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A DERIVATION OF THE VLASOV–NAVIER–STOKES MODEL FOR AEROSOL FLOWS FROM KINETIC THEORY*

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Abstract. This article proposes a derivation of the Vlasov–Navier–Stokes system for spray/aerosol flows. The distribution function of the dispersed phase is governed by a Vlasov-equation, while the velocity field of the propellant satisfies the Navier–Stokes equations for incompressible fluids. The dynamics of the dispersed phase and of the propellant are coupled through the drag force exerted by the propellant on the dispersed phase. We present a formal derivation of this model from a multiphase Boltzmann system for a binary gaseous mixture, involving the droplets/dust particles in the dispersed phase as one species, and the gas molecules as the other species. Under suitable assumptions on the collision kernels, we prove that the sequences of solutions to the multiphase Boltzmann system converge to distributional solutions to the Vlasov-Navier–Stokes equation in some appropriate distinguished scaling limit. Specifically, we assume (a) that the mass ratio of the gas molecules to the dust particles/droplets is small, (b) that the thermal speed of the dust particles/droplets is much smaller than that of the gas molecules and (c) that the mass density of the gas and of the dispersed phase are of the same order of magnitude. The class of kernels modelling the interaction between the dispersed phase and the gas includes, among others, elastic collisions and inelastic collisions of the type introduced in [F. Charles: in "Proceedings of the 26th International Symposium on Rarefied Gas Dynamics", AIP Conf. Proc. 1084:409–414, 2008].

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1. Introduction

An aerosol (or a spray) is a complex fluid consisting of a *dispersed phase*, for instance solid particles or liquid droplets, immersed in a gas, sometimes referred to as the *propellant*.

An important class of models for the dynamics of aerosol/spray flows consists of

(a) a kinetic equation for the dispersed phase, and

(b) a fluid equation for the background gas.

The kinetic equation for the dispersed phase and the fluid equation for the background gas are coupled through the drag force exerted by the gas on the droplets/particles.

A well-known example of this class of models is the (incompressible) Vlasov–Navier–

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Stokes system:

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \dfrac{\kappa}{m_p} \operatorname{div}_v ((v-u)F) = 0, \\ \rho_g (\partial_t u + u \cdot \nabla_x u) + \nabla_x p = \rho_g \nu \Delta_x u + \kappa \displaystyle\int_{\mathbf{R}^3} (v-u) F \,\mathrm{d}v, \\ \operatorname{div}_x u = 0. \end{cases} \tag{1.1}$$

The unknowns in this system are $F \equiv F(t,x,v) \geq 0$, the distribution function of the dispersed phase, i.e. the number density of particles or droplets with velocity v located at the position x at time t, and $u \equiv u(t,x) \in \mathbf{R}^3$, the velocity field in the gas. The parameters κ, m_p, ρ_g and ν are positive constants. Specifically, κ is the friction coefficient of the gas on the dispersed phase, m_p is the mass of a particle or droplet, and ρ_g is the gas density, while ν is the kinematic viscosity of the gas. The last equation in the system above indicates that the gas flow is considered as incompressible[1]. The scalar pressure field $p \equiv p(t,x) \in \mathbf{R}$ is instantaneously coupled to the unknowns F and u by the Poisson equation

$$-\Delta_x p = \rho_g \operatorname{trace}((\nabla_x u)^2) - \kappa \operatorname{div}_x \int_{\mathbf{R}^3} (v-u) F \,\mathrm{d}v.$$

The mathematical theory of the Vlasov–Navier–Stokes system has been discussed in [34]. Various asymptotic limits of the Vlasov-Stokes system that are of great interests in the modeling of aerosol or spray flows have been investigated in the mathematical literature: see for instance [17, 23, 24].

Our aim in the present work is different: we are concerned in deriving models such as the Vlasov–Navier–Stokes system from a more microscopic description of aerosol flows.

Perhaps the most natural idea for doing so is to view the Vlasov–Navier–Stokes system as a mean field model governing the limit of the solid particles (or droplets) phase space empirical measure as the particle number tends to infinity and the particle radius vanishes in some appropriate distinguished scaling.

Derivations of the Stokes and Navier–Stokes equation with a force term including the drag force exerted by the particles on the fluid (known as the Brinkman force) from a system consisting of a large number of particles immersed in a viscous fluid can be found in [1, 16]. While these papers obtain the same Navier-Stokes equation as in the coupled system above (more precisely, its steady variant), they assume that the phase space distribution of particles or droplets is given, and therefore do not derive the full Vlasov–Navier–Stokes system. The reason for this shortcoming is the following: both references [1, 16] use the method of homogenization of elliptic operators with holes of finite capacity pioneered by Khruslov and his school — see for instance [11, 30]. Unfortunately, these methods assume that the minimal distance between particles remains uniformly much larger than the particle radius $r \ll 1$ — specifically, of the order of $r^{1/3}$ in space dimension 3. While this assumption can be imposed if the distribution of particles is given, such a control on the distance between neighboring particles is probably not nicely propagated by the particle dynamics and most likely hard to establish (see however [26] for interesting ideas in this direction). Even if such a control could be

[1]It is well known that the motion of a gas at a very low Mach number is governed by the equations of incompressible fluid mechanics, even though a gas is a compressible fluid. A formal justification for this fact can be found on pp. 11–12 in [29].

established, configurations of N particles with such a uniform control on the minimal distance between neighboring particles are of vanishing probability in the large N limit: see for instance Proposition 4 in [25]. Worse, the coupled dynamics of finitely many rigid spheres immersed in a Navier–Stokes flow may not be defined for all positive times: see [14, 19].

In view of all these difficulties, we have chosen another route to derive coupled systems such as the Vlasov–Navier–Stokes system from a more microscopic model. Specifically, we start from a coupled system of Boltzmann equations for the solid particles or droplets and for the gas molecules.

One might object that the Boltzmann equation is a first principle equation neither for the solid particles nor for the gas molecules. In addition, the idea to treat the gas molecules and the particles in the dispersed phase on equal footing is most unnatural. On the other hand, the system of Newton's equations written for each solid particle immersed in an incompressible Navier–Stokes fluid cannot be considered as a first principle model for aerosol flows either. Indeed, it is only in some very special asymptotic limit that the dynamics of a gas is governed by the incompressible Navier-Stokes equations.

On the other hand, using a system of Boltzmann equations for the dispersed phase and the gas allows considering distributions of solid particles or droplets without any constraint on the minimal distance between neighbouring particles. In fact collisions between particles in the dispersed phase are described by a collision integral, in the same way as collisions between gas molecules.

Another benefit in this approach is the great variety of models describing the interaction between the dispersed phase and the propellant. In the present work, this interaction is described in terms of a general class of Boltzmann type collision integrals, assumed to satisfy a few assumptions discussed in Section 3 below. We have focused our attention on two examples of such collision integrals; in one case, collisions are assumed to be elastic, while the other example is based on the diffuse reflection of gas molecules on the surface of dust particles or droplets, which is an inelastic process.

For that reason, we believe that the idea of starting from the kinetic theory of multicomponent gases may provide an interesting alternative to the traditional arguments used in deriving the various dynamical models appearing in the theory of aerosol flows.

This approach should not be confused with the more detailed analysis of rarefied gas flows past an immersed body (see for instance [31–33]). It has been known for a long time that the motion of an immersed body in a viscous fluid involves nonlocal effects in the time variable: see [5] for a short, yet detailed presentation of the Boussinesq-Basset force. Similar effects can be observed in the case of a solid particle immersed in a rarefied gas and have been recently studied: see [2] and the references therein. Since our description of the interaction between the dispersed phase and the propellant is based on collision integrals, it does not include such effects. On the other hand, our purpose is not to focus on the details of the interaction between a single dust particle or droplet with the propellant, but rather to investigate the collective behavior of the dispersed phase. Whether the system of Boltzmann equations for a 2-component gas can be justified from a more detailed, microscopic model, such as the dynamics of a system of solid particles immersed in a rarefied gas, seems to be a very interesting problem, albeit a very difficult one.

Our derivation of dynamical equations for aerosol flows from the kinetic theory of multicomponent gas is systematic yet formal, in the sense of the derivations of fluid dynamic equations from the Boltzmann equation in [3].

2. Boltzmann equations for multicomponent gases

Consider a binary mixture consisting of microscopic gas molecules and much bigger solid dust particles or liquid droplets. For simplicity, we henceforth assume that the dust particles or droplets are identical (in particular, the spray is monodisperse: all particles have the same mass), and that the gas is monatomic. We denote from now on by $F \equiv F(t,x,v) \geq 0$ the distribution function of dust particles or droplets, and by $f \equiv f(t,x,w) \geq 0$ the distribution function of gas molecules. These distribution functions satisfy the system of Boltzmann equations

$$\begin{aligned}(\partial_t + v\cdot\nabla_x)F &= \mathcal{D}(F,f) + \mathcal{B}(F),\\ (\partial_t + w\cdot\nabla_x)f &= \mathcal{R}(f,F) + \mathcal{C}(f).\end{aligned} \tag{2.1}$$

The terms $\mathcal{B}(F)$ and $\mathcal{C}(f)$ are the Boltzmann collision integrals for pairs of dust particles or liquid droplets and for pairs of gas molecules respectively. The terms $\mathcal{D}(F,f)$ and $\mathcal{R}(f,F)$ are Boltzmann type collision integrals describing the deflection of dust particles or liquid droplets due to the impingement of gas molecules, and the slowing down of gas molecules caused by collisions with dust particles or liquid droplets respectively.

2.1. Outline of the paper. Our main result, i.e. is the derivation of the Vlasov–Navier–Stokes system from the system of Boltzmann equations (2.1), is stated as Theorem 4.1 in Section 4. Unfortunately, even a formal derivation as in Theorem 4.1 cannot be formulated before several preliminaries have been presented in detail. Roughly speaking, the principle of this derivation is based on the following key ideas.

(a) The deflection collision integral $\mathcal{D}(F,f)$, appropriately scaled, converges to the term

$$\frac{\kappa}{m_p}\operatorname{div}_v((v-u)F)$$

in the Vlasov equation in system (1.1); see Proposition 4.2 in Section 4.2.2.

(b) The first order moment of the collision integral $\mathcal{R}(f,F)$, i.e.

$$\int v\mathcal{R}(f,F)\,\mathrm{d}v,$$

appropriately scaled, converges to the friction term

$$\kappa\int_{\mathbf{R}^3}(v-u)F\,\mathrm{d}v;$$

see Proposition 4.3 in Section 4.2.3

(c) Finally, the incompressible Navier–Stokes system is derived from the Boltzmann equation for the propellant (the second equation in (2.1)) by the formal procedure described in detail in [3]. In particular, the asymptotic form of propellant distribution function is established in Proposition 4.1 in Section 4.2.1, the incompressibility condition in Proposition 4.4 in Section 4.2.4, while the derivations of the convection and the diffusion term in the Navier–Stokes equation in (1.1) are presented in Propositions 4.6 and 4.5, to be found in Sections 4.2.6 and 4.2.5 respectively.

Otherwise, the outline of this paper is the following: Section 2 introduces the system of Boltzmann equations used as the starting point in our derivation. In particular, the fundamental conservation laws and basic properties of this system are recalled in Section 2, along with the dimensionless form of the equations and the definition of the scaling parameters involved. Section 3 studies in detail the main properties of the collision kernel describing the interaction of gas molecules with dust particles or droplets. The proof of Theorem 4.1 occupies most of Section 4.

2.2. Fundamental conservation laws for multicomponent Boltzmann systems. Before describing in detail the collision integrals introduced above, we recall their fundamental properties.

Collisions between gas molecules are assumed to be elastic, so that the Boltzmann collision integral $\mathcal{C}(f)$ satisfies the following local conservation laws of mass, momentum and energy: for each measurable f defined a.e. on $\mathbf{R}^3$ and rapidly decaying as $|w|\to\infty$,

$$\int_{\mathbf{R}^3}\mathcal{C}(f)(w)\begin{pmatrix}1\\ w\\ |w|^2\end{pmatrix}\mathrm{d}w=0. \tag{2.2}$$

Collisions between dust particles or liquid droplets may not be perfectly elastic, so that the Boltzmann collision integral $\mathcal{B}(F)$ satisfies only the local conservation laws of mass and momentum: for each measurable F defined a.e. on $\mathbf{R}^3$ and rapidly decaying as $|v|\to\infty$,

$$\int_{\mathbf{R}^3}\mathcal{B}(F)(v)\begin{pmatrix}1\\ v\end{pmatrix}\mathrm{d}v=0. \tag{2.3}$$

Collisions between gas molecules and dust particles or liquid droplets obviously preserve the nature of the colliding objects. Therefore, the collision integrals $\mathcal{D}(F,f)$ and $\mathcal{R}(f,F)$ satisfy the following local conservation of particle number per species: for each measurable F and f defined a.e. on $\mathbf{R}^3$ and rapidly decaying at infinity,

$$\int_{\mathbf{R}^3}\mathcal{D}(F,f)(v)\,\mathrm{d}v=\int_{\mathbf{R}^3}\mathcal{R}(f,F)(w)\,\mathrm{d}w=0. \tag{2.4}$$

These collision integrals satisfy the local balance of momentum in the aerosol, i.e.

$$m_p\int_{\mathbf{R}^3}\mathcal{D}(F,f)(v)v\,\mathrm{d}v+m_g\int_{\mathbf{R}^3}\mathcal{R}(f,F)(w)w\,\mathrm{d}w=0, \tag{2.5}$$

where m_g is the mass of gas molecules and m_p the mass of dust particles or liquid droplets.

If the collisions between gas molecules and droplets or dust particles are elastic, these collision integrals satisfy in addition the local balance of energy in the aerosol, i.e.

$$m_p\int_{\mathbf{R}^3}\mathcal{D}(F,f)(v)\tfrac{1}{2}|v|^2\,\mathrm{d}v+m_g\int_{\mathbf{R}^3}\mathcal{R}(f,F)(w)\tfrac{1}{2}|w|^2\,\mathrm{d}w=0.$$

2.3. Dimensionless Boltzmann systems. We assume for simplicity that the aerosol is enclosed in a periodic box of size $L>0$, i.e. $x\in\mathbf{R}^3/L\mathbf{Z}^3$. Henceforth, the distribution functions F and f are viewed as $L\mathbf{Z}^3$-periodic functions of the position x. The system of Boltzmann equations (2.1) involves an important number of physical parameters, which are listed in the table below.

We first define a dimensionless position variable:

$$\hat{x}:=x/L,$$

Parameter	Definition
L	size of the container (periodic box)
$\mathcal{N}_p$	number of particles/L^3
$\mathcal{N}_g$	number of gas molecules/L^3
V_p	thermal speed of particles
V_g	thermal speed of gas molecules
S_{pp}	average particle/particle cross-section
S_{pg}	average particle/gas cross-section
S_{gg}	average molecular cross-section
$\eta = m_g/m_p$	mass ratio (molecules/particles)
$\epsilon = V_p/V_g$	thermal speed ratio (particles/molecules)

TABLE 2.1. *the physical parameters for binary gas mixtures*

together with dimensionless velocity variables for each species:

$$\hat{v} := v/V_p, \quad \hat{w} := w/V_g.$$

In other words, the velocity of each species is measured in terms of the thermal speed of the particles in the species under consideration.

Next, we define a time variable, which is adapted to the individual motion of the typical particle of the slowest species, i.e. the dust particles or droplets:

$$\hat{t} := tV_p/L.$$

Finally, we define dimensionless distribution functions for each particle species:

$$\hat{F}(\hat{t},\hat{x},\hat{v}) := V_p^3 F(t,x,v)/\mathcal{N}_p, \qquad \hat{f}(\hat{t},\hat{x},\hat{w}) := V_g^3 f(t,x,w)/\mathcal{N}_g.$$

The definition of dimensionless collision integrals is more complex and involves the average collision cross-sections S_{pp}, S_{pg}, S_{gg}, whose definition is recalled below.

