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CUTOFF ESTIMATES FOR THE LINEARIZED BECKER–DÖRING EQUATIONS*

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Abstract. This paper continues the authors' previous study [R. Murray and R. Pego, SIAM J. Math. Anal., 48:2819–2842, 2016] of the trend toward equilibrium of the Becker–Döring equations with subcritical mass, by characterizing certain fine properties of solutions to the linearized equation. In particular, we partially characterize the spectrum of the linearized operator, showing that it contains the entire imaginary axis in polynomially weighted spaces. Moreover, we prove detailed cutoff estimates that establish upper and lower bounds on the lifetime of a class of perturbations to equilibrium.

Keywords. coagulation-fragmentation equations; spectrum; cutoff estimates.

AMS subject classifications. 34D05; 47D06; 82C05.

1. Introduction

This work considers the Becker-Döring equations, which are given by the infinite sequence of differential equations

$$\begin{aligned} \frac{d}{dt}c_i(t) &= J_{i-1}(t) - J_i(t), \quad i = 2,3,\ldots, \\ \frac{d}{dt}c_1(t) &= -J_1(t) - \sum_{i=1}^{\infty} J_i(t), \end{aligned} \tag{1.1}$$

where the J_i can be written as

$$J_i(t) = a_i c_1(t) c_i(t) - b_{i+1} c_{i+1}(t), \tag{1.2}$$

and where $(a_i),(b_i)$ are fixed, positive sequences, known as the coagulation and fragmentation coefficients respectively. These equations are a well-known model for certain physical phenomena occuring in phase transitions, such as condensation in alloys and polymers. In this context, $c_i(t)$ typically represents the density of particles of size i ("i-particles") in some units. The Becker–Döring Equations (1.1) describe the evolution of the discrete size distribution (c_i) under mean-field assumptions which state that i-particles aggregate with 1-particles (monomers) to form $i+1$-particles at rate $a_i c_1(t)$ per particle, and $i+1$-particles break in two pieces, monomers and i-particles, at rate b_{i+1} per particle. The first moment $\mu = \sum_{i=1}^{\infty} i c_i(t)$ corresponds to the total mass in the system and is formally conserved in time, due to the evolution equation for $c_1(t)$ in equation (1.1).

The quantity $J_i(t)$ is the net reaction rate for i-particles to become $i+1$-particles, and this vanishes in equilibrium. Under typical assumptions on the rate coefficients

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(described below), it is known [1] that there is a critical mass $\mu_{\text{crit}} \le \infty$ such that for positive initial data (c_i^0) with subcritical mass, meaning

$$\sum_{i=1}^{\infty} i c_i^0 =: \mu < \mu_{\text{crit}}, \tag{1.3}$$

the solution $(c_i(t))$ converges strongly to the unique equilibrium solution (Q_i) with the same mass, determined by the condition of detailed balance that says $J_i = 0$ for all i. Here by strong convergence we mean that

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} i|c_i(t) - Q_i| = 0. \tag{1.4}$$

One can write $Q_i = \tilde{Q}_i z^i$, where $\tilde{Q}_i$ is determined by the recursion

$$\tilde{Q}_1 = 1, \qquad b_{i+1}\tilde{Q}_{i+1} = a_i \tilde{Q}_i \tilde{Q}_1, \quad i = 1, 2, \ldots, \tag{1.5}$$

and $z = Q_1$ is determined by the requirement that

$$\sum_{i=1}^{\infty} i Q_i = \sum_{i=1}^{\infty} i \tilde{Q}_i z^i = \mu. \tag{1.6}$$

Various authors have sought to establish uniform convergence rates in the limit (1.4). The previous works [5, 9] focused on convergence rates in the setting where the initial data decays exponentially fast. More recent works [4, 13] have focused on convergence rates when the initial data decays only algebraically fast. In particular, in [13] the present authors proved the following result.

THEOREM 1.1 ([13]). *Assume the model coefficients in equation (1.2) satisfy conditions (1.17)-(1.20) below. Let $(c_i(t))$ be a solution of the Becker–Döring Equations (1.1) with subcritical mass, and let its deviation from equilibrium be represented by $(h_i(t))$, defined so that*

$$c_i = Q_i(1 + h_i). \tag{1.7}$$

Let m and k be real numbers satisfying $m > 0$ and $k > m + 2$. Then there exists positive constants $\delta_{k,m}, C_{k,m}$ so that if $\|h(0)\|_{X_{1+k}} < \delta_{k,m}$ then

$$\|h(t)\|_{X_{1+m}} \le C_{k,m}(1+t)^{-(k-m-1)}\|h(0)\|_{X_{1+k}} \quad \textit{for all } t \ge 0. \tag{1.8}$$

Here we are writing

$$X_k := \left\{ (h_i) : \|h\|_{\ell^1(Q_i i^k)} := \sum_{i=1}^{\infty} Q_i i^k |h_i| < \infty, \quad \sum_{i=1}^{\infty} Q_i i h_i = 0 \right\}, \quad k \ge 1, \tag{1.9}$$

with norm $\|\cdot\|_{X_k} = \|\cdot\|_{\ell^1(Q_i i^k)}$.

Theorem 1.1 was derived from analysis conducted on the linearized equation

$$\frac{d}{dt} h = Lh, \tag{1.10}$$

where the operator L is defined in weak form on suitable spaces by the requirement that for all suitable test sequences (ϕ_i),

$$\sum_{i=1}^{\infty} Q_i (Lh)_i \phi_i = \sum_{i=1}^{\infty} Q_i Q_1 a_i (h_{i+1} - h_i - h_1)(\phi_1 + \phi_i - \phi_{i+1}). \tag{1.11}$$

In strong form L may be expressed by (after using equation (1.5))

$$Q_i(Lh)_i = \begin{cases} Q_i Q_1 a_i (h_{i+1} - h_i - h_1) - Q_i b_i (h_1 + h_{i-1} - h_i) & \text{for } i > 1, \\ Q_1^2 a_1 (h_2 - 2h_1) + \sum_{i=1}^{\infty} Q_i Q_1 a_i (h_{i+1} - h_i - h_1) & \text{for } i = 1. \end{cases} \tag{1.12}$$

In particular, it was first shown that e^{Lt} is uniformly bounded in X_1, after which bounds of the type (1.8) were obtained for the linearized equation via interpolation theory. The linearized estimates were then extended, after some technicalities, to the non-linear setting.

1.1. Spectral properties of L. A natural question is whether the bounds in this theorem are optimal. More generally, one would hope for a more detailed understanding of the dynamics as solutions converge to equilibrium. The first theorem in the present work seeks to address these questions by giving additional information about the spectrum of the linearized operator.

THEOREM 1.2. *Suppose, in addition to the assumptions (1.17)-(1.20), that*

$$a_i - a_{i-1} = o(1), \quad b_i - b_{i-1} = o(1) \quad \text{and} \quad a_i \to \infty.$$

Then $\lambda\mathbf{i}$ is in the approximate point spectrum of the operator L in the space X_k for all $\lambda \in \mathbb{R}$ and all $k \geq 1$, where $\mathbf{i} = \sqrt{-1}$.

