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REGULARLY DECOMPOSABLE TENSORS AND CLASSICAL SPIN STATES*

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Abstract. A spin-j state can be represented by a symmetric tensor of order $N=2j$ and dimension 4. Here, j can be a positive integer, which corresponds to a boson; j can also be a positive half-integer, which corresponds to a fermion. In this paper, we introduce regularly decomposable tensors and show that a spin-j state is classical if and only if its representing tensor is a regularly decomposable tensor. In the even-order case, a regularly decomposable tensor is a completely decomposable tensor but not vice versa; a completely decomposable tensors is a sum-of-squares (SOS) tensor but not vice versa; an SOS tensor is a positive semi-definite (PSD) tensor but not vice versa. In the odd-order case, the first row tensor of a regularly decomposable tensor is regularly decomposable and its other row tensors are induced by the regular decomposition of its first row tensor. We also show that complete decomposability and regular decomposability are invariant under orthogonal transformations, and that the completely decomposable tensor cone and the regularly decomposable tensor cone are closed convex cones. Furthermore, in the even-order case, the completely decomposable tensor cone and the PSD tensor cone are dual to each other. The Hadamard product of two completely decomposable tensors is still a completely decomposable tensor. Since one may apply the positive semi-definite programming algorithm to detect whether a symmetric tensor is an SOS tensor or not, this gives a checkable necessary condition for classicality of a spin-j state. Further research issues on regularly decomposable tensors are also raised.

Keywords. positive semi-definite tensors; sum-of-squares tensors; quantum entanglement; spin states; bosons; fermions; classicality.

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1. Introduction

A geometrical picture of quantum states often helps getting some insight on underlying physical properties. For arbitrary pure spin states, such a geometrical representation was developed by Ettore Majorana [1]: a spin-j state is visualized as $N=2j$ points on the unit sphere S^2, called in this context the Bloch sphere. The advantage of such a picture is a direct interpretation of certain unitary operations: namely, if a quantum spin-j state is mapped to another one by a unitary operation that correspond to a $(2j+1)$-dimensional representation of a spatial rotation, its Majorana points are mapped to points obtained by that spatial rotation. Recently a tensor representation of

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an arbitrary mixed or pure spin-j state was proposed that generalizes this picture [2]. It consists of a real symmetric tensor of order $N = 2j$ and dimension 4. A spin-j state corresponds to a boson if j is a positive integer, and corresponds to a fermion if j is a positive half-integer. Thus, a boson corresponds to an even-order four dimensional tensor, while a fermion corresponds to an odd order four dimensional tensor.

The geometrical picture is particularly useful when it comes to studying classicality properties of spin states. In quantum optics, coherent states are quantum states that behave most classically, in that they minimize the uncertainty relation between position and momentum. Coherent states can also be defined in the context of spins. Statistical mixtures of coherent states can thus be considered the "least quantum" states. The set of classical spin states was introduced in [3] as the convex hull of the set of coherent spin states. It can be interpreted (see e.g. [4]) as the set of fully separable states in the symmetric sector of the tensor product of $2j$ spins-1/2. The above geometric picture easily allows one to characterize coherent spin states: a coherent spin-j state can be represented by $N = 2j$ points located at the same position on the Bloch sphere. The characterization of classical states is less easy to obtain, but the tensorial picture helps to get some results on this issue. For instance, in [4] it was shown that when j is an integer, i.e., N is an even number, a classical spin-j state is such that its representing tensor is positive semi-definite (PSD) in the sense of [5] (see Section 2).

Positive semi-definiteness of the tensor representation is a necessary and sufficient condition of classicality in the case $j = 1$ [6]. It is only a necessary condition for classicality of a spin-j state, and only if j is a positive integer, as pointed out in [4]. A natural question is therefore whether it is possible to formulate a necessary and sufficient condition for classicality of a spin-j state in terms of its tensor representation, first in the case where j is a positive integer, i.e., the boson case, and then in the case where j is a half-integer, i.e., the fermion case. The aim of this paper is to introduce tools in order to reformulate these two questions from a mathematical perspective.

The PSD condition can be expressed in terms of tensor eigenvalues. A tensor is PSD if and only if its smallest H-eigenvalue or Z-eigenvalue is nonnegative [5]. This links classicality of a spin-j state (with j as a positive integer) with the smallest tensor eigenvalue of its representing tensor. This result echoes the result of [7], which stated that the geometric measure of entanglement of a pure state is equal to the largest tensor eigenvalue. Note that tensor eigenvalues have found applications in different areas of physics [8–11]. To go beyond the PSD condition for classicality, we have to consider stronger properties. A property stronger than positive semi-definiteness is the sum-of-squares (SOS) property. SOS tensors were introduced in [12, 13]. According to the Hilbert theory [14], an SOS tensor is a PSD tensor but not vice versa. Both PSD and SOS tensors have been studied intensively in recent years. Some references on PSD and SOS tensors include [15–22]. One can show (see below) that when j is an integer, if a spin-j state is classical, then its representing tensor is an SOS tensor in the sense of [12, 13, 15]. But this is still a necessary condition. A property stronger than the SOS property is complete decomposability. Completely decomposable tensors were introduced and studied in [23, 24]. An even-order completely decomposable tensor is an SOS tensor but not vice versa [23, 24]. Again, when j is an integer, if a spin-j state is classical, then its representing tensor is a completely decomposable tensor, and this is still a necessary condition.

In this paper, we introduce regularly decomposable tensors. A regularly decomposable tensor is a completely decomposable tensor but not vice versa. Furthermore, we define regularly decomposable tensors also in the odd-order case. In the odd-order case, the first row tensor of a regularly decomposable tensor is regularly decomposable and its other row tensors are induced by the regular decomposition of its first row tensor. We show that in both the odd-order (fermion) and even-order (boson) cases a spin-j state is classical if and only if its representing tensor is a regularly decomposable tensor. Thus, it is important to study properties of regularly decomposable tensors and completely decomposable tensors, as well as some further properties of PSD tensors and SOS tensors.

