathbf{x}))^t$ is the representation of F in the embedding coordinates.

The point integral method is closely related with the graph Laplacian. Graph Laplacian is a discrete object which reveals many properties of graphs [9]. When there is no boundary, it is proved that [4,18,20,35] that the graph Laplacian with the Gaussian weights well approximates the Laplace-Beltrami operator under the assumption that vertices of the graph sample the underlying manifold. However, graph Laplacian does not give a method to deal with the boundary condition. Near the boundary, it was observed [5,20] that the graph Laplacian is dominated by the first order derivative and thus fails to be true Laplacian.

Based on the integral approximation (1.1), Neumann boundary condition is easy to handle while Dirichlet boundary condition is more involved. Volume constraint gives a powerful tool to deal with the Dirichlet boundary condition in the nonlocal diffusion problem [11]. By integrating volume constraint in the point integral method, we get a numerical method for elliptic equation with Dirichlet boundary condition on point cloud. In this paper, we focus on the Poisson equation with Dirichlet boundary condition, i.e.,

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \mathcal{M}, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M}, \end{cases} \tag{1.3}$$

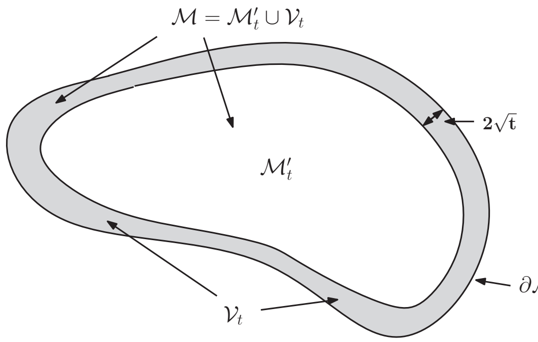


FIG. 1.1. *Computational domain for volume constraint.*

To correctly enforce the Dirichlet boundary condition, we use the volume constraint which was proposed by Du et.al. [11] in the nonlocal diffusion problem. The main idea in the volume constraint is to extend the boundary condition to a small region adjacent to the boundary to remove the boundary term in (1.1). Integrating the volume constriant and the integral approximation (1.1), we get following integral equation to approximate the Dirichlet problem (1.3),

$$\begin{cases} \frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y} = \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y}, & \mathbf{x}\in\mathcal{M}_t', \\ u(\mathbf{x}) = 0, & \mathbf{x}\in\mathcal{V}_t. \end{cases} \tag{1.4}$$

Here, $\mathcal{M}_t'$ and $\mathcal{V}_t$ are subsets of $\mathcal{M}$ which are defined as

$$\mathcal{M}_t' = \left\{\mathbf{x}\in\mathcal{M} : B\left(\mathbf{x}, 2\sqrt{t}\right)\cap\partial\mathcal{M} = \emptyset\right\}, \quad \mathcal{V}_t = \mathcal{M}\backslash\mathcal{M}_t'.$$

The thickness of $\mathcal{V}_t$ is $2\sqrt{t}$ which implies that $|\mathcal{V}_t| = O(\sqrt{t})$. The relation of $\mathcal{M}$, $\partial\mathcal{M}$, $\mathcal{M}_t'$ and $\mathcal{V}_t$ are sketched in Figure 1.1.

The integral Equation (1.4) is easy to discretized on point cloud. Assume we are given a set of sample points $P = \{\mathbf{p}_i : \mathbf{p}_i \in \mathcal{M},\ i = 1,\cdots,n\}$ sampling the submanifold $\mathcal{M}$ and one vector $\mathbf{V} = (V_1,\cdots,V_n)^t$ where V_i is the volume weight of $\mathbf{p}_i$ in $\mathcal{M}$. In addition, we assume that the point set P is a good sample of manifold $\mathcal{M}$ in the sense that the integral on $\mathcal{M}$ can be well approximated by the summation over P, see Section 2.

Then, (1.4) can be easily discretized to get following linear system

$$\begin{cases} \frac{1}{t}\sum_{\mathbf{p}_j\in P} R_t(\mathbf{p}_i,\mathbf{p}_j)(u_i-u_j)V_j = \sum_{\mathbf{p}_j\in P} \bar{R}_t(\mathbf{p}_i,\mathbf{p}_j)f(\mathbf{p}_j)V_j, & \mathbf{p}_i\in\mathcal{M}_t', \\ u_i = 0, & \mathbf{p}_i\in\mathcal{V}_t. \end{cases} \tag{1.5}$$

This is the discretization of the Poisson Equation (1.3) given by volume constraint in the point integral method on point cloud.

In [25, 34], Li et.al. proposed to use the Robin boundary condition to approximate the Dirichlet boundary condition. Comparing with this approach, volume constraint is much easier to analyze and has many good properties, for example, the discrete operator is symmetric and positive definite, the maximum principle is preserved.

In the algorithm presented in this paper, we use the first order quadrature rule to discretize the integral equation. This simple quadrature makes the algorithm very robust and applicable in high dimensional problems, such as semi-supervised learning, image processing. On the other hand, low order accuracy in quadrature makes the convergence of the algorithm very slow. Recently, on the approximations of nonlocal diffusion models, high order discretizations are discussed [13, 36, 37]. We can also use higher order quadrature to improve the accuracy. However, the analysis of convergence may be more involved.

The rest of the paper is organized as following. The main theorem is presented in Section 2. In Section 3, we prove several stability results which will be used in the analysis of the convergence. The convergence is proved in Section 4. Several numerical results are presented in Section 5. And one application in semi-supervised learning is given in Section 6. Finally, in Section 7, some conclusions are made.

2. Main results

Before proving the convergence of (1.5), we need to clarify the meaning of the convergence between the point cloud $(P, \mathbf{V})$ and the manifold $\mathcal{M}$. In this paper, we consider the convergence in the sense that $h(P, \mathbf{V}, \mathcal{M}) \to 0$ where $h(P, \mathbf{V}, \mathcal{M})$ is the *integral accuracy index* defined as following,

DEFINITION 2.1 (Integral Accuracy Index). *For the point cloud $(P, \mathbf{V})$ which samples the manifold $\mathcal{M}$, the integral accuracy index $h(P, \mathbf{V}, \mathcal{M})$ is defined as*

$$h(P, \mathbf{V}, \mathcal{M}) = \sup_{f \in C^1(\mathcal{M})} \frac{\left| \int_{\mathcal{M}} f(\mathbf{y}) \mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_i \in P} f(\mathbf{p}_i) V_i \right|}{|\mathrm{supp}(f)| \|f\|_{C^1(\mathcal{M})}}.$$

where $\|f\|_{C^1(\mathcal{M})} = \|f\|_\infty + \|\nabla f\|_\infty$ and $|\mathrm{supp}(f)|$ is the volume of the support of f.

In some sense, $h(P, \mathbf{V}, \mathcal{M})$ is a measure of how well the point cloud sample the underlying manifold. We say that the point cloud $(P, \mathbf{V})$ converges to the manifold $\mathcal{M}$ if $h(P, \mathbf{V}, \mathcal{M}) \to 0$. In the convergence analysis, we assume that $h(P, \mathbf{V}, \mathcal{M})$ is small enough. Here, "small enough" means that less than a generic constant which only depends on $\mathcal{M}$.

To get the convergence, we also need some assumptions on the regularity of the submanifold $\mathcal{M}$ and the integral kernel function R.

ASSUMPTION 2.1.

- Smoothness of the manifold: $\mathcal{M}, \partial\mathcal{M}$ are both compact and C^∞ smooth k-dimensional submanifolds isometrically embedded in a Euclidean space $\mathbb{R}^d$.
- Assumptions on the kernel function $R(r)$:
  - (a) Smoothness: $R \in C^2(\mathbb{R}^+)$;
  - (b) Nonnegativity: $R(r) \geq 0$ for any $r \geq 0$.
  - (c) Compact support: $R(r) = 0$ for $\forall r > 1$;
  - (d) Nondegeneracy: $\exists \delta_0 > 0$ so that $R(r) \geq \delta_0$ for $0 \leq r \leq \frac{1}{2}$.

REMARK 2.1. In the nondegeneracy assumption, 1/2 may be replaced by a positive number θ_0 with $0 < \theta_0 < 1$. Similar assumptions on the kernel function is also used in analysis the nonlocal diffusion problem [12].

Under above assumptions, we have the main result of this paper.

THEOREM 2.1. *Let $u(\mathbf{x})$ be solution of* (1.3) *and $\mathbf{u} = [u_1, \cdots, u_n]^t$ be solution of* (1.5) *and $f \in C^1(\mathcal{M})$ in both problems. There exists $C > 0$ only depends on $\mathcal{M}$, such* *that*

$$\|u - u_{t,h}\|_{H^1(\mathcal{M}'_t)} \le C\left(t^{1/4} + \frac{h(P,\mathbf{V},\mathcal{M})}{t^{3/2}}\right)\|f\|_{C^1(\mathcal{M})}$$

where

$$u_{t,h}(\mathbf{x}) = \begin{cases} \dfrac{1}{w_{t,h}(\mathbf{x})}\left(\displaystyle\sum_{\mathbf{p}_j\in P} R_t(\mathbf{x},\mathbf{p}_j)u_jV_j + t\sum_{\mathbf{p}_j\in P}\bar{R}_t(\mathbf{x},\mathbf{p}_j)f(\mathbf{p}_j)V_j\right), & \mathbf{x}\in\mathcal{M}'_t, \\ 0, & \mathbf{x}\in\mathcal{V}_t. \end{cases} \tag{2.1}$$

and $w_{t,h}(\mathbf{x}) = \sum_{\mathbf{p}_j\in P} R_t(\mathbf{x},\mathbf{p}_j)V_j$.

3. Stability analysis

To prove the convergence, we need some stability results which are listed in this section. The first lemma is about the coercivity of the integral operator and the proof can be found in [24,33].

LEMMA 3.1. *For any function $u \in L^2(\mathcal{M})$, there exists a constant $C > 0$ only depends on $\mathcal{M}$, such that*

$$\frac{1}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \ge C\int_{\mathcal{M}} |\nabla v|^2 \mathrm{d}\mathbf{x},$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y},$$

and $w_t(\mathbf{x}) = \displaystyle\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}$.

Following corollary directly follows from Lemma 3.1.

COROLLARY 3.1. *For any function $u \in L_2(\mathcal{M}'_t)$, there exists a constant $C > 0$ only depends on $\mathcal{M}$, such that*

$$\frac{1}{t}\int_{\mathcal{M}'_t}\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} + \frac{1}{t}\int_{\mathcal{M}'_t} u^2(\mathbf{x})\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}$$
$$\ge C\int_{\mathcal{M}'_t} |\nabla v|^2 \mathrm{d}\mathbf{x},$$

where

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y},$$

and $w_t(\mathbf{x}) = \displaystyle\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}$.

Proof. Let

$$\tilde{u}(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x}\in\mathcal{M}'_t, \\ 0, & \mathbf{x}\in\mathcal{V}_t. \end{cases}$$

Using Lemma 3.1,

$$\begin{aligned}
&\int_{\mathcal{M}'_t} |\nabla v|^2 \mathrm{d}\mathbf{x} \le \int_{\mathcal{M}} |\nabla v|^2 \mathrm{d}\mathbf{x} \\
&\le \frac{C}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(\tilde{u}(\mathbf{x})-\tilde{u}(\mathbf{y}))^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \frac{C}{t}\int_{\mathcal{M}'_t}\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} + \frac{C}{t}\int_{\mathcal{M}'_t} u^2(\mathbf{x}) \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right) \mathrm{d}\mathbf{x}.
\end{aligned}$$

□

Using Lemma 3.1, we can also get following lemma.