We recall that the approximate point spectrum of a closed operator A is defined as the set where either $\lambda - A$ is is not injective or the range of $\lambda - A$ is not closed. Alternatively, the approximate point spectrum may be characterized by the existence of a sequence of *approximate eigenvectors*; namely λ is in the approximate point spectrum if and only if there exists some sequence $\|x_n\| = 1$ with $(A - \lambda)x_n \to 0$ (see [8, Section IV.1]).

In the previous theorem we write $\mathbf{i} = \sqrt{-1}$ so that we may use i freely as an index throughout the work. We remark that the assumptions of this theorem are satisfied by a wide class of coefficients used in applications, see e.g. the coefficients (1.21).

When the conditions of Theorem 1.2 are satisfied, previous results already give a significant amount of information about the spectrum of L in X_k. In particular, the right half plane is in the resolvent set of L, since it follows from Proposition 3.2, Lemma 4.7 and Theorem 2.9 in [13] that L generates a uniformly bounded semigroup on X_k. Furthermore, on the zero mass subspace of $\ell^2(Q_i)$, it is shown in [5] that L is a relatively bounded (with bound < 1) perturbation of a diagonal operator with coefficients growing like a_i. This implies that L has compact resolvent in that space (see [11, Theorem IV.3.17], e.g.), and consequently (since L is self-adjoint) L has eigenvalues on the real axis going to negative infinity. As $\ell^2(Q_i)$ is continuously embedded in X_k, we have that L has eigenvalues on the real axis going to negative infinity in X_k as well.

Theorem 1.2 highlights significant differences between the operator in exponentially weighted spaces as opposed to polynomially weighted ones. In exponentially weighted spaces one finds that the operator L generates an analytic semigroup with uniform decay, and can even be self-adjoint [5]. On the other hand, in polynomially weighted spaces, the operator L only generates a bounded C_0 semigroup. The hypotheses of

Theorem 1.2 cover most physically relevant cases, for which $a_i \sim i^\alpha$ with $\alpha \in (0,1)$. In these cases, Theorem 1.2 shows the operator L actually has approximate point spectrum at every point on the imaginary axis. In particular, the presence of this approximate point spectrum implies that L does not generate an analytic semigroup and that one cannot obtain uniform decay rates in these spaces.

It is our viewpoint that this spectral phenomenon may give insight into some of the difficulties encountered in the study of coagulation-fragmentation equations in a more general setting. Here, in the most natural space $\ell^1(i)$, the operator L has spectrum on the imaginary axis, suggesting that systems of this type should be treated like hyperbolic equations. Indeed, many of the techniques used in studying coagulation-fragmentation equations (such as entropy methods and limits of regularizations [1]) originate in the study of hyperbolic equations. We believe that some of the obstacles for analyzing well-posedness and long-time behavior for coagulation-fragmentation systems may directly relate to "hyperbolic" aspects of the equations.

1.2. Persistence of certain initial data. The next theorem seeks to give more detailed information on the dynamics of the linearized system with 'pulse-like' initial data supported far from the origin, with $0 \ll N_1 < i < N_2$. Below, the *support* of h^0 is the set $\{i : h_i^0 \neq 0\}$.

THEOREM 1.3. *Suppose, in addition to the assumptions (1.17)-(1.20), that $a_i/i^\alpha \to 1$ with $\alpha \in (0,1)$. Let $h^0 = (h_i^0)$ be a non-negative sequence satisfying $\sum Q_i i h_i^0 = 1$ and let*

$$h(t) = e^{Lt} h^0$$

be the solution of equation (1.10) in $\ell^1(Q_i i)$ with initial data h^0. Then there exists $N^* > 0$ with the following property: for any $\varepsilon > 0$, there exist K^* and $\delta > 0$ such that whenever h^0 is supported in $\{N_1 < i < N_2\}$ with $N^* < N_1 < N_2$, then for all times $t < T = \delta N_1^{1-\alpha}$ we have that

$$\sum_{i=1}^{\infty} Q_i i h_i(t) \chi(i,t) > 1 - \varepsilon,$$

where χ has the form

$$\chi(x,t) = \begin{cases} 1 & \text{if } A(N_1, 2t) - K^* < x < A(N_2, t/2) + K^*, \\ 0 & \text{otherwise,} \end{cases}$$

where $A(x,t)$ is the solution of

$$\frac{\partial}{\partial t} A = -(z_s - z) A^\alpha, \quad A(x,0) = x. \tag{1.13}$$

In equation (1.13), z_s is the critical monomer density, given in the definition (1.19) below. The explicit solution of equation (1.13) is

$$A(x,t) = (x^{1-\alpha} - (z_s - z)(1-\alpha)t)^{1/(1-\alpha)}. \tag{1.14}$$

The intuition behind the result in Theorem 1.3 can be explained as follows. After writing $u_i = Q_i i h_i$, we can formally approximate $u(t)$ by solving a model advection-diffusion equation of the form

$$u_t = p(x) u_x + q(x) u_{xx},$$

where $p(x)$ grows like $(z_s - z)x^\alpha$ and $q(x)$ grows like zx^α. If we neglect the diffusion term, we find that the solution is constant along the characteristic curves precisely given by A, and χ then describes how "mass" travels through the model system.

The situation is slightly more complicated in reality, since positivity of the u_i may not be preserved. Of course there are error terms in approximating the evolution of (u_i) by this advection-diffusion equation, but these error terms go to zero for large-enough cluster size i. Thus the result of Theorem 1.3 essentially tells one that this advection mechanism is sufficient to prevent decay of the solution, at least when initial data consists of a pulse well-separated from $i=1$.

We also remark that we do not in fact solve the advection-diffusion equation in proving Theorem 1.3; instead we opt to construct a supersolution and then work exclusively with the discrete equations.

One natural application of the previous theorem is the following corollary.

COROLLARY 1.1. *Suppose, in addition to the assumptions (1.17)-(1.20), that* $a_i/i^\alpha \to 1$ *with* $\alpha \in (0,1)$. *Let* N^* *be given as in the statement of Theorem 1.3. Then for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any* $N > N^*$ *there exists an* $(h_i^0) \in X_1$ *with support in* $\{i < N\}$ *and with* $\|h^0\|_{X_1} = 1$ *satisfying*

$$\|e^{Lt}h^0\|_{X_1} \geq 1 - \varepsilon \tag{1.15}$$

for all $t < \delta N^{1-\alpha}$. *In particular,* $\|e^{Lt}\|_{\mathcal{L}(X_1)} \geq 1$ *for all* $t \geq 0$.

On the other hand, for any $\varepsilon > 0$, *there exists a* $\delta > 0$ *so that for any* $(h_i^0) \in X_1$ *with support in* $\{i < N\}$ *satisfying* $\|h^0\|_{X_1} = 1$ *we have that*

$$\|e^{Lt}h^0\|_{X_1} \leq \varepsilon \tag{1.16}$$

for all $t > \delta N$.

This corollary implies that the estimates in [13] are optimal in the sense that we cannot expect any uniform decay estimates for data in the natural space X_1. We remark that the upper bound (1.16) on perturbation lifetimes is a direct consequence of the decay estimates in exponentially weighted spaces derived in [5], and may not be sharp. The novel contribution here is the lower bound (1.15).