The remaining part of this paper is organized as follows. In Section 2, we review the definitions of PSD, SOS and completely decomposable tensors, and define regularly decomposable tensors. In Section 3, we show that in both the odd-order (fermion) and even-order (boson) cases a spin-j state is classical if and only if its representing tensor is a regularly decomposable tensor. Some properties of completely decomposable tensors and regularly decomposable tensors and their implications in physics are studied in Section 4. Some further research issues on regularly decomposable tensors are raised in Section 5.

2. PSD, SOS, completely decomposable and regularly decomposable tensors

In this paper, for a vector $\mathbf{x}\in\Re^{n+1}$, we denote it as $\mathbf{x}=(x_0,x_1,\dots,x_n)^\top$. Later, in physical applications, we will have $n=3$. Here, we assume that $n\geq 2$. Denote the zero vector in $\Re^{n+1}$ by $\mathbf{0}$.

Let $\mathcal{A}=(a_{i_1\dots i_m})$ be an mth order $(n+1)$-dimensional real tensor. We say that $\mathcal{A}$ is a symmetric tensor if the entries $a_{i_1\dots i_m}$ are invariant under permutation of their indices. Denote $T_{m,n+1}$ as the set of all mth order $(n+1)$-dimensional real tensors, and $S_{m,n+1}$ as the set of all mth order $(n+1)$-dimensional real symmetric tensors. Then $T_{m,n+1}$ is a linear space, and $S_{m,n+1}$ is a linear subspace of $T_{m,n+1}$. Denote the zero tensor in $S_{m,n+1}$ by $\mathcal{O}$.

Let $\mathcal{A}=(a_{i_1\dots i_m})\in T_{m,n+1}$ and $\mathcal{B}=(b_{i_1\dots i_p})\in T_{p,n+1}$. The outer product of $\mathcal{A}$ and $\mathcal{B}$, denoted as $\mathcal{C}=\mathcal{A}\otimes\mathcal{B}$, is a real tensor in $T_{m+p,n+1}$, defined by $\mathcal{C}=(a_{i_1\dots i_m}b_{i_{m+1}\dots i_{m+p}})$. We also denote $\mathcal{A}^{\otimes 2}=\mathcal{A}\otimes\mathcal{A}$, $\mathcal{A}^{\otimes(k+1)}=\mathcal{A}^{\otimes k}\otimes\mathcal{A}$ for $k\geq 2$. A *symmetric rank-one* tensor is defined as a symmetric tensor in $S_{m,n+1}$ of the form $\alpha\mathbf{x}^{\otimes m}$, where $\alpha\in\Re$ and $\mathbf{x}\in\Re^{n+1}$.

Let $\mathcal{A}=(a_{i_1\dots i_m})$ and $\mathcal{B}=(b_{i_1\dots i_m})$ in $S_{m,n+1}$. The inner product of $\mathcal{A}$ and $\mathcal{B}$, denoted as $\mathcal{A}\bullet\mathcal{B}$, is a scalar, defined by

$$\mathcal{A}\bullet\mathcal{B}=\sum_{i_1,\dots,i_m=0}^{n} a_{i_1\dots i_m}b_{i_1\dots i_m}.$$

Let $\mathcal{A}=(a_{i_1\dots i_m})\in S_{m,n+1}$ and $\mathbf{x}\in\Re^{n+1}$. Then we have

$$\mathcal{A}\bullet\mathbf{x}^{\otimes m}\equiv\sum_{i_1,\dots,i_m=0}^{n} a_{i_1\dots i_m}x_{i_1}\dots x_{i_m}.$$

If for any $\mathbf{x}\in\Re^{n+1}$, we have $\mathcal{A}\bullet\mathbf{x}^{\otimes m}\geq 0$, then we say that $\mathcal{A}$ is a **positive semi-definite (PSD)** tensor. If for any $\mathbf{x}\in\Re^{n+1},\mathbf{x}\neq\mathbf{0}$, we have $\mathcal{A}\bullet\mathbf{x}^{\otimes m}>0$, then we say that $\mathcal{A}$ is a **positive definite (PD)** tensor. Clearly, if m is odd, then the only PSD tensor is the zero tensor, and there is no PD tensor. Thus, we only discuss even-order PSD and PD tensors.

Suppose that $m=2l$ is even. Let $\mathcal{A}\in S_{m,n+1}$. If there are symmetric tensors $\mathcal{A}^{(1)},\dots,\mathcal{A}^{(r)}\in S_{l,n+1}$ such that for all $\mathbf{x}\in\Re^{n+1}$,

$$\mathcal{A}\bullet\mathbf{x}^{\otimes m}=\sum_{k=1}^{r}\left(\mathcal{A}^{(k)}\bullet\mathbf{x}^{\otimes l}\right)^2,$$

then $\mathcal{A}$ is called a **sum-of-squares (SOS)** tensor. Then, for any $\mathbf{x}\in\Re^{n+1}$, we have $\mathcal{A}\bullet\mathbf{x}^{\otimes m}\ge 0$. Thus, an SOS tensor is always a PSD tensor, but not vice versa. By the Hilbert theory [14], only in the following three cases: 1) $m=2$, 2) $n=1$, 3) $m=4$ and $n=2$, a PSD tensor is always an SOS tensor; otherwise, there are always PSD tensors which are not SOS tensors. David Hilbert [14] stated this in the language of polynomials. But the meanings are the same.