LEMMA 3.2. *For any function $u \in L_2(\mathcal{M})$ with $u(\mathbf{x}) = 0$ in $\mathcal{V}_t$, there exists a constant $C > 0$ independent on t*

$$\frac{1}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \ge C\|u\|^2_{L_2(\mathcal{M})},$$

as long as t small enough.

Proof. Let

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}.$$

Since $u(\mathbf{x}) = 0,\ \mathbf{x} \in \mathcal{V}_t$, for any $\mathbf{x} \in \partial\mathcal{M}$,

$$v(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y} = 0.$$

By Lemma 3.1 and the Poincare inequality, there exists a constant $C > 0$, such that

$$\int_{\mathcal{M}} |v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \le \int_{\mathcal{M}} |\nabla v(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \le \frac{C}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R_t\left(\mathbf{x},\mathbf{y}\right)\left(u(\mathbf{x})-u(\mathbf{y})\right)^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.$$

Let $\delta = \frac{w_{\min}}{2w_{\max}+w_{\min}}$, where $w_{\min} = \min_{\mathbf{x}\in\mathcal{M}} w_t(\mathbf{x})$ and $w_{\max} = \max_{\mathbf{x}\in\mathcal{M}} w_t(\mathbf{x})$. If u is smooth and close to its smoothed version v, in particular,

$$\int_{\mathcal{M}} v^2(\mathbf{x})\mathrm{d}\mathbf{x} \ge \delta^2 \int_{\mathcal{M}} u^2(\mathbf{x})\mathrm{d}\mathbf{x}, \tag{3.1}$$

then the proof is completed.

Now consider the case where (3.1) does not hold. Note that we now have

$$\begin{aligned}
\|u-v\|_{L^2(\mathcal{M})} &\ge \|u\|_{L_2(\mathcal{M})} - \|v\|_{L_2(\mathcal{M})} > (1-\delta)\|u\|_{L_2(\mathcal{M})} \\
&> \frac{1-\delta}{\delta}\|v\|_{L_2(\mathcal{M})} = \frac{2w_{\max}}{w_{\min}}\|v\|_{L_2(\mathcal{M})}.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\frac{1}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&= \frac{2}{t}\int_{\mathcal{M}} u(\mathbf{x})\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{t}\left(\int_{\mathcal{M}} u^2(\mathbf{x}) w_t(\mathbf{x})\mathrm{d}\mathbf{x} - \int_{\mathcal{M}} u(\mathbf{x})v(\mathbf{x})w_t(\mathbf{x})\mathrm{d}\mathbf{x}\right)\\
&= \frac{2}{t}\left(\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 w_t(\mathbf{x})\mathrm{d}\mathbf{x} + \int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))v(\mathbf{x})w_t(\mathbf{x})\mathrm{d}\mathbf{x}\right)\\
&\geq \frac{2}{t}\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 w_t(\mathbf{x})\mathrm{d}\mathbf{x} - \frac{2}{t}\left(\int_{\mathcal{M}} v^2(\mathbf{x})w_t(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{1/2}\left(\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2 w_t(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{1/2}\\
&\geq \frac{2w_{\min}}{t}\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2\mathrm{d}\mathbf{x} - \frac{2w_{\max}}{t}\left(\int_{\mathcal{M}} v^2(\mathbf{x})\mathrm{d}\mathbf{x}\right)^{1/2}\left(\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2\mathrm{d}\mathbf{x}\right)^{1/2}\\
&\geq \frac{w_{\min}}{t}\int_{\mathcal{M}} (u(\mathbf{x})-v(\mathbf{x}))^2\mathrm{d}\mathbf{x} \geq \frac{w_{\min}}{t}(1-\delta)^2\int_{\mathcal{M}} u^2(\mathbf{x})\mathrm{d}\mathbf{x}.
\end{aligned}
$$

This completes the proof for the theorem. ∎

COROLLARY 3.2. *For any function $u \in L_2(\mathcal{M}_t')$, there exists a constant $C > 0$ independent on t, such that*

$$
\begin{aligned}
&\frac{1}{t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} + \int_{\mathcal{M}_t'} u^2(\mathbf{x})\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\\
&\geq C\|u\|^2_{L_2(\mathcal{M}_t')},
\end{aligned}
$$

as long as t small enough.

Proof. Consider

$$
\tilde{u}(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \mathcal{M}_t',\\ 0, & \mathbf{x} \in \mathcal{V}_t. \end{cases}
$$

and apply Lemma 3.2. ∎

THEOREM 3.1. *Let $u_t(\mathbf{x}) \in L^2(\mathcal{M})$ be solution of following integral equation*

$$
\begin{cases} \dfrac{1}{t}\displaystyle\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x})-u_t(\mathbf{y}))\mathrm{d}\mathbf{y} = r(\mathbf{x}), & \mathbf{x} \in \mathcal{M}_t'\\ u_t(\mathbf{x}) = 0, & \mathbf{x} \in \mathcal{V}_t \end{cases} \tag{3.2}
$$

There exists $C > 0$ only depends on $\mathcal{M}$ and $\partial\mathcal{M}$, such that

$$
\|u_t\|_{H^1(\mathcal{M}_t')} \leq C\|r\|_{L^2(\mathcal{M}_t')} + Ct\|\nabla r\|_{L^2(\mathcal{M}_t')}.
$$

Proof. First of all, we have

$$
\begin{aligned}
&\frac{1}{t}\int_{\mathcal{M}_t'} u_t(\mathbf{x})\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x})-u_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\\
&= \frac{1}{t}\int_{\mathcal{M}_t'} u_t(\mathbf{x})\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x})-u_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\\
&\quad + \frac{1}{t}\int_{\mathcal{M}_t'} u_t(\mathbf{x})\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x})-u_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\\
&= \frac{1}{2t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x})-u_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} + \frac{1}{t}\int_{\mathcal{M}_t'} u_t^2(\mathbf{x})\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Now we can get L^2 estimate of u_t. Using Corollary 3.2, we have

$$
\|u_t\|^2_{2,\mathcal{M}_t'} \leq \frac{C}{t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x})-u_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} + \frac{C}{t}\int_{\mathcal{M}_t'} |u_t(\mathbf{x})|^2\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}
$$

$$\leq \left| \frac{C}{t} \int_{\mathcal{M}_t'} u_t(\mathbf{x}) \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x}) - u_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \right|$$
$$\leq C\|u_t\|_{2,\mathcal{M}_t'}\|r\|_{2,\mathcal{M}_t'}$$

This gives that

$$\|u_t\|_{L^2(\mathcal{M}_t')} \leq C\|r\|_{L^2(\mathcal{M}_t')}. \tag{3.3}$$

Next, we turn to estimate the L^2 norm of ∇u_t in $\mathcal{M}_t'$. Using the integral Equation (3.2), u_t has following expression

$$u_t(\mathbf{x}) = \frac{1}{w_t(\mathbf{x})} \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})u_t(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{t}{w_t(\mathbf{x})} r(\mathbf{x}), \quad \mathbf{x} \in \mathcal{V}_t. \tag{3.4}$$

Then $\|\nabla u_t\|_{2,\mathcal{M}_t'}^2$ can be bounded as following

$$\|\nabla u_t\|_{2,\mathcal{M}_t'}^2 \leq C \left\| \nabla \left(\frac{1}{w_t(\mathbf{x})} \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})u_t(\mathbf{y})\mathrm{d}\mathbf{y} \right) \right\|_{2,\mathcal{M}_t'}^2 + Ct^2 \left\| \nabla \left(\frac{r(\mathbf{x})}{w_t(\mathbf{x})} \right) \right\|_{2,\mathcal{M}_t'}^2 \tag{3.5}$$

Corollary 3.1 gives a bound the first term of (3.5).

$$\begin{aligned} &\left\| \nabla \left(\frac{1}{w_t(\mathbf{x})} \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})u_t(\mathbf{y})\mathrm{d}\mathbf{y} \right) \right\|_{2,\mathcal{M}_t'}^2 \\ \leq &\frac{C}{t} \int_{\mathcal{M}_t'} \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x}) - u_t(\mathbf{y}))^2\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} + \frac{C}{t} \int_{\mathcal{M}_t'} |u_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x}. \end{aligned} \tag{3.6}$$

The second terms of (3.5) can be bounded by direct calculation.

$$\begin{aligned} \left\| \nabla \left(\frac{r(\mathbf{x})}{w_t(\mathbf{x})} \right) \right\|_{2,\mathcal{M}_t'}^2 &\leq C \left\| \frac{\nabla r(\mathbf{x})}{w_t(\mathbf{x})} \right\|_{2,\mathcal{M}_t'}^2 + C \left\| \frac{r(\mathbf{x})\nabla w_t(\mathbf{x})}{(w_t(\mathbf{x}))^2} \right\|_{2,\mathcal{M}_t'}^2 \\ &\leq C \|\nabla r(\mathbf{x})\|_{2,\mathcal{M}_t'}^2 + \frac{C}{t} \|r(\mathbf{x})\|_{2,\mathcal{M}_t'}^2 . \end{aligned} \tag{3.7}$$

Now we have the bound of $\|\nabla u_t\|_{2,\mathcal{M}_t'}$ by combining (3.5), (3.6), and (3.7)

$$\begin{aligned} \|\nabla u_t\|_{2,\mathcal{M}_t'}^2 \leq &\frac{C}{t} \int_{\mathcal{M}_t'} \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x}) - u_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\ &+ \frac{C}{t} \int_{\mathcal{M}_t'} |u_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x} + Ct^2 \|\nabla r(\mathbf{x})\|_{2,\mathcal{M}_t'}^2 + Ct \|r(\mathbf{x})\|_{2,\mathcal{M}_t'}^2 . \end{aligned} \tag{3.8}$$

Then the bound of $\|\nabla u_t\|_{2,\mathcal{M}_t'}$ can be obtained also from (3.8)

$$\begin{aligned} &\|\nabla u_t\|_{2,\mathcal{M}_t'}^2 \\ &\leq \frac{C}{t} \int_{\mathcal{M}_t'} \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x}) - u_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} + Ct \|r(\mathbf{x})\|_{2,\mathcal{M}_t'}^2 \\ &\quad + \frac{C}{t} \int_{\mathcal{M}_t'} |u_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} \right) \mathrm{d}\mathbf{x} + Ct^2 \|\nabla r(\mathbf{x})\|_{2,\mathcal{M}_t'}^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \left|\frac{C}{t}\int_{\mathcal{M}_t'} u_t(\mathbf{x})\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u_t(\mathbf{x})-u_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\right| \\
&\quad + Ct^2\left\|\nabla r(\mathbf{x})\right\|_{2,\mathcal{M}_t'}^2 + Ct\left\|r(\mathbf{x})\right\|_{2,\mathcal{M}_t'}^2 \\
&\leq \|u_t\|_{2,\mathcal{M}_t'}\|r\|_{2,\mathcal{M}_t'} + Ct^2\left\|\nabla r(\mathbf{x})\right\|_{2,\mathcal{M}_t'}^2 + Ct\left\|r(\mathbf{x})\right\|_{2,\mathcal{M}_t'}^2 \\
&\leq C\|r\|_{L^2(\mathcal{M}_t')}^2 + Ct^2\left\|\nabla r(\mathbf{x})\right\|_{2,\mathcal{M}_t'}^2 .
\end{aligned}
$$

Then we have

$$
\|\nabla u_t\|_{2,\mathcal{M}_t'} \leq C\|r\|_{L^2(\mathcal{M}_t')} + Ct\left\|\nabla r(\mathbf{x})\right\|_{2,\mathcal{M}_t'} . \tag{3.9}
$$

The proof is completed by putting (3.3) and (3.9) together. □

4. Convergence analysis (Proof of Theorem 2.1)

The main purpose of this section is to prove that the solution of (1.5) converges to the solution of the original Poisson Equation (1.3), i.e. Theorem 2.1 in Section 2. To prove this theorem, we split it to two parts. First, we prove that the solution of the integral Equation (1.4) converges to the solution of the Poisson Equation (1.3), which is given in Theorem 4.1. Then we prove Theorem 4.2 to show that the solution of (1.5) converges to the solution of (1.4).