1.3. Cutoff phenomenon. The result of Corollary 1.1 is analogous to the *cutoff phenomenon* in the theory of Markov chains. In short, a Markov chain is said to exhibit a cutoff phenomenon if typical states remain far from equilibrium up to well-quantified time after which a rapid transition to equilibrium occurs. See [6] for a detailed introduction to the subject. Examples of Markov chains exhibiting such behavior include card shuffling[1] [2] and random walks on a hypercube [7].

The case of random walks on a hypercube gives an illustrative example (here we follow the presentation in lecture notes by N. Berestycki [3]; the original treatment was due to Diaconis et al. [7]). The random walk on the hypercube $H_n = \{0,1\}^n$ is the process which at rate 1 selects a random coordinate and changes its value. Clearly the equilibrium distribution is one where each state has probability 2^{-n}. Given that we start at time 0 with a vector of n zeros, we can treat each coordinate as an independent arrival process, and hence the probability of that any given coordinate is 1 at time t is given by

$$\mathbb{P}(N_{t/n} = 1 \bmod 2) = \frac{1}{2}(1 - e^{-2t/n})$$

[1] This is easily remembered by the rule of thumb given in [2] that a deck of cards is not random until shuffled 7 times.

for N_t a Poisson process with rate 1. This implies that the probability of finding any vector with k ones is given by $2^{-n}(1-e^{-2t/n})^k(1+e^{-2t/n})^{n-k}$. After summing, one finds that the ℓ^1 distance $d(\cdot)$ between the probability density at time t and the equilibrium density is given by

$$d(t)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}|(1-e^{-2t/n})(1+e^{-2t/n})-1|.$$

Careful computations reveal that for any small $\varepsilon>0$,

$$\lim_{n\to\infty} d((1/4-\varepsilon)n\log n)=1,$$

while

$$\lim_{n\to\infty} d((1/4+\varepsilon)n\log n)=0.$$

This sharp transition from non-random (when $d=1$) to completely random (where $d=0$) at a well-quantified time (in this case $\frac{1}{4}n\log n$) is known as the cutoff phenomenon.

If we view the linearized Becker–Döring equations as analogous to a continuous time Markov chain, then the inequalities in Corollary 1.1 precisely describe a type of cutoff phenomenon, where the cutoff time depends on the support of the initial data.

We remark that in studying cutoff phenomena, the norm that is used is often critically important. For example, in the case of random walks on a hypercube, if deviations from equilibrium are measured in ℓ^2 then there is no cutoff (in fact the transition matrix is symmetric), but if measured in ℓ^1 a cutoff phenomenon occurs [10, 15]. The analogy continues to hold for the linearized Becker–Döring equations: e^{Lt} is exponentially decaying (and L is self-adjoint) in exponentially-weighted ℓ^2 spaces but displays a persistence phenomenon in polynomially-weighted ℓ^1 spaces.

The analogies actually go even deeper. In [10], Jonsson and Trefethen suggest that for the random walk on a hypercube the 1-pseudospectra help explain the cutoff phenomenon. In the Becker–Döring case, Theorem 1.2 establishes the existence of approximate point spectrum (in the 1-norm) on the imaginary axis. Jonsson and Trefethen also explain the cutoff phenomenon for the random walk on a hypercube case in terms of the overlap of two sliding Gaussians, or in other words in terms of an advection phenomenon. Our results above show that the cutoff times in the Becker–Döring case are similarly explained by advection. In short, the linearized Becker–Döring equations exhibit many of the same features seen in Markov chains that exhibit cutoff.

At this point, we are careful to remark that our cutoff result is not sharp, in the sense that the upper and lower bounds do not match. More delicate analysis would be required to obtain sharp results in this direction.

We also note that our results in this work do not address the cutoff phenomenon in the context of nonlinear Becker–Döring dynamics. Our reason for focusing on the linear case is that it highlights some fundamental obstacles to uniform rates of convergence and the differences between the dynamics in different function spaces, without getting too bogged down in technicalities.

1.4. Assumptions on the coefficients. Consistent with other works on the Becker–Döring equations [9, 13], we will make the following standard assumptions on the model coefficients $(a_i),(b_i)$:

$$a_i>C_1>0 \qquad \text{for all } i\geq 1, \tag{1.17}$$

$$\lim_{i\to\infty}\frac{a_{i+1}}{a_i}=1, \tag{1.18}$$

$$\lim_{i\to\infty}\frac{a_i}{b_i}=:\frac{1}{z_s}\in(0,\infty) \tag{1.19}$$

$$a_i,b_i\le C_2 i \qquad \text{for all } i\ge 1. \tag{1.20}$$

These assumptions are satisfied by many of the coefficients proposed for physical phenomena. For example, these assumptions are satisfied by the coefficients proposed in [14]

$$a_i=i^\alpha,\quad b_i=a_i\left(z_s+\frac{q}{i^{1-\beta}}\right),\quad \alpha\in(0,1],\quad \beta\in[0,1],\quad q>0. \tag{1.21}$$

Using the recursion (1.5) and the assumptions (1.18) and (1.19), it is straightforward to show that

$$\lim_{i\to\infty}\frac{Q_{i+1}}{Q_i}=\frac{z}{z_s}. \tag{1.22}$$

In fact, if we write $Q_i=\tilde{Q}_i z^i$, then z_s and μ_s are given by

$$z_s=\sup\left\{\zeta:\sum_{i=1}^{\infty} i\tilde{Q}_i\zeta^i<\infty\right\},\quad \mu_s=\sup\left\{\sum_{i=1}^{\infty} i\tilde{Q}_i\zeta^i:\zeta<z_s\right\}.$$

Hence the restriction to subcritical data, that is the condition (1.3), implies that $\frac{z}{z_s}<1$, which in turn implies that the Q_i are exponentially decaying.

2. Approximate spectrum

The aim of this section is to prove Theorem 1.2. The main idea is to construct approximate eigenvectors using wide pulses of constant modulus with support far away from $i=1$. A simple version of the theorem, in the case $\lambda=0$, can be found in the first author's thesis, see Theorem 8.2.10 in [12].

Proof. **(Proof of Theorem 1.2.)** Define $\tilde{h}_i$ so that

$$iQ_i\tilde{h}_i=\begin{cases}0 & \text{if } i<N_1\\ \exp\left(\lambda\mathbf{i}(z_s-z)^{-1}\sum_{j=N_1}^{i}a_j^{-1}\right) & \text{if } N_1\le i\le N_2\\ 0 & \text{if } N_2<i\end{cases} \tag{2.1}$$

where $N_1<N_2$ are constants to be determined. Clearly

$$\sum_{i=1}^{\infty} Q_i i^k|\tilde{h}_i|=\sum_{i=N_1}^{N_2} i^{k-1}. \tag{2.2}$$