The collision integrals $\mathcal{B}(F)$, $\mathcal{C}(f)$, $\mathcal{D}(F,f)$ and $\mathcal{R}(f,F)$ are given by expressions of the form

$$\begin{aligned}
\mathcal{B}(F)(v) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} F(v')F(v'_*)\Pi_{pp}(v,\mathrm{d}v'\,\mathrm{d}v'_*) \\
&\quad - F(v)\int_{\mathbf{R}^3} F(v_*)|v-v_*|\Sigma_{pp}(|v-v_*|)\,\mathrm{d}v_*, \\
\mathcal{C}(f)(w) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} f(w')f(w'_*)\Pi_{gg}(w,\mathrm{d}w'\,\mathrm{d}w'_*) \\
&\quad - f(w)\int_{\mathbf{R}^3} f(w_*)|w-w_*|\Sigma_{gg}(|w-w_*|)\,\mathrm{d}w_*, \\
\mathcal{D}(F,f)(v) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} F(v')f(w')\Pi_{pg}(v,\mathrm{d}v'\,\mathrm{d}w') \\
&\quad - F(v)\int_{\mathbf{R}^3} f(w)|v-w|\Sigma_{pg}(|v-w|)\,\mathrm{d}w, \\
\mathcal{R}(f,F)(w) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} F(v')f(w')\Pi_{gp}(w,\mathrm{d}v'\,\mathrm{d}w') \\
&\quad - f(w)\int_{\mathbf{R}^3} F(v)|v-w|\Sigma_{pg}(|v-w|)\,\mathrm{d}v.
\end{aligned} \tag{2.6}$$

In these expressions, $\Pi_{pp}, \Pi_{gg}, \Pi_{pg}, \Pi_{gp}$ are nonnegative, measure-valued measurable functions defined a.e. on $\mathbf{R}^3$, while $\Sigma_{pp}, \Sigma_{gg}, \Sigma_{pg}$ are nonnegative measurable functions defined a.e. on $\mathbf{R}_+$.

The quantities Π and Σ are related by the following identities:

$$\begin{aligned}\int_{\mathbf{R}^3_v} \mathrm{d}v\, \Pi_{pp}(v, \mathrm{d}v'\,\mathrm{d}v'_*) &= |v'-v'_*|\Sigma_{pp}(|v'-v'_*|)\,\mathrm{d}v'\,\mathrm{d}v'_*,\\ \int_{\mathbf{R}^3_w} \mathrm{d}w\, \Pi_{gg}(w, \mathrm{d}w'\,\mathrm{d}w'_*) &= |w'-w'_*|\Sigma_{gg}(|w'-w'_*|)\,\mathrm{d}w'\,\mathrm{d}w'_*,\\ \int_{\mathbf{R}^3_v} \mathrm{d}v\, \Pi_{pg}(v, \mathrm{d}v'\,\mathrm{d}w') &= |v'-w'|\Sigma_{pg}(|v'-w'|)\,\mathrm{d}v'\,\mathrm{d}w',\\ \int_{\mathbf{R}^3_w} \mathrm{d}w\, \Pi_{gp}(w, \mathrm{d}v'\,\mathrm{d}w') &= |v'-w'|\Sigma_{pg}(|v'-w'|)\,\mathrm{d}v'\,\mathrm{d}w'.\end{aligned} \tag{2.7}$$

In each one of these identities, the left hand side is to be understood as an integral with respect to the unprimed variable (v for the 1st and 3rd identities, w for the 2nd and the 4th) of a nonnegative measurable function with values in the set of Borel measures on $\mathbf{R}^3 \times \mathbf{R}^3$. The identities (2.7) imply the conservation of mass for each species of particles for all the collision integrals appearing in (2.6). These conservation laws have been stated above: see the first lines of equations (2.2) and (2.3), and (2.4).

We refer to formula (3.6) in [28] for this general presentation of collision integrals. Specific examples of these collision integrals will be discussed in Section 2.4 below

According to formula (2.2) in [28], Σ_{pp}, Σ_{gg} and Σ_{pg} have the dimensions of areas. The corresponding dimensionless quantities are

$$\begin{aligned}\hat{\Sigma}_{pp}(|\hat{v}|) &= \Sigma_{pp}(V_p|\hat{v}|)/S_{pp},\\ \hat{\Sigma}_{gg}(|\hat{w}|) &= \Sigma_{gg}(V_g|\hat{w}|)/S_{gg},\\ \hat{\Sigma}_{pg}(|\hat{z}|) &= \Sigma_{pg}(V_g|\hat{z}|)/S_{pg}.\end{aligned}$$

Likewise, we define the dimensionless measure-valued collision kernels by the formulas

$$\begin{aligned}\hat{\Pi}_{pp}(\hat{v}, \mathrm{d}\hat{v}'\,\mathrm{d}\hat{v}'_*) &= \Pi_{pp}(v, \mathrm{d}v'\,\mathrm{d}v'_*)/S_{pp}V_p^4,\\ \hat{\Pi}_{gg}(\hat{w}, \mathrm{d}\hat{w}'\,\mathrm{d}\hat{w}'_*) &= \Pi_{gg}(w, \mathrm{d}w'\,\mathrm{d}w'_*)/S_{gg}V_g^4,\\ \hat{\Pi}_{pg}(\hat{v}, \mathrm{d}\hat{v}'\,\mathrm{d}\hat{w}') &= \Pi_{pg}(v, \mathrm{d}v'\,\mathrm{d}w')/S_{pg}V_g^4,\\ \hat{\Pi}_{gp}(\hat{w}, \mathrm{d}\hat{v}'\,\mathrm{d}\hat{w}') &= \Pi_{gp}(w, \mathrm{d}v'\,\mathrm{d}w')/S_{pg}V_gV_p^3.\end{aligned}$$

We henceforth define the dimensionless collision integrals as follows:

$$\begin{aligned}\hat{\mathcal{B}}(\hat{F})(\hat{v}) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} \hat{F}(\hat{v}')\hat{F}(\hat{v}'_*)\hat{\Pi}_{pp}(\hat{v}, \mathrm{d}\hat{v}'\,\mathrm{d}\hat{v}'_*)\\ &\quad - \hat{F}(\hat{v})\int_{\mathbf{R}^3} \hat{F}(\hat{v}_*)|\hat{v}-\hat{v}_*|\hat{\Sigma}_{pp}(|\hat{v}-\hat{v}_*|)\,\mathrm{d}\hat{v}_*,\\ \hat{\mathcal{C}}(\hat{f})(\hat{w}) &= \iint_{\mathbf{R}^3\times\mathbf{R}^3} \hat{f}(\hat{w}')\hat{f}(\hat{w}'_*)\hat{\Pi}_{gg}(\hat{w}, \mathrm{d}\hat{w}'\,\mathrm{d}\hat{w}'_*)\\ &\quad - \hat{f}(\hat{w})\int_{\mathbf{R}^3} \hat{f}(\hat{w}_*)|\hat{w}-\hat{w}_*|\hat{\Sigma}_{gg}(|\hat{w}-\hat{w}_*|)\,\mathrm{d}\hat{w}_*,\end{aligned}$$

while

$$
\begin{aligned}
\hat{\mathcal{D}}(\hat{F},\hat{f})(\hat{v}) = & \iint_{\mathbf{R}^3\times\mathbf{R}^3} \hat{F}(\hat{v}')f(\hat{w}')\hat{\Pi}_{pg}(\hat{v},\mathrm{d}\hat{v}'\,\mathrm{d}\hat{w}') \\
& - \hat{F}(\hat{v})\int_{\mathbf{R}^3} \hat{f}(\hat{w})\left|\tfrac{V_p}{V_g}\hat{v}-\hat{w}\right|\hat{\Sigma}_{pg}\left(\left|\tfrac{V_p}{V_g}\hat{v}-\hat{w}\right|\right)\mathrm{d}\hat{w}, \\
\hat{\mathcal{R}}(\hat{f},\hat{F})(\hat{w}) = & \iint_{\mathbf{R}^3\times\mathbf{R}^3} \hat{F}(\hat{v}')\hat{f}(\hat{w}')\hat{\Pi}_{gp}(\hat{w},\mathrm{d}\hat{v}'\,\mathrm{d}\hat{w}') \\
& - \hat{f}(\hat{w})\int_{\mathbf{R}^3} \hat{F}(\hat{v})\left|\tfrac{V_p}{V_g}\hat{v}-\hat{w}\right|\hat{\Sigma}_{pg}\left(\left|\tfrac{V_p}{V_g}\hat{v}-\hat{w}\right|\right)\mathrm{d}\hat{v}.
\end{aligned}
$$

With the dimensionless quantities so defined, we arrive at the following dimensionless form of the multicomponent Boltzmann system:

$$
\begin{cases}
\partial_{\hat{t}}\hat{F} + \hat{v}\cdot\nabla_{\hat{x}}\hat{F} = \mathcal{N}_g S_{pg} L \dfrac{V_g}{V_p}\hat{\mathcal{D}}(\hat{F},\hat{f}) + \mathcal{N}_p S_{pp} L\hat{\mathcal{B}}(\hat{F}), \\
\partial_{\hat{t}}\hat{f} + \dfrac{V_g}{V_p}\hat{w}\cdot\nabla_{\hat{x}}\hat{f} = \mathcal{N}_p S_{pg} L\dfrac{V_g}{V_p}\hat{\mathcal{R}}(\hat{f},\hat{F}) + \mathcal{N}_g S_{gg} L\dfrac{V_g}{V_p}\hat{\mathcal{C}}(\hat{f}).
\end{cases} \tag{2.8}
$$

Throughout the present study, we shall always assume that

$$\mathcal{N}_p S_{pp} L \ll 1. \tag{2.9}$$

In other words, the collision integral for dust particles or droplets $\mathcal{N}_p S_{pp} L\hat{\mathcal{B}}(\hat{F})$ is considered as formally negligible, and will be henceforth systematically discarded in the equations.

Besides, the thermal speed V_p of dust particles or droplets is in general smaller than the thermal speed V_g of gas molecules; thus we denote their ratio by

$$\epsilon = \frac{V_p}{V_g} \in [0,1]. \tag{2.10}$$

Recalling that the mass ratio $[0,1] \ni \eta = m_g/m_p$ is supposed to be extremely small, since the particles are usually much bigger than the molecules, we also assume

$$\eta = \frac{\mathcal{N}_p}{\mathcal{N}_g} \in [0,1]. \tag{2.11}$$

This assumption on the ratio of the number of particles to the number of molecules defines a scaling such that the mass density of the gas is of the same order of magnitude as the mass density of droplets.

Finally, we shall assume that

$$\mathcal{N}_p S_{pg} L = \epsilon, \quad \text{and} \quad \mathcal{N}_g S_{gg} L = 1/\epsilon. \tag{2.12}$$

Under these assumptions,

$$\mathcal{N}_g S_{pg} L \frac{V_g}{V_p} = \frac{\mathcal{N}_g}{\mathcal{N}_p}(\mathcal{N}_p S_{pg} L)\frac{V_g}{V_p} = \frac{1}{\eta},$$

$$(\mathcal{N}_p S_{pg} L)\frac{V_g}{V_p} = (\mathcal{N}_g S_{gg} L)\frac{V_g}{V_p} = 1,$$

so that we arrive at the scaled system

$$\begin{cases} \partial_{\hat{t}}\hat{F}+\hat{v}\cdot\nabla_{\hat{x}}\hat{F}=\frac{1}{\eta}\hat{\mathcal{D}}(\hat{F},\hat{f}), \\ \partial_{\hat{t}}\hat{f}+\frac{1}{\epsilon}\hat{w}\cdot\nabla_{\hat{x}}\hat{f}=\hat{\mathcal{R}}(\hat{f},\hat{F})+\frac{1}{\epsilon^2}\hat{\mathcal{C}}(\hat{f}). \end{cases} \tag{2.13}$$

Henceforth, we drop hats on all dimensionless quantities and variables introduced in this section. Only dimensionless variables, distribution functions and collision integrals will be considered from now on. We also use V,W as dummy variables in the gain part of the collision operators $\mathcal{D}$ and $\mathcal{R}$, in order to avoid confusion.

We define therefore the (ϵ- and η-dependent) dimensionless collision integrals

$$\begin{aligned} \mathcal{C}(f)(w)=&\iint_{\mathbf{R}^3\times\mathbf{R}^3} f(w')f(w'_*)\Pi_{gg}(w,\mathrm{d}w'\,\mathrm{d}w'_*) \\ &-f(w)\int_{\mathbf{R}^3} f(w_*)|w-w_*|\Sigma_{gg}(|w-w_*|)\,\mathrm{d}w_*, \end{aligned} \tag{2.14}$$

$$\begin{aligned} \mathcal{D}(F,f)(v)=&\iint_{\mathbf{R}^3\times\mathbf{R}^3} F(V)f(W)\Pi_{pg}(v,\mathrm{d}V\,\mathrm{d}W) \\ &-F(v)\int_{\mathbf{R}^3} f(w)\,|\epsilon v-w|\,\Sigma_{pg}\,(|\epsilon v-w|)\,\mathrm{d}w, \end{aligned} \tag{2.15}$$

$$\begin{aligned} \mathcal{R}(f,F)(w)=&\iint_{\mathbf{R}^3\times\mathbf{R}^3} F(V)f(W)\Pi_{gp}(w,\mathrm{d}V\,\mathrm{d}W) \\ &-f(w)\int_{\mathbf{R}^3} F(v)\,|\epsilon v-w|\,\Sigma_{pg}\,(|\epsilon v-w|)\,\mathrm{d}v, \end{aligned} \tag{2.16}$$

with Σ_{gg}, Σ_{pg} satisfying equations (2.7). Notice that the scattering kernels Π_{pg} and Π_{gp} depend in fact on ϵ and η. Whenever necessary (for instance in describing the asymptotic behavior of these kernels in the small ϵ and η limit), we shall denote them $\Pi_{pg}^{\epsilon,\eta}$ and $\Pi_{gp}^{\epsilon,\eta}$ respectively.

The scaled Boltzmann system (2.13) is then recast as

$$\begin{cases} \partial_t F+v\cdot\nabla_x F=\frac{1}{\eta}\mathcal{D}(F,f), \\ \partial_t f+\frac{1}{\epsilon}w\cdot\nabla_x f=\mathcal{R}(f,F)+\frac{1}{\epsilon^2}\mathcal{C}(f). \end{cases} \tag{2.17}$$

2.4. Explicit formulas for the collision integrals

In the previous section, we have introduced a general setting for the various collisional processes involved in gas-particle mixtures. The explicit formulas for the main examples of collision integrals considered in this work are given in the next three sections.

2.4.1. The Boltzmann collision integral for gas molecules

The dimensionless collision integral $\mathcal{C}(f)$ is given by the formula

$$\mathcal{C}(f)(w)=\iint_{\mathbf{R}^3\times\mathbf{S}^2}(f(w')f(w'_*)-f(w)f(w_*))c(w-w_*,\omega)\,\mathrm{d}w_*\mathrm{d}\omega, \tag{2.18}$$

for each measurable f defined a.e. on $\mathbf{R}^3$ and rapidly decaying at infinity, where

$$\begin{aligned} w' &\equiv w'(w,w_*,\omega) := w-(w-w_*)\cdot\omega\omega, \\ w'_* &\equiv w'_*(w,w_*,\omega) := w_*+(w-w_*)\cdot\omega\omega, \end{aligned} \tag{2.19}$$

(see formulas (3.11) and (4.16) in chapter II of [7]). The collision kernel c is of the form

$$c(w-w_*,\omega) = |w-w_*|\sigma_{gg}(|w-w_*|,|\cos(\widehat{w-w_*,\omega})|), \tag{2.20}$$

where σ_{gg} is the dimensionless differential cross-section of gas molecules. In other words,

$$\Sigma_{gg}(|z|) = 4\pi\int_0^1 \sigma_{gg}(|z|,\mu)\,\mathrm{d}\mu,$$

while

$$\Pi_{gg}(w,\cdot) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} \mathrm{d}w_*\mathrm{d}\omega\,\delta_{w'(w,w_*,\omega)}\otimes\delta_{w'_*(w,w_*,\omega)}c(w-w_*,\omega). \tag{2.21}$$

The left hand side is to be understood as a function of w with values in the set of positive Borel measures on $\mathbf{R}^3\times\mathbf{R}^3$, while the right hand side is a linear superposition of the positive Borel measures $\delta_{w'(w,w_*,\omega)}\otimes\delta_{w'(w,w_*,\omega)}$ on $\mathbf{R}^3\times\mathbf{R}^3$ obtained by integrating over w_*,ω while w is kept fixed.