THEOREM 4.1. *Let $u(\mathbf{x})$ be solution of* (1.3) *and $u_t(\mathbf{x})$ be solution of* (1.4). *There exists $C>0$ only depends on $\mathcal{M}$ and $\partial\mathcal{M}$, such that*

$$
\|u-u_t\|_{H^1(\mathcal{M}_t')} \leq Ct^{1/4}\|f\|_{H^1(\mathcal{M})}.
$$

THEOREM 4.2. *Let $u_t(\mathbf{x})$ be the solution of the problem* (1.4) *and $\mathbf{u}=(u_1,\cdots,u_n)$ be the solution of the problem* (1.5). *If $f\in C^1(\mathcal{M})$ in both problems, then there exists constants $C>0$ depending only on $\mathcal{M}$ and $\partial\mathcal{M}$ so that*

$$
\|u_{t,h}-u_t\|_{H^1(\mathcal{M}_t')} \leq \frac{Ch}{t^{3/2}}\|f\|_{C^1(\mathcal{M})},
$$

as long as t and $\frac{h}{\sqrt{t}}$ are both small enough. Here

$$
u_{t,h}(\mathbf{x}) = \begin{cases} \dfrac{1}{w_{t,h}(\mathbf{x})}\left(\displaystyle\sum_{\mathbf{p}_j\in P} R_t(\mathbf{x},\mathbf{p}_j)u_jV_j + t\sum_{\mathbf{p}_j\in P}\bar{R}_t(\mathbf{x},\mathbf{p}_j)f(\mathbf{p}_j)V_j\right), & \mathbf{x}\in\mathcal{M}_t', \\ 0, & \mathbf{x}\in\mathcal{V}_t. \end{cases}
$$

and $w_{t,h}(\mathbf{x})=\sum_{\mathbf{p}_j\in P}R_t(\mathbf{x},\mathbf{p}_j)V_j$.

4.1. Proof of Theorem 4.1.

To prove the convergence of the integral Equation (1.4), we need following theorem about the consistency which has been proved in [24].

THEOREM 4.3. *Let $u(\mathbf{x})$ be the solution of the problem* (1.3). *Let $u\in H^3(\mathcal{M})$ and*

$$
r(\mathbf{x}) = \frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y} - \int_{\mathcal{M}}\bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y}.
$$

There exists constants C,T_0 depending only on $\mathcal{M}$ and $\partial\mathcal{M}$, so that for any $t\leq T_0$,

$$
\|r(\mathbf{x})\|_{L^2(\mathcal{M}_t')} \leq Ct^{1/2}\|u\|_{H^3(\mathcal{M})}, \tag{4.1}
$$

$$\|\nabla r(\mathbf{x})\|_{L^2(\mathcal{M}_t')} \le C\|u\|_{H^3(\mathcal{M})}. \tag{4.2}$$

Now, we can prove Theorem 4.1.

Proof. Let $e_t(\mathbf{x}) = u(\mathbf{x}) - u_t(\mathbf{x})$, first of all, we have

$$\begin{aligned}
&\frac{1}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x}) - e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\
=&\frac{1}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x}) - e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\
&+ \frac{1}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x}) - e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\
=&\frac{1}{2t}\int_{\mathcal{M}_t'} \int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x}) - e_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \\
&+ \frac{1}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x}) - e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}.
\end{aligned} \tag{4.3}$$

The second term can be calculated as

$$\begin{aligned}
&\frac{1}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x}) - e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x} \\
=& \frac{1}{t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} - \frac{1}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}.
\end{aligned} \tag{4.4}$$

Here we use the definition of e_t and the volume constraint condition $u_t(\mathbf{x}) = 0,\ \mathbf{x} \in \mathcal{V}_t$ to get that $e_t(\mathbf{x}) = u(\mathbf{x}),\ \mathbf{x} \in \mathcal{V}_t$.

The first term is positive which is good for us. We only need to bound the second term of (4.4) to show that it can be controlled by the first term. First, the second term can be bounded as following

$$\begin{aligned}
&\frac{1}{t}\left|\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\right| \\
\le& \frac{1}{t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})| \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)^{1/2} \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})|u(\mathbf{y})|^2\mathrm{d}\mathbf{y}\right)^{1/2}\mathrm{d}\mathbf{x} \\
\le& \frac{1}{t}\left(\int_{\mathcal{M}_t'} \frac{1}{2}|e_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} + 2\int_{\mathcal{M}_t'} \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})|u(\mathbf{y})|^2\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\right) \\
\le& \frac{1}{2t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} + \frac{2}{t}\int_{\mathcal{V}_t} |u(\mathbf{y})|^2 \left(\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\right)\mathrm{d}\mathbf{y} \\
\le& \frac{1}{2t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} + \frac{C}{t}\int_{\mathcal{V}_t} |u(\mathbf{y})|^2\mathrm{d}\mathbf{y} \\
\le& \frac{1}{2t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} + C\sqrt{t}\|f\|_{H^1(\mathcal{M})}^2.
\end{aligned} \tag{4.5}$$

Here we use Lemma A.1 in Appendix A to get the last inequality.

By substituting (4.5), (4.4) in (4.3), we get

$$\left|\frac{1}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x}) \int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x}) - e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\right|$$

$$\geq \frac{1}{2t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$
$$+\frac{1}{2t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})|^2\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} - C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t}. \quad (4.6)$$

This is the key estimate we used to get convergence.

Notice that $e_t(\mathbf{x})$ satisfying an integral equation,

$$\frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))\mathrm{d}\mathbf{y} = r(\mathbf{x}), \quad \forall \mathbf{x}\in\mathcal{M}_t', \quad (4.7)$$

where $r(\mathbf{x}) = \frac{1}{t}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y} - \int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y}$.

From Theorem 4.3, we know that

$$\|r(\mathbf{x})\|_{L^2(\mathcal{M}_t')} \leq Ct^{1/2}\|u\|_{H^3(\mathcal{M})} \leq C\sqrt{t}\|f\|_{H^1(\mathcal{M})}, \quad (4.8)$$
$$\|\nabla r(\mathbf{x})\|_{L^2(\mathcal{M}_t')} \leq C\|u\|_{H^3(\mathcal{M})} \leq C\|f\|_{H^1(\mathcal{M})}. \quad (4.9)$$

Now we can get L^2 estimate of e_t. Using Corollary 3.2, we have

$$\begin{aligned}
\|e_t\|_{2,\mathcal{M}_t'}^2 \leq& \frac{C}{t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
&+\frac{C}{t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})|^2\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}\\
\text{(from (4.6))}\quad \leq& \left|\frac{C}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x})\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\right| + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t}\\
\text{(from (4.7))}\quad \leq& C\|e_t\|_{2,\mathcal{M}_t'}\|r\|_{2,\mathcal{M}_t'} + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t}\\
\text{(from (4.8))}\quad \leq& C\|f\|_{H^1(\mathcal{M})}\|e_t\|_{2,\mathcal{M}_t'}\sqrt{t} + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t}. \quad (4.10)
\end{aligned}$$

This gives that

$$\|e_t\|_{2,\mathcal{M}_t'} \leq Ct^{1/4}\|f\|_{H^1(\mathcal{M})}. \quad (4.11)$$

Next, we turn to estimate the L^2 norm of ∇e_t in $\mathcal{M}_t'$. Using the integral Equation (4.7), e_t has following expression

$$\begin{aligned}
e_t(\mathbf{x}) &= \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})e_t(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{t}{w_t(\mathbf{x})}r(\mathbf{x})\\
&= \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})e_t(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{1}{w_t(\mathbf{x})}\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y} + \frac{t}{w_t(\mathbf{x})}r(\mathbf{x}). \quad (4.12)
\end{aligned}$$

Then $\|\nabla e_t\|_{2,\mathcal{M}_t'}^2$ can be bounded as following

$$\begin{aligned}
\|\nabla e_t\|_{2,\mathcal{M}_t'}^2 \leq& C\left\|\nabla\left(\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})e_t(\mathbf{y})\mathrm{d}\mathbf{y}\right)\right\|_{2,\mathcal{M}_t'}^2\\
&+C\left\|\nabla\left(\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right)\right\|_{2,\mathcal{M}_t'}^2 + Ct^2\left\|\nabla\left(\frac{r(\mathbf{x})}{w_t(\mathbf{x})}\right)\right\|_{2,\mathcal{M}_t'}^2. \quad (4.13)
\end{aligned}$$

Corollary 3.1 gives a bound the first term of (4.13).

$$\begin{aligned}&\left\|\nabla\left(\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{M}_t'}R_t(\mathbf{x},\mathbf{y})e_t(\mathbf{y})\mathrm{d}\mathbf{y}\right)\right\|_{2,\mathcal{M}_t'}^2\\ \leq&\frac{C}{t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'}R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))^2\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}+\frac{C}{t}\int_{\mathcal{M}_t'}|e_t(\mathbf{x})|^2\left(\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}.\end{aligned}\tag{4.14}$$

The second and third terms of (4.13) can be bounded by direct calculation.

$$\begin{aligned}\left\|\nabla\left(\frac{r(\mathbf{x})}{w_t(\mathbf{x})}\right)\right\|_{2,\mathcal{M}_t'}^2&\leq C\left\|\frac{\nabla r(\mathbf{x})}{w_t(\mathbf{x})}\right\|_{2,\mathcal{M}_t'}^2+C\left\|\frac{r(\mathbf{x})\nabla w_t(\mathbf{x})}{(w_t(\mathbf{x}))^2}\right\|_{2,\mathcal{M}_t'}^2\\&\leq C\left\|\nabla r(\mathbf{x})\right\|_{2,\mathcal{M}_t'}^2+\frac{C}{t}\left\|r(\mathbf{x})\right\|_{2,\mathcal{M}_t'}^2\\&\leq C\|f\|_{H^1(\mathcal{M})}^2,\end{aligned}\tag{4.15}$$

and

$$\begin{aligned}&\left|\nabla\left(\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right)\right|\\ \leq&\left|\frac{\nabla w_t(\mathbf{x})}{(w_t(\mathbf{x}))^2}\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right|+\left|\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{V}_t}\nabla R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right|\\ \leq& C\|f\|_{H^1(\mathcal{M})}\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}+C\|f\|_{H^1(\mathcal{M})}\int_{\mathcal{V}_t}\left|R_t'(\mathbf{x},\mathbf{y})\right|\mathrm{d}\mathbf{y}.\end{aligned}\tag{4.16}$$