Furthermore, for $N_1<i<N_2$, letting $w_i:=\exp\left(\lambda\mathbf{i}(z_s-z)^{-1}\sum_{j=N_1}^{i}a_j^{-1}\right)$,

$$Q_i i^k((L\tilde{h})_i-\lambda\mathbf{i}\tilde{h}_i)=i^kQ_i\left(b_i(\tilde{h}_{i-1}-\tilde{h}_i)+a_iQ_1(\tilde{h}_{i+1}-\tilde{h}_i)-\lambda\mathbf{i}\tilde{h}_i\right)$$

$$=i^{k-1}w_i\left(b_i\left(\frac{Q_i i\exp\left(-\frac{\lambda\mathbf{i}}{(z_s-z)a_i}\right)}{Q_{i-1}(i-1)}-1\right)+a_iQ_1\left(\frac{Q_i i\exp\left(\frac{\lambda\mathbf{i}}{(z_s-z)a_{i+1}}\right)}{Q_{i+1}(i+1)}-1\right)-\lambda\mathbf{i}\right)$$

$$=i^{k-1}w_i\left(-b_i+a_{i-1}Q_1\frac{i\exp\left(-\frac{\lambda\mathbf{i}}{(z_s-z)a_i}\right)}{(i-1)}-a_iQ_1+b_{i+1}\frac{i\exp\left(\frac{\lambda\mathbf{i}}{(z_s-z)a_{i+1}}\right)}{(i+1)}-\lambda\mathbf{i}\right),$$

where we have used equations (1.12) and (1.5). By using the Taylor expansion of $\exp(w)$ near $w=1$, and recalling that $\frac{i}{i+1}=1+O(i^{-1})$, we may use the definition (1.19) to find that

$$\begin{aligned}&Q_i i^k((L\tilde{h})_i-\lambda\mathbf{i}\tilde{h}_i)\\ &=i^{k-1}w_i\left(a_{i-1}Q_1-b_i+b_{i+1}-a_iQ_1+\lambda\mathbf{i}\left(\frac{-a_{i-1}Q_1}{a_i(z_s-z)}+\frac{b_{i+1}}{a_{i+1}(z_s-z)}-1\right)\right.\\ &\qquad\left.+\,O\left(\frac{1}{a_i}\right)+O\left(\frac{a_i}{i}\right)\right).\end{aligned}$$

Since $a_i-a_{i-1}=o(1)$, we remark that (by properties of Cesáro means)

$$\frac{a_i}{i}=\frac{a_1+\sum_{j=1}^{i-1}(a_{j+1}-a_j)}{i}\to 0. \tag{2.3}$$

Hence, using the assumptions that $a_i-a_{i-1}=o(1)$, $b_i-b_{i-1}=o(1)$ and $a_i\to\infty$ we find that

$$Q_i i^k((L\tilde{h})_i-\lambda\mathbf{i}\tilde{h}_i)=i^{k-1}w_i\left(\lambda\mathbf{i}\left(\frac{-a_{i-1}Q_1}{a_i(z_s-z)}+\frac{b_{i+1}}{a_{i+1}(z_s-z)}-1\right)+o(1)\right)$$

Recalling the assumptions (1.18) and (1.19), and that $Q_1=z$, we find that for any $\delta>0$ we may choose N_1 large enough that

$$Q_i i^k|(L\tilde{h})_i-\lambda\mathbf{i}\tilde{h}_i|\le\delta i^{k-1} \tag{2.4}$$

for all $N_1<i<N_2$.

On the other hand, for any $i>1$ we have, by the limits (1.19) and (1.22), that

$$\begin{aligned}Q_i i^k|(L\tilde{h})_i-\lambda\mathbf{i}\tilde{h}_i|&=Q_i i^k\left|b_i(\tilde{h}_{i-1}-\tilde{h}_i)+a_iQ_1(\tilde{h}_{i+1}-\tilde{h}_i)-\lambda\mathbf{i}\tilde{h}_i\right|\\ &\le i^k\left(\left|\frac{b_iQ_i}{Q_{i-1}(i-1)}\right|+\left|\frac{b_i}{i}\right|+\left|\frac{a_iQ_1Q_i}{Q_{i+1}(i+1)}\right|+\left|\frac{a_iQ_1}{i}\right|+\left|\frac{\lambda}{i}\right|\right)\le Ci^{k-1}a_i,\end{aligned}$$

where C is independent of i,N_1, and N_2. Equation 2.3 then implies that for any $\delta>0$ we may choose a N_1 large enough that for all $i>1$ we have

$$Q_i i^k|(L\tilde{h})_i-\lambda\mathbf{i}\tilde{h}_i|<\delta i^k. \tag{2.5}$$

Similarly, for $i=1$ by the limits (1.18) and (1.22) we find that

$$\begin{aligned}|Q_1(L\tilde{h})_1-\lambda\mathbf{i}\tilde{h}_1|&=\left|\sum_{j=1}^{\infty}a_jQ_jQ_1(\tilde{h}_{j+1}-\tilde{h}_j)\right|\\ &\le C\sum_{j=N_1}^{N_2}\frac{a_j}{j},\end{aligned}$$

with C independent of N_1, N_2. Hence, by equation (2.3), for any $\delta > 0$ we may again choose N_1 large enough that

$$|Q_1(L\tilde{h})_1 - \lambda \mathbf{i}\tilde{h}_1| < \delta(N_2 - N_1). \tag{2.6}$$

Choosing $N_2 = 2N_1$ it is straightforward to show that

$$C_1 N_1^k < \sum_{i=N_1}^{N_2} i^{k-1} \leq C_2 N_1^k. \tag{2.7}$$

By inequalities (2.4), (2.5), (2.6) and (2.7), we have that

$$\begin{aligned}\sum_{i=1}^{\infty} Q_i i^k |(L\tilde{h})_i - \lambda \mathbf{i}\tilde{h}_i| \leq & |Q_1(L\tilde{h})_1 - \lambda \mathbf{i}\tilde{h}_1| + |Q_{N_1-1}(L\tilde{h})_{N_1-1} - \lambda \mathbf{i}\tilde{h}_{N_1-1}| \\ & + |Q_{N_1}(L\tilde{h})_{N_1} - \lambda \mathbf{i}\tilde{h}_{N_1}| + |Q_{N_2}(L\tilde{h})_{N_2} - \lambda \mathbf{i}\tilde{h}_{N_2}| \\ & + |Q_{N_2+1}(L\tilde{h})_{N_2+1} - \lambda \mathbf{i}\tilde{h}_{N_2+1}| + \sum_{i=N_1+1}^{N_2-1} Q_i i^k |(L\tilde{h})_i - \lambda \mathbf{i}\tilde{h}_i| \\ \leq & \delta N_1 + 4\delta N_1^k + \delta C_2 N_1^k,\end{aligned}$$

where $\delta \to 0$ as $N_1 \to \infty$. On the other hand, by equations (2.2) and (2.7) we have that

$$\sum_{i=1}^{\infty} Q_i i^k |\tilde{h}_i| \geq C_1 N_1^k.$$

Hence we find that

$$\frac{\sum_{i=1}^{\infty} Q_i i^k |(L\tilde{h})_i - \lambda \mathbf{i}\tilde{h}_i|}{\sum_{i=1}^{\infty} Q_i i^k |\tilde{h}_i|} \to 0 \quad \text{as } N_1 \to \infty. \tag{2.8}$$

Using the sequence (2.1), we can construct two pulses supported on $[N_1, N_2]$ and $[\tilde{N}_1, \tilde{N}_2]$ with $N_2 < \tilde{N}_1$ and so that the left-hand side of equation (2.8) is arbitrarily small for each pulse. By then summing the two pulses together (after scaling so that the mass constraint is satisfied), we obtain the desired result. This completes the proof. ∎

3. Cutoff phenomenon

The aim of this section is to prove Theorem 1.3. Before we begin the proof, we will give some definitions and recall key facts.