Let $\mathcal{A}\in S_{m,n+1}$. Here, m can be either even or odd. If there are vectors $\mathbf{u}^{(1)},\dots,\mathbf{u}^{(r)}\in\Re^{n+1}$ such that

$$\mathcal{A}=\sum_{k=1}^{r}\left(\mathbf{u}^{(k)}\right)^{\otimes m},\tag{2.1}$$

then we say that $\mathcal{A}$ is a **completely decomposable tensor**. If all the vectors $\mathbf{u}^{(1)},\dots,\mathbf{u}^{(r)}\in\Re^{n+1}$ are nonnegative vectors, then $\mathcal{A}$ is called a **completely positive** tensor [20, 34]. Actually, all odd-order symmetric tensors are completely decomposable tensors [23]. Thus, the concept of completely decomposable tensors is not useful for odd order. However, if $m=2l$ is even, and $\mathcal{A}$ is a completely decomposable tensor as defined by equation (2.1), then by letting $\mathcal{A}^{(k)}=\left(\mathbf{u}^{(k)}\right)^{\otimes l}$, we see that $\mathcal{A}$ is an SOS tensor. On the other hand, by the examples given in [23, 24], an SOS tensor may not be a completely decomposable tensor.

In order to define regularly decomposable tensors, we still need two more concepts: regular vectors and row-tensors.

DEFINITION 2.1. *Let* $\mathbf{x}=(x_0,x_1,\dots,x_n)^\top\in\Re^{n+1}$. *We say that* $\mathbf{x}$ *is a* **regular vector** *if* $x_0\ne 0$ *and* $x_0^2=x_1^2+\dots+x_n^2$.

DEFINITION 2.2. *For any* $\mathcal{A}=(a_{i_1\dots i_m})\in S_{m,n+1}$, *define its* i*th* **row tensor** $\mathcal{A}_i$ *as a symmetric tensor in* $S_{m-1,n+1}$, *by* $\mathcal{A}_i=(a_{ii_2\dots i_m})$, *for* $i=0,\dots,n$.

We can then define regularly decomposable tensors as follows:

DEFINITION 2.3. (i.) *Let the order* $m=2l$ *be even and* $\mathcal{A}\in S_{m,n+1}$. *If* $\mathcal{A}$ *is a completely decomposable tensor defined by equation (2.1), where* $\mathbf{u}^{(1)},\dots,\mathbf{u}^{(r)}$ *are regular vectors, then we say that* $\mathcal{A}$ *is a* **regularly decomposable tensor** *of even order.*
(ii.) *Let the order* $m=2l+1$ *be odd and* $\mathcal{A}\in S_{m,n+1}$. *If* $\mathcal{A}_0\in S_{2l,n+1}$ *is a regularly decomposable tensor with the regular decomposition*

$$\mathcal{A}_0=\sum_{k=1}^{r}\left(\mathbf{u}^{(k)}\right)^{\otimes 2l},\tag{2.2}$$

where $\mathbf{u}^{(k)}=\left(u_0^{(k)},\dots,u_n^{(k)}\right)^\top$, $k=1,\dots,r$, *are regular vectors, and the other row tensors of* $\mathcal{A}$ *are induced by this regular decomposition,*

$$\mathcal{A}_i=\sum_{k=1}^{r}\frac{u_i^{(k)}}{u_0^{(k)}}\left(\mathbf{u}^{(k)}\right)^{\otimes 2l},\tag{2.3}$$

for $i=1,\ldots,n$, then we say that $\mathcal{A}$ is a **regularly decomposable tensor** *of odd order.*

Clearly an even-order regularly decomposable tensor is a completely decomposable tensor but not vice versa.

THEOREM 2.1. *A regularly decomposable tensor $\mathcal{A}=(a_{i_1\ldots i_m})\in S_{m,n+1}$ can be written as*

$$\mathcal{A}=\sum_{k=1}^{r}\alpha_k\left(\mathbf{v}^{(k)}\right)^{\otimes m}, \tag{2.4}$$

where $\alpha_k>0$ and $\mathbf{v}^{(k)}=\left(1,v_1^{(k)},\ldots,v_n^{(k)}\right)^\top$,

$$\sum_{i=1}^{n}\left(v_i^{(k)}\right)^2=1, \tag{2.5}$$

for $k=1,\ldots,r$. Furthermore, we have

$$a_{00i_3\ldots i_m}=\sum_{i=1}^{n}a_{iii_3\ldots i_m} \tag{2.6}$$

for $m\geq 2$ and all $i_3,\ldots,i_m=0,1,\ldots,n$.

Proof. Suppose that m is even, and $\mathcal{A}$ is defined by equation (2.1), where $\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}$ are regular vectors. Let

$$\mathbf{v}^{(k)}=\frac{\mathbf{u}^{(k)}}{u_0^{(k)}}, \tag{2.7}$$

for $k=1,\ldots,r$. Then we see that $\mathcal{A}$ can be expressed by equation (2.4), where $\alpha_k=\left(u_0^{(k)}\right)^m>0$ and $\mathbf{v}^{(k)}=\left(1,v_1^{(k)},\ldots,v_n^{(k)}\right)^\top$ satisfy (2.5) for $k=1,\ldots,r$. Suppose that $m=2l+1$ is odd, and $\mathcal{A}_0$ is defined by (2.2), where $\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(r)}$ are regular vectors and the other row tensors of $\mathcal{A}$ are defined by equation (2.3). Then we see that $\mathcal{A}$ can also be expressed by equation (2.4), where $\alpha_k=\left(u_0^{(k)}\right)^{2l}>0$, $\mathbf{v}^{(k)}=\left(1,v_1^{(k)},\ldots,v_n^{(k)}\right)^\top$, still defined by (2.7), satisfies (2.5) for $k=1,\ldots,r$. By these, we see that equation (2.6) is satisfied. ∎

Suppose that $\mathcal{A}=(a_{i_1\ldots i_m})\in S_{m,n+1}$ satisfies (2.6). Then we call $\mathcal{A}$ a **regular symmetric tensor**. If moreover $a_{00\ldots 0}=1$ we call $\mathcal{A}$ a **regular normalized symmetric tensor**. In the next section we will see that an important research issue is to determine whether a given regular symmetric tensor is a regularly decomposable tensor or not.