Then the second term of (4.14) has following bound

$$\begin{aligned}&\left\|\nabla\left(\frac{1}{w_t(\mathbf{x})}\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})u(\mathbf{y})\mathrm{d}\mathbf{y}\right)\right\|_{2,\mathcal{M}_t'}^2\\ \leq& C\|f\|_{H^1(\mathcal{M})}^2\int_{\mathcal{M}_t'}\left(\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)^2\mathrm{d}\mathbf{x}+C\|f\|_{H^1(\mathcal{M})}^2\int_{\mathcal{M}_t'}\left(\int_{\mathcal{V}_t}\left|R_t'(\mathbf{x},\mathbf{y})\right|\mathrm{d}\mathbf{y}\right)^2\mathrm{d}\mathbf{x}\\ \leq& C\|f\|_{H^1(\mathcal{M})}^2\int_{\mathcal{M}_t'}\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}+C\|f\|_{H^1(\mathcal{M})}^2\int_{\mathcal{M}_t'}\int_{\mathcal{V}_t}\left|R_t'(\mathbf{x},\mathbf{y})\right|\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\\ \leq& C\|f\|_{H^1(\mathcal{M})}^2|\mathcal{V}_t|\leq C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t}.\end{aligned}\tag{4.17}$$

Now we have the bound of $\|\nabla e_t\|_{2,\mathcal{M}_t'}$ by combining (4.13), (4.15), (4.14) and (4.17)

$$\begin{aligned}\|\nabla e_t\|_{2,\mathcal{M}_t'}^2\leq&\frac{C}{t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'}R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\&+\frac{C}{t}\int_{\mathcal{M}_t'}|e_t(\mathbf{x})|^2\left(\int_{\mathcal{V}_t}R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x}+C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t}.\end{aligned}\tag{4.18}$$

Then the bound of $\|\nabla e_t\|_{2,\mathcal{M}_t'}$ can be obtained also from (4.18)

$$\|\nabla e_t\|_{2,\mathcal{M}_t'}^2\leq\frac{C}{t}\int_{\mathcal{M}_t'}\int_{\mathcal{M}_t'}R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))^2\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

$$
\begin{aligned}
&\quad + \frac{C}{t}\int_{\mathcal{M}_t'} |e_t(\mathbf{x})|^2 \left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}\right)\mathrm{d}\mathbf{x} + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t} \\
\text{(from (4.6))}\quad &\le \left|\frac{C}{t}\int_{\mathcal{M}_t'} e_t(\mathbf{x})\int_{\mathcal{M}} R_t(\mathbf{x},\mathbf{y})(e_t(\mathbf{x})-e_t(\mathbf{y}))\mathrm{d}\mathbf{y}\mathrm{d}\mathbf{x}\right| + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t} \\
\text{(from (4.7))}\quad &\le \|e_t\|_{2,\mathcal{M}_t'}\|r\|_{2,\mathcal{M}_t'} + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t} \\
\text{(from (4.8))}\quad &\le C\|f\|_{H^1(\mathcal{M})}\|e_t\|_{2,\mathcal{M}_t'}\sqrt{t} + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t} \\
\text{(from (4.11))}\quad &\le Ct^{3/4}\|f\|_{H^1(\mathcal{M})} + C\|f\|_{H^1(\mathcal{M})}^2\sqrt{t}. \qquad (4.19)
\end{aligned}
$$

Then we have

$$
\|\nabla e_t\|_{2,\mathcal{M}_t'} \le Ct^{1/4}\|f\|_{H^1(\mathcal{M})}. \qquad (4.20)
$$

The proof is completed by putting (4.11) and (4.20) together. □

4.2. Proof of Theorem 4.2. To prove Theorem 4.2, we only need to prove following consistency result.

THEOREM 4.4. *Let $u_t(\mathbf{x})$ be the solution of the problem (1.4) and $\mathbf{u}$ be the solution of the problem (1.5). If $f \in C^1(\mathcal{M})$, in both problems, then there exists constants $C > 0$ depending only on $\mathcal{M}$ and $\partial\mathcal{M}$ so that*

$$
\|L_t\left(u_{t,h}-u_t\right)\|_{L^2(\mathcal{M}_t')} \le \frac{Ch}{t^{3/2}}\|f\|_{C^1(\mathcal{M})}, \qquad (4.21)
$$

$$
\|\nabla L_t\left(u_{t,h}-u_t\right)\|_{L^2(\mathcal{M}_t')} \le \frac{Ch}{t^2}\|f\|_{C^1(\mathcal{M})}. \qquad (4.22)
$$

as long as t and $\frac{h}{\sqrt{t}}$ are small enough. $u_{t,h}$ is defined in (2.1) and L_t is an integral operator defined as

$$
L_t u(\mathbf{x}) = \frac{1}{t}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y} + \frac{u(\mathbf{x})}{t}\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}. \qquad (4.23)
$$

Using Theorem 3.1 and above theorem, we have

$$
\begin{aligned}
\|u_t - u_{t,h}\|_{H^1(\mathcal{M}_t')} &\le C\|L_t\left(u_{t,h}-u_t\right)\|_{L^2(\mathcal{M}_t')} + Ct\|\nabla L_t\left(u_{t,h}-u_t\right)\|_{L^2(\mathcal{M}_t')} \\
&\le \frac{Ch}{t^{3/2}}\|f\|_{C^1(\mathcal{M})}
\end{aligned}
$$

which proves Theorem 4.2.

Next, we only need to prove Theorem 4.4. First, we need a technical lemma which is stated as following

LEMMA 4.1. *Let $\mathbf{u} = [u_1, \cdots, u_n]^t$ be the solution of the problem (1.5) with $f \in C(\mathcal{M})$ and $u_{t,h}$ be associate smooth function defined in (2.1). Then there exists $C > 0$ such that*

$$
\|u_{t,h}(\mathbf{x})\|_{L^2(\mathcal{M}_t')} \le C\|f\|_\infty,
$$

$$
\|\nabla u_{t,h}(\mathbf{x})\|_{L^2(\mathcal{M}_t')} \le \frac{C}{\sqrt{t}}\|f\|_\infty.
$$

The proof of this lemma is put in Appendix B.

Now, we are ready to prove Theorem 4.4.

Proof. **(Proof of Theorem 4.4.)** First, we split $L_t\left(u_{t,h}-u_t\right)$ to three terms, for any $\mathbf{x}\in\mathcal{M}_t'$,

$$
\begin{aligned}
& L_t\left(u_{t,h}-u_t\right) \\
&= L_t(u_{t,h}) - L_{t,h}(u_{t,h}) + L_{t,h}(u_{t,h}) - L_t u_t \\
&= \left(L_t'(u_{t,h}) - L_{t,h}'(u_{t,h})\right) + \left(\frac{u_{t,h}(\mathbf{x})}{t}\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_i\in\mathcal{V}_t} R_t(\mathbf{x},\mathbf{p}_i)V_i\right)\right) \\
&\quad + \left(\int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y}) - \sum_{\mathbf{p}_j\in\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{p}_j)f(\mathbf{p}_j)V_j\right).
\end{aligned} \tag{4.24}
$$

where L_t', $L_{t,h}$ and $L_{t,h}'$ are given as

$$L_t'u(\mathbf{x}) =\frac{1}{t}\int_{\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{y})(u(\mathbf{x})-u(\mathbf{y}))\mathrm{d}\mathbf{y}, \tag{4.25}$$

$$(L_{t,h}u)(\mathbf{x}) =\frac{1}{t}\sum_{\mathbf{p}_j\in\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{p}_j)(u(\mathbf{x})-u(\mathbf{p}_j))V_j + \frac{u(\mathbf{x})}{t}\sum_{\mathbf{p}_j\in\mathcal{V}_t} R_t(\mathbf{p}_i,\mathbf{p}_j)V_j, \tag{4.26}$$

$$(L_{t,h}'u)(\mathbf{x}) =\frac{1}{t}\sum_{\mathbf{p}_j\in\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{p}_j)(u(\mathbf{x})-u(\mathbf{p}_j))V_j. \tag{4.27}$$

To get the last equality, we use that u_t solves Equation (1.4) and $u_{t,h}$ solves

$$L_{t,h}u_{t,h}(\mathbf{x}) = \sum_{\mathbf{p}_j\in\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{p}_j)f(\mathbf{p}_j)V_j, \quad \mathbf{x}\in\mathcal{M}_t'. \tag{4.28}$$

The second and third terms are easy to bound. By using Lemma 4.1 and $(P,\mathbf{V})$ is h-integrable approximation of $\mathcal{M}$, we have

$$\left\|\frac{u_{t,h}(\mathbf{x})}{t}\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_i\in\mathcal{V}_t} R_t(\mathbf{x},\mathbf{p}_i)V_i\right)\right\|_{L^2(\mathcal{M}_t')} \leq\frac{Ch}{t^{3/2}}\|f\|_\infty, \tag{4.29}$$

$$\left\|\nabla\left(\frac{u_{t,h}(\mathbf{x})}{t}\left(\int_{\mathcal{V}_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_i\in\mathcal{V}_t} R_t(\mathbf{x},\mathbf{p}_i)V_i\right)\right)\right\|_{L^2(\mathcal{M}_t')} \leq\frac{Ch}{t^{2}}\|f\|_\infty \tag{4.30}$$

and

$$\left\|\int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j\in\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{p}_j)f(\mathbf{p}_j)V_j\right\|_{L^2(\mathcal{M}_t')} \leq \frac{Ch}{\sqrt{t}}\|f\|_{C^1(\mathcal{M})}, \tag{4.31}$$

$$\left\|\nabla\left(\int_{\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j\in\mathcal{M}} \bar{R}_t(\mathbf{x},\mathbf{p}_j)f(\mathbf{p}_j)V_j\right)\right\|_{L^2(\mathcal{M}_t')} \leq \frac{Ch}{t}\|f\|_{C^1(\mathcal{M})}. \tag{4.32}$$

The first term of (4.24) is much more complicated to bound. We split it further to two terms. Denote

$$a_{t,h}(\mathbf{x}) = \frac{1}{w_{t,h}(\mathbf{x})}\sum_{\mathbf{p}_j\in\mathcal{M}_t'} R_t(\mathbf{x},\mathbf{p}_j)u_jV_j, \tag{4.33}$$

$$c_{t,h}(\mathbf{x}) = \frac{t}{w_{t,h}(\mathbf{x})} \sum_{\mathbf{p}_j \in \mathcal{M}'_t} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f(\mathbf{p}_j) V_j, \tag{4.34}$$

and then $u_{t,h}(\mathbf{x}) = a_{t,h}(\mathbf{x}) + c_{t,h}(\mathbf{x}),\ \mathbf{x} \in \mathcal{M}'_t$.