If more to one's taste, one can equivalently formulate this equality by applying both sides to a test function $\chi\in C_c(\mathbf{R}^3\times\mathbf{R}^3)$:

$$\begin{aligned} &\iint_{\mathbf{R}^3\times\mathbf{R}^3} \chi(W,W_*)\Pi_{gg}(w,\mathrm{d}W\mathrm{d}W_*) \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} \langle\delta_{w'(w,w_*,\omega)}\otimes\delta_{w'_*(w,w_*,\omega)},\chi\rangle c(w-w_*,\omega)\,\mathrm{d}w_*\mathrm{d}\omega \\ &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} \chi(w'(w,w_*,\omega),w'_*(w,w_*,\omega))c(w-w_*,\omega)\,\mathrm{d}w_*\mathrm{d}\omega. \end{aligned}$$

One recognizes in the last right hand side of the equalities above the usual expression for the gain term in the Boltzmann collision integral for identical particles interacting by elastic collisions.

We recall that the collision integral $\mathcal{C}$ satisfies the conservation of mass, momentum and kinetic energy (2.2) — see formulas (1.16)-(1.18) in chapter II of [6].

We assume that the molecular interaction is defined in terms of a hard potential satisfying Grad's cutoff assumption. In other words, we assume that there exists $c_*>1$ and $\gamma\in[0,1]$ such that

$$\begin{aligned} &0 < c(z,\omega) \le c_*(1+|z|)^\gamma, && \text{for a.e. } (z,\omega)\in\mathbf{R}^3\times\mathbf{S}^2, \\ &\int_{\mathbf{S}^2} c(z,\omega)\,\mathrm{d}\omega \ge \frac{1}{c_*}\frac{|z|}{1+|z|}, && \text{for a.e. } z\in\mathbf{R}^3. \end{aligned} \tag{2.22}$$

Next we discuss the properties of the linearization about a Maxwellian equilibrium state of the collision integral $\mathcal{C}$. By scaling and Galilean invariance, one can consider the Maxwellian distribution

$$M(w) := \tfrac{1}{(2\pi)^{3/2}}e^{-|w|^2/2} \tag{2.23}$$

without loss of generality. The linearized collision integral is defined as

$$\mathcal{L}\phi := -M^{-1}D\mathcal{C}(M)\cdot(M\phi), \tag{2.24}$$

where D is the functional derivative. In other words,

$$\mathcal{L}\phi(w) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} (\phi(w)+\phi(w_*)-\phi(w')-\phi(w'_*))c(w-w_*,\omega)M(w_*)\mathrm{d}w_*\mathrm{d}\omega.$$

The following result is a theorem of Hilbert in the case of hard sphere collisions, extended by Grad to the case of hard cutoff potentials (see [7], especially Theorem I on p.186 and Theorem II on p.187).

Theorem 2.1. *The linearized collision integral $\mathcal{L}$ is an unbounded operator on $L^2(M\mathrm{d}v)$ with domain $\mathrm{Dom}\,\mathcal{L} = L^2((\bar{c}\star M)^2 M\mathrm{d}v)$, where*

$$\bar{c}(z) := \int_{\mathbf{S}^2} c(z,\omega)\mathrm{d}\omega.$$

Moreover, $\mathcal{L} = \mathcal{L}^* \geq 0$, with nullspace

$$\mathrm{Ker}\,\mathcal{L} = \mathrm{Span}\{1, w_1, w_2, w_3, |w|^2\}. \tag{2.25}$$

Finally, $\mathcal{L}$ is a Fredholm operator, so that

$$\mathrm{Im}\,\mathcal{L} = \mathrm{Ker}\,\mathcal{L}^{\perp}.$$

Defining by

$$A(w) := w\otimes w - \tfrac{1}{3}|w|^2 I \tag{2.26}$$

the traceless component of the tensor $w\otimes w$, we see that $A \perp \mathrm{Ker}\,\mathcal{L}$ in $L^2(M\mathrm{d}v)$. Since $\mathcal{L}$ satisfies the Fredholm alternative, there exists a unique $\tilde{A} \in \mathrm{Dom}\,\mathcal{L}$ such that

$$\mathcal{L}\tilde{A} = A, \qquad \tilde{A} \perp \mathrm{Ker}\,\mathcal{L}. \tag{2.27}$$

Using the symmetry properties of the collision integral and the rotation invariance of the Maxwellian distribution (2.23), one can show that the matrix field $\tilde{A}$ is of the form

$$\tilde{A}(w) = \alpha(|w|)A(w), \tag{2.28}$$

where α is a measurable function such that

$$\int_{\mathbf{R}^3} \alpha(|w|)^2 |w|^4 (\bar{c}\star M(w))^2 M(w)\mathrm{d}w < \infty.$$

See [15] for a complete proof of this statement.

In the sequel, we shall assume for simplicity that the molecular interaction is such that

$$\alpha \in L^\infty(\mathbf{R}_+).$$

It is a well known fact that, in the case of Maxwell molecules, that is, in the case where the collision kernel is of the form

$$c(z,\omega) = C(|\cos(\widehat{v-v_*,\omega})|),$$

then α is a positive constant. (See for instance the discussion between formulas (3.15) and (3.17) in chapter V of [6].)

2.4.2. The collision integrals $\mathcal{D}$ and $\mathcal{R}$ for elastic collisions. For each measurable F and f defined a.e. on $\mathbf{R}^3$ and rapidly decaying at infinity, the dimensionless collision integrals $\mathcal{D}(F,f)$ and $\mathcal{R}(f,F)$ are given by the formulas

$$\mathcal{D}(F,f)(v) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} (F(v'')f(w'') - F(v)f(w))b(\epsilon v - w,\omega)\,\mathrm{d}w\mathrm{d}\omega,$$
$$\mathcal{R}(f,F)(w) = \iint_{\mathbf{R}^3\times\mathbf{S}^2} (f(w'')F(v'') - f(w)F(v))b(\epsilon v - w,\omega)\,\mathrm{d}v\mathrm{d}\omega,$$

where

$$\begin{aligned} v'' &\equiv v''(v,w,\omega) := v - \frac{2\eta}{1+\eta}\left(v - \frac{1}{\epsilon}w\right)\cdot\omega\omega, \\ w'' &\equiv w''(v,w,\omega) := w - \frac{2}{1+\eta}(w - \epsilon v)\cdot\omega\omega, \end{aligned} \tag{2.29}$$

(see formula (5.10) in chapter II of [7]). The collision kernel b is of the form

$$b(\epsilon v - w,\omega) = |\epsilon v - w|\sigma_{pg}(|\epsilon v - w|, |\cos(\widehat{\epsilon v - w,\omega})|), \tag{2.30}$$

where σ_{pg} is the dimensionless differential cross-section of gas molecules. In other words,

$$\Sigma_{pg}(|z|) = 4\pi\int_0^1 \sigma_{pg}(|z|,\mu)\,\mathrm{d}\mu, \tag{2.31}$$

while

$$\begin{aligned} \Pi_{pg}(v,\cdot) &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} \mathrm{d}w\,\mathrm{d}\omega\, b(\epsilon v - w,\omega)\delta_{v''(v,w,\omega)}\otimes\delta_{w''(v,w,\omega)}, \\ \Pi_{gp}(w,\cdot) &= \iint_{\mathbf{R}^3\times\mathbf{S}^2} \mathrm{d}v\,\mathrm{d}\omega\, b(\epsilon v - w,\omega)\delta_{v''(v,w,\omega)}\otimes\delta_{w''(v,w,\omega)}, \end{aligned} \tag{2.32}$$

where the equalities (2.32) are to be understood in the same way as equation (2.21).

One should keep in mind that the velocity of each species is measured in units of the thermal speed of that species. This accounts for the appearance of the thermal speed ratio ϵ in the formulas above. Moreover, the reduced mass of the dust particles or droplets and gas molecules defined by formula (5.2) in chapter II of [7] is

$$\frac{m_p m_g}{m_p + m_g} = \frac{m_g}{1+\eta} = \frac{m_p\eta}{1+\eta}.$$

These formulas explain how the mass ratio η appears in the definition of v'' and w'' above.

We recall that the operators $\mathcal{D}$ and $\mathcal{R}$ defined in this subsection satisfy separately the conservation of the number of particles and molecules (2.4), and jointly the conservation of momentum (involving both operators):

$$\epsilon\int_{\mathbf{R}^3} \mathcal{D}(F,f)(v)v\,\mathrm{d}v + \eta\int_{\mathbf{R}^3} \mathcal{R}(f,F)(w)w\,\mathrm{d}w = 0. \tag{2.33}$$

This last identity is a dimensionless version of (2.5).

These properties can be easily checked using the formulas

$$\epsilon v'' + \eta w'' = \epsilon v + \eta w, \quad \epsilon v'' - w'' = R_\omega(\epsilon v - w), \tag{2.34}$$

where R_ω is the reflection defined by $R_\omega w = w - 2(w\cdot\omega)\omega$ for each $\omega \in \mathbf{S}^2$. Indeed these formulas show that $(v,w)\mapsto(v'',w'')$ is a linear involution for each $\omega\in\mathbf{S}^2$.

As in the case of the molecular collision kernel c, we assume that b is a cutoff kernel associated with a hard potential, i.e. we assume that there exists $b_*>1$ and $\beta^*\in[0,1]$ such that

$$\begin{aligned} &0<b(z,\omega)\le b_*(1+|z|)^{\beta^*}, && \text{for a.e. } (z,\omega)\in\mathbf{R}^3\times\mathbf{S}^2,\\ &\int_{\mathbf{S}^2} b(z,\omega)\,\mathrm{d}\omega \ge \frac{1}{b_*}\frac{|z|}{1+|z|}, && \text{for a.e.} z\in\mathbf{R}^3. \end{aligned} \tag{2.35}$$

We also assume that (for a.e. $\omega\in\mathcal{S}^2$)

$$b(\cdot,\omega)\in C^1(\mathbf{R}^3\setminus\{0\}), \text{ and } \sup_{\omega\in\mathbf{S}^2}|\partial_z b(z,\omega)|\le C(1+|z|). \tag{2.36}$$

2.4.3. An inelastic model of collision integrals $\mathcal{D}$ and $\mathcal{R}$. Dust particles or droplets are macroscopic objects when compared to gas molecules. This suggests using the classical models of gas-surface interaction to describe the impingement of gas molecules on dust particles or droplets. Perhaps the simplest such model of collisions has been introduced by F. Charles in [8], with a detailed discussion in Section 1.3 of [9] and in [10]. We briefly recall this model below.

First, the (dimensional) particle-molecule cross-section is

$$S_{pg}=\pi(r_g+r_p)^2,$$

where r_g is the molecular radius and r_p the radius of dust particles or droplets. Then, the dimensionless particle-molecule cross-section is

$$\Sigma_{pg}(|\epsilon v-w|)=1.$$

The formulas for S_{pg} and Σ_{pg} correspond to a binary collision between two balls of radius r_p and r_g.

Next, the measure-valued functions Π_{pg} and Π_{gp} are defined as follows:

$$\begin{aligned} \Pi_{pg}(v,\mathrm{d}V\,\mathrm{d}W) &:= K_{pg}(v,V,W)\,\mathrm{d}V\,\mathrm{d}W,\\ \Pi_{gp}(w,\mathrm{d}V\,\mathrm{d}W) &:= K_{gp}(w,V,W)\,\mathrm{d}V\,\mathrm{d}W, \end{aligned} \tag{2.37}$$

where,

$$\begin{aligned} K_{pg}(v,V,W) := {} & \tfrac{1}{2\pi^2}\left(\tfrac{1+\eta}{\eta}\right)^4\beta^4\epsilon^3\exp\left(-\tfrac12\beta^2\left(\tfrac{1+\eta}{\eta}\right)^2\left|\epsilon v-\frac{\epsilon V+\eta W}{1+\eta}\right|^2\right)\\ & \times\int_{\mathbf{S}^2}(n\cdot(\epsilon V-W))_+\left(n\cdot\left(\frac{\epsilon V+\eta W}{1+\eta}-\epsilon v\right)\right)_+ dn, \end{aligned} \tag{2.38}$$

$$\begin{aligned} K_{gp}(w,V,W) := {} & \tfrac{1}{2\pi^2}(1+\eta)^4\beta^4\exp\left(-\tfrac12\beta^2(1+\eta)^2\left|w-\frac{\epsilon V+\eta W}{1+\eta}\right|^2\right)\\ & \times\int_{\mathbf{S}^2}(n\cdot(\epsilon V-W))_+\left(n\cdot\left(w-\frac{\epsilon V+\eta W}{1+\eta}\right)\right)_+ dn. \end{aligned} \tag{2.39}$$

In these formulas

$$\beta=\sqrt{\frac{m_g}{k_B T_{surf}}}$$

where k_B is the Boltzmann constant and T_{surf} the surface temperature of the particles.

Thus, defining

$$P[\lambda](\xi,n):=\tfrac{1}{2\pi}\lambda^4\exp(-\tfrac{1}{2}\lambda^2|\xi|^2)(\xi\cdot n)_+, \tag{2.40}$$

for each $\lambda>0$ and $n\in\mathbf{S}^2$, we see that the integral kernels K_{pg} and K_{gp} are given in terms of P by the expressions

$$K_{pg}(v,V,W)=\tfrac{1}{\pi}\epsilon^3\int P[\beta\tfrac{1+\eta}{\eta}]\left(\tfrac{\epsilon V+\eta W}{1+\eta}-\epsilon v,n\right)((\epsilon V-W)\cdot n)_+dn,$$
$$K_{gp}(w,V,W)=\tfrac{1}{\pi}\int P[\beta(1+\eta)]\left(w-\tfrac{\epsilon V+\eta W}{1+\eta},n\right)((\epsilon V-W)\cdot n)_+dn.$$

3. Assumptions on Π_{pg} and Π_{gp}

In the sequel, we shall state a theorem which holds for all collision integrals satisfying a few assumptions introduced below.

We recall that Π_{pg} and Π_{gp} are nonnegative measure-valued functions of the variable $v\in\mathbf{R}^3$ and $w\in\mathbf{R}^3$ resp., which depend in general on the small parameters ϵ and η (see formulas (2.29), (2.32), and (2.37)-(2.39)). We do not make this dependence explicit, unless if necessary (as in Assumptions (H4)-(H5) below). In this case, we write $\Pi_{pg}^{\epsilon,\eta}$ and $\Pi_{gp}^{\epsilon,\eta}$ instead of Π_{pg} and Π_{gp}.

Assumption (H1). There exists a nonnegative measurable function

$$q\equiv q(r)\le C(1+r) \quad \text{for some } C>0$$

such that the measure-valued functions Π_{pg} and Π_{gp} satisfy

$$\int_{\mathbf{R}^3}\Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\mathrm{d}v=\int_{\mathbf{R}^3}\Pi_{gp}(w,\mathrm{d}V\mathrm{d}W)\mathrm{d}w=q(|\epsilon V-W|)\mathrm{d}V\mathrm{d}W.$$

Note that Assumption (H1) is coherent with the fact that, in the last two lines of equations (2.7), the same cross-section Σ_{pg} appears (and thus with the conservation of mass).

Assumption (H2). There exists a function $Q\equiv Q(r)\in C(\mathbf{R}_+^*)$ satisfying

$$Q\ge 0, \quad \text{and } Q(r)+|Q'(r)|\le C(1+r) \text{ for some } C>0,$$

such that the measure-valued functions Π_{pg} and Π_{gp} satisfy

$$\epsilon\int_{\mathbf{R}^3}\mathrm{d}v\,(v-V)\Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)=-\eta\int_{\mathbf{R}^3}\mathrm{d}w\,(w-W)\Pi_{gp}(w,\mathrm{d}V\mathrm{d}W)$$
$$=-\frac{\eta}{1+\eta}(\epsilon V-W)Q(|\epsilon V-W|)\mathrm{d}V\mathrm{d}W.$$

This assumption implies the conservation of momentum between molecules and particles.