We recall that L generates a semigroup of contractions on $\ell^2(Q_i)$ (see Section 2 in [5]). We also recall that L generates a bounded semigroup on X_1, namely the zero mass subspace of $\ell^1(Q_i i)$ (see Theorem 2.11 in [13]). In the proof, it will be necessary to consider semigroups on the space $\ell^1(Q_i i)$, not X_1. To this end, note that $\xi_i = i/\sum_{i=1}^{\infty} Q_i i^2$ is an eigenvector (with eigenvalue 0) of the operator L, normalized in $\ell^1(Q_i i)$. Furthermore, the linear mapping

$$h \mapsto \mu(h) = \sum_{i=1}^{\infty} Q_i i h_i$$

is continuous on the space $\ell^1(Q_i i)$. Hence for $h \in \ell^1(Q_i i)$ we can write

$$e^{Lt} h = e^{Lt}(h - \xi\mu(h)) + \xi\mu(h).$$

Because $h-\xi\mu(h)\in X_1$, it is then straightforward to estimate

$$\begin{aligned}\|e^{Lt}h\|_{\ell^1(Q_ii)} &\le \|e^{Lt}(h-\xi\mu(h))\|_{\ell^1(Q_ii)}+\|\xi\mu(h)\|_{\ell^1(Q_ii)}\\ &\le M\|(h-\xi\mu(h))\|_{\ell^1(Q_ii)}+C\|h\|_{\ell^1(Q_ii)}\le C\|h\|_{\ell^1(Q_ii)}.\end{aligned}$$

We also remark that e^{Lt} preserves mass, in the sense that for any $h\in\ell^1(Q_ii)$

$$\sum_{i=1}^{\infty}Q_ii(e^{Lt}h)_i=\sum_{i=1}^{\infty}Q_iih_i. \tag{3.1}$$

The strategy in proving Theorem 1.3 is to approximate the operator L by a tridiagonal operator $\tilde{L}$ which is in "divergence" form in mass-weighted variables. We recall equation (1.12), which states that for $i>1$,

$$Q_i(Lh)_i=Q_iQ_1a_i(h_{i+1}-h_i-h_1)-Q_ib_i(h_i-h_{i-1}-h_1),$$

while for $i=1$,

$$Q_1(Lh)_1=Q_1^2a_1(h_2-2h_1)+\sum_{i=1}^{\infty}Q_iQ_1a_i(h_{i+1}-h_i-h_1).$$

We define the operator $\tilde{L}$ by

$$(\tilde{L}h)_i:=\begin{cases}Q_1a_i(h_{i+1}-h_i)-\frac{(i-1)}{i}b_i(h_i-h_{i-1}) & \text{for } i>1\\ Q_1a_1(h_2-h_1) & \text{for } i=1.\end{cases}$$

PROPOSITION 3.1. *The operator $L-\tilde{L}$ is a bounded operator in $\ell^2(Q_i)$ and in $\ell^1(Q_i)$.*

Proof. In the $\ell^2(Q_i)$ case we compute

$$\begin{aligned}\|(L-\tilde{L})h\|_{\ell^2(Q_i)} &\le \left(\sum_{i=1}^{\infty}Q_i\left((b_i-Q_1a_i)h_1+\frac{b_i}{i}h_{i-1}-\frac{b_i}{i}h_i\right)^2\right)^{1/2}\\ &\quad +Q_1^{1/2}\left|\sum_{i=1}^{\infty}Q_ia_i(h_{i+1}-h_i-h_1)\right|+Q_1^{1/2}|Q_1a_1h_1|\\ &\le C\|h\|_{\ell^2(Q_i)},\end{aligned}$$

where we have used the assumption that $b_i/i\to 0$, the Cauchy–Schwarz inequality, and the fact that $\sum_{i=1}^{\infty}Q_ia_i^2<\infty$. In the $\ell^1(Q_ii)$ case we compute

$$\begin{aligned}\|(L-\tilde{L})h\|_{\ell^1(Q_ii)} &\le \sum_{i=1}^{\infty}Q_ii\left|(b_i-Q_1a_i)h_1+\frac{b_i}{i}h_{i-1}-\frac{b_i}{i}h_i\right|\\ &\quad +Q_1\left|\sum_{i=1}^{\infty}Q_ia_i(h_{i+1}-h_i-h_1)\right|+Q_1|Q_1a_1h_1|\\ &\le C\|h\|_{\ell^1(Q_ii)},\end{aligned}$$

where we have used the fact that $\sum_{i=1}^{\infty}Q_iia_i<\infty$ and that $\frac{a_i}{i}$ and $\frac{b_i}{i}$ go to zero. ∎

It will be more convenient throughout the proof to work with the mass-weighted variables defined by $v_i = Q_i i h_i$, so that

$$\sum_{i=1}^{\infty} Q_i i |h_i| = \sum_{i=1}^{\infty} |v_i|.$$

We can write $v = \mathcal{I}h$ in terms of the diagonal operator $\mathcal{I}$ with entries $Q_i i$. Then we can express the operator L in these new coordinates via similarity transformation as

$$\boldsymbol{L}v := \mathcal{I}L\mathcal{I}^{-1}v.$$

Written explicitly, the operator $\boldsymbol{L}$ is given by

$$(\boldsymbol{L}v)_i = \begin{cases} i(a_{i-1}Q_{i-1} - a_iQ_i)v_1 - (a_iQ_1 + b_i)v_i + a_{i-1}Q_1\frac{i}{i-1}v_{i-1} + b_{i+1}\frac{i}{i+1}v_{i+1}, & i > 1, \\ \sum_{i=1}^{\infty}\frac{b_{i+1}}{i+1}v_{i+1} - \frac{a_iQ_1}{i}v_i - a_iQ_iv_1, & i = 1. \end{cases} \tag{3.2}$$

Similarly, in these coordinates, letting

$$\tilde{\boldsymbol{L}}v := \mathcal{I}\tilde{L}\mathcal{I}^{-1}v,$$

we find that

$$(\tilde{\boldsymbol{L}}v)_i = \begin{cases} a_{i-1}Q_1v_{i-1} - a_iQ_1v_i + b_{i+1}\frac{i}{i+1}v_{i+1} - b_i\frac{i-1}{i}v_i & \text{for } i > 1, \\ -a_1Q_1v_1 + b_2\frac{1}{2}v_2 & \text{for } i = 1. \end{cases} \tag{3.3}$$

These expressions show that $\tilde{\boldsymbol{L}}v$ takes the form of a discrete "divergence," a fact that will be useful below.