First we upper bound $\|L'_t(u_{t,h}) - L'_{t,h}(u_{t,h})\|_{L^2(\mathcal{M}'_t)}$. For $c_{t,h}$, we have

$$\begin{aligned}
&\left|\left(L'_t c_{t,h} - L'_{t,h} c_{t,h}\right)(\mathbf{x})\right| \\
&= \frac{1}{t}\left|\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})(c_{t,h}(\mathbf{x}) - c_{t,h}(\mathbf{y}))\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j \in \mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)(c_{t,h}(\mathbf{x}) - c_{t,h}(\mathbf{p}_j))V_j\right| \\
&\le \frac{1}{t}|c_{t,h}(\mathbf{x})|\left|\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j \in \mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)V_j\right| \\
&\quad + \frac{1}{t}\left|\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})c_{t,h}(\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j \in \mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)c_{t,h}(\mathbf{p}_j)V_j\right| \\
&\le \frac{Ch}{t^{3/2}}|c_{t,h}(\mathbf{x})| + \frac{Ch}{t^{3/2}}\|c_{t,h}\|_{C^1(\mathcal{M}'_t)} \\
&\le \frac{Ch}{t^{3/2}} t\|f\|_\infty + \frac{Ch}{t^{3/2}}(t\|f\|_\infty + t^{1/2}\|f\|_\infty) \le \frac{Ch}{t}\|f\|_\infty.
\end{aligned}$$

For $a_{t,h}$, we have

$$\begin{aligned}
&\int_{\mathcal{M}'_t} (a_{t,h}(\mathbf{x}))^2 \left|\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j \in \mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)V_j\right|^2 \mathrm{d}\mathbf{x} \\
&\le \frac{Ch^2}{t}\int_{\mathcal{M}'_t} (a_{t,h}(\mathbf{x}))^2\,\mathrm{d}\mathbf{x} \\
&\le \frac{Ch^2}{t}\int_{\mathcal{M}'_t} \left(\frac{1}{w_{t,h}(\mathbf{x})}\sum_{\mathbf{p}_j \in \mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)u_jV_j\right)^2 \mathrm{d}\mathbf{x} \\
&\le \frac{Ch^2}{t}\int_{\mathcal{M}'_t} \left(\sum_{\mathbf{p}_j \in \mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)u_j^2V_j\right)\left(\sum_{\mathbf{p}_j \in \mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)V_j\right)\mathrm{d}\mathbf{x} \\
&\le \frac{Ch^2}{t}\left(\sum_{\mathbf{p}_j \in \mathcal{M}'_t} u_j^2V_j \int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)\mathrm{d}\mathbf{x}\right) \le \frac{Ch^2}{t}\sum_{\mathbf{p}_j \in \mathcal{M}'_t} u_j^2V_j.
\end{aligned} \tag{4.35}$$

Let

$$\begin{aligned}
A = C_t &\int_{\mathcal{M}'_t} \frac{1}{w_{t,h}(\mathbf{y})} R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right) R\left(\frac{|\mathbf{p}_i-\mathbf{y}|^2}{4t}\right)\mathrm{d}\mathbf{y} \\
&- C_t \sum_{\mathbf{p}_j \in \mathcal{M}'_t} \frac{1}{w_{t,h}(\mathbf{p}_j)} R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t}\right) R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right) V_j.
\end{aligned}$$

We have $|A| < \frac{Ch}{t^{1/2}}$ for some constant C independent of t. In addition, notice that only when $|\mathbf{x}-\mathbf{p}_i|^2 \le 16t$ is $A \ne 0$, which implies

$$|A| \le \frac{1}{\delta_0}|A| R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{32t}\right).$$

Then we have

$$\begin{aligned}
&\int_{\mathcal{M}'_t}\left|\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})a_{t,h}(\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j\in\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)a_{t,h}(\mathbf{p}_j)V_j\right|^2 \mathrm{d}\mathbf{x}\\
&=\int_{\mathcal{M}'_t}\left(\sum_{\mathbf{p}_i\in\mathcal{M}'_t} C_t u_i V_i A\right)^2 \mathrm{d}\mathbf{x}\\
&\le\frac{Ch^2}{t}\int_{\mathcal{M}'_t}\left(\sum_{\mathbf{p}_i\in\mathcal{M}'_t} C_t |u_i| V_i R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{32t}\right)\right)^2 \mathrm{d}\mathbf{x}\\
&\le\frac{Ch^2}{t}\int_{\mathcal{M}'_t}\left(\sum_{\mathbf{p}_i\in\mathcal{M}'_t} C_t R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{32t}\right) u_i^2 V_i\right)\left(\sum_{\mathbf{p}_i\in\mathcal{M}'_t} C_t R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{32t}\right) V_i\right) \mathrm{d}\mathbf{x}\\
&\le\frac{Ch^2}{t}\sum_{\mathbf{p}_i\in\mathcal{M}'_t} u_i^2 V_i\left(\int_{\mathcal{M}'_t} C_t R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{32t}\right)\mathrm{d}\mathbf{x}\right) \le \frac{Ch^2}{t}\left(\sum_{\mathbf{p}_i\in\mathcal{M}'_t} u_i^2 V_i\right). \qquad (4.36)
\end{aligned}$$

Combining Equation (4.35), (4.36) and Theorem B.1,

$$\begin{aligned}
\|L'_t a_{t,h} - L'_{t,h} a_{t,h}\|_{L^2(\mathcal{M})} &= \left(\int_{\mathcal{M}'_t}\left|\left(L'_t(a_{t,h}) - L'_{t,h}(a_{t,h})\right)(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right)^{1/2}\\
&\le \frac{Ch}{t^{3/2}}\left(\sum_{\mathbf{p}_i\in\mathcal{M}'_t} u_i^2 V_i\right)^{1/2} \le \frac{Ch}{t^{3/2}}\|f\|_\infty.
\end{aligned}$$

Assembling the parts together, we have the following upper bound.

$$\begin{aligned}
&\|L'_t u_{t,h} - L'_{t,h} u_{t,h}\|_{L^2(\mathcal{M}'_t)}\\
&\le \|L'_t a_{t,h} - L'_{t,h} a_{t,h}\|_{L^2(\mathcal{M}'_t)} + \|L'_t c_{t,h} - L'_{t,h} c_{t,h}\|_{L^2(\mathcal{M}'_t)}\\
&\le \frac{Ch}{t^{3/2}}\|f\|_\infty + \frac{Ch}{t}\|f\|_\infty \le \frac{Ch}{t^{3/2}}\|f\|_\infty. \qquad (4.37)
\end{aligned}$$

The complete L^2 estimate follows from Equation (4.29), (4.31) and (4.37).

Next, we turn to upper bound $\|\nabla(L'_t u_t - L'_{t,h} u_{t,h})\|_{L^2(\mathcal{M}'_t)}$. Consider $\|\nabla(L'_t a_{t,h} - L'_{t,h} a_{t,h})\|_{L_2(\mathcal{M}'_t)}$, it can be splited into the summation of three terms. Next, we estimate these three terms separately. The first term is

$$\begin{aligned}
&\int_{\mathcal{M}'_t} |\nabla a_{t,h}(\mathbf{x})|^2 \left|\int_{\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y} - \sum_{\mathbf{p}_j\in\mathcal{M}'_t} R_t(\mathbf{x},\mathbf{p}_j)V_j\right|^2 \mathrm{d}\mathbf{x}\\
&\le \frac{Ch^2}{t}\int_{\mathcal{M}'_t} |\nabla a_{t,h}(\mathbf{x})|^2 \mathrm{d}\mathbf{x}\\
&\le \frac{Ch^2}{t}\left(\int_{\mathcal{M}'_t}\left|\frac{1}{w_{t,h}(\mathbf{x})}\sum_{\mathbf{p}_j\in\mathcal{M}'_t} \nabla R_t(\mathbf{x},\mathbf{p}_j) u_j V_j\right|^2 \mathrm{d}\mathbf{x}\right.
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathcal{M}_t'}\left|\frac{\nabla w_{t,h}(\mathbf{x})}{w_{t,h}^2(\mathbf{x})}\sum_{\mathbf{p}_j\in\mathcal{M}_t'}R_t(\mathbf{x},\mathbf{p}_j)u_jV_j\right|^2\mathrm{d}\mathbf{x}\Bigg)\\
&\le\frac{Ch^2}{t^2}\int_{\mathcal{M}_t'}\left|\sum_{\mathbf{p}_j\in\mathcal{M}_t'}R_{2t}(\mathbf{x},\mathbf{p}_j)u_jV_j\right|^2\mathrm{d}\mathbf{x}\\
&\le\frac{Ch^2}{t^2}\int_{\mathcal{M}_t'}\left(\sum_{\mathbf{p}_j\in\mathcal{M}_t'}R_{2t}(\mathbf{x},\mathbf{p}_j)u_j^2V_j\right)\left(\sum_{\mathbf{p}_j\in\mathcal{M}_t'}R_{2t}(\mathbf{x},\mathbf{p}_j)V_j\right)\mathrm{d}\mathbf{x}\\
&\le\frac{Ch^2}{t^2}\left(\sum_{\mathbf{p}_j\in\mathcal{M}_t'}u_j^2V_j\int_{\mathcal{M}_t'}R_{2t}(\mathbf{x},\mathbf{p}_j)\mathrm{d}\mathbf{x}\right)\le\frac{Ch^2}{t^2}\sum_{\mathbf{p}_j\in\mathcal{M}_t'}u_j^2V_j \qquad (4.38)
\end{aligned}
$$

where $R_{2t}(\mathbf{x},\mathbf{p}_j)=C_tR\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{8t}\right)$. Here we use the assumption that $R(s)>\delta_0$ for all $0\le s\le 1/2$.

The second term is

$$
\begin{aligned}
&\int_{\mathcal{M}_t'}|a_{t,h}(\mathbf{x})|^2\left|\int_{\mathcal{M}_t'}\nabla R_t(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{y}-\sum_{\mathbf{p}_j\in\mathcal{M}_t'}\nabla R_t(\mathbf{x},\mathbf{p}_j)V_j\right|^2\mathrm{d}\mathbf{x}\\
&\le\frac{Ch^2}{t^2}\int_{\mathcal{M}_t'}|a_{t,h}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}\le\frac{Ch^2}{t^2}\sum_{\mathbf{p}_j\in\mathcal{M}_t'}u_j^2V_j. \qquad (4.39)
\end{aligned}
$$

Let

$$
\begin{aligned}
B=\;&C_t\int_{\mathcal{M}_t'}\frac{1}{w_{t,h}(\mathbf{y})}\nabla R\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}\right)R\left(\frac{|\mathbf{p}_i-\mathbf{y}|^2}{4t}\right)\mathrm{d}\mathbf{y}\\
&-C_t\sum_{\mathbf{p}_j\in\mathcal{M}_t'}\frac{1}{w_{t,h}(\mathbf{p}_j)}\nabla R\left(\frac{|\mathbf{x}-\mathbf{p}_j|^2}{4t}\right)R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right)V_j.
\end{aligned}
$$

We have $|B|<\frac{Ch}{t^{1/2}}$ for some constant C independent of t. In addition, notice that only when $|\mathbf{x}-\mathbf{x}_i|^2\le 16t$ is $B\neq 0$, which implies

$$
|B|\le\frac{1}{\delta_0}|B|R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{32t}\right).
$$

Then we have the upper bound of the third term

$$
\begin{aligned}
&\int_{\mathcal{M}_t'}\left|\int_{\mathcal{M}_t'}\nabla R_t(\mathbf{x},\mathbf{y})a_{t,h}(\mathbf{y})\mathrm{d}\mathbf{y}-\sum_{\mathbf{p}_j\in\mathcal{M}_t'}\nabla R_t(\mathbf{x},\mathbf{p}_j)a_{t,h}(\mathbf{p}_j)V_j\right|^2\mathrm{d}\mathbf{x}\\
&=\int_{\mathcal{M}_t'}\left(\sum_{\mathbf{p}_i\in\mathcal{M}_t'}C_tu_iV_iB\right)^2\mathrm{d}\mathbf{x}\\
&\le\frac{Ch^2}{t^2}\int_{\mathcal{M}_t'}\left(\sum_{\mathbf{p}_i\in\mathcal{M}_t'}C_t|u_i|V_iR\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{32t}\right)\right)^2\mathrm{d}\mathbf{x}
\end{aligned}
$$

$$\leq \frac{Ch^2}{t^2}\left(\sum_{\mathbf{p}_i\in\mathcal{M}_t'} u_i^2 V_i\right). \tag{4.40}$$

Combining Equation (4.38), (4.39) and (4.40), we have

$$\begin{aligned}
&\|\nabla(L_t' a_{t,h} - L_{t,h}' a_{t,h})\|_{L^2(\mathcal{M}_t')}\\
&= \left(\int_{\mathcal{M}_t'} \left|\left(L_t(a_{t,h}) - L_{t,h}(a_{t,h})\right)(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}\right)^{1/2}\\
&\leq \frac{Ch}{t^2}\left(\sum_{\mathbf{p}_i\in\mathcal{M}_t'} u_i^2 V_i\right)^{1/2} \leq \frac{Ch}{t^2}\|f\|_\infty.
\end{aligned}$$

Using a similar argument, we obtain

$$\|\nabla(L_t' c_{t,h} - L_{t,h}' c_{t,h})\|_{L^2(\mathcal{M}_t')} \leq \frac{Ch}{t^{3/2}}\|f\|_\infty,$$

and thus

$$\|\nabla(L_t' u_{t,h} - L_{t,h}' u_{t,h})\|_{L^2(\mathcal{M}_t')} \leq \frac{Ch}{t^2}\|f\|_\infty. \tag{4.41}$$

At last, we complete the proof using (4.30), (4.32) and (4.41). □

5. Numerical experiments

In this section, we present several numerical results to show the convergence of the Point Integral method with volume constraint, PIM_VC for short, from point clouds.