Assumption (H3). There exists a constant $C>0$ such that the measure-valued function Π_{pg} satisfies

$$\int_{\mathbf{R}^3} \mathrm{d}v \left| \epsilon v - \frac{\epsilon V + \eta W}{1+\eta} \right|^2 \Pi_{pg}(v, \mathrm{d}V\mathrm{d}W) \le C\eta^2 (1+|\epsilon V - W|^2) q(|\epsilon V - W|) \mathrm{d}V\mathrm{d}W,$$

where q is the function appearing in Assumption (H1).

Assumption (H4). The limiting measure $\Pi_{gp}^{0,0}$ satisfies the following invariance[2] property:

$$\mathcal{T}_R \# \Pi_{gp}^{0,0} = \Pi_{gp}^{0,0} \qquad \text{for each } R \in O_3(\mathbf{R}),$$

where

$$\mathcal{T}_R : (w,V,W) \mapsto (Rw, V, RW). \tag{3.1}$$

Besides, for each $\Phi := \Phi(w,W)$ such that $|\Phi(w,W)| \le C(1+|w|^2+|W|^2)M(W)$,

$$\int_{\mathbf{R}^3} (1+|V|^2)^{-p} \left| \iint_{\mathbf{R}^3\times\mathbf{R}^3} \Phi(w,W)(\Pi_{gp}^{\epsilon,\eta}(w,\mathrm{d}V\mathrm{d}W) - \Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W))\,\mathrm{d}w \right| \to 0$$

for some $p>3$, as $\epsilon,\eta \to 0$. Moreover,

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathrm{d}w\,(1+|w|^2+|W|^2)M(W)\Pi_{gp}^{0,0}(w,dVdW) \in L^1((1+V^2)^{-3}dV).$$

Assumption (H5). For all $h \in L^2(M(w)\mathrm{d}w)$,

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} (1+|W|^2)(1+|V|^2)^{-p}(1+|w|^2)M(W)|h(W)|\Pi_{gp}^{\epsilon,\eta}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w \le C||h||_{L^2(M(w)\mathrm{d}w)},$$

where C does not depend on η and ϵ (for η and ϵ close to 0).

Finally, for the sake of being complete, we repeat the assumptions made on the molecular collision model — specifically

Assumption (H6). The collision kernel $c \equiv c(z,\omega)$ is of the form (2.20) and satisfies

$$0 < c(z,\omega) \le c_*(1+|z|)^\gamma, \qquad \text{for a.e. } (z,\omega) \in \mathbf{R}^3 \times \mathbf{S}^2,$$
$$\int_{\mathbf{S}^2} c(z,\omega)\,\mathrm{d}\omega \ge \frac{1}{c_*}\frac{|z|}{1+|z|}, \qquad \text{for a.e. } z \in \mathbf{R}^3,$$

for some $c_* > 1$ and some $\gamma \in [0,1]$.

Our assumption on the radial function α defined in equation (2.28) can be recast as follows.

Assumption (H7). The radial function $\alpha \equiv \alpha(|w|)$ such that

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} (\alpha(|w|)w_1w_2 + \alpha(|w_*|)w_{*1}w_{*2} - \alpha(|w'|)w'_1w'_2 - \alpha(|w'_*|)w'_{*1}w'_{*2})\,c(w-w_*,\omega)\,M(w_*)\mathrm{d}w_*\mathrm{d}\omega = w_1w_2$$

[2]The notation $\mathcal{T}\#m$ designates the push-forward of the measure m by the transformation $\mathcal{T}$.

satisfies the condition $\alpha\in L^\infty(\mathbf{R}_+)$.

The main result in [15] implies that the condition above defines a unique radial function (up to equality a.e.), and that the element α of $L^\infty(\mathbf{R}_+)$ so defined satisfies

$$\mathcal{L}(\alpha A_{kl})=A_{kl} \quad \text{for all } k,l=1,2,3.$$

We next prove that the elastic and inelastic models previously introduced (in Sections 2.4.2 and 2.4.3 resp.) satisfy the Assumptions (H1)-(H5).

3.1. Verification of (H1)-(H5) for the elastic collision model.

PROPOSITION 3.1. *For each collision kernel b of the form (2.30) satisfying (2.35), let the quantities Σ_{pg}, Π_{pg} and Π_{gp} be defined by equations (2.29), (2.31) and (2.32). Then, Assumptions (H1)-(H5) are satisfied, with*

$$q(|\epsilon v-w|)=4\pi\int_0^1 |\epsilon v-w|\,\sigma_{pg}(|\epsilon v-w|,\mu)\,\mathrm{d}\mu, \tag{3.2}$$

$$Q(|\epsilon v-w|)=8\pi\int_0^1 |\epsilon v-w|\,\sigma_{pg}(|\epsilon v-w|,\mu)\mu^2\,\mathrm{d}\mu, \tag{3.3}$$

and

$$C=1. \tag{3.4}$$

Proof. For each continuous and compactly supported test functions $\phi\equiv\phi(v,V,W)$ and $\psi\equiv\psi(w,V,W)$, one has

$$\begin{aligned}&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}\phi(v,V,W)\Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}v\\ &=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\phi(v,v'',w'')b(\epsilon v-w,\omega)\,\mathrm{d}\omega\mathrm{d}w\mathrm{d}v\\ &=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\phi(v'',v,w)b(\epsilon v-w,\omega)\,\mathrm{d}\omega\mathrm{d}w\mathrm{d}v,\end{aligned} \tag{3.5}$$

where the last equality follows from the fact that the map $(v,w)\mapsto(v'',w'')$ is a linear involution for each $\omega\in\mathbf{S}^2$. By the same token

$$\begin{aligned}&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}\psi(w,V,W)\Pi_{gp}(w,\mathrm{d}V\mathrm{d}W)\mathrm{d}w\\ &=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\psi(w'',v,w)b(\epsilon v-w,\omega)\mathrm{d}\omega\mathrm{d}w\mathrm{d}v.\end{aligned} \tag{3.6}$$

Observing that

$$\int_{\mathbf{S}^2} b(\epsilon v-w,\omega)\,\mathrm{d}\omega=4\pi|\epsilon v-w|\int_0^1\sigma_{pg}(|\epsilon v-w|,\mu)\,\mathrm{d}\mu,$$

one arrives at Assumption (H1) with q defined by equation (3.2).

Then we see that

$$\begin{aligned}\epsilon(v-v'')\cdot\omega&=-\eta(w-w'')\cdot\omega\\ &=-\frac{2\eta}{1+\eta}(w-\epsilon v)\cdot\omega=\frac{2\eta}{1+\eta}(w''-\epsilon v'')\cdot\omega,\end{aligned}$$

and that

$$\int_{\mathbf{S}^2}((\epsilon v''-w'')\cdot\omega)\,b(\epsilon v-w,\omega)\omega\,\mathrm{d}\omega$$
$$=4\pi|\epsilon v-w|(\epsilon v''-w'')\int_0^1\sigma_{pg}(|\epsilon v-w|,\mu)\mu^2\,\mathrm{d}\mu,$$

and conclude that Assumption (H2) holds with Q defined by equation (3.3).

Observing that

$$\left|\epsilon v-\frac{\epsilon v''+\eta w''}{1+\eta}\right|^2=\left(\frac{\eta}{1+\eta}\right)^2|\epsilon v''-w''|^2$$
$$=\left(\frac{\eta}{1+\eta}\right)^2|\epsilon v-w|^2\le\eta^2|\epsilon v-w|^2,$$

shows that Assumption (H3) holds with $C=1$.

Next, one has

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}\phi(w,V,W)\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}\phi(w-2w\cdot\omega\omega,v,w)b(w,\omega)\,\mathrm{d}v\mathrm{d}w\mathrm{d}\omega,$$

which obviously implies the relation

$$\Pi_{gp}^{0,0}=T_R\#\Pi_{gp}^{0,0}.$$

Besides, for each $p>3$ and each continuous $\Phi\equiv\Phi(w,W)$ such that

$$|\Phi(w,W)|\le C(1+|w|^2+|W|^2)M(W)$$

one has

$$\int_{\mathbf{R}^3}(1+|V|^2)^{-p}\left|\iint_{\mathbf{R}^3\times\mathbf{R}^3}\Phi(w,W)(\Pi_{gp}^{\epsilon,\eta}(w,\mathrm{d}V\mathrm{d}W)-\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W))\,\mathrm{d}w\right|$$
$$=\int_{\mathbf{R}^3}(1+|v|^2)^{-p}\left|\iint_{\mathbf{R}^3\times\mathbf{S}^2}(\Phi(w'',w)b(\epsilon v-w,\omega)-\Phi(\tilde{w},w)b(w,\omega))\,\mathrm{d}v\mathrm{d}w\mathrm{d}\omega\right|,$$

where

$$\tilde{w}=w-2(w\cdot\omega)\omega.$$

By continuity of b and Φ, we see that

$$\Phi(w'',w)b(\epsilon v-w,\omega)\to\Phi(\tilde{w},w)b(w,\omega)$$

as $\epsilon,\eta\to 0$. Then, using the estimate

$$|\Phi(w'',w)|b(\epsilon v-w,\omega)\le C(1+|w|^2)^{3/2}(1+|v|^2)^{3/2}M(w),$$

we conclude that

$$\int_{\mathbf{R}^3}(1+|V|^2)^{-p}\left|\iint\Phi(w,W)(\Pi_{gp}^{\epsilon,\eta}(w,dVdW)dw-\Pi_{gp}^{0,0}(w,dVdW)dw)\right|\to 0$$

as $\epsilon,\eta \to 0$ by dominated convergence. Finally we observe that

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(1+|w|^2+|W|^2)(1+V^2)^{-3}M(W)\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}(1+|w-2(w\cdot\omega)\omega|^2+|w|^2)(1+|v|^2)^{-3}M(w)b(-w,\omega)\,\mathrm{d}\omega\mathrm{d}w\mathrm{d}v$$
$$\le C\iint_{\mathbf{R}^3\times\mathbf{R}^3}(1+|w|^2)(1+v^2)^{-3}M(w)|w|\,\mathrm{d}w\mathrm{d}v<\infty,$$

so that Assumption (H4) is satisfied.

Finally, for each $h\in L^2(M(w)\mathrm{d}w)$ and each $\epsilon,\eta>0$ small enough,

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(1+|W|^2)(1+|V|^2)^{-p}(1+|w|^2)M(W)|h(W)|\Pi_{gp}^{\epsilon,\eta}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}(1+|w|^2)(1+|v|^2)^{-p}(1+|w''|^2)M(w)|h(w)|b(\epsilon v-w,\omega)\,\mathrm{d}\omega\mathrm{d}w\mathrm{d}v$$
$$\le C\iint_{\mathbf{R}^3\times\mathbf{R}^3}(1+|w|^2)(1+|v|^2)^{-p}(1+|v|^2+|w|^2)(1+|v|+|w|)M(w)|h(w)|\,\mathrm{d}w\mathrm{d}v$$
$$\le C\int_{\mathbf{R}^3}(1+|w|^2)^{5/2}M(w)|h(w)|\mathrm{d}w\le C||h||_{L^2(M(w)\mathrm{d}w)}$$

by the Cauchy-Schwarz inequality. Hence Assumption (H5) is also verified. ∎

3.2. Verification of (H1)-(H5) for the inelastic collision model.

PROPOSITION 3.2. *The scattering kernels Π_{pg} and Π_{gp} defined by (2.37)-(2.39) satisfy Assumptions (H1)-(H5), with*

$$q(|\epsilon v-w|)=|\epsilon v-w|,\quad Q(|\epsilon v-w|)=\frac{\sqrt{2\pi}}{3\beta}+|\epsilon v-w|,$$

and

$$C=\frac{16}{\beta^2}.$$

As in the previous section, we explicitly mention the ϵ,η-dependence of the scattering kernels whenever needed, in which case we use the notation $K_{pg}^{\epsilon,\eta},K_{gp}^{\epsilon,\eta}$ to designate K_{pg} and K_{gp} respectively.

Proof. Setting successively $a=\frac{\epsilon V+\eta W}{1+\eta}-\epsilon v$ and $b=\beta\left(\frac{1+\eta}{\eta}\right)a$, we see that

$$\int_{\mathbf{R}^3}K_{pg}(v,V,W)\,\mathrm{d}v$$
$$=\iint_{\mathbf{R}^3\times\mathbf{S}^2}\frac{\beta^4}{2\pi^2}\left(\tfrac{1+\eta}{\eta}\right)^4\exp\left(-\tfrac{1}{2}\beta^2\left(\tfrac{1+\eta}{\eta}\right)^2|a|^2\right)(a\cdot n)_+\left((\epsilon V-W)\cdot n\right)_+\,\mathrm{d}a\mathrm{d}n$$
$$=\frac{1}{2\pi^2}\iint_{\mathbf{R}^3\times\mathbf{S}^2}\exp(-\tfrac{1}{2}|b|^2)(b\cdot n)_+\left((\epsilon V-W)\cdot n\right)_+\,\mathrm{d}b\mathrm{d}n$$
$$=\tfrac{1}{\pi}\int_{\mathbf{S}^2}\left((\epsilon V-W)\cdot n\right)_+\,\mathrm{d}n=|\epsilon V-W|.$$

Likewise, setting $b' = \beta(1+\eta)\left(w - \frac{\epsilon V + \eta W}{1+\eta}\right)$, we see that

$$\begin{aligned}&\int_{\mathbf{R}^3} K_{gp}(w,V,W)\,\mathrm{d}w\\&=\frac{1}{2\pi^2}\iint_{\mathbf{R}^3\times\mathbf{S}^2}\exp(-\tfrac{1}{2}|b'|^2)(b'\cdot n)_+((\epsilon V - W)\cdot n)_+\,\mathrm{d}b'\mathrm{d}n = |\epsilon V - W|,\end{aligned}$$

so that Assumption (H1) is verified.

With $b = \beta\left(\frac{1+\eta}{\eta}\right)\left(\frac{\epsilon V+\eta W}{1+\eta} - \epsilon v\right)$ as above

$$\begin{aligned}&\epsilon\int_{\mathbf{R}^3}(v-V)K_{pg}(v,V,W)\,\mathrm{d}v\\&=\frac{1}{\pi}\frac{\eta}{1+\eta}\int_{\mathbf{S}^2}\left((W-\epsilon V) - \frac{1}{2\pi\beta}\int_{\mathbf{R}^3} be^{-|b|^2/2}(b\cdot n)_+\,\mathrm{d}b\right)((\epsilon V - W)\cdot n)_+\,\mathrm{d}n\\&=-\frac{1}{\pi}\frac{\eta}{1+\eta}\left(\int_{\mathbf{S}^2}(\epsilon V - W)((\epsilon V - W)\cdot n)_+\,\mathrm{d}n + \frac{\sqrt{2\pi}}{2\beta}\int n((\epsilon V - W)\cdot n)_+\,\mathrm{d}n\right)\\&=-\frac{\eta}{1+\eta}(\epsilon V - W)\left(|\epsilon V - W| + \frac{(2\pi)^{1/2}}{3\beta}\right).\end{aligned}$$

Likewise, setting $b' = \beta(1+\eta)\left(w - \frac{\epsilon V+\eta W}{1+\eta}\right)$ as above, we see that

$$\begin{aligned}&-\eta\int_{\mathbf{R}^3}(w-W)K_{gp}(w,V,W)\,\mathrm{d}w\\&=-\frac{1}{\pi}\frac{\eta}{1+\eta}\int\left(\frac{(2\pi)^{1/2}}{2\beta}n + (\epsilon V - W)\right)((\epsilon V - W)\cdot n)_+\,\mathrm{d}n\\&=-\frac{\eta}{1+\eta}(\epsilon V - W)\left(|\epsilon V - W| + \frac{(2\pi)^{1/2}}{3\beta}\right)\end{aligned}$$

so that Assumption (H2) is also satisfied.