3. Regularly decomposable tensors and classicality of spin states

Several definitions of classicality of a quantum state exist in the literature, based e.g. on the positivity of the Wigner function, or the absence of entanglement in the case of multi-partite systems [7, 25–28]. In [3] a suitable definition of classicality of spin states was introduced. Firstly, pure classical spin states are defined as angular-momentum coherent states, also called "SU(2)-coherent states", and in the following also simply "coherent states". Their properties are well-known from work in quantum optics [29,30] and quantum-chaos [31]. For being self-contained, we briefly review them here.

SU(2)-coherent states can be labeled by a complex label α, related by stereographic projection to polar and azimuthal angles θ and ϕ, $\alpha=\tan(\theta/2)e^{\imath\phi}$ with $\theta\in[0,\pi]$ and $\phi\in[0,2\pi[$. Let $\mathbf{J}\equiv(J_x,J_y,J_z)$ denote the angular momentum vector, and $|j,m\rangle$ the joint-eigenbasis states of the angular momentum component J_z and the total angular momentum $\mathbf{J}^2\equiv J_x^2+J_y^2+J_z^2$, with $J_z|j,m\rangle=m|j,m\rangle$, $\mathbf{J}^2|j,m\rangle=j(j+1)|j,m\rangle$. The components J_x and J_y are related to the ladder operators $J_\pm$ by $J_\pm=J_x\pm\imath J_y$ and $J_\pm|j,m\rangle=\sqrt{j(j+1)-m(m\pm1)}|j,m\pm1\rangle$, where $\imath=\sqrt{-1}$ is the imaginary unit. The coherent states can be written as

$$|\alpha\rangle=\sum_{m=-j}^{j}\sqrt{\binom{2j}{j+m}}\left(\cos\frac{\theta}{2}\right)^{j+m}\left(\sin\frac{\theta}{2}e^{\imath\phi}\right)^{j-m}|j,m\rangle. \tag{3.1}$$

For $\theta=0$ or $\theta=\pi$, $|\alpha\rangle=|j,j\rangle$ or $|j,-j\rangle$ respectively, i.e. the angular momentum states with largest or smallest J_z-component are always coherent states. Geometrically, a coherent state $|\alpha\rangle$ with $\alpha=\tan(\theta/2)e^{\imath\phi}$ is associated to a direction $\hat{\mathbf{n}}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ on the Bloch sphere. Coherent states have the important property that the quantum uncertainty of the rescaled angular momentum vector $\mathbf{J}/j$ of a spin-j is minimal for all pure quantum states, $(\langle\alpha|\mathbf{J}^2|\alpha\rangle-\langle\alpha|\mathbf{J}|\alpha\rangle^2)/j^2=1/j$. The uncertainty vanishes in the classical limit of a large spin, $j\to\infty$. The coherent states come as closely as possible to the ideal of a classical phase space point, i.e. represent as best as allowed by the laws of quantum mechanics an angular momentum pointing in a precise direction,

$$\langle\alpha|\mathbf{J}|\alpha\rangle=j(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)=j\hat{\mathbf{n}}. \tag{3.2}$$

Another important feature of coherent states is that they remain coherent under unitary transformations of the form $U=e^{-\imath\gamma\hat{\mathbf{n}}\cdot\mathbf{J}}$. Such unitary transformations arise from the dynamics of the angular momentum in a magnetic field (assuming that the angular momentum is associated with a magnetic moment). Classically, the spin precesses around the axis given by the magnetic field, and this is reproduced by the behavior of the coherent state. One can see this most easily for $\hat{\mathbf{n}}=\hat{e}_z=(0,0,1)$, i.e. a magnetic field in the z-direction, in which case $U=e^{-\imath\gamma J_z}$ can be immediately applied to the basis states $|j,m\rangle$ and gives rise to additional phase factors $e^{-\imath\gamma m}$, i.e. $\phi\mapsto\phi+\gamma$, and correspondingly the expectation value $\langle\alpha|\mathbf{J}|\alpha\rangle$ is rotated by the angle γ about the $z-$axis. In general, the mapping $|\alpha\rangle\mapsto|\tilde{\alpha}\rangle=e^{-\imath\gamma\hat{\mathbf{n}}\cdot\mathbf{J}}|\alpha\rangle$ leads to an expectation value $\langle\tilde{\alpha}|\mathbf{J}|\tilde{\alpha}\rangle=R(\hat{\mathbf{n}},\gamma)\langle\alpha|\mathbf{J}|\alpha\rangle$, where $R(\hat{\mathbf{n}},\gamma)$ is a 3×3 orthonormal matrix representing rotation about the axis $\hat{\mathbf{n}}$ with a rotation angle γ. Due to equation (3.2), it is clear that all coherent states can be obtained by an appropriate unitary transformation of the form $U=e^{-\imath\gamma\hat{\mathbf{n}}\cdot\mathbf{J}}$ acting on the state $|j,j\rangle$ associated with the direction $\hat{e}_z$.

The quantum state of any physical system with finite dimensional Hilbert space can be represented by a density operator (also called density matrix) ρ, a positive semidefinite hermitian operator with $\mathrm{tr}\rho=1$. If λ_i and $|\psi_i\rangle$ are respectively the eigenvalues and eigenvectors of ρ, one has the eigendecomposition $\rho=\sum_i\lambda_i|\psi_i\rangle\langle\psi_i|$. The density matrix ρ can therefore be interpreted as representing a quantum state which is in some pure state $|\psi_i\rangle$ with probability λ_i. The condition $\mathrm{tr}\rho=1$ ensures that the probabilities are normalized to 1; it is however possible to work with unnormalized density matrices by relaxing the constraint on $\mathrm{tr}\rho$. In the present paper we will follow that option. As most equations we consider are linear in ρ, this just means that we may forget about an overall normalization constant.