The numerical experiments were carried out in unit disk. We discretize unit disk with 684, 2610, 10191 and 40269 points respectively and check the convergence of the point integral method with volume constraint.In the experiments, the volume weight vector $\mathbf{V}$ is estimated using the method proposed in [27]. First, we locally approximate the tangent space at each point and then project the nearby points onto the tangent space over which a Delaunay triangulation is computed in the tangent space. The volume weight is estimated as the volume of the Voronoi cell of that point.

Given a sampling P on $\mathcal{M}$, let δ_i be the average distance from $p_i \in P$ to its 10 nearest neighbors in P and δ is the average of δ_i over all points $p_i \in P$. In this experiment, the parameter t is set to be $(0.75\delta)^2$.

$\lvert P\rvert$	684	2610	10191	40269
PIM_Robin	0.1500	0.0428	0.0140	0.0052
PIM_VC	0.3046	0.0747	0.0201	0.0067

TABLE 5.1. *l^2 error with different number of points. PIM_Robin: Point Integral method with Robin boundary; PIM_VC: Point Integral method with volume constraint. The exact solution is* $\cos 2\pi\sqrt{x^2+y^2}$.

Table 5.1 gives the l^2 error of different methods with 684, 2610, 10191 and 40269 points. The exact solution is $\cos 2\pi\sqrt{x^2+y^2}$. PIM_Robin is the Point Integral method and using Robin boundary to approximate the Dirichlet boundary condition. PIM_VC is the Point Integral method and using volume constraint to enforce the Dirichlet boundary condition. These two methods both converge. The rates of convergence are very close and the error of PIM_VC is a little larger than the error of PIM_Robin.

6. Semi-supervised Learning

In this section, we briefly describe the algorithm of semi-supervised learning based on the method proposed by Zhu et al. [39]. We plug into the algorithm the aforementioned approach of volume constraint, and apply them to several well-known data sets, and compare their performance.

Assume we are given a point set $X = \{\mathbf{x}_1, \cdots, \mathbf{x}_m, \mathbf{x}_{m+1}, \cdots, \mathbf{x}_n\} \subset \mathbb{R}^d$, and a label set $\{1, 2, \cdots, l\}$, and the label assignment on the first m points $L : \{\mathbf{x}_1, \cdots, \mathbf{x}_m\} \to \{1, 2, \cdots, l\}$. In a typical setting, m is much smaller than n. The purpose of the semi-supervised learning is to extend the label assignment L to the entire X, namely, infer the labels for the unlabeled points.

Think of the label points as the boundary $B = \{\mathbf{x}_1, \cdots, \mathbf{x}_m\}$. For the label i, we set up the Dirichlet boundary $\mathbf{g}^i$ as follows. If a point $\mathbf{x}_j \in B$ is labeled as i, set $\mathbf{g}^i(\mathbf{x}_j) = 1$, and otherwise set $\mathbf{g}^i(\mathbf{x}_j) = 0$. In [39], the harmonic extension $\mathbf{u}^i$ of $\mathbf{g}^i$ is computed by solving

$$\left\{ \begin{array}{rcll} \frac{1}{t} \sum_{\mathbf{x}_j \in X} R_t(\mathbf{x}_j, \mathbf{x}_{j'})(u_j^i - u_{j'}^i) V_{j'} & = & 0, & \mathbf{x}_j \in X \backslash B, \\ u_j^i & = & \mathbf{g}^i(\mathbf{x}_j), & \mathbf{x}_j \in B. \end{array} \right. \tag{6.1}$$

In this way, we obtain a set of l harmonic functions $\mathbf{u}^1, \mathbf{u}^2, \cdots, \mathbf{u}^l$. We label $\mathbf{x}_j$ using k where $k = \arg\max_{i \le l} \mathbf{u}^i(\mathbf{x}_j)$.

In [39], they gave an explanation of (6.1) based on random walk and graph Laplacian. In this paper, we call this method graph Laplacian method, GLM for short. Using the analysis in this paper, we know that (6.1) can not correctly enforce the Dirichlet boundary condition. To get the correct harmonic extension, we need to use the volume constraint and extend the boundary to a small volume, i.e., solving following alternative linear system,

$$\left\{ \begin{array}{rcll} \frac{1}{t} \sum_{\mathbf{x}_j \in X} R_t(\mathbf{x}_j, \mathbf{x}_{j'})(u_j^i - u_{j'}^i) V_{j'} & = & 0, & \mathbf{x}_j \in X_t, \\ u_j^i & = & \bar{\mathbf{g}}^i(\mathbf{x}_j), & \mathbf{x}_j \in B_t. \end{array} \right. \tag{6.2}$$

where $B_t = \{\mathbf{x} \in X : \min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\| \le 2\sqrt{t}\}$ and $\bar{\mathbf{g}}^i$ is an extension of $\mathbf{g}^i$ on B_t, for any $\mathbf{x} \in B_t$

$$\bar{\mathbf{g}}^i(\mathbf{x}) = \mathbf{g}^i(\mathbf{x}_j), \quad \mathbf{x}_j = \arg\min_{\mathbf{y} \in B} \|\mathbf{x} - \mathbf{y}\|.$$

This alternative method is called volume constraint method, VCM for short. In Algorithm 1, we summarize these two algorithm, GLM and VCM.

6.1. Experiments.

We now apply the above algorithms, GLM and VCM, to a couple of well-known data sets: MNIST and 20 Newsgroups.

MNIST: In this experiment, we use the MNIST of dataset of handwritten digits [7], which contains 6×10^4 gray scale digit images (28×28 resolution) with labels. We view digits $0 \sim 9$ as ten classes. Each digit can be seen as a point in a common 784-dimensional Euclidean space. We randomly choose 16k images. Specifically, there are 1606, 1808, 1555, 1663, 1552, 1416, 1590, 1692, 1521 and 1597 digits in $0 \sim 9$ class respectively.

To set the parameter t, for each point $\mathbf{x}_i$, we compute the standard Euclidean distance between $\mathbf{x}_i$ and its 10 nearest neighbors. Denote h_i as the average of the

Algorithm 1 Semi-Supervised Learning with harmonic extension

Require: A point set $X = \{\mathbf{x}_1, \cdots, \mathbf{x}_m, \mathbf{x}_{m+1}, \cdots, \mathbf{x}_n\} \subset \mathbb{R}^d$ and a partial label assignment $L : \{\mathbf{x}_1, \cdots, \mathbf{x}_m\} \to \{1, 2, \cdots, l\}$
Ensure: A complete label assignment $L : X \to \{1, 2, \cdots, l\}$
  for $i = 1 : l$ **do**
    for $j = 1 : m$ **do**
      Set $\mathbf{g}^i(\mathbf{x}_j) = 1$ if $L(\mathbf{x}_j) = i$, and otherwise set $\mathbf{g}^i(\mathbf{x}_j) = 0$.
    end for
    Compute the harmonic extension $\mathbf{u}^i$ of $\mathbf{g}^i$ by solving (6.2) in VCM and (6.1) in GLM.
  end for
  for $j = m + 1 : n$ **do**
    $L(\mathbf{x}_j) = k$ where $k = \arg\max_{i \leq l} \mathbf{u}^i(\mathbf{x}_j)$.
  end for

distances for $\mathbf{x}_i$ to its 10 nearest neighbors. Let h be the average of h_i's over all points and set $t = h^2$.

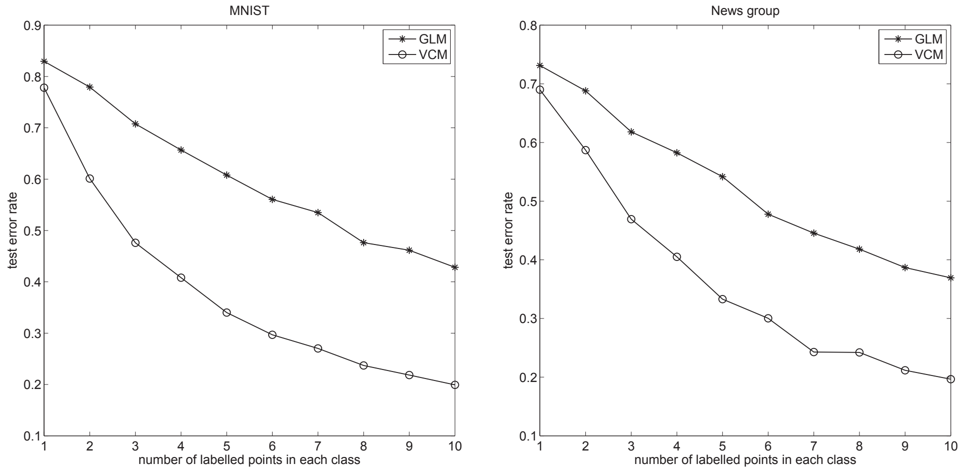
For a particular trial, we choose k ($k = 1, 2, \cdots, 10$) images randomly from each class to assemble the labelled set B and assume all the other images are unlabeled. For each fixed k, we do 100 trials. The average error of the tests is presented in Figure 6.1 (a). It is quite clear that VCM outperforms GLM.

Newsgroup: In this experiment, we use the 20-newsgroups dataset, which is a classic dataset in text classification. We only choose the articles from topic *rec* containing four classes from the version 20-news-18828. We use Rainbow (version:20020213) to pre-process the dataset and finally vectorize them. The following command-line options are required[1]: (1)- -*skip-header*: to avoid lexing headers; (2)- -*use-stemming*: to modify lexed words with the 'Porter' stemmer; (3)- -*use-stoplist*: to toss lexed words that appear in the SMART stoplist; (4)- -*prune-vocab-by-doc-count=5*: to remove words that occur in 5 or fewer documents; Then, we use TF-IDF algorithm to normalize the word count matrix. Finally, we obtain 3970 documents (990 from rec.autos, 994 from rec.motorcycles, 994 from rec.sport.baseball and 999 from rec.sport.hockey) and a list of 8014 words. Each document will be treated as a point in a 8014-dimensional space.