Still with $b = \beta\left(\frac{1+\eta}{\eta}\right)\left(\frac{\epsilon V+\eta W}{1+\eta} - \epsilon v\right)$, one has

$$\begin{aligned}&\int_{\mathbf{R}^3}\left|\frac{\epsilon V+\eta W}{1+\eta} - \epsilon v\right|^2 K_{pg}(v,V,W)\,\mathrm{d}v\\&=\left(\frac{\eta}{1+\eta}\right)^2\frac{1}{\pi\beta^2}\int_{\mathbf{S}^2}\left(\frac{1}{2\pi}\int_{\mathbf{R}^3}|b|^2\exp(-\tfrac{1}{2}|b|^2)(b\cdot n)_+\,\mathrm{d}b\right)\mathrm{d}n\,|\epsilon V - W|\\&\le\frac{16}{\beta^2}\left(\frac{\eta}{1+\eta}\right)^2|\epsilon V - W|,\end{aligned}$$

so that Assumption (H3) is satisfied.

Observe that

$$\Pi_{gp}^{0,0}(w,\mathrm{d}V,\mathrm{d}W) = K^{0,0}(w,W)\,\mathrm{d}V\mathrm{d}W$$

with

$$\begin{aligned}K^{0,0}(w,W) &= \tfrac{1}{2\pi^2}\beta^4\exp(-\tfrac{1}{2}\beta^2|w|^2)\int_{\mathbf{S}^2}(-W\cdot n)_+(w\cdot n)_+\,\mathrm{d}n\\&= K^{0,0}(Rw,RW)\end{aligned}$$

for each $R\in O_3(\mathbf{R})$. Hence $\mathcal{T}_R\#\Pi^{0,0}_{gp}=\Pi^{0,0}_{gp}$, which is the first property in Assumption (H4).

On the other hand

$$\begin{aligned}&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(1+|V|^2)^{-3}(1+|w|^2+|W|^2)M(W)\Pi^{0,0}_{gp}(w,\mathrm{d}V,\mathrm{d}W)\,\mathrm{d}w\\&=\int_{\mathbf{R}^3}\frac{\mathrm{d}V}{(1+|V|^2)^3}\iint_{\mathbf{R}^3\times\mathbf{R}^3}(1+|w|^2+|W|^2)M(W)K^{0,0}(w,W)\,\mathrm{d}w\mathrm{d}W<\infty\end{aligned}$$

since

$$0\le K^{0,0}(w,W)\le\tfrac{2}{\pi}\beta^4|w||W|\exp(-\tfrac{1}{2}\beta^2|w|^2).$$

Hence the third property in Assumption (H4) is verified.

Let $\Phi\equiv\Phi(w,W)$ be such that $|\Phi(w,W)|\le C(1+|w|^2+|W|^2)M(W)$. Then,

$$\begin{aligned}\frac{2\pi^2}{\beta^4}|\Phi(w,W)|K^{\epsilon,\eta}_{gp}(w,V,W)&\le C(|V|+|W|+|w|)(|V|+|W|)\\&\times(1+|w|^2+|W|^2)M(W)\exp\left(-\tfrac{1}{2}\beta^2\left|w-\frac{\epsilon V+\eta W}{1+\eta}\right|^2\right).\end{aligned}$$

Since

$$\begin{aligned}&W|^2+\beta^2\left|w-\frac{\epsilon}{1+\eta}V-\frac{\eta}{1+\eta}W\right|^2\\&\ge\min(1,\beta^2/2)\left(|W|^2+\left|w-\frac{\eta}{1+\eta}W\right|^2-|V|^2\right)\\&\ge\min(1,\beta^2/2)(|W|^2+|w|^2-2|V|^2),\end{aligned}$$

we see that

$$\begin{aligned}\frac{2\pi^2}{\beta^4}|\Phi(w,W)|K^{\epsilon,\eta}_{gp}(w,V,W)&\le C(|V|+|W|+|w|)(|V|+|W|)\\&\times(1+|w|^2+|W|^2)\exp(\mu|V|^2)\exp(-\tfrac{1}{2}\mu(|w|^2+|W|^2)),\end{aligned}$$

with

$$\mu:=\min(1,\beta^2/2).$$

Therefore

$$\begin{aligned}&\iint_{\mathbf{R}^3\times\mathbf{R}^3}\Phi(w,W)K^{\epsilon,\eta}_{gp}(w,V,W)\,\mathrm{d}w\mathrm{d}W\\&\to\iint_{\mathbf{R}^3\times\mathbf{R}^3}\Phi(w,W)K^{0,0}_{gp}(w,V,W)\,\mathrm{d}w\mathrm{d}W\end{aligned}$$

for each $V\in\mathbf{R}^3$ by dominated convergence. Setting $y=w-\frac{\epsilon V+\eta W}{1+\eta}$, one has

$$\frac{2\pi^2}{\beta^4}(1+|V|^2)^{-p}\left|\iint_{\mathbf{R}^3\times\mathbf{R}^3}\Phi(w,W)K^{\epsilon,\eta}_{gp}(w,V,W)\,\mathrm{d}w\mathrm{d}W\right|$$

$$
\begin{aligned}
&=(1+|V|^2)^{-p}\left|\iint_{\mathbf{R}^3\times\mathbf{R}^3}\Phi\left(y+\frac{\epsilon V+\eta W}{1+\eta},W\right)(1+\eta)^4\exp\left(-\tfrac{1}{2}\beta^2(1+\eta)^2|y|^2\right)\right.\\
&\qquad\left.\times\int_{\mathbf{R}^3}((\epsilon V-W)\cdot n)_+(-y\cdot n)_+\,\mathrm{d}n\mathrm{d}y\mathrm{d}W\right|\\
&\le C(1+|V|^2)^{-p}\iint_{\mathbf{R}^3\times\mathbf{R}^3}(1+|y|^2+|W|^2+|V|^2)M(W)\\
&\qquad\times\exp(-\tfrac{1}{2}\beta^2|y|^2)|y|(|V|+|W|)\,\mathrm{d}y\mathrm{d}W
\end{aligned}
$$

which is integrable in $V\in\mathbf{R}^3$ for $p>3$. Therefore,

$$
\int_{\mathbf{R}^3}(1+|V|^2)^{-p}\left|\iint\Phi(w,W)(\Pi_{gp}^{\epsilon,\eta}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w-\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w)\right|\to 0
$$

as $\epsilon,\eta\to 0$ for all $p>3$ by dominated convergence. This completes the verification of Assumption (H4).

Using again the substitution $y=w-\frac{\epsilon V+\eta W}{1+\eta}$, one has

$$
\begin{aligned}
&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(1+|W|^2)M(W)|h(W)|(1+|V|^2)^{-p}(1+|w|^2)\Pi_{gp}^{\epsilon,\eta}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\\
&=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(1+|W|^2)M(W)|h(W)|(1+|V|^2)^{-p}(1+|w|^2)\\
&\quad\times\frac{\beta^4}{2\pi^2}(1+\eta)^4\exp\left(-\tfrac{1}{2}\beta^2(1+\eta)^2\left|w-\frac{\epsilon V+\eta W}{1+\eta}\right|^2\right)\\
&\quad\times\int_{\mathbf{S}^2}(\epsilon V-W)\cdot n)_+\left(\left(\frac{\epsilon V+\eta W}{1+\eta}-w\right)\cdot n\right)_+\mathrm{d}n\mathrm{d}V\mathrm{d}W\mathrm{d}w\\
&\le C\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(1+|W|^2)(1+|V|^2)^{-p}(1+|V|^2+|W|^2+|y|^2)M(W)|h(W)|\\
&\quad\times|V+W||y|\exp\left(-\tfrac{1}{2}\beta^2|y|^2\right)\mathrm{d}V\mathrm{d}W\mathrm{d}y\le C\|h\|_{L^2(M(w)\mathrm{d}w)},
\end{aligned}
$$

which is precisely Assumption (H5). □

4. Passage to the limit

In this section, we use the material presented in Sections 2-3 to state and prove the main result in this paper, i.e. the derivation of the Vlasov–Navier–Stokes model for thin sprays from the system of Boltzmann equations for a binary mixture of gas molecules and dust particles or droplets.

4.1. Statement of the main result. We henceforth consider a sequence of solutions $f_n\equiv f_n(t,x,w)$, and $F_n\equiv F_n(t,x,v)$ to the system of kinetic-fluid equations (2.17), with sequences $\epsilon_n,\eta_n\to 0$ in the place of the parameters $\epsilon,\eta>0$:

$$
\begin{aligned}
&\partial_t F_n+v\cdot\nabla_x F_n=\frac{1}{\eta_n}\mathcal{D}(F_n,f_n),\\
&\partial_t f_n+\frac{1}{\epsilon_n}w\cdot\nabla_x f_n=\mathcal{R}(f_n,F_n)+\frac{1}{\epsilon_n^2}\mathcal{C}(f_n),
\end{aligned}
\tag{4.1}
$$

where $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{R}$ are defined by equations (2.18)-(2.20), (2.15) and (2.16).

THEOREM 4.1. *Assume that the scattering kernels $\Pi_{pg}^{\epsilon_n,\eta_n}$ and $\Pi_{gp}^{\epsilon_n,\eta_n}$ in equations (2.15)-(2.16) satisfy Assumptions (H1)-(H5), while the molecular collision kernel c in equations (2.18)-(2.20) satisfies Assumptions (H6)-(H7).*

Assume that the scaling parameters ϵ_n *and* η_n *satisfy the Vlasov–Navier–Stokes scaling assumption:*

$$\epsilon_n \to 0, \quad \text{and} \quad \eta_n/\epsilon_n^2 \to 0. \tag{4.2}$$

Let $g_n \equiv g_n(t,x,w) \geq 0$ *and* $F_n \equiv F_n(t,x,v) \geq 0$ *be sequences of smooth (at least* C^1*) functions, and let*

$$f_n(t,x,w) := M(w)(1+\epsilon_n g_n(t,x,w)), \tag{4.3}$$

where M *is the Maxwellian distribution (2.23). Assume that*

$$F_n \rightharpoonup F \text{ in } L^\infty_{loc} \text{ weak-*}, \quad \text{and that } g_n \rightharpoonup g \text{ in } L^2_{loc}(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3) \text{ weak}$$

for some $F \in L^\infty_{loc}(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ *and* $g \in L^2_{loc}(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$*.*

Assume that

(a) the pair (F_n, f_n) *is a solution to system (4.1), with* $\mathcal{C}, \mathcal{D}, \mathcal{R}$ *defined by equations (2.18)-(2.20), (2.15) and (2.16)*

(b) there exists $p > 3$ *such that*

$$\sup_{n \geq 1} \sup_{(t,x,v) \in [0,R] \times [-R,R]^3 \times \mathbf{R}^3} (1+|v|^2)^p F_n(t,x,v) \leq C_R < \infty$$

for each $R > 0$*,*

(c) the sequence

$$\int_{\mathbf{R}^3} g_n(t,x,w)^2 M(w) \, \mathrm{d}w$$

is bounded in $L^1_{loc}(\mathbf{R}_+^* \times \mathbf{R}^3)$*,*

(d) the sequence of velocity averages of g_n

$$\int_{\mathbf{R}^3} g_n \phi(w) M(w) \, \mathrm{d}w \to \int_{\mathbf{R}^3} g \phi(w) M(w) \, \mathrm{d}w \tag{4.4}$$

strongly in $L^2_{loc}(\mathbf{R}_+^* \times \mathbf{R}^3)$ *for each* $\phi \in C_c(\mathbf{R}^3)$*.*

Then there exist L^∞ *functions* $\rho \equiv \rho(t,x) \in \mathbf{R}$ *and* $\theta \equiv \theta(t,x) \in \mathbf{R}$*, and a* L^∞ *vector field* $u \equiv u(t,x) \in \mathbf{R}^3$ *s.t.*

$$\begin{aligned} g(t,x,w) = \rho(t,x) + u(t,x) \cdot w + \theta(t,x) \tfrac{1}{2}(|w|^2 - 3) \\ \text{for a.e. } (t,x,w) \in \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3, \end{aligned} \tag{4.5}$$

and the pair (F, u) *satisfies the Vlasov–Navier–Stokes system*

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = \kappa \operatorname{div}_v((v-u)F), \\ \operatorname{div}_x u = 0, \\ \partial_t u + \operatorname{div}_x(u \otimes u) = \nu \Delta_x u - \nabla_x p + \kappa \displaystyle\int (v-u) F \, \mathrm{d}v, \end{cases} \tag{4.6}$$

in the sense of distributions, with

$$\nu := \tfrac{1}{10} \int \tilde{A} : \mathcal{L} \tilde{A} M(w) \, \mathrm{d}w > 0, \quad \kappa := \tfrac{1}{3} \int Q(|w|) |w|^2 M(w) \, \mathrm{d}w > 0, \tag{4.7}$$

where Q is defined in Assumption (H2), while $\tilde{A}$,$\mathcal{L}$ are defined by equations (2.27), (2.24).

Some comments on the assumptions before giving the proof of Theorem 4.1.

Remark on the Vlasov–Navier–Stokes scaling assumption. Since the propellant is in Maxwellian equilibrium at leading order, its thermal speed is given by the formula $V_g := \sqrt{3k_BT_g/m_g}$ (where k_B is the Boltzmann constant) and T_g is the temperature in the propellant (here $T_g = 1$ to leading order). The thermal speed of the dispersed phase is defined as the square-root of the variance of the velocity under the distribution F, i.e.

$$V_p^2 := \left(\iint |v|^2 F\,\mathrm{d}x\mathrm{d}v - \left|\iint vF\,\mathrm{d}x\mathrm{d}v\right|^2\right) \Big/ \iint F\,\mathrm{d}x\mathrm{d}v$$

and one can define a notion of temperature in the dispersed phase by the formula $T_p := m_pV_p^2/3k_B$. Then

$$\epsilon^2 = (V_p/V_g)^2 = \eta T_p/T_g.$$

Thus the Vlasov–Navier–Stokes scaling condition $\eta \ll \epsilon^2$ implies that $T_g \ll T_p$. Since the Vlasov–Navier–Stokes system describes a situation where the dispersed phase is not in equilibrium with the propellant, one should not expect T_p and T_g to be of the same order of magnitude. In addition, the effect of collisions between the heavy particles is negligible, so that there is no thermalization effect in the dispersed phase. This is obviously consistent with the fact that the dynamics of the dispersed phase is described by a Vlasov equation. For that reason, the temperature T_p defined above contains little physical information.

Remark on the assumptions on the class of solutions considered. We have not tried to optimize the assumptions bearing on the solutions to the coupled Boltzmann system.

Assuming a uniform control of g_n^2M is fairly natural, since this quantity appears naturally in the entropy estimate implied by Boltzmann's H Theorem — specifically, the quantity

$$\iint g^2(t,x,v)M(v)\,\mathrm{d}x\mathrm{d}v$$

is the Hessian of the relative entropy

$$\iint (f(x,v)\ln(f(x,v)/M(v)) - f(x,v) + M(v))\,\mathrm{d}x\mathrm{d}v$$

at $f = M$ computed on an increment of f of the form Mg. See section 3 in [4], and Proposition 2.3 in [22] for a detailed discussion of this point.

As for F_n, we have chosen a L^∞ setting since, in the limit, F will satisfy a Vlasov equation which propagates L^∞ estimates over finite time intervals.

Finally, only the averages with respect to w of g_n are required to converge strongly in L^2_{loc} and a.e., as in [3]. No such assumption is required on the averages of F_n since no quadratic term in F appears in the limit.

Although these estimates may seem relatively realistic, one should bear in mind that, even in the case of the Boltzmann equation for a single species of particles, the only global existence theory without restriction on the size of the initial data involves the notion of renormalized solutions due to R. DiPerna and P.-L. Lions [18]. Most likely, the DiPerna-Lions theory should be adapted to the case of systems of Boltzmann equations of the form (4.1). For such solutions, one expects a control of g_n that is slightly weaker than Assumption (c) in Theorem 4.1. Likewise, the L^∞ bound on F_n (implied by the assumption that $F_n \rightharpoonup F$ in L^∞_{loc} weak-* in Theorem 4.1) is a rather optimistic assumption, if compared with the much weaker estimate provided by the DiPerna-Lions theory of renormalized solutions.