For the remainder of the section, we will consider initial data $(h^0) \in \ell^1(Q_i i)$, and set

$$h = e^{Lt}h^0, \qquad u^0 = \mathcal{I}h^0 = (Q_i i h_i^0), \qquad u(t) = \mathcal{I}h(t) = e^{\boldsymbol{L}t}u^0.$$

Clearly

$$\sum_{i=1}^{\infty}|u_i| = \sum_{i=1}^{\infty} Q_i i |h_i|.$$

Given initial data u^0 as in the assumptions, we let

$$v(t) := e^{\tilde{\boldsymbol{L}}t}u^0 = \mathcal{I}e^{\tilde{L}t}\mathcal{I}^{-1}u^0,$$

and note that

$$\frac{d}{dt}\sum_{j=1}^{i} v_i = \sum_{j=1}^{i}(\tilde{\boldsymbol{L}}v)_i = -a_iQ_1v_i + b_{i+1}v_{i+1}\frac{i}{i+1}. \tag{3.4}$$

Setting

$$V_i := \sum_{j=1}^{i} v_i,$$

and with the convention that $V_0=0$, we find then that

$$\frac{d}{dt}V_i=(\mathbb{L}V)_i, \tag{3.5}$$

where the operator $\mathbb{L}$ is defined by

$$(\mathbb{L}V)_i:=-a_iQ_1(V_i-V_{i-1})+b_{i+1}(V_{i+1}-V_i)\frac{i}{i+1}.$$

The following proposition establishes a basic property of the V_i.

PROPOSITION 3.2. *If u^0 has compact support and $\sum_{i=1}^{\infty}Q_i i h_i^0=1$, then $\lim_{i\to\infty}V_i(t)=1$ for all t.*

Proof. As L generates a semigroup of contractions on $\ell^2(Q_i)$, and as $\tilde{L}$ is a bounded perturbation of L in $\ell^2(Q_i)$, we then have that $e^{\tilde{L}t}$ also generates a semigroup on $\ell^2(Q_i)$ with bound Ce^{Ct}. Thus for any $t>0$, $e^{\tilde{L}t}h^0$ is an element of $\ell^2(Q_i)$. In other words, we have that

$$\sum_{i=1}^{\infty}\frac{v_i^2(t)}{Q_i i^2}=\sum_{i=1}^{\infty}Q_ih_i^2(t)<Ce^{Ct}.$$

In particular, $|v_i|<CiQ_i^{1/2}$ for all $t\in[0,T]$. Thus for any $T>0$, and any $t\in[0,T]$ we have that

$$|(\mathbb{L}V)_i|\le\left|-a_iQ_1v_i(t)+b_{i+1}v_{i+1}(t)\frac{i}{i+1}\right|\le Ci^{1+\alpha}(Q_i)^{1/2}.$$

As the Q_i decay exponentially, see equation (1.22), this goes to zero as $i\to\infty$. This implies that, uniformly for $t\in[0,T]$,

$$\lim_{i\to\infty}|V_i(t)-V_i(0)|\le\lim_{i\to\infty}\int_0^t|(\mathbb{L}V(s))_i|\,ds=0.$$

As $\lim_{i\to\infty}V_i(0)=1$ (since $\sum_{i=1}^{\infty}Q_iih_i^0=1$), we then have that uniformly for $t\in[0,T]$,

$$\lim_{i\to\infty}V_i(t)=1. \tag{3.6}$$

□

Next we prove a minimum principle for $\mathbb{L}$.

PROPOSITION 3.3. *$\mathbb{L}$ satisfies a minimum principle in the sense that if*

$$\partial_tW_i-(\mathbb{L}W)_i\ge0, \qquad \textit{for all } i\in\mathbb{N},\ t\in[0,T], \tag{3.7}$$

then for any N

$$\min_{i\in1\ldots N,t\in[0,T]}W_i(t)\ge\min\left(\min_{i\in1\ldots N}W_i(0),\ \min_{t\in[0,T]}W_N(t),\ \min_{t\in[0,T]}W_1(t)\right).$$

Proof. Suppose that there exists some $j\in2\ldots N-1$ and $\hat{t}\in(0,T]$ so that $W_j(\hat{t})\le W_i(t)$ for all $i\in1\ldots N$ and $t\in[0,T]$. Then $\partial_tW_j(\hat{t})\le0$, which in turn implies that $(\mathbb{L}W(\hat{t}))_j\le0$. However, the form of the differences in $\mathbb{L}W$ (see equation (3.5)), and the fact that $W_j(\hat{t})$ is a minimizer, in turn implies that $W_{j-1}(\hat{t})=W_{j+1}(\hat{t})=W_j(\hat{t})$. By repeating this argument, we find that $W_N(\hat{t})=W_1(\hat{t})=W_j(\hat{t})$, which implies that the bound holds. ∎

Exactly the same proof gives a maximum principle as well, in the sense that if

$$\partial_t W_i-(\mathbb{L}W)_i\le 0, \qquad \text{for all } i\in\mathbb{N},\ t\in[0,T]$$

then for any N

$$\max_{i\in 1\dots N, t\in[0,T]} W_i(t)\le \max\left(\max_{i\in 1\dots N} W_i(0),\ \max_{t\in[0,T]} W_N(t),\ \max_{t\in[0,T]} W_1(t)\right).$$

With these comparison principles in hand, we now turn to proving Theorem 1.3. This is accomplished by constructing supersolutions, and then using the minimum principle to obtain good controls on the support of $e^{\tilde{\boldsymbol{L}}t}u^0$. The proof is completed by using Duhamel's formula to establish suitable estimates on $(e^{\boldsymbol{L}t}-e^{\tilde{\boldsymbol{L}}t})u^0$.

Proof. **(Proof of Theorem 1.3).**

Step 1: Supersolutions. Define the function

$$W^1(x,t):=\begin{cases}\exp\left(\frac{x-A(N_1,2t)}{D}\right) & \text{for } x<A(N_1,2t),\\ 1 & \text{otherwise,}\end{cases} \tag{3.8}$$

where $D>0$ is a constant that will be determined later.

We then claim that there exists some N^* (independent of N_1) so that $W_i^1(t):=W^1(i,t)$ is a supersolution (i.e., the inequality (3.7) is satisfied) for all $t>0$ satisfying $A(N_1,2t)>N^*$. Clearly for $i>A(N_1,2t)+1$, $(\mathbb{L}W^1)_i=0$ and $\partial_t(W_i^1)=0$, and hence the inequality (3.7) is satisfied trivially. In the region $i<A(N_1,2t)-1$ we compute:

$$\begin{aligned}\partial_t W_i^1-(\mathbb{L}W^1)_i&=\frac{2(z_s-z)A(N_1,2t)^\alpha W_i^1}{D}+a_iQ_1W_i^1(1-e^{-D^{-1}})-W_i^1\frac{b_{i+1}i}{i+1}(e^{D^{-1}}-1)\\ &\ge W_i^1\left(\frac{2(z_s-z)A(N_1,2t)^\alpha}{D}+\frac{a_iQ_1-\frac{b_{i+1}i}{i+1}}{D}-\frac{Ci^\alpha}{D^2}\right)\\ &=W_i^1\left(\frac{2(z_s-z)A(N_1,2t)^\alpha}{D}+\frac{i^\alpha(z-z_s)}{D}+\frac{o(i^\alpha)}{D}-\frac{Ci^\alpha}{D^2}\right),\end{aligned}$$

where on the last line we have used the assumption that $a_i/i^\alpha\to 1$, as well as equation (1.19). We note that as long as $A(N_1,2t)>N^*$, with N^* independent of N_1,N_2 (and D), the $o(i^\alpha)$ term will be dominated by $(z_s-z)A(N_1,2t)^\alpha/2D$ for all $i<A(N_1,2t)$. Hence for D chosen large enough that $\frac{C}{D}<\frac{z_s-z}{2}$, and for $i<A(N_1,2t)-1$ we will have that

$$\partial_t W_i^1-(\mathbb{L}W^1)_i\ge 0.$$

It only remains to prove the boundary cases. If $A(N_1,2t)<i<A(N_1,2t)+1$, then

$$\partial_t W_i^1-(\mathbb{L}W^1)_i=-(\mathbb{L}W^1)_i=a_iQ_1(W_i^1-W_{i-1}^1)\ge 0.$$