To deal with text-kind data, we define a new distance introduced by Zhu et al. [39]: the distance between $\mathbf{x}_i$ and $\mathbf{x}_j$ is $d(\mathbf{x}_i, \mathbf{x}_j) = 1 - \cos\alpha$, where α is the angle between $\mathbf{x}_i$ and $\mathbf{x}_j$ in Euclidean space. Under this new distance, we ran the same experiment with the same parameter as we process the above MNIST dataset. The result of the tests for 20-newsgroups is shown in Figure 6.1 (b). In this example, VCM also gives better result than GLM.

Based on the analysis in this paper, GLM does not enforce the Dirichlet boundary condition correctly while VCM gives a correct boundary condition, such that VCM gives better results in above two examples. It is also found that if the number of labeled points is increased, GLM becomes comparable to VCM. This also can be explained qualitatively using volume constraint. When the labeled points become more and more, some of them may concentrate together such that the boundary condition is actually enforced in an area instead of isolated points. If the size of these areas is comparable with the support of the kernel function, based on the theory of volume constraint, the

[1]all the following options are offered by Rainbow



(a) (b)

FIG. 6.1. *(a) the error rates of digit recognition with a 16000-size subset of MNIST dataset; (b) the error rates of text classification with 20-newsgroups.*

boundary condition is correctly enforced even in GLM.

7. Conclusion

In this paper, we combine the point integral method and the volume constraint [11] to get a numerical method to solve Laplace equation with Dirichlet boundary condition on point cloud. The convergence is proved. Our study shows that Point Integral method together with the volume constraint gives an efficient numerical approach to solve the Poisson equation with Dirichlet boundary on point cloud.

Moreover, volume constraint provides a general method to enforce the Dirichlet boundary condition for different types of PDEs, not only for Laplace equation. The other advantage of the volume constraint approach is that it is relatively easy to prove the convergence of the eigenvalue problem of Laplace-Beltrami operator with Dirichlet boundary condition on point cloud. The progress will be reported in our subsequent papers.

Appendix A. One basic estimates.

LEMMA A.1. *Let $u(\mathbf{x})$ be the solution of (1.3) and $f \in H^1(\mathcal{M})$, then there is a generic constant $C > 0$ and $T_0 > 0$ only depend on $\mathcal{M}$ and $\partial\mathcal{M}$, for any $t < T_0$,*

$$\int_{\mathcal{V}_t} |u(\mathbf{y})|^2 \mathrm{d}\mathbf{y} \le C t^{3/2} \|f\|_{H^1(\mathcal{M})}^2.$$

Proof. Both $\mathcal{M}$ and $\partial\mathcal{M}$ are compact and C^∞ smooth. Consequently, it is well known that both $\mathcal{M}$ and $\partial\mathcal{M}$ have positive reaches, which means that there exists $T_0 > 0$ only depends on $\mathcal{M}$ and $\partial\mathcal{M}$, if $t < T_0$, $\mathcal{V}_t$ can be parametrized as $(\mathbf{z}(\mathbf{y}), \tau) \in \partial\mathcal{M} \times [0, 1]$, where $\mathbf{y} = \mathbf{z}(\mathbf{y}) + \tau(\mathbf{z}'(\mathbf{y}) - \mathbf{z}(\mathbf{y}))$ and $\left|\det\left(\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}(\mathbf{z}(\mathbf{y}),\tau)}\right)\right| \le C\sqrt{t}$ and $C > 0$ is a constant only depends on $\mathcal{M}$ and $\partial\mathcal{M}$. Here $\mathbf{z}'(\mathbf{y})$ is the intersection point between $\partial\mathcal{M}'$ and the line determined by $\mathbf{z}(\mathbf{y})$ and $\mathbf{y}$. The parametrization is illustrated in Figure A.1.

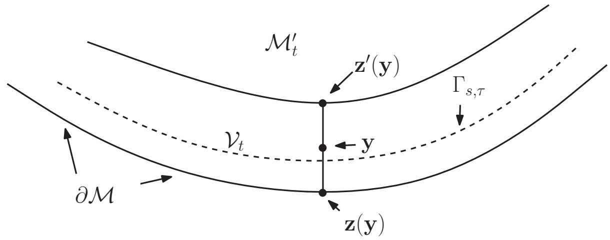


FIG. A.1. *Parametrization of* $\mathcal{V}_t$

First, we have

$$\begin{aligned}\int_{\mathcal{V}_t}|u(\mathbf{y})|^2\mathrm{d}\mathbf{y} &= \int_{\mathcal{V}_t}|u(\mathbf{y})-u(\mathbf{z}(\mathbf{y}))|^2\mathrm{d}\mathbf{y}\\ &= \int_{\mathcal{V}_t}\left|\int_0^1\frac{\mathrm{d}}{\mathrm{d}s}u(\mathbf{y}+s(\mathbf{z}(\mathbf{y})-\mathbf{y}))\mathrm{d}s\right|^2\mathrm{d}\mathbf{y}\\ &= \int_{\mathcal{V}_t}\left|\int_0^1(\mathbf{z}(\mathbf{y})-\mathbf{y})\cdot\nabla u(\mathbf{y}+s(\mathbf{z}(\mathbf{y})-\mathbf{y}))\mathrm{d}s\right|^2\mathrm{d}\mathbf{y}\\ &\le Ct\int_{\mathcal{V}_t}\int_0^1|\nabla u(\mathbf{y}+s(\mathbf{z}(\mathbf{y})-\mathbf{y}))|^2\,\mathrm{d}s\mathrm{d}\mathbf{y}\\ &\le Ct\sup_{0\le s\le 1}\int_{\mathcal{V}_t}|\nabla u(\mathbf{y}+s(\mathbf{z}(\mathbf{y})-\mathbf{y}))|^2\,\mathrm{d}\mathbf{y}.\end{aligned}$$

Here, we use the fact that $\|\mathbf{z}(\mathbf{y})-\mathbf{y}\|_2\le 2\sqrt{t}$ to get the second last inequality.

Then, the proof can be completed by following estimation.

$$\begin{aligned}&\int_{\mathcal{V}_t}|\nabla u(\mathbf{y}+s(\mathbf{z}(\mathbf{y})-\mathbf{y}))|^2\,\mathrm{d}\mathbf{y}\\ &\le C\sqrt{t}\int_0^1\int_{\partial\mathcal{M}}|\nabla u(\mathbf{z}(\mathbf{y})+(1-s)\tau(\mathbf{z}'(\mathbf{y})-\mathbf{z}(\mathbf{y})))|^2\,\mathrm{d}\mathbf{z}(\mathbf{y})\mathrm{d}\tau\\ &\le C\sqrt{t}\sup_{0\le\tau\le 1}\int_{\partial\mathcal{M}}|\nabla u(\mathbf{z}+(1-s)\tau(\mathbf{z}'-\mathbf{z}))|^2\,\mathrm{d}\mathbf{z}\\ &\le C\sqrt{t}\sup_{0\le\tau\le 1}\int_{\Gamma_{s,\tau}}|\nabla u(\tilde{\mathbf{z}})|^2\,\mathrm{d}\tilde{\mathbf{z}}\\ &\le C\sqrt{t}\|u\|^2_{H^2(\mathcal{M})}\le C\sqrt{t}\|f\|^2_{H^1(\mathcal{M})},\end{aligned}$$

where $\Gamma_{s,\tau}$ is a $k-1$ dimensional manifold given by $\Gamma_{s,\tau}=\{\mathbf{z}+(1-s)\tau(\mathbf{z}'-\mathbf{z}):\mathbf{z}\in\partial\mathcal{M}\}$. We use the trace theorem to get the second last inequality and the last inequality is due to that u is the solution of the Poisson Equation (1.3). ∎

Appendix B. Proof of Lemma 4.1.

First, we need one technical lemma.

LEMMA B.1. *For any* $\mathbf{u}=(u_1,\cdots,u_n)^t$ *with* $u_i=0$, $\mathbf{p}_i\in\mathcal{V}_t$, *there exist constants* $C>0$, $C_0>0$ *independent on* t *so that for sufficient small* t *and* $\frac{h}{\sqrt{t}}$

$$\frac{1}{t}\sum_{\mathbf{p}_i\in\mathcal{M}}\sum_{\mathbf{p}_j\in\mathcal{M}}R_t(\mathbf{p}_i,\mathbf{p}_j)(u_i-u_j)^2V_iV_j\ge C\Big(1-\frac{C_0h}{\sqrt{t}}\Big)\sum_{\mathbf{p}_j\in\mathcal{M}}u_i^2V_i.$$

Proof. First, we introduce a smooth function u that approximates $\mathbf{u}$ at the samples P.

$$u(\mathbf{x}) = \frac{1}{w_{t',h}(\mathbf{x})}\sum_{i=1}^{n} R_{t'}\left(\mathbf{x},\mathbf{p}_i\right) u_i V_i, \quad \mathbf{x}\in\mathcal{M}, \tag{B.1}$$

where $w_{t',h}(\mathbf{x}) = C_t\sum_{i=1}^n R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{4t'}\right)V_i$ and $t' = t/18$. Using the condition that $u_i = 0,\ \mathbf{p}_i\in\mathcal{V}_t$ and $t'=t/18$, we know that

$$u(\mathbf{x}) = 0, \quad \mathbf{x}\in\mathcal{V}_{t'}. \tag{B.2}$$

Then using Lemma 3.2, we have

$$\int_{\mathcal{M}} |u(\mathbf{x})|^2 \mathrm{d}\mathbf{x} \le \frac{C}{t}\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{y})\left(u(\mathbf{x})-u(\mathbf{y})\right)^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}.$$

On the other hand

$$\begin{aligned}
&\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{y})\left(u(\mathbf{x})-u(\mathbf{y})\right)^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
=&\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{y})\left(\frac{1}{w_{t',h}(\mathbf{x})}\sum_{i=1}^n R_{t'}(\mathbf{x},\mathbf{p}_i)u_iV_i - \frac{1}{w_{t',h}(\mathbf{y})}\sum_{j=1}^n R_{t'}(\mathbf{p}_j,\mathbf{y})u_jV_j\right)^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
=&\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{y})\left(\frac{1}{w_{t',h}(\mathbf{x})w_{t',h}(\mathbf{y})}\sum_{i,j=1}^n R_{t'}(\mathbf{x},\mathbf{p}_i)R_{t'}(\mathbf{p}_j,\mathbf{y})V_iV_j(u_i-u_j)\right)^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
\le&\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{y})\frac{1}{w_{t',h}(\mathbf{x})w_{t',h}(\mathbf{y})}\sum_{i,j=1}^n R_{t'}(\mathbf{x},\mathbf{p}_i)R_{t'}(\mathbf{p}_j,\mathbf{y})V_iV_j(u_i-u_j)^2 \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
=&\sum_{i,j=1}^n\left(\int_{\mathcal{M}}\int_{\mathcal{M}}\frac{1}{w_{t',h}(\mathbf{x})w_{t',h}(\mathbf{y})}R_{t'}(\mathbf{x},\mathbf{p}_i)R_{t'}(\mathbf{p}_j,\mathbf{y})R_{t'}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\right)V_iV_j(u_i-u_j)^2.
\end{aligned} \tag{B.3}$$