In short, the assumptions in Theorem 4.1 are probably more restrictive than those which could be deduced from a variant of the DiPerna-Lions theory adapted to the system (4.1). For this reason the mathematical value of a formal limit theorem such as Theorem 4.1 can be questioned.

However, it can be argued that the moment approach in the formal limit theorem [3] is the basis of the rigorous proofs of the hydrodynamic limit of the Boltzmann equation leading to the incompressible Navier–Stokes equation in [21, 22]. Perhaps the hydrodynamic limit for the propellant is the most difficult part in Theorem 4.1, and a rigorous derivation of the formal limit discussed here can be obtained along the lines of [21, 22]. That the discussion specific to the interaction of the propellant with the dispersed phase, i.e. steps 2, 3 and 7 in the proof of Theorem 4.1, can be isolated from the Navier–Stokes limit for the propellant suggests that this derivation could be made rigorous in the not too distant future.

4.2. Proof of Theorem 4.1. The proof of Theorem 4.1 is based on the formal derivation of the incompressible fluid dynamic limit of the Boltzmann equation formulated in [3]. However, the interaction with the dust particles/droplets involves very serious complications.

This proof is split in several steps, referred to as Propositions 4.1 to 4.7, and a final part in which all the convergences of the different terms appearing in equation (4.1) are established.

4.2.1. Step 1: Asymptotic form of the molecular distribution function.

PROPOSITION 4.1. *Under the assumptions of Theorem 4.1, there exist L^∞ functions $\rho \equiv \rho(t,x) \in \mathbf{R}$ and $\theta \equiv \theta(t,x) \in \mathbf{R}$, and a L^∞ vector field $u \equiv u(t,x) \in \mathbf{R}^3$ s.t. equation (4.5) holds.*

Proof. Since $\mathcal{C}$ is a quadratic operator, its Taylor expansion terminates at order 2, i.e.

$$\begin{aligned}\mathcal{C}(M(1+\epsilon_n g_n)) &= \mathcal{C}(M) + \epsilon_n D\mathcal{C}(M)\cdot(Mg_n) + \epsilon_n^2 \mathcal{C}(Mg_n)\\ &= -\epsilon_n M\mathcal{L}g_n + \epsilon_n^2 M\mathcal{Q}(g_n),\end{aligned}$$

where $\mathcal{L}\phi$ is defined by equation (2.24) and

$$\mathcal{Q}(\phi) := M^{-1}\mathcal{C}(M\phi). \tag{4.8}$$

Then the kinetic equation for the propellant (second line of equation (4.1)) can be recast in terms of the fluctuation of the distribution function g_n:

$$\partial_t g_n + \frac{1}{\epsilon_n} w\cdot\nabla_x g_n + \frac{1}{\epsilon_n^2}\mathcal{L}g_n = \frac{1}{\epsilon_n}M^{-1}\mathcal{R}(M(1+\epsilon g_n), F_n) + \frac{1}{\epsilon_n}\mathcal{Q}(g_n). \tag{4.9}$$

Multiplying each side of this equation by ϵ_n^2 leads to the equality

$$\mathcal{L}g_n = \epsilon_n(M^{-1}\mathcal{R}(M(1+\epsilon g_n),F_n) + \mathcal{Q}(g_n)) - \epsilon_n^2\partial_t g_n - \epsilon_n w\cdot\nabla_x g_n. \tag{4.10}$$

The two last terms of this identity clearly converge to 0 in the sense of distributions since $g_n \rightharpoonup g$ weakly in L^2_{loc}.

Next, for each test function $\phi \in C_c(\mathbf{R}^3)$,

$$\int_{\mathbf{R}^3} \mathcal{Q}(g_n)(w)\phi(w)\,\mathrm{d}w = \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} \big(M^{-1}(w')\phi(w') - M^{-1}(w)\phi(w)\big) \times M(w_*)g_n(w_*)M(w)g_n(w)c(w-w_*,\omega)\,\mathrm{d}\omega\mathrm{d}w_*\mathrm{d}w,$$

where w', w'_* are defined by equation (2.19). By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left|\int_{\mathbf{R}^3} \mathcal{Q}(g_n)\phi(w)\,\mathrm{d}w\right| \\ &\le C\iint_{\mathbf{R}^3\times\mathbf{R}^3} M(w_*)g_n(w_*)M(w)g_n(w)(1+|w|+|w_*|)\,\mathrm{d}w_*\mathrm{d}w \\ &\le C\int_{\mathbf{R}^3} M(w)g_n(w)^2\,\mathrm{d}w \int_{\mathbf{R}^3} M(w)(1+|w|)^2\,\mathrm{d}w,\end{aligned}$$

so that

$$\int_{\mathbf{R}^3} \mathcal{Q}(g_n)\phi(w)\,\mathrm{d}w \text{ is bounded in } L^1_{loc}(\mathbf{R}^*_+\times\mathbf{R}^3)$$

for each $\phi\in C_c(\mathbf{R}^3)$, and $\epsilon_n\mathcal{Q}(g_n)\to 0$ in the sense of distributions.

Likewise, for each $\phi\in C_c(\mathbf{R}^3)$, we deduce from Assumption (H1) that

$$\begin{aligned}&\int_{\mathbf{R}^3} \mathcal{R}(f_n,F_n)M^{-1}(w)\phi(w)\,\mathrm{d}w \\ &= \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} (M^{-1}(w)\phi(w) - M^{-1}(W)\phi(W))f_n(W)F_n(V)\Pi_{gp}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w,\end{aligned}$$

so that

$$\begin{aligned}&\left|\int_{\mathbf{R}^3} \mathcal{R}(f_n,F_n)M^{-1}(w)\phi(w)\,\mathrm{d}w\right| \\ &\le C\iint_{\mathbf{R}^3\times\mathbf{R}^3} F_n(V)f_n(W)q(|\epsilon_n V - W|)\,\mathrm{d}V\mathrm{d}W \\ &\le C\int_{\mathbf{R}^3} M(W)(1+\epsilon_n g_n)(W)(1+|W|)\,\mathrm{d}W,\end{aligned}$$

which is bounded in $L^2_{loc}(\mathbf{R}^*_+\times\mathbf{R}^3)$, according to (H1) and Assumption (b) in Theorem 4.1. Therefore $\epsilon_n\mathcal{R}(f_n,F_n)M^{-1}(w)\to 0$ in the sense of distributions.

Finally, for each test function $\phi\in C_c(\mathbf{R}^3)$, one has[3]:

$$(\mathcal{L}g_n|\phi)_{L^2(M\,\mathrm{d}v)} = (g_n|\mathcal{L}\phi)_{L^2(M\,\mathrm{d}v)} \to (g|\mathcal{L}\phi)_{L^2(M\,\mathrm{d}v)} = (\mathcal{L}g|\phi)_{L^2(M\,\mathrm{d}v)}$$

[3]We use the notation

$$(\phi|\psi)_{L^2(M,\mathrm{d}v)} := \int_{\mathbf{R}^3} \overline{\phi(v)}\psi(v)M(v)\,\mathrm{d}v, \quad \text{for each } \phi,\psi\in L^2(M\,\mathrm{d}v).$$

since $\mathcal{L}\phi \in L^2(M\,\mathrm{d}v)$ and $g_n \rightharpoonup g$ in $L^2_{loc}(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$ weak. Hence

$$\mathcal{L}g_n \to \mathcal{L}g = 0 \text{ in the sense of distributions.}$$

According to equation (2.25), g is of the form (4.5). ∎

4.2.2. Step 2: Asymptotic deflection term. The following proposition is the key observation in this work. Because the mass ratio of the gas molecules to the particles in the dispersed phase is assumed to be small, the heavier particles are only slightly deflected upon colliding with the lighter gas molecules. It explains how the collision integral $\mathcal{D}(F,f)$ in the kinetic equation for the distribution function of the dispersed phase converges to the acceleration term which appears in the Vlasov equation. This result is reminiscent of Theorem 4.3 in [12].

PROPOSITION 4.2. *Under the assumptions of Theorem 4.1,*

$$\frac{1}{\eta_n}\mathcal{D}(F_n,f_n) \to \kappa \operatorname{div}((v-u)F) \quad \text{in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3),$$

with κ defined in equation (4.7). More precisely, for each $\phi \equiv \phi(v) \in C^2(\mathbf{R}^3)$ such that $\nabla\phi$ and $\nabla^2\phi \in L^\infty(\mathbf{R}^3)$, one has

$$-\frac{1}{\eta_n}\int_{\mathbf{R}^3} \mathcal{D}(F_n,f_n)\phi(v)\,\mathrm{d}v \to \kappa \int_{\mathbf{R}^3} F(v)\nabla\phi(v)\cdot(v-u)\,\mathrm{d}v$$

in $\mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3)$.

Proof. Using Assumption (H1) or equation (2.7) and the Taylor expansion at order 2 for the C^2 function ϕ, one has

$$\begin{aligned}
&\frac{1}{\eta_n}\int_{\mathbf{R}^3} \mathcal{D}(F_n,f_n)(v)\phi(v)\,\mathrm{d}v \\
&= \frac{1}{\eta_n}\iint_{\mathbf{R}^3\times\mathbf{R}^3} F_n(V)f_n(W)\int_{\mathbf{R}^3}(\phi(v)-\phi(V))\Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}v \\
&= \frac{1}{\eta_n}\iint_{\mathbf{R}^3\times\mathbf{R}^3} F_n(V)f_n(W)\nabla\phi(V)\cdot\int(v-V)\Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}v \\
&\quad + \frac{1}{\eta_n}\iint_{\mathbf{R}^3\times\mathbf{R}^3} F_n(V)f_n(W)\int H(v,V):(v-V)^{\otimes 2}\Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}v \\
&=: I_n + J_n,
\end{aligned}$$

where

$$H(v,V) := \int_0^1 (1-t)\nabla^2\phi((1-t)V+tv)\,\mathrm{d}t.$$

We first treat the term I_n. According to Assumption (H2)

$$I_n = -\int_{\mathbf{R}^3} F_n(V)\nabla\phi(V)\cdot\frac{K_n(V)}{1+\eta_n}\,\mathrm{d}V,$$

where

$$K_n(V) := \frac{1}{\epsilon_n}\int_{\mathbf{R}^3} f_n(W)(\epsilon_n V - W)Q(|\epsilon_n V - W|)\,\mathrm{d}W.$$

Hence

$$I_n = I_n^1 + I_n^2 + I_n^3 + I_n^4 + I_n^5,$$

with

$$\begin{aligned}
I_n^1 &= -\epsilon_n \int_{\mathbf{R}^3} F_n(V) \frac{\nabla\phi(V)}{1+\eta_n} \cdot \int_{\mathbf{R}^3} M(W) g_n(W) V Q(|\epsilon_n V - W|) \,\mathrm{d}W \mathrm{d}V, \\
I_n^2 &= \int_{\mathbf{R}^3} F_n(V) \frac{\nabla\phi(V)}{1+\eta_n} \cdot \int_{\mathbf{R}^3} M(W) g_n(W) W (Q(|\epsilon_n V - W|) - Q(|W|) \,\mathrm{d}W, \\
I_n^3 &= \int_{\mathbf{R}^3} F_n(V) \frac{\nabla\phi(V)}{1+\eta_n} \cdot \int_{\mathbf{R}^3} M(W) g_n(W) W Q(|W|) \,\mathrm{d}W \mathrm{d}V, \\
I_n^4 &= -\int_{\mathbf{R}^3} F_n(V) \frac{\nabla\phi(V)}{1+\eta_n} \cdot \int_{\mathbf{R}^3} M(W) V Q(|\epsilon_n V - W|) \,\mathrm{d}W \mathrm{d}V, \\
I_n^5 &= \frac{1}{\epsilon_n} \int F_n(V) \frac{\nabla\phi(V)}{1+\eta_n} \cdot \int M(W) W (Q(|\epsilon_n V - W|) - Q(|W|)) \,\mathrm{d}W \mathrm{d}V.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\left| \int_{\mathbf{R}^3} M(W) g_n(W) V Q(|\epsilon_n V - W|) \,\mathrm{d}W \right| \\
&\le C \int_{\mathbf{R}^3} (1 + \epsilon_n |V| + |W|) M(W) |g_n(W)| |V| \,\mathrm{d}W \\
&\le C|V|(1+|V|) \sqrt{\int_{\mathbf{R}^3} M g_n^2 \,\mathrm{d}W}
\end{aligned}$$

by the Cauchy-Schwarz inequality, so that

$$I_n^1 \to 0 \quad \text{in } L^2_{loc}(\mathbf{R}_*^+ \times \mathbf{R}^3).$$

Then,

$$\begin{aligned}
&\left| \int_{\mathbf{R}^3} M(W) g_n(W) W (Q(|\epsilon_n V - W|) - Q(|W|)) \,\mathrm{d}W \right| \\
&\le C\epsilon_n \int_{\mathbf{R}^3} M(W) |g_n(W)| |W| |V| (1 + \epsilon_n |V| + |W|) \,\mathrm{d}W \\
&\le C\epsilon_n (1+|V|^2) \sqrt{\int_{\mathbf{R}^3} M(W) g_n^2 \,\mathrm{d}W},
\end{aligned}$$

so that

$$I_n^2 \to 0 \quad \text{in } L^2_{loc}(\mathbf{R}_*^+ \times \mathbf{R}^3).$$

By Assumption (d) in Theorem 4.1

$$\begin{aligned}
&\int_{\mathbf{R}^3} M(W) g_n(W) W Q(|W|) \,\mathrm{d}W \to \int_{\mathbf{R}^3} M(W) g(W) W Q(|W|) \,\mathrm{d}W \\
&= \tfrac{1}{3} u \int_{\mathbf{R}^3} M(W) |W|^2 Q(|W|) \,\mathrm{d}W = \kappa u
\end{aligned}$$

in $L^1_{loc}(\mathbf{R}^+_* \times \mathbf{R}^3)$, and therefore

$$I_n^3 \to \kappa u \cdot \int_{\mathbf{R}^3} F(V)\nabla\phi(V)\,\mathrm{d}V \quad \text{in } \mathcal{D}'(\mathbf{R}^+_* \times \mathbf{R}^3).$$

Then

$$\begin{aligned}
&\left|\int_{\mathbf{R}^3} M(W)V(Q(|\epsilon_n V - W|) - Q(|W|))\,\mathrm{d}W\right| \\
&\le C\epsilon_n \int_{\mathbf{R}^3} M(W)|V|^2(1+\epsilon_n|V|+|W|)\,\mathrm{d}W \\
&\le C\epsilon_n |V|^2(1+|V|),
\end{aligned}$$

so that

$$\int_{\mathbf{R}^3} F_n(V)\frac{\nabla\phi(V)}{1+\eta_n}\cdot\int_{\mathbf{R}^3} M(W)V(Q(|\epsilon_n V - W|) - Q(|W|))\,\mathrm{d}W\mathrm{d}V \to 0$$

locally uniformly on $\mathbf{R}^+_* \times \mathbf{R}^3$, and

$$I_n^4 \to -\int_{\mathbf{R}^3} F(V)\nabla\phi(V)\cdot V\int_{\mathbf{R}^3} M(W)Q(|W|)\,\mathrm{d}W\mathrm{d}V \text{ in } \mathcal{D}'(\mathbf{R}^+_* \times \mathbf{R}^3).$$