The density operator of an arbitrary spin-j quantum state can be written in the form of a diagonal representation,

$$\rho = \int_{S^2} d\alpha P(\alpha)|\alpha\rangle\langle\alpha|, \tag{3.3}$$

where $P(\alpha)$ is known as the (Glauber-Sudarshan) $P-$function [30], and $d\alpha = \sin\theta d\theta d\phi$ is the integration measure over the unit sphere S^2 in three dimensions. Classically mixing states, i.e. drawing randomly pure states according to a classical probability distribution, should not increase the non-classicality of a state. Hence, a spin-state is called classical, if and only if a decomposition of ρ in the form of equation (3.3) exists with $P(\alpha) \geq 0$, in which case $P(\alpha)$ can be interpreted as classical probability density of finding the pure SU(2)-coherent state $|\alpha\rangle$ in the mixture. Since by definition classical states form a convex set, Caratheodory's theorem implies immediately that a classical state can be written as a finite convex sum of projectors onto coherent states,

$$\rho = \sum_{i=1}^{(N+1)^2+1} w_i |\alpha_i\rangle\langle\alpha_i|, \tag{3.4}$$

where $w_i \geq 0$. Equation (3.4) is the general definition of a classical spin state adopted in [3], and we will base the rest of the paper on it.

A single spin-1/2 is equivalent to a qubit, i.e. a quantum-mechanical two state system. The two states "spin-up" and "spin-down", namely $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$ in the above $|j,m\rangle$ notation, are also called "computational-basis". Denoted as $|0\rangle$ and $|1\rangle$ in quantum-information theory, they are represented as column-vectors $(1,0)^T$ and $(0,1)^T$. In this basis, the density operator can be represented by a 2×2 complex hermitian matrix with $\mathrm{tr}\rho = 1$ that can be expanded over the Pauli-matrix basis,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\rho = \frac{1}{2}\sum_{i=0}^{3} \sigma_i a_i. \tag{3.5}$$

The four components a_i, $i \in \{0,1,2,3\}$ form an order-1 tensor $\mathcal{A}$ of dimension 4. The Pauli matrices $(\sigma_1,\sigma_2,\sigma_3) \equiv \boldsymbol{\sigma}$ are matrix representations of the components of the operator $2\mathbf{J}$ in the "spin-up" and "spin-down" computational-basis. We have $\mathrm{tr}\rho = a_0$. The vector $\mathbf{v} \equiv (a_1,a_2,a_3)^T \in \Re^3$ is the so-called Bloch-vector. It satisfies $||\mathbf{v}||_2 \leq a_0$ in order to guarantee the positivity of ρ. In particular, $||\mathbf{v}||_2 = a_0$ signals pure states (i.e. rank-1 states), and $||\mathbf{v}||_2 < a_0$ mixed states (rank-2 states). Due to the orthonormality of the Pauli-matrix basis, $\mathbf{v}$ can be obtained from a given state as $\mathbf{v} = \mathrm{tr}\rho\boldsymbol{\sigma}$. In particular, for a SU(2)-coherent state $|\alpha\rangle$, one finds $\mathbf{v} = \langle\alpha|2\mathbf{J}|\alpha\rangle = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, as evidenced by equation (3.2). The Bloch picture is particularly useful for visualizing unitary operations: due to the rotation properties of the coherent states under a unitary operation, if $\tilde{\rho} = U\rho U^\dagger$, the corresponding Bloch-vector $\tilde{\mathbf{v}}$ of $\tilde{\rho}$ is obtained by rotation of the original Bloch-vector, namely $\tilde{\mathbf{v}} = R(\hat{\mathbf{n}},\gamma)\mathbf{v}$. As the zero-component of tensor $\mathcal{A}$ has to remain unchanged due to the conservation of the trace under unitary operations, $\tilde{a}_0 = a_0$, the transformation of $\mathcal{A}$ reads $\tilde{a}_i = \mathcal{R}_{ij}a_j$ with

$$\mathcal{R}_{00} = 1,\ \mathcal{R}_{0i} = \mathcal{R}_{i0} = 0\ (i \in \{1,2,3\}) \text{ and } \mathcal{R}_{ij} = R(\hat{\mathbf{n}},\gamma)_{ij}\ (i,j \in \{1,2,3\}). \tag{3.6}$$

In [2] the Bloch-vector of a spin-1/2 was generalized to a Bloch-tensor of a spin-j. A spin-j can be composed from $N=2j$ spins-1/2. The total spin is then the sum of the N spins-1/2, i.e. $\mathbf{J}=\sum_{i=1}^N \boldsymbol{\sigma}^{(i)}/2$. In general, combining two spins j_1 and j_2 gives rise to total spins j ranging from $|j_1-j_2|$ to j_1+j_2. A spin $j=N/2$ is hence the maximum total spin achievable with N spins-1/2. All basis states $|j,m\rangle$ can be created by acting with the ladder operator J_- on the state $|j,j\rangle$, which in turn is the state $|\frac{1}{2},\frac{1}{2}\rangle^{\otimes N}$ of all spins-up in the full Hilbert space of N spins-1/2. Since both $|j,j\rangle$ and J_- are fully symmetric under the exchange of all spins-1/2, all $|j,m\rangle$ states lie in the fully symmetric subspace $\mathcal{H}_S$ of the total Hilbert-space $\mathcal{H}=\mathbb{C}^{2^N}$. A projector $\mathcal{P}_S$ onto that subspace can be obtained as

$$\mathcal{P}_S \equiv \sum_{k=0}^{N} \left|D_N^{(k)}\right\rangle \left\langle D_N^{(k)}\right|, \tag{3.7}$$

where the so-called Dicke states $|D_N^{(k)}\rangle$ are defined as

$$\left|D_N^{(k)}\right\rangle = \mathcal{N}\sum_{\pi} |\underbrace{0\ldots0}_{k}\underbrace{1\ldots1}_{N-k}\rangle, \quad k=0,\ldots N,$$

$\mathcal{N}$ is a normalization constant, and the sum is over all permutations of the spin-1/2 states, written here as tensor product of the computational basis states $|0\rangle$ and $|1\rangle$ of each spin-1/2. The Dicke states are in 1-1 correspondence with the $|j,m\rangle$ states, with $j=N/2$ and $m=k-N/2$.