Denote

$$A = \int_{\mathcal{M}}\int_{\mathcal{M}}\frac{1}{w_{t',h}(\mathbf{x})w_{t',h}(\mathbf{y})}R_{t'}(\mathbf{x},\mathbf{p}_i)R_{t'}(\mathbf{p}_j,\mathbf{y})R_{t'}(\mathbf{x},\mathbf{y})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$$

and then notice only when $|\mathbf{p}_i-\mathbf{p}_j|^2\le 36t'$ is $A\neq 0$. For $|\mathbf{p}_i-\mathbf{p}_j|^2\le 36t'$, we have

$$\begin{aligned}
A \le&\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{p}_i)R_{t'}(\mathbf{p}_j,\mathbf{y})R_{t'}(\mathbf{x},\mathbf{y})R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{72t'}\right)^{-1}R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{72t'}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
\le&\frac{CC_t}{\delta_0}\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{p}_i)R_{t'}(\mathbf{p}_j,\mathbf{y})R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{72t'}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\\
\le&CC_t\int_{\mathcal{M}}\int_{\mathcal{M}} R_{t'}(\mathbf{x},\mathbf{p}_i)R_{t'}(\mathbf{p}_j,\mathbf{y})R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{72t'}\right)\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\le CC_tR\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right).
\end{aligned} \tag{B.4}$$

Combining Equation (B.3), (B.4) and Lemma 3.2, we obtain

$$\frac{C}{t}\sum_{\mathbf{p}_i,\mathbf{p}_j\in\mathcal{M}} R_t\left(\mathbf{p}_i,\mathbf{p}_j\right)(u_i-u_j)^2V_iV_j \geq \int_{\mathcal{M}}|u(\mathbf{x})|^2\mathrm{d}\mathbf{x}. \tag{B.5}$$

Denote

$$\begin{aligned} B = &\int_{\mathcal{M}}\frac{C_t}{w_{t',h}^2(\mathbf{x})}R\left(\frac{|\mathbf{x}-\mathbf{p}_i|^2}{4t'}\right)R\left(\frac{|\mathbf{x}-\mathbf{p}_l|^2}{4t'}\right)\mathrm{d}\mathbf{x}\\ &-\sum_{j=1}^n\frac{C_t}{w_{t',h}^2(\mathbf{p}_j)}R\left(\frac{|\mathbf{p}_j-\mathbf{p}_i|^2}{4t'}\right)R\left(\frac{|\mathbf{p}_j-\mathbf{p}_l|^2}{4t'}\right)V_j \end{aligned}$$

and then $|B|\leq\frac{Ch}{t^{1/2}}$. At the same time, notice that only when $|\mathbf{p}_i-\mathbf{p}_l|^2<16t'$ is $B\neq 0$. Thus we have

$$|B|\leq\frac{1}{\delta_0}|A|R(\frac{|\mathbf{p}_i-\mathbf{p}_l|^2}{32t'}),$$

and

$$\begin{aligned} \left|\int_{\mathcal{M}}u^2(\mathbf{x})\mathrm{d}\mathbf{x}-\sum_{j=1}^n u^2(\mathbf{p}_j)V_j\right| &\leq \sum_{i,l=1}^n|C_tu_iu_lV_iV_l||A|\\ &\leq\frac{Ch}{t^{1/2}}\sum_{i,l=1}^n\left|C_tR\left(\frac{|\mathbf{p}_i-\mathbf{p}_l|^2}{32t'}\right)u_iu_lV_iV_l\right|\\ &\leq\frac{Ch}{t^{1/2}}\sum_{i,l=1}^n C_tR\left(\frac{|\mathbf{p}_i-\mathbf{p}_l|^2}{32t'}\right)u_i^2V_iV_l\leq\frac{Ch}{t^{1/2}}\sum_{i=1}^n u_i^2V_i. \end{aligned} \tag{B.6}$$

Now combining Equation (B.5) and (B.6), we have for small t

$$\begin{aligned} \sum_{i=1}^n u^2(\mathbf{p}_i)V_i &= \int_{\mathcal{M}}u^2(\mathbf{x})\mathrm{d}\mathbf{x}+\frac{Ch}{t^{1/2}}\sum_{i=1}^n u_i^2V_i\\ &\leq\frac{CC_t}{t}\sum_{i,j=1}^n R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right)(u_i-u_j)^2V_iV_j+\frac{Ch}{t}\sum_{i=1}^n u_i^2V_i. \end{aligned}$$

Here we use the fact that for $t=18t'$

$$R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t'}\right)\leq\frac{1}{\delta_0}R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t}\right).$$

Let $\delta=\frac{w_{\min}}{2w_{\max}+w_{\min}}$ with $w_{\min}=\min_{\mathbf{x}}w_{t,h}(\mathbf{x})$ and $w_{\max}=\max_{\mathbf{x}}w_{t,h}(\mathbf{x})$. If $\sum_{i=1}^n u^2(\mathbf{p}_i)V_i\geq\delta^2\sum_{i=1}^n u_i^2V_i$, then we have completed the proof. Otherwise, we have

$$\sum_{i=1}^n(u_i-u(\mathbf{p}_i))^2V_i=\sum_{i=1}^n u_i^2V_i+\sum_{i=1}^n u(\mathbf{p}_i)^2V_i-2\sum_{i=1}^n u_iu(\mathbf{p}_i)V_i\geq(1-\delta)^2\sum_{i=1}^n u_i^2V_i.$$

This enables us to prove the lemma as follows.

$$C_t\sum_{i,j=1}^n R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t'}\right)(u_i-u_j)^2V_iV_j$$

$$
\begin{aligned}
&= 2C_t \sum_{i,j=1}^n R\left(\frac{|\mathbf{p}_i-\mathbf{p}_j|^2}{4t'}\right) u_i(u_i-u_j)V_iV_j \\
&= 2\sum_{i=1}^n (u_i-u(\mathbf{p}_i))^2 w_{t,h}(\mathbf{p}_i)V_i + 2\sum_{i=1}^n u(\mathbf{p}_i)(u_i-u(\mathbf{p}_i))w_{t,h}(\mathbf{p}_i)V_i \\
&\geq 2\sum_{i=1}^n (u_i-u(\mathbf{p}_i))^2 w_{t,h}(\mathbf{p}_i)V_i \\
&\quad -2\left(\sum_{i=1}^n u^2(\mathbf{p}_i)w_{t,h}(\mathbf{p}_i)V_i\right)^{1/2}\left(\sum_{i=1}^n (u_i-u(\mathbf{p}_i))^2 w_{t,h}(\mathbf{p}_i)V_i\right)^{1/2} \\
&\geq 2w_{\min}\sum_{i=1}^n (u_i-u(\mathbf{p}_i))^2 V_i - 2w_{\max}\left(\sum_{i=1}^n u^2(\mathbf{p}_i)V_i\right)^{1/2}\left(\sum_{i=1}^n (u_i-u(\mathbf{p}_i))^2 V_i\right)^{1/2} \\
&\geq 2(w_{\min}(1-\delta)^2 - w_{\max}\delta(1-\delta))\sum_{i=1}^n u_i^2 V_i \geq w_{\min}(1-\delta)^2\sum_{i=1}^n u_i^2V_i.
\end{aligned}
$$

∎

One direct corollary of above lemma is the boundness of $\left(\sum_{\mathbf{p}_i\in\mathcal{M}} u_i^2V_i\right)^{1/2}$.

COROLLARY B.1. *Suppose* $\mathbf{u}=(u_1,\cdots,u_n)^t$ *with* $u_i=0,\ \mathbf{p}_i\in\mathcal{V}_t$ *solves the problem* (1.5) *with* $f\in C(\mathcal{M})$.

Then there exists a constant $C>0$ *such that*

$$
\left(\sum_{\mathbf{p}_i\in\mathcal{M}} u_i^2V_i\right)^{1/2} \leq C\|f\|_\infty,
$$

provided t *and* $\frac{h}{\sqrt{t}}$ *are small enough.*

Proof. From the elliptic property of $\mathcal{L}$, we have

$$
\begin{aligned}
\sum_{\mathbf{p}_i\in\mathcal{M}} u_i^2V_i &\leq \sum_{\mathbf{p}_i\in\mathcal{M}}\left(\sum_{\mathbf{p}_j\in\mathcal{M}}\bar{R}_t(\mathbf{p}_i,\mathbf{p}_j)f(\mathbf{p}_j)V_j\right)u_iV_i \\
&\leq \left(\sum_{\mathbf{p}_i\in\mathcal{M}} u_i^2V_i\right)^{1/2}\left(\sum_{\mathbf{p}_i\in\mathcal{M}}\left(\|f\|_\infty\sum_{\mathbf{p}_j\in\mathcal{M}}\bar{R}_t(\mathbf{p}_i,\mathbf{p}_j)V_j\right)^2 V_i\right)^{1/2} \\
&\leq C\left(\sum_{\mathbf{p}_i\in\mathcal{M}} u_i^2V_i\right)^{1/2}\|f\|_\infty.
\end{aligned}
$$

This proves the lemma. ∎

Now, we can give the proof of Lemma 4.1.

Proof. First, $\|u_{t,h}(\mathbf{x})\|_{L^2(\mathcal{M}_t')}$ is bounded.

$$
\|u_{t,h}(\mathbf{x})\|^2_{L^2(\mathcal{M}_t')} = \int_{\mathcal{M}_t'}\frac{1}{w_{t,h}^2(\mathbf{x})}\left(\sum_{\mathbf{p}_j\in P}R_t(\mathbf{x},\mathbf{p}_j)u_jV_j - t\sum_{\mathbf{p}_j\in P}\bar{R}_t(\mathbf{x},\mathbf{p}_j)f_jV_j\right)^2\mathrm{d}\mathbf{x}
$$

$$
\begin{aligned}
&\leq C \int_{\mathcal{M}_t'} \left(\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j V_j \right)^2 \mathrm{d}\mathbf{x} + Ct^2 \int_{\mathcal{M}_t'} \left(\sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) f_j V_j \right)^2 \mathrm{d}\mathbf{x} \\
&\leq C \int_{\mathcal{M}_t'} \left(\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) V_j \right) \left(\sum_{\mathbf{p}_j \in P} R_t(\mathbf{x}, \mathbf{p}_j) u_j^2 V_j \right) \mathrm{d}\mathbf{x} \\
&\qquad + Ct^2 \|f\|_\infty^2 \int_{\mathcal{M}_t'} \left(\sum_{\mathbf{p}_j \in P} \bar{R}_t(\mathbf{x}, \mathbf{p}_j) V_j \right)^2 \mathrm{d}\mathbf{x} \\
&\leq C \sum_{\mathbf{p}_j \in P} u_j^2 V_j \left(\int_{\mathcal{M}_t'} R_t(\mathbf{x}, \mathbf{p}_j) \mathrm{d}\mathbf{x} \right) + Ct^2 \|f\|_\infty^2 \\
&\leq C \sum_{\mathbf{p}_j \in P} u_j^2 V_j + Ct^2 \|f\|_\infty^2 \leq C \|f\|_\infty^2.
\end{aligned}
$$

Moreover, from the definition of $u_{t,h}$, (2.1), we can see that all derivatives are applied on the kernel functions. The kernel functions are smooth functions, it gives one factor of $\frac{1}{\sqrt{t}}$ after derivative. Based on this observation, we can get the bound of $\|\nabla u_{t,h}(\mathbf{x})\|_{L^2(\mathcal{M}_t')}$. ∎

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