Finally

$$\begin{aligned}
&\left|\int_{\mathbf{R}^3} M(W)W\left(\frac{Q(|\epsilon_n V - W|) - Q(|W|)}{\epsilon_n} + \frac{W}{|W|}\cdot VQ'(|W|)\right)\mathrm{d}W\right| \\
&\le \int_{\mathbf{R}^3} M(W)|W||V|\int_0^1 |Q'(|\theta\epsilon_n V - W|) - Q'(|W|)|\,\mathrm{d}\theta\mathrm{d}W \\
&\le C|V|(1+|V|)
\end{aligned}$$

and

$$\int_{\mathbf{R}^3} M(W)|W||V|\int_0^1 |Q'(|\theta\epsilon_n V - W|) - Q'(|W|)|\,\mathrm{d}\theta\mathrm{d}W \to 0$$

for all $V \in \mathbf{R}^3$ by dominated convergence. With Assumption (b) in Theorem 4.1, we see that

$$I_n^5 + \int_{\mathbf{R}^3} F_n(V)\frac{\nabla\phi(V)}{1+\eta_n}\cdot\int_{\mathbf{R}^3} M(W)\frac{W}{|W|}W\cdot VQ'(|W|)\,\mathrm{d}W\mathrm{d}V \to 0$$

locally uniformly on $\mathbf{R}^+_* \times \mathbf{R}^3$, and therefore

$$I_n^5 \to -\int_{\mathbf{R}^3} F(V)\nabla\phi(V)\cdot\int_{\mathbf{R}^3} M(W)\frac{W}{|W|}W\cdot VQ'(|W|)\,\mathrm{d}W\mathrm{d}V \text{ in } \mathcal{D}'(\mathbf{R}^+_* \times \mathbf{R}^3).$$

By isotropy, one has

$$\int_{\mathbf{R}^3} M(W)\frac{W}{|W|}W\cdot VQ'(|W|)\,\mathrm{d}W = \tfrac{1}{3}V\int_{\mathbf{R}^3} M(W)|W|Q'(|W|)\,\mathrm{d}W,$$

so that

$$I_n^4 + I_n^5 \to -\int_{\mathbf{R}^3} F(V)\nabla\phi(V)\cdot V\int_{\mathbf{R}^3} M(W)(Q(|W|)+\tfrac{1}{3}|W|Q'(|W|))\,\mathrm{d}W\mathrm{d}V$$

in $\mathcal{D}'(\mathbf{R}_*^+ \times \mathbf{R}^3)$. On the other hand, we observe that

$$W \cdot \nabla M(W) = -|W|^2 M(W)$$

so that

$$\begin{aligned}\int_{\mathbf{R}^3} M(W)|W|^2 Q(|W|)\,\mathrm{d}W &= -\int_{\mathbf{R}^3} W \cdot \nabla M(W) Q(|W|)\,\mathrm{d}W \\ &= \int_{\mathbf{R}^3} M(W)\,\mathrm{div}(WQ(|W|))\,\mathrm{d}W \\ &= \int_{\mathbf{R}^3} M(W)(3Q + |W|Q')(|W|)\,\mathrm{d}W.\end{aligned}$$

Hence

$$\begin{aligned}I_n^4 + I_n^5 \to &-\int_{\mathbf{R}^3} F(V)\nabla\phi(V)\cdot V \int_{\mathbf{R}^3} \tfrac{1}{3}|W|^2 M(W) Q(|W|)\,\mathrm{d}W\,\mathrm{d}V \\ &= -\kappa \int_{\mathbf{R}^3} F(V)\nabla\phi(V)\cdot V\,\mathrm{d}V\end{aligned}$$

in $\mathcal{D}'(\mathbf{R}_*^+ \times \mathbf{R}^3)$. Therefore

$$I_n \to -\kappa \int_{\mathbf{R}^3} F(V)\nabla\phi(V)\cdot(V-u)\,\mathrm{d}V \quad \text{in } \mathcal{D}'(\mathbf{R}_*^+ \times \mathbf{R}^3),$$

with

$$\kappa = \tfrac{1}{3}\int_{\mathbf{R}^3} M(W)Q(|W|)|W|^2\,\mathrm{d}W > 0.$$

Next we treat the term J_n. One has

$$|J_n| \le \frac{1}{2\eta_n}\|\nabla^2\phi\|_{L^\infty} \iint_{\mathbf{R}^3\times\mathbf{R}^3} F_n(V) f_n(W) \int_{\mathbf{R}^3} |v-V|^2 \Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}v.$$

With $U = \frac{\epsilon_n V + \eta_n W}{1+\eta_n}$, one has

$$\begin{aligned}\epsilon_n^2|v-V|^2 &\le 2|\epsilon_n v - U|^2 + 2|U - \epsilon_n V|^2 \\ &= 2|\epsilon_n v - U|^2 + \frac{2\eta_n^2}{(1+\eta_n)^2}|\epsilon_n V - W|^2.\end{aligned}$$

According to Assumption (H3),

$$\int_{\mathbf{R}^3} |v-V|^2 \Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}v \le \frac{2C}{\epsilon_n^2}\eta_n^2(1+|\epsilon_n V - W|^2) q(|\epsilon_n V - W|),$$

so that

$$|J_n| \le \frac{C}{\epsilon_n^2}\eta_n ||\nabla^2\phi||_{L^\infty} \iint_{\mathbf{R}^3\times\mathbf{R}^3} F_n(V) f_n(W)(1+|\epsilon_n V - W|^2) q(|\epsilon_n V - W|)\,\mathrm{d}V\mathrm{d}W.$$

By (H1) and Assumption (b) in Theorem 4.1,

$$\begin{aligned}&\iint_{\mathbf{R}^3\times\mathbf{R}^3} F_n(V) f_n(W)(1+|\epsilon_n V - W|)^2 q(|\epsilon_n V - W|)\,\mathrm{d}V\mathrm{d}W \\ &\le C C_R \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{(1+\epsilon_n|V|+|W|)^3}{(1+|V|)^p} M(W)(1+\epsilon_n g_n)(W)\,\mathrm{d}V\mathrm{d}W\end{aligned}$$

for $(t,x)\in[0,R]\times[-R,R]^3$. By Assumption (c) in Theorem 4.1 and the Cauchy-Schwarz inequality, the right hand side is bounded in $L^2_{loc}(\mathbf{R}_*^+\times\mathbf{R}^3)$. Hence

$$J_n\to 0 \quad \text{in } L^2_{loc}(\mathbf{R}_*^+\times\mathbf{R}^3)$$

since $\eta_n/\epsilon_n^2\to 0$, which concludes the proof of Proposition 4.2. □

4.2.3. Step 3: Asymptotic friction term.

PROPOSITION 4.3. *Under the assumptions of Theorem 4.1,*

$$\frac{1}{\epsilon_n}\int_{\mathbf{R}^3} w\mathcal{R}(f_n,F_n)\,\mathrm{d}w\to\kappa\int_{\mathbf{R}^3}(v-u)F\,\mathrm{d}v \qquad \text{in } \mathcal{D}'(\mathbf{R}_+^*\times\mathbf{R}^3),$$

with κ defined by formula (4.7).

Proof. By Assumptions (H1)-(H2),

$$\begin{aligned}
&\frac{1}{\epsilon_n}\int_{\mathbf{R}^3} w\mathcal{R}(f_n,F_n)\,\mathrm{d}w\\
&=\frac{1}{\epsilon_n}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(w-W)f_n(W)F_n(V)\Pi_{gp}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\\
&=-\frac{1}{\eta_n}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(v-V)F_n(V)f_n(W)\Pi_{pg}(v,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}v\\
&=-\frac{1}{\eta_n}\int_{\mathbf{R}^3} v\mathcal{D}(F_n,f_n)\,\mathrm{d}v.
\end{aligned}$$

Proposition 4.2 then implies that

$$-\frac{1}{\eta_n}\int_{\mathbf{R}^3}\phi(v)\mathcal{D}(F_n,f_n)\,\mathrm{d}v\to\kappa\int_{\mathbf{R}^3}F(V)\nabla\phi(V)\cdot(V-u)\,\mathrm{d}V \tag{4.11}$$

in $\mathcal{D}'(\mathbf{R}_+^*\times\mathbf{R}^3)$ for each test function $\phi\equiv\phi(v)$ satisfying

$$\phi\in C^2(\mathbf{R}^3) \quad \text{and } \nabla\phi,\nabla^2\phi\in L^\infty(\mathbf{R}^3).$$

Setting $\phi(v)=v$ in equation (4.11) leads to the conclusion. □

4.2.4. Step 4: Incompressibility condition.

PROPOSITION 4.4. *Under the assumptions of Theorem 4.1, the velocity field u satisfies the incompressibility condition*

$$\operatorname{div}_x u=0 \tag{4.12}$$

in the sense of distributions on $\mathbf{R}_+^*\times\mathbf{R}^3$.

Proof. For each $\phi:=\phi(w)\in L^1(M\,\mathrm{d}v)$, we set

$$\langle\phi\rangle:=\int_{\mathbf{R}^3}\phi(w)M(w)\,\mathrm{d}w. \tag{4.13}$$

Multiplying both sides of equation (4.9) by $\epsilon_n M(w)$ and integrating in w shows that

$$\epsilon_n\partial_t\langle g_n\rangle+\operatorname{div}_x\langle wg_n\rangle=0,$$

according to equation (2.2). Since $g_n \rightharpoonup g$ in $L^2(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3)$ weak and satisfies Assumption (c) in Theorem 4.1,

$$\langle g_n\rangle \to \langle g\rangle \quad \text{ and } \langle wg_n\rangle \to \langle wg\rangle \text{ in } L^2_{loc}(\mathbf{R}_+^* \times \mathbf{R}^3) \text{ weak.}$$

Hence

$$\operatorname{div}_x\langle wg_n\rangle = -\epsilon_n \partial_t \langle g_n\rangle \to 0 \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3),$$

so that

$$\operatorname{div}_x\langle wg\rangle = 0.$$

According to Proposition 4.1, one has $\langle wg\rangle = u$, so that condition (4.12) holds. ∎

4.2.5. Step 5: Viscosity term.

Proposition 4.5. *Under the assumptions of Theorem 4.1,*

$$\langle \tilde{A}(w)w \cdot \nabla_x g\rangle = \nu(\nabla_x u + (\nabla_x u)^T),$$

where $\tilde{A}$ is defined in equation (2.27), and ν is defined in equation (4.7).

Proof. By Proposition 4.1, one has

$$\langle \tilde{A}(w)w \cdot \nabla_x g\rangle = \langle \tilde{A}(w) \otimes A(w)\rangle : \nabla_x u$$

since the tensor field $w \mapsto A(w)w$ is odd. By Lemma 4.4 in [4] (see formula (4.13a)), one has

$$\langle \tilde{A}_{ij} A_{kl}\rangle = \nu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{1}{3}\delta_{ij}\delta_{kl}),$$

with

$$\nu := \tfrac{1}{10}\langle \tilde{A} : \mathcal{L}\tilde{A}\rangle > 0$$

(see formula (4.10) in [4]). Formulas (4.10)-(4.13a) in [4] are based on elementary symmetry arguments — most notably the fact that $A(Rw) = RA(w)R^T$ for each $R \in O_3(\mathbf{R})$. Complete proofs of these formulas can be found in Lemma 4.3 of [20]. Hence

$$\langle \tilde{A}(w)w \cdot \nabla_x g\rangle = \nu(\nabla_x u + (\nabla_x u)^T - \tfrac{2}{3}(\operatorname{div}_x u)I).$$

Since the velocity field u is divergence-free by Proposition 4.4, this concludes the proof. ∎

4.2.6. Step 6: Convection term.

Proposition 4.6. *Under the assumptions of Theorem 4.1,*

$$\langle \tilde{A}(w)\mathcal{Q}(g)\rangle = A(u)$$

where $\tilde{A}$ is defined in equation (2.27), while $\mathcal{Q}$ is defined in equation (4.8).

Proof. By Proposition 4.1, $g(t,x,\cdot) \in \operatorname{Ker}\mathcal{L}$ for a.e. $(t,x) \in \mathbf{R}_+^* \times \mathbf{R}^3$. According to formula (60) in [3], one has

$$\mathcal{Q}(g(t,x,\cdot)) = \tfrac{1}{2}\mathcal{L}(g(t,x,\cdot)^2), \quad \text{ for a.e. } (t,x) \in \mathbf{R}_+^* \times \mathbf{R}^3.$$

Since $\mathcal{L}$ is self-adjoint on $L^2(M\,\mathrm{d}w)$ by Theorem 2.1 and $g^2\in\mathrm{Dom}\mathcal{L}$, one has

$$\langle\tilde{A}(w)\mathcal{Q}(g)\rangle=\langle\tilde{A}(w)\tfrac{1}{2}\mathcal{L}(g^2)\rangle=\tfrac{1}{2}\langle(\mathcal{L}\tilde{A})g^2\rangle=\tfrac{1}{2}\langle Ag^2\rangle.$$

Eliminating the odd component of g^2 since $w\mapsto A(w)$ is even, one finds that

$$\langle Ag^2\rangle=\langle A\otimes w\otimes w\rangle:(u\otimes u)+\left\langle A\left(\rho+\theta\tfrac{1}{2}(|w|^2-3)\right)^2\right\rangle.$$

First

$$\left\langle A\left(\rho+\theta\tfrac{1}{2}(|w|^2-3)\right)^2\right\rangle=\tfrac{1}{3}\left\langle \mathrm{trace}(A)\left(\rho+\theta\tfrac{1}{2}(|w|^2-3)\right)^2\right\rangle I=0$$

because $A(Rw)=RA(w)A^T$ and $\mathrm{trace}(A)=0$ — see Lemma 4.2 in [20] for a detailed proof.

Then

$$\langle A\otimes w\otimes w\rangle_{ijkl}=\langle A_{ij}A_{kl}\rangle=\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\tfrac{2}{3}\delta_{ij}\delta_{kl},$$

by Lemma 4.2 in [20], so that

$$\langle A\otimes w\otimes w\rangle:(u\otimes u)=2u\otimes u-\tfrac{2}{3}|u|^2I.$$

This concludes the proof. ∎

4.2.7. Step 7: Asymptotic friction flux.

Proposition 4.7. *Under the assumptions of Theorem 4.1,*

$$\int\tilde{A}(w)\mathcal{R}(f_n,F_n)(w)dw\to 0\qquad in\ \mathcal{D}'(\mathbf{R}_+^*\times\mathbf{R}^3).$$

Proof. First, we deduce from (H1) that

$$\int_{\mathbf{R}^3}\tilde{A}(w)\mathcal{R}(M,F_n)(w)\,\mathrm{d}w$$
$$=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}F_n(V)M(W)(\tilde{A}(w)-\tilde{A}(W))\Pi_{gp}^{\epsilon_n,\eta_n}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w.$$

Then

$$\left|\int_{\mathbf{R}^3}\left(\tilde{A}\mathcal{R}(M,F_n)-\iint_{\mathbf{R}^3\times\mathbf{R}^3}F(V)M(W)(\tilde{A}(w)-\tilde{A}(W))\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\right)\mathrm{d}w\right|$$
$$\le\left|\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}F_n(V)M(W)(\tilde{A}(w)-\tilde{A}(W))(\Pi_{gp}^{\epsilon_n,\eta_n}-\Pi_{gp}^{0,0})(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\right|$$
$$+\left|\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(F_n(V)-F(V))M(W)(\tilde{A}(w)-\tilde{A}(W))\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\right|.$$

The first term on the right hand side vanishes in the sense of distributions on $\mathbf{R}_+^*\times\mathbf{R}^3$ because of the second part of Assumption (H4) and the fact that the radial function α in equation (2.28) belongs to $L^\infty(\mathbf{R}_+)$. The second term on the right hand side also vanishes in the sense of distributions on $\mathbf{R}_+^*\times\mathbf{R}^3$ because of the last part of Assumption (H4).

According to the first part of Assumption (H4),

$$\begin{aligned}&\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} F(V)M(W)(\tilde{A}(w)-\tilde{A}(W))\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\\ &=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} F(V)M(W)(\tilde{A}(w)-\tilde{A}(W))\mathcal{T}_R\#\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\\ &=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} F(V)M(W)(\tilde{A}(Rw)-\tilde{A}(RW))\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\end{aligned}$$

for each $R\in O_3(\mathbf{R})$, where $\mathcal{T}_R$ is defined in equation (3.1). Because of equation (2.28),

$$\tilde{A}(Rw)=R\tilde{A}(w)R^T,\qquad \text{for each } R\in O_3(\mathbf{R}).$$

Thus, for each $R\in O_3(\mathbf{R})$,

$$\mathcal{A}:=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} F(V)M(W)(\tilde{A}(Rw)-\tilde{A}(RW))\Pi_{gp}^{0,0}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w=R\mathcal{A}R^T$$

a.e. on $\mathbf{R}_+^*\times\mathbf{R}^3$. At this point, we use the following classical lemma.