It was shown in [2] that a tight frame of matrices $S_{i_1\ldots i_N}$ can be obtained by projecting $\boldsymbol{\sigma}_{i_1 i_2\ldots i_N}\equiv\sigma_{i_1}\otimes\sigma_{i_2}\ldots\otimes\sigma_{i_N}$ into $\mathcal{H}_S$. More precisely, the $S_{i_1 i_2\ldots i_N}$ are the $(N+1)$-dimensional blocks spanned by the $|D_N^{(k)}\rangle$ $(k=0,1,\ldots,N)$ of the matrix $\mathcal{P}_S\boldsymbol{\sigma}_{i_1 i_2\ldots i_N}\mathcal{P}_S^\dagger$, i.e. in terms of matrix elements

$$\langle D_N^{(k)}|S_{i_1 i_2\ldots i_N}|D_N^{(l)}\rangle = \langle D_N^{(k)}|\boldsymbol{\sigma}_{i_1 i_2\ldots i_N}|D_N^{(l)}\rangle. \tag{3.8}$$

By definition, there are 4^N matrices $S_{i_1 i_2\ldots i_N}$. However, since they are invariant under permuation of indices, many of them coincide. $S_{0\ldots0}$ is the identity matrix acting on $\mathcal{H}_S$. Due to the tight-frame property, one can expand any density operator of a spin-j as

$$\rho = \sum_{i_1,\ldots,i_N=0}^{n} \frac{1}{2^N} a_{i_1 i_2\ldots i_N} S_{i_1 i_2\ldots i_N}, \tag{3.9}$$

with real and permutationally invariant coefficients

$$a_{i_1 i_2\ldots i_N} = \operatorname{tr}(\rho S_{i_1 i_2\ldots i_N}). \tag{3.10}$$

Therefore, each density matrix ρ corresponds to a 4-dimensional tensor $\mathcal{A}_{N,4}=(a_{i_1 i_2\ldots i_N})$. Note that there are other ways than equation (3.10) to choose the $a_{i_1,\ldots,i_N}$ as the $S_{i_1\ldots i_N}$ form an overcomplete basis.

The representing tensor of a coherent state is particularly simple: Since any spin-j coherent state $|\alpha\rangle$ can be obtained by acting with $U=e^{-\imath\gamma\hat{\mathbf{n}}\cdot\mathbf{J}}$ on $|j,j\rangle=|\frac{1}{2},\frac{1}{2}\rangle^{\otimes N}$, a spin-$j$ coherent state is simply a tensor product of spin-1/2 coherent states, $|\alpha\rangle_j=$

$|\alpha\rangle_{1/2}\otimes\ldots\otimes|\alpha\rangle_{1/2}$, where we have added a subscript indicating the total spin-quantum number. Since it is a symmetric state ($\mathcal{P}_S|\alpha\rangle=|\alpha\rangle$) we have

$$\langle\alpha|S_{i_1i_2\ldots i_N}|\alpha\rangle=\langle\alpha|\mathcal{P}_S\boldsymbol{\sigma}_{i_1i_2\ldots i_N}\mathcal{P}_S^\dagger|\alpha\rangle=\langle\alpha|\otimes\ldots\otimes\langle\alpha|\sigma_{i_1}\otimes\sigma_{i_2}\ldots\sigma_{i_N}|\alpha\rangle\otimes\ldots\otimes|\alpha\rangle \tag{3.11}$$

$$=v_{i_1}v_{i_2}\ldots v_{i_N}. \tag{3.12}$$

As a consequence, $\rho=|\alpha\rangle\langle\alpha|$ has the tensor representation $a_{i_1\ldots i_N}=v_{i_1}\ldots v_{i_N}$, i.e. the representing tensor $\mathcal{A}$ of $\rho=|\alpha\rangle\langle\alpha|$ is a rank-1 tensor with $v_0=1$ and $||\mathbf{v}||=1$.

For an arbitrary density matrix ρ, the tensor $\mathcal{A}_{N,4}$ enjoys useful properties. Firstly, the $a_{i_1i_2\ldots i_N}$ in equation (3.10) are such that

$$a_{00i_3\ldots i_N}=\sum_{i=1}^{3}a_{iii_3\ldots i_N}. \tag{3.13}$$

To see this, let $|\alpha\rangle$ be a coherent state. Since its representing tensor is $a_{i_1\ldots i_N}=v_{i_1}\ldots v_{i_N}$, and $\mathbf{v}^2=v_0^2=1$, we have

$$v_0v_0v_{i_3}\ldots v_{i_N}=\sum_{a=1}^{3}v_av_av_{i_3}\ldots v_{i_N}, \tag{3.14}$$

which is equation (3.13) for coherent states. Due to the linearity of the decomposition (3.3) of ρ in terms of coherent states, equation (3.13) for arbitrary states follows.