In the case $A(N_1,2t)-1<i<A(N_1,2t)$, then

$$\partial_t W_i^1-(\mathbb{L}W^1)_i$$

$$
\begin{aligned}
&= \frac{2(z_s - z)A(N_1, 2t)^\alpha W_i^1}{D} + a_i Q_1 W_i^1 (1 - e^{-D^{-1}}) - \frac{b_{i+1} i}{i+1}(W_{i+1}^1 - W_i^1) \\
&\geq \frac{2(z_s - z)A(N_1, 2t, N_1)^\alpha W_i^1}{D} + a_i Q_1 W_i^1 (1 - e^{-D^{-1}}) - W_i^1 \frac{b_{i+1} i}{i+1}(e^{D^{-1}} - 1) \\
&= W_i^1 \left(\frac{2(z_s - z)A(N_1, 2t)^\alpha}{D} + \frac{i^\alpha (z - z_s)}{D} + \frac{o(i^\alpha)}{D} - \frac{C i^\alpha}{D^2} \right) \geq 0.
\end{aligned}
$$

Hence (W^1) is a supersolution in the sense that it satisfies the inequality (3.7).

Similarly, if we define

$$
W^2(x,t) := \begin{cases} 1 & \text{if } x < A(N_2, t/2), \\ \exp\left(\frac{A(N_2, t/2) - x}{D} \right) & \text{otherwise.} \end{cases}
$$

we can use exactly the same type of estimates to show that that $W_i^2(t) := W^2(i,t)$ is also a supersolution in the sense that the inequality (3.7) is satisfied, as long as $A(N_1, 2t) > N^*$.

Step 2: Support bounds for v. We claim that $W_i^1(t) \geq V_i(t)$ for all $i \in \mathbb{N}$ and $t > 0$ such that $A(N_1, 2t) > N^*$. First, we note that since $u(0)$ is zero for $i < N_1$, then $V(0)$ is also zero for $i < N_1$. Also, since $u(0)$ is positive, $V(0)$ is increasing, and by Proposition 3.2 we have that $V_i(0) \leq 1$. Thus because $A(N_1, 0) = N_1$, the definition of $W^1(0)$ implies that $W^1(0) \geq V(0)$. Furthermore, by equations (3.6) and (3.8) we have that

$$
\lim_{i \to \infty} W_i^1(t) - V_i(t) = 0,
$$

for all $t > 0$ such that $A(N_1, 2t) > N^*$. Since $W^1 - V$ is a supersolution, the minimum principle then implies that

$$
\min_{i \in \mathbb{N}, t \in [0,T]} W_i^1(t) - V_i(t) \geq \min \left(0, \min_{t \in [0,T]} W_1^1(t) - V_1(t) \right).
$$

Suppose for the sake of contradiction that

$$
\min_{t \in [0,T]} W_1^1(t) - V_1(t) = W_1^1(\hat{t}) - V_1(\hat{t}) < 0
$$

for some $\hat{t} \in (0, T]$. Clearly $\partial_t (W_1^1 - V_1)(\hat{t}) \leq 0$, and as $W^1 - V$ is a supersolution then $(\mathbb{L}(W^1 - V))_1 \leq 0$. The definition of $\mathbb{L}$, along with the fact that $W^1 - V$ is minimized at $j = 1$, $t = \hat{t}$, then gives that

$$
\begin{aligned}
0 &\geq (\mathbb{L}(W^1 - V))_1 = -a_i Q_1 (W_1^1 - V_1) + b_{i+1}((W_2^1 - V_2) - (W_1^1 - V_1)) \frac{i}{i+1} \\
&\geq -a_i Q_1 (W_1^1 - V_1) > 0,
\end{aligned}
$$

which is a contradiction. This then implies that

$$
\min_{i \in \mathbb{N}, t \in [0,T]} W_i^1(t) - V_i(t) \geq 0,
$$

which is the desired conclusion.

This readily implies that for any ε there exists a K^* (which depends only upon D and ε) so that for all t small enough that $A(N_1,2t)>N^*$ we have that

$$\sum_{i=1}^{\lceil A(N_1,2t)-K^*\rceil} v_i(t)=V_{\lceil A(N_1,2t)-K^*\rceil}\le W^1_{\lceil A(N_1,2t)-K^*\rceil}<\frac{\varepsilon}{2}.$$

On the other hand, $W^2(0)\ge 1-V(0)$. Hence by the minimum principle, $W^2(t)\ge 1-V(t)$ for all $t>0$ such that $A(N_1,2t)>N^*$, which thus implies, for all such t, that

$$\sum_{i=\lceil A(N_2,t/2)+K^*\rceil}^{\infty} v_i(t)<\frac{\varepsilon}{2}.$$

Next, we observe that the coefficient of v_i in $(\tilde{\boldsymbol{L}}v)_i$ is negative, whereas all of the other coefficients are positive, see equation (3.3). This readily implies that if $v^0\ge 0$ then $v_i(t)\ge 0$ for all $i\in\mathbb{N},t\ge 0$. In turn, we may use the previous inequality to deduce that for all $t>0$ such that $A(N_1,2t)>N^*$,

$$\|v-\chi v\|_{\ell^1}<\varepsilon.$$

Step 3: Duhamel estimates. By Duhamel's formula, one has that

$$v(t)=e^{\tilde{\boldsymbol{L}}t}u^0=e^{\boldsymbol{L}t}u^0+\int_0^t e^{\boldsymbol{L}(t-s)}(\tilde{\boldsymbol{L}}-\boldsymbol{L})v(s)\,ds. \tag{3.9}$$

This formula is valid because $\tilde{\boldsymbol{L}}-\boldsymbol{L}$ is bounded in ℓ^1 (see Proposition 3.1) and because $\boldsymbol{L}$ generates a semigroup on ℓ^1. See [8, Corollary III.1.7] for a precise statement of Duhamel's formula in this setting.