Lemma 4.1. *Let $\mathcal{M}=\mathcal{M}^T\in M_3(\mathbf{R})$ satisfy*

$$R\mathcal{M}=\mathcal{M}R \text{ for each } R\in O_3(\mathbf{R}).$$

Then $\mathcal{M}$ is of the form

$$\mathcal{M}=\lambda I,\qquad \text{with } \lambda=\tfrac{1}{3}\operatorname{trace}\mathcal{M}.$$

(The proof of this lemma is an easy exercise in linear algebra; alternately, it is a special case of Lemma 4.1 in [20] for $m=2$ and in the case of a constant tensor field, i.e. $T(\xi)\equiv T(0)$.)

As a consequence,

$$\mathcal{A}(t,x)=\tfrac{1}{3}\operatorname{trace}(\mathcal{A}(t,x))I=0,$$

since

$$\operatorname{trace}\mathcal{A}=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3} F(V)M(W)\operatorname{trace}(\tilde{A}(w)-\tilde{A}(W))\Pi_{gp}^{0,0}(dwdVdW)=0.$$

Hence

$$\int_{\mathbf{R}^3}\tilde{A}(w)\mathcal{R}(M,F_n)(w)\,\mathrm{d}w\to 0\qquad \text{in } \mathcal{D}(\mathbf{R}_+^*\times\mathbf{R}^3). \tag{4.14}$$

Next, we deduce from (H1) that

$$\begin{aligned}&\left|\int_{\mathbf{R}^3}\mathcal{R}(Mg_n,F_n)\tilde{A}(w)\,\mathrm{d}w\right|\\ &=\left|\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(\tilde{A}(w)-\tilde{A}(W))M(W)g_n(W)F_n(V)\Pi_{gp}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\right|\\ &\le C_K\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{R}^3}(|w|^2+|W|^2)M(W)|g_n(W)|(1+|V|^2)^{-p}\Pi_{gp}(w,\mathrm{d}V\mathrm{d}W)\,\mathrm{d}w\end{aligned}$$

$$\leq CC_K \|g\|_{L^2(M\,\mathrm{d}w)} \tag{4.15}$$

for all $(t,x) \in [0,K] \times [-K,K]^3$, by (H5) and Assumptions (c) in Theorem 4.1.

The conclusion follows from (4.14)-(4.15), from Assumption (c) in Theorem 4.1 showing the last right hand side of inequality (4.15) is bounded in $L^2_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^3)$, and from the identity

$$\int_{\mathbf{R}^3} \tilde{A}(w)\mathcal{R}(f_n,F_n)(w)\,\mathrm{d}w = \int_{\mathbf{R}^3} \tilde{A}(w)\mathcal{R}(M,F_n)(w)\,\mathrm{d}w \\ + \epsilon_n \int_{\mathbf{R}^3} \tilde{A}(w)\mathcal{R}(Mg_n,F_n)(w)\,\mathrm{d}w.$$

□

4.2.8. Step 8: End of the proof of Theorem 4.1. First, we recall that $\mathcal{L}$ is self-adjoint in $L^2(M\,\mathrm{d}w)$ according to Theorem 2.1. Hence

$$\frac{1}{\epsilon_n}\langle A(w)g_n\rangle = \frac{1}{\epsilon_n}\langle(\mathcal{L}\tilde{A})(w)g_n\rangle = \left\langle \tilde{A}(w)\frac{1}{\epsilon_n}\mathcal{L}g_n\right\rangle.$$

Following the same procedure as in [3], we use the Boltzmann equation for g_n in the form (4.10) to express the term $\frac{1}{\epsilon_n}\mathcal{L}g_n$:

$$\frac{1}{\epsilon_n}\langle A(w)g_n\rangle = \langle\tilde{A}(w)\mathcal{Q}(g_n)\rangle - \langle\tilde{A}(w)(\epsilon_n\partial_t + w\cdot\nabla_x)g_n\rangle \\ + \langle\tilde{A}(w)M^{-1}\mathcal{R}(f_n,F_n)\rangle. \tag{4.16}$$

We first pass to the limit in the term $\langle\tilde{A}(w)(\epsilon_n\partial_t + w\cdot\nabla_x)g_n\rangle$ in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$. Since the function α in equation (2.28) is bounded, one has

$$\int_{\mathbf{R}^3} (1+|w|)^2|\tilde{A}(w)|^2 M(w)\,\mathrm{d}w < \infty.$$

By Assumption (c) in Theorem 4.1 and the Cauchy-Schwarz inequality,

$$\langle\tilde{A}g_n\rangle \to \langle\tilde{A}g\rangle \text{ and } \langle w\tilde{A}g_n\rangle \to \langle w\tilde{A}g\rangle \text{ in } L^2_{loc}(\mathbf{R}^*_+ \times \mathbf{R}^3) \text{ weak.}$$

Hence

$$\langle\tilde{A}(w)(\epsilon_n\partial_t + w\cdot\nabla_x)g_n\rangle = \epsilon_n\partial_t\langle\tilde{A}g_n\rangle + \mathrm{div}_x\langle w\tilde{A}g_n\rangle \to \mathrm{div}_x\langle w\tilde{A}g\rangle$$

in $\mathcal{D}'(\mathbf{R}^*_+ \times \mathbf{R}^3)$. By Proposition 4.5,

$$\langle\tilde{A}(w)(\epsilon_n\partial_t + w\cdot\nabla_x)g_n\rangle \to \nu(\nabla_x u + (\nabla_x u)^T) \text{ in } \mathcal{D}'(\mathbf{R}^*_+ \times \mathbf{R}^3). \tag{4.17}$$

Next we use the identity

$$\langle\tilde{A}\mathcal{Q}(g_n)\rangle = \iint_{\mathbf{R}^3\times\mathbf{R}^3} P(w,w_*)M(w_*)g_n(w_*)M(w)g_n(w)\mathrm{d}w\mathrm{d}w_*$$

where

$$P(w,w_*) := \int_{\mathbf{S}^2} (\tilde{A}(w') - \tilde{A}(w))c(w-w_*,\omega)\,\mathrm{d}\omega.$$

Obviously

$$\langle \tilde{A}\mathcal{Q}(g_n)\rangle = \int_{\mathbf{R}^3} h_n(t,x,w)M(w)g_n(w)\mathrm{d}w$$

with

$$h_n(t,x,w) := \int_{\mathbf{R}^3} P(w,w_*)M(w_*)g_n(t,x,w_*)\,\mathrm{d}w_*.$$

One has

$$|P(w,w_*)| \le C(1+|w|^3+|w_*|^3)$$

because of the growth Assumption (2.22) on the collision kernel, and the assumption that the function α in equation (2.28) is bounded on $\mathbf{R}_+^*$. Assumption (c) in Theorem 4.1 implies that

$$\sup_{n\ge 1}\iiint_{[0,R]\times[-R,R]^3\times\mathbf{R}^3} M(w_*)g_n(t,x,w_*)^2\,\mathrm{d}w_*\mathrm{d}x\mathrm{d}t < \infty$$

so that, by the Cauchy-Schwarz inequality,

$$\int_{|w_*|>R} |P(w,w_*)||g_n(t,x,w_*)|M(w_*)\,\mathrm{d}w_* \to 0 \text{ in } L^2_{loc}(\mathbf{R}_+^*\times\mathbf{R}^3\times\mathbf{R}^3)$$

uniformly in $n\ge 1$ as $R\to\infty$. Therefore, we deduce from Assumption (d) in Theorem 4.1 that

$$h_n(t,x,w) \to \int_{\mathbf{R}^3} P(w,w_*)M(w_*)g(t,x,w_*)\,\mathrm{d}w_* =: h(t,x,w)$$

in $L^2_{loc}(\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3)$. In particular, by weak-strong continuity of the pointwise product, one has

$$\int_{|w|\le K} h_n(t,x,w)M(w)g_n(t,x,w)\mathrm{d}w \to \int_{|w|\le K} h(t,x,w)M(w)g(t,x,w)\mathrm{d}w$$

in $\mathcal{D}'(\mathbf{R}_+\times\mathbf{R}^3)$ for all $K>0$. On the other hand

$$M(w)h_n(t,x,w)^2 \le C(1+|w|^3)^2 M(w)\int_{\mathbf{R}^3} M(w_*)g_n(t,x,w_*)^2\,\mathrm{d}w_*,$$

so that, by the Cauchy-Schwarz inequality,

$$\begin{aligned}&\int_{|w|>K} h_n(t,x,w)M(w)g_n(t,x,w)\mathrm{d}w\\ &\le \sqrt{C}\left(\int_{|w|>K}(1+|w|^3)^2M(w)\,\mathrm{d}w\right)^{1/2}\int_{\mathbf{R}^3} M(\xi)g_n(t,x,\xi)^2\,\mathrm{d}\xi \to 0\end{aligned}$$

in $L^2_{loc}(\mathbf{R}_+^*\times\mathbf{R}^3)$ as $K\to+\infty$ uniformly in $n\ge 1$, according to Assumption (c) in Theorem 4.1. Hence

$$\langle \tilde{A}\mathcal{Q}(g_n)\rangle(t,x) = \int_{\mathbf{R}^3} h_n(t,x,w)M(w)g_n(t,x,w)\mathrm{d}w$$

$$\to \int_{\mathbf{R}^3} h(t,x,w)M(w)g(t,x,w)\mathrm{d}w = \langle \tilde{A}\mathcal{Q}(g)\rangle(t,x) = A(u)(t,x) \quad (4.18)$$

in $\mathcal{D}'(\mathbf{R}_+ \times \mathbf{R}^3)$, where the last equality follows from Proposition 4.6.

Since the last term on the right hand side of equation (4.16) vanishes by Proposition 4.7, we conclude that

$$\frac{1}{\epsilon_n}\langle A(w)g_n\rangle \to A(u) - \nu\left((\nabla_x u) + (\nabla_x u)^T\right) \qquad \text{in } \mathcal{D}'(\mathbf{R}_+ \times \mathbf{R}^3).$$

In particular,

$$\begin{aligned} &\operatorname{div}_x \frac{1}{\epsilon_n}\langle A(w)g_n\rangle \to \operatorname{div}_x(u\otimes u) - \tfrac{1}{3}\nabla_x|u|^2 - \nu\Delta_x u - \nu\nabla_x \operatorname{div}_x u \\ &= \operatorname{div}_x(u\otimes u) - \nu\Delta_x u - \tfrac{1}{3}\nabla_x|u|^2 \end{aligned}$$

in $\mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3)$, by the divergence-free condition in Proposition 4.4. Hence, for each divergence-free, compactly supported, smooth vector field $\xi \equiv \xi(x) \in \mathbf{R}^3$,

$$\begin{aligned} \int_{\mathbf{R}^3} \frac{1}{\epsilon_n}\langle w\otimes w g_n\rangle(t,x) : \nabla\xi(x)\,\mathrm{d}x &= \int_{\mathbf{R}^3} \frac{1}{\epsilon_n}\langle A(w)g_n\rangle(t,x) : \nabla\xi(x)\,\mathrm{d}x \\ &\to \int_{\mathbf{R}^3} (u\otimes u - \nu\nabla_x u)(t,x) : \nabla\xi(x)\,\mathrm{d}x \end{aligned}$$

in $\mathcal{D}'(\mathbf{R}_+^*)$.

We recall that the momentum balance law for the Boltzmann equation for gas molecules is

$$\partial_t\langle w g_n\rangle + \frac{1}{\epsilon_n}\operatorname{div}_x\langle w^{\otimes 2} g_n\rangle = \frac{1}{\epsilon_n}\langle w M^{-1}\mathcal{R}(f_n, F_n)\rangle. \quad (4.19)$$

By Proposition 4.1,

$$\langle w g_n\rangle \to \langle w g\rangle = u \qquad \text{in } L^2(\mathbf{R}_+^* \times \mathbf{R}^3) \text{ weak,}$$

while

$$\frac{1}{\epsilon_n}\langle w M^{-1}\mathcal{R}(f_n, F_n)\rangle \to \kappa \int (v-u)F dv \qquad \text{in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3).$$

Thus, for each divergence-free, compactly supported, smooth vector field $\xi \equiv \xi(x) \in \mathbf{R}^3$, passing to the limit in the weak formulation (in x) of the momentum balance law (4.19), i.e.

$$\begin{aligned} &\partial_t \int_{\mathbf{R}^3} \xi(x)\cdot\langle w g_n\rangle(t,x)\,\mathrm{d}x - \frac{1}{\epsilon_n}\int_{\mathbf{R}^3}\langle A(w)g_n\rangle(t,x) : \nabla\xi(x)\,\mathrm{d}x \\ &= \frac{1}{\epsilon_n}\int_{\mathbf{R}^3} \xi(x)\cdot\langle w M^{-1}\mathcal{R}(f_n,F_n)\rangle(t,x)\,\mathrm{d}x, \end{aligned}$$

results in the equality

$$\begin{aligned} \partial_t \int_{\mathbf{R}^3} u(t,x)\cdot\xi(x)\,\mathrm{d}x &= \int_{\mathbf{R}^3} (u\otimes u - \nu\nabla_x u)(t,x) : \nabla\xi(x)\,\mathrm{d}x \\ &\quad + \kappa \iint_{\mathbf{R}^3\times\mathbf{R}^3} \xi(x)\cdot(v - u(t,x))F(t,x,v)\,\mathrm{d}v\mathrm{d}x. \end{aligned}$$

By de Rham's characterization of currents homologous to 0 (see Thm. 17' in [13]), there exists $p \in \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3)$ such that

$$\partial_t u + \mathrm{div}_x(u \otimes u - \nu \nabla_x u) - \kappa \int_{\mathbf{R}^3} (v-u) F \, \mathrm{d}v = -\nabla_x p.$$

Finally, we recall the equation for the distribution function of the dispersed phase:

$$\partial_t F_n + v \cdot \nabla_x F_n = \frac{1}{\eta_n} \mathcal{D}(F_n, f_n).$$

The assumptions on the convergence of F_n in Theorem 4.1 imply that

$$\partial_t F_n + v \cdot \nabla_x F_n \to \partial_t F + v \cdot \nabla_x F \quad \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3).$$

Applying Proposition 4.2 shows that

$$\partial_t F + v \cdot \nabla_x F = \kappa \, \mathrm{div}_v((v-u)F),$$

and this concludes the proof of Theorem 4.1.

5. Conclusions and perspectives

We conclude this paper with a few remarks on the method presented here, and on the assumptions used in Theorem 4.1.

Let us first discuss the class of collision interactions considered in this work.

We have assumed that intermolecular collisions correspond to cut-off hard potentials, which is natural. However, our assumption that the radial function α in equation (2.28) is bounded could be a significant restriction to the class of intermolecular potentials considered. At present, this assumption is known to be satisfied only in the case of cut-off Maxwell molecules, when α is a constant. It would be natural to expect that the growth of α at infinity is such that

$$\alpha(|w|) \sim (\bar{c} \star M(w))^{-1} \quad \text{ as } |w| \to \infty,$$

however, we are not aware of any result of this type in the existing literature, and we have not been able to prove it, even in the simplest case of hard sphere collisions. Perhaps the assumption that α is bounded can be relaxed at the expense of more technical proofs.

Likewise, we have considered in this paper only the case of a monatomic propellant; however, this assumption could certainly be relaxed and more realistic models of propellant could be handled with the same methods.

Concerning collisions between gas molecules and dust particles/droplets, the hard spheres model for the collision cross-section may be the best choice when the detail of the interaction is not known, because the dust particles/droplets, though tiny, are macroscopic objects if compared to gas molecules. Hard spheres clearly belong to the class of cross-sections included in the assumptions of our theorem.

Otherwise, it would be more realistic to include polydispersion in our model of aerosol/spray — i.e. to assume that the dust particles/droplets are distributed in size, and to include aggregation and fragmentation effects in the equation for the distribution function of the dispersed phase. Such a generalization of the model considered in this work would be extremely natural, although significantly more technical, and we have avoided these effects in the present paper for the sake of simplicity.

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