Secondly, by equations (3.3), (3.10) and (3.11), we have

$$\begin{aligned}a_{00\ldots0}&=\mathrm{tr}(\rho S_{00\ldots0})\\&=\mathrm{tr}\left(\int_{S^2}d\alpha P(\alpha)|\alpha\rangle\langle\alpha|S_{00\ldots0}\right)\\&=\int_{S^2}d\alpha P(\alpha)\langle\alpha|S_{00\ldots0}|\alpha\rangle\\&=\int_{S^2}d\alpha P(\alpha)\langle\alpha|\boldsymbol{\sigma}_{00\ldots0}|\alpha\rangle\\&=\int_{S^2}d\alpha P(\alpha),\end{aligned} \tag{3.15}$$

so that $a_{00\ldots0}=1$ if the state is normalized. Finally, as shown in [2], the $a_{i_1i_2\ldots i_N}$ are unique if they are restricted to real numbers, invariant under permutation of the indices, and verifying the condition equation (3.13). There is therefore a mapping from the density matrices ρ of a spin-j state to 4-dimensional real symmetric normalized tensors of order $N=2j$, $\mathcal{A}_{N,4}=(a_{i_1i_2\ldots i_N})\in S_{N,4}$. We call this tensor the “representing tensor” of the state ρ.

Hence, by equation (3.4), a spin-j state is classical if and only if there are positive weights $w_k>0$ for $k=1,\ldots,r$, and vectors $\mathbf{v}^{(k)}=\left(1,v_1^{(k)},v_2^{(k)},v_3^{(k)}\right)^\top\in\Re^4$, satisfying

$$\left(v_1^{(k)}\right)^2+\left(v_2^{(k)}\right)^2+\left(v_3^{(k)}\right)^2=1, \tag{3.16}$$

for $k=1,\ldots,r$, such that the representing tensor $\mathcal{A}=(a_{i_1\ldots i_N})\in S_{N,4}$ of that spin-j state satisfies

$$\mathcal{A}=\sum_{k=1}^{r}w_k\left(\mathbf{v}^{(k)}\right)^{\otimes N}, \tag{3.17}$$

i.e., $\mathcal{A}$ is a regularly decomposable tensor.

Based upon the above discussions and Theorem 2.1, we have the following theorem.

THEOREM 3.1. *The tensor $\mathcal{A}=(a_{i_1\ldots i_N})\in S_{N,4}$ representing a spin-j state (with $N=2j$) is a regular symmetric tensor. A spin-j state is classical if and only if its representing tensor is a regularly decomposable tensor.*

Thus, the physical problem of determining whether a spin-j state is classical or not is equivalent to a mathematical problem to determine whether its representing tensor is a regularly decomposable tensor or not.

4. Properties of completely decomposable and regularly decomposable tensors

There is already substantial literature on PSD tensors and SOS tensors, including [5, 15–24]. There are only two papers on completely decomposable tensors [23, 24]. Regularly decomposable tensors are introduced in this paper. By the discussion in the last section, we see that regularly decomposable tensors play a significant role for the classicality of spin states. Thus, in this section, we discuss properties of completely decomposable tensors and regularly decomposable tensors.

4.1. Invariance of complete decomposability and regular decomposability. Any measure of entanglement should be invariant under local unitary transformations (see e.g. [32]). Hence, also the set of fully separable states must be invariant under local unitary transformations. Correspondingly, the classicality of a spin-j state should be invariant under rotations of the coordinate system. For a physical system in three spatial dimensions, such a rotation is represented by the 3×3 orthogonal transformation matrix $R(\hat{\mathbf{n}},\gamma)$ introduced above that acts on a vector of spatial coordinates x_1,x_2,x_3. The corresponding transformation of a covariant tensor (i.e. a tensor that transforms as the coordinates) of dimension 4 and order m is given by its inner product with $\mathcal{R}^{\otimes m}$, where $\mathcal{R}$ is defined by equation (3.6). More generally, we expect the regular decomposability of a tensor to be a property invariant under orthogonal transformations described by an $(n+1)\times(n+1)$ matrix

$$\mathcal{R}=\begin{pmatrix} 1 & \mathbf{0}^\top \\ \mathbf{0} & R \end{pmatrix},$$

where $\mathbf{0}$ is the zero vector in $\Re^n$, and R is now an $n\times n$ orthogonal matrix. Then

$$R\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}=\begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \\ y_n \end{pmatrix}$$

and

$$\mathcal{R}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{pmatrix}=\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-1} \\ y_n \end{pmatrix}$$

with $x_0 = y_0$. We call such an orthogonal matrix a **normalized orthogonal matrix**. Denote $\mathcal{R} = (r_{li})$. As in [5], for any symmetric tensor $\mathcal{A} = (a_{i_1 \ldots i_m}) \in S_{m,n+1}$, let $\mathcal{B} = (b_{l_1 \ldots l_m}) \equiv \mathcal{R}^m \mathcal{A} \in S_{m,n+1}$ be defined by

$$b_{l_1 \ldots l_m} = \sum_{i_1, \ldots, i_m = 0}^{n} a_{i_1 \ldots i_m} r_{l_1 i_1} \ldots r_{l_m i_m}$$

for $l_1, \ldots, l_m = 0, \ldots, n$. By [5], $\mathcal{A}$ and $\mathcal{B}$ have the same E-eigenvalues and Z-eigenvalues. In particular, when m is even, $\mathcal{A}$ is PSD if and only if $\mathcal{B}$ is PSD. By [13], when m is even, $\mathcal{A}$ is SOS if and only if $\mathcal{B}$ is SOS. This shows that the PSD property and the SOS property can represent physical properties, as they are invariant under orthogonal transformation.