We seek to derive some simple bounds on the last term, which will provide the estimates we need. To that end, we estimate

$$\begin{aligned}\left\|\int_0^t e^{\boldsymbol{L}(t-s)}(\tilde{\boldsymbol{L}}-\boldsymbol{L})v(s)\,ds\right\|_{\ell^1}&\le M\int_0^t\|(\tilde{\boldsymbol{L}}-\boldsymbol{L})v(s)\|_{\ell^1}\,ds\\ \le M\int_0^t\sum_{i=2}^{\infty}\left|v_1 i(-a_{i-1}Q_{i-1}+a_iQ_i)+\frac{b_i}{i}v_i-\frac{a_{i-1}Q_1}{i-1}v_{i-1}\right|ds&\\ +M\int_0^t\left|-a_1Q_1v_1+\sum_{i=1}^{\infty}\left(\frac{b_{i+1}}{i+1}v_{i+1}-\frac{a_iQ_1}{i}v_i-a_iQ_iv_1\right)\right|ds,&\end{aligned}$$

where we have used that $e^{\boldsymbol{L}t}$ has uniformly bounded operator norm in ℓ^1 (since e^{Lt} is uniformly bounded in $\ell^1(Q_i i)$). We remark that by our assumption that $a_i/i^\alpha\to 1$, and the fact that the Q_i decay exponentially, we have that

$$\sum_{i=2}^{\infty}\left|v_1 i(-a_{i-1}Q_{i-1}+a_iQ_i)+\frac{b_i}{i}v_i-\frac{a_{i-1}Q_1}{i-1}v_{i-1}\right|\le C\sum_{i=1}^{\infty}i^{\alpha-1}|v_i|.$$

Then using the facts that $V\le W^1$, $v_i(t)\ge 0$, and that $0<\alpha<1$, we can estimate

$$\int_0^t\sum_{i=1}^{\infty}i^{\alpha-1}|v_i|\le\int_0^t\sum_{i=1}^{\lceil A(N_1,2s)/2\rceil}i^{\alpha-1}|v_i|+\sum_{\lceil A(N_1,2s)/2\rceil}^{\infty}i^{\alpha-1}|v_i|\,ds$$

$$\leq C\int_0^t \exp\left(\frac{-A(N_1,2s)}{2D}\right)+\left(\frac{A(N_1,2s)}{2}\right)^{\alpha-1} ds.$$

By the change of variables $s\mapsto A(N_1,2s)$, using equation (1.14) to write $2(z_s-z)ds=-A^{-\alpha}dA$, and assuming $t\leq T$ where $N_T:=A(N_1,2T)>1$, we find that

$$\int_0^t \sum_{i=1}^\infty i^{\alpha-1}|v_i| \leq C\int_{N_T}^{N_1}\left(A^{-\alpha}\exp\left(\frac{-A}{2D}\right)+A^{-1}\right)dA$$
$$\leq C\left(N_T^{1-\alpha}\exp\left(\frac{-N_T}{2D}\right)\left(\frac{N_1}{N_T}-1\right)+\log\left(\frac{N_1}{N_T}\right)\right).$$

Similarly, we note that

$$\left|-a_1Q_1v_1+\sum_{i=1}^\infty \frac{b_{i+1}}{i+1}v_{i+1}-\frac{a_iQ_1}{i}v_i-a_iQ_iv_1\right|\leq C\sum_{i=1}^\infty i^{\alpha-1}|v_i|,$$

and hence we find that

$$\left\|\int_0^t e^{\boldsymbol{L}(t-s)}(\tilde{\boldsymbol{L}}-\boldsymbol{L})v(s)\,ds\right\|_{\ell^1}\leq C\left(\left(\frac{N_1}{N_T}-1\right)+\log\left(\frac{N_1}{N_T}\right)\right),$$

where C is independent of v. Hence, recalling equation (1.14), we find that for any $\varepsilon>0$ there exists a δ so that for $t<\delta N_1^{1-\alpha}$ we have that

$$\left\|\int_0^t e^{\boldsymbol{L}(t-s)}(\tilde{\boldsymbol{L}}-\boldsymbol{L})v(s)\,ds\right\|_{\ell^1}<\varepsilon.$$

In turn, by Duhamel's formula (3.9), this immediately implies that for all $t<\delta N_1^{1-\alpha}$,

$$\|u-v\|_{\ell^1}\leq\varepsilon.$$

Step 4: Cutoff estimates for u. Using the triangle inequality, and the support estimates on v from Step 3, we find that

$$\|u-\chi u\|_{\ell^1}\leq\|u-v\|_{\ell^1}+\|v-\chi v\|_{\ell^1}+\|\chi(v-u)\|_{\ell^1}\leq 3\varepsilon$$

for any $t<\delta N_1^{1-\alpha}$. This, in light of equation (3.1), finishes the proof of Theorem 1.3. □

Finally, we prove Corollary 1.1, as a natural consequence of Theorem 1.3.

Proof. **(Proof of Corollary 1.1.)** Let

$$u_i^1=\begin{cases}2/N & \text{for } i\in[N/4,N/2],\\ 0 & \text{otherwise.}\end{cases}$$
$$u_i^2=\begin{cases}2/N & \text{for } i\in[3N/4,N],\\ 0 & \text{otherwise.}\end{cases}$$

Let χ^1 and χ^2 be the functions associated with u^1 and u^2 from Theorem 1.3. By the form of A, for some $\hat{\delta}>0$ independent of N we know that $\chi^1\chi^2=0$ for all $t<\hat{\delta}N^{1-\alpha}$.

Let $u^0=u^1-u^2$. By the form of A, that is equation (1.14), along with Theorem 1.3, we know that

$$\|e^{\boldsymbol{L}t}u^0\|_{\ell^1}\geq\sum_{i=1}^\infty \chi^1|(e^{\boldsymbol{L}t}(u^1-u^2))_i|+\sum_{i=1}^\infty\chi^2|(e^{\boldsymbol{L}t}(u^1-u^2))_i|$$

$$\geq \sum_{i=1}^{\infty} \chi^1|(e^{\boldsymbol{L}t}u^1)_i| - \chi^1|(e^{\boldsymbol{L}t}u^2)_i| + \chi^2|(e^{\boldsymbol{L}t}u^2)_i| - \chi^2|(e^{\boldsymbol{L}t}u^1)_i|$$
$$\geq 1-\varepsilon.$$

This proves the first part of the corollary.

For the second part, we recall (see Section 3 in [5]) that e^{Lt} generates a semigroup on the space

$$Y_\eta := \left\{ h : \|h\|_\eta < \infty, \quad \|h\|_\eta := \sum_{i=1}^{\infty} Q_i e^{\eta i} h_i, \quad \sum_{i=1}^{\infty} Q_i i h_i = 0 \right\}$$

as long as η is sufficiently small, and that e^{Lt} satisfies the bound

$$\|e^{Lt}h^0\|_\eta \leq Ce^{-\lambda t}\|h^0\|_\eta$$

for some $\lambda > 0$, and for all $h \in Y_\eta$.

We then note that any element of X_1 with support in $\{i < N\}$ will also be an element of Y_η, and will satisfy

$$\|h^0\|_\eta = \sum_{i=1}^{N} Q_i e^{i\eta}|h_i^0| \leq e^{N\eta} \sum_{i=1}^{\infty} Q_i i |h_i^0| = e^{N\eta}.$$

Hence we find that

$$\|e^{Lt}h^0\|_1 \leq C\|e^{Lt}h^0\|_\eta \leq Ce^{-\lambda t}\|h^0\|_\eta \leq Ce^{-\lambda t + N\eta}.$$

The result then follows. □

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