Theorem 4.1. *Let $\mathcal{R}$ be a normalized orthogonal matrix, $\mathcal{A}, \mathcal{B} \in S_{m,n+1}, \mathcal{B} = \mathcal{R}^m \mathcal{A}$. Then $\mathcal{A}$ is completely decomposable if and only if $\mathcal{B}$ is completely decomposable, and $\mathcal{A}$ is regularly decomposable if and only if $\mathcal{B}$ is regularly decomposable.*

Proof. Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m}) \in S_{m,n+1}$ is completely decomposable, $\mathcal{B} = (b_{k_1 \ldots k_m}) \in S_{m,n+1}$, $\mathcal{B} = \mathcal{R}^m \mathcal{A}$, where $\mathcal{R} = (r_{li})$ is an $(n+1) \times (n+1)$ orthogonal matrix. Then there are vectors $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)} \in \Re^{n+1}$, where $\mathbf{u}^{(k)} = (u_0^{(k)}, \ldots, u_n^{(k)})^\top$ for $k = 1, \ldots, r$, such that

$$\mathcal{A} = \sum_{k=1}^{r} \left(\mathbf{u}^{(k)} \right)^{\otimes m},$$

i.e., for $i_1, \ldots, i_m = 0, \ldots, n$,

$$a_{i_1 \ldots i_m} = \sum_{k=1}^{r} u_{i_1}^{(k)} \ldots u_{i_m}^{(k)}.$$

Then, for $l_1, \ldots, l_m = 0, \ldots, n$, we have

$$\begin{aligned} b_{l_1 \ldots l_m} &= \sum_{i_1, \ldots, i_m = 0}^{n} a_{i_1 \ldots i_m} r_{l_1 i_1} \ldots r_{l_m i_m} \\ &= \sum_{k=1}^{r} \sum_{i_1, \ldots, i_m = 0}^{n} u_{i_1}^{(k)} \ldots u_{i_m}^{(k)} r_{l_1 i_1} \ldots r_{l_m i_m} \\ &= \sum_{k=1}^{r} v_{l_1}^{(k)} \ldots v_{l_m}^{(k)}, \end{aligned}$$

where for $k = 1, \ldots, r, l = 0, \ldots, n$,

$$v_l^{(k)} = \sum_{i=0}^{n} r_{li} u_i^{(k)}.$$

This implies that

$$\mathcal{B} = \sum_{k=1}^{r} \left(\mathbf{v}^{(k)} \right)^{\otimes m},$$

where $\mathbf{v}^{(k)} = (v_0^{(k)}, \ldots, v_n^{(k)})^\top$ for $k = 1, \ldots r$. This implies that $\mathcal{B}$ is completely decomposable. By [5], if $\mathcal{B} = \mathcal{R}^m \mathcal{A}$, then $\mathcal{A} = (\mathcal{R}^\top)^m \mathcal{B}$. Thus, if $\mathcal{B}$ is completely decomposable, then $\mathcal{A}$ is also completely decomposable.

Assume that m is even, $\mathcal{A}$ is regularly decomposable and $\mathcal{R}$ is a normalized orthogonal matrix. Then, we may assume that in the above discussion, vectors $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}$ are regular. Since $\mathbf{v}^{(k)} = \mathcal{R}\mathbf{u}^{(k)}$ for $k = 1, \ldots, r$, and $\mathcal{R}$ is a normalized orthogonal matrix, we may conclude that $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(r)}$ are also regular. This implies that $\mathcal{B}$ is also regularly decomposable. By [5], if $\mathcal{B} = \mathcal{R}^m \mathcal{A}$, then $\mathcal{A} = (\mathcal{R}^\top)^m \mathcal{B}$. Thus, if $\mathcal{B}$ is regularly decomposable, then $\mathcal{A}$ is also regularly decomposable.

Now assume that m is odd, $\mathcal{A}$ is regularly decomposable and $\mathcal{R}$ is a normalized orthogonal matrix. Then $\mathcal{B}_0$ is also regularly decomposable. As $\mathcal{A}_i$ for $i = 1, \ldots, n$, are induced from the regular decomposition of $\mathcal{A}_0$, we may see that $\mathcal{B}_i$ for $i = 1, \ldots, n$, are induced from the regular decomposition of $\mathcal{B}_0$. This implies that $\mathcal{B}$ is also regularly decomposable. Similarly, if $\mathcal{B}$ is regularly decomposable, then $\mathcal{A}$ is also regularly decomposable. ∎

The proof of this theorem can be simplified by applying Theorem 2.2 of [13], Theorem 2.4 and the definition of the normalized orthogonal matrices in this paper.

These show that complete decomposability and regular decomposability are invariant under normalized orthogonal transformation.

4.2. Hadamard products. For any two tensors $\mathcal{A} = (a_{i_1 \cdots i_m})$, $\mathcal{B} = (b_{i_1 \cdots i_m}) \in T_{m,n+1}$, their **Hadamard product**, denoted as $\mathcal{A} \circ \mathcal{B}$, is defined by

$$\mathcal{A} \circ \mathcal{B} = (a_{i_1 \cdots i_m} b_{i_1 \cdots i_m}) \in T_{m,n+1}. \tag{4.1}$$

In matrix theory, the Hadamard product of two PSD symmetric matrices is also a PSD symmetric matrix. This is no longer true for tensors. In [18], an example was given that the Hadamard product of two PSD Hankel tensors may not be PSD. Hankel tensors are symmetric tensors. Thus, the Hadamard product of two PSD symmetric tensors may not be PSD. In [13], an example was given that the Hadamard product of two SOS tensors may not be an SOS tensor. However, we have the following proposition:

PROPOSITION 4.1. *Suppose that $\mathcal{A} = (a_{i_1 \cdots i_m})$, $\mathcal{B} = (b_{i_1 \cdots i_m}) \in S_{m,n+1}$ are completely decomposable tensors. Then their Hadamard product $\mathcal{A} \circ \mathcal{B}$ is also a completely decomposable tensor.*

Proof. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are completely decomposable. Then there are vectors $\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(r)}, \mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(p)} \in \Re^{n+1}$, such that

$$\mathcal{A} = \sum_{k=1}^{r} \left(\mathbf{u}^{(k)}\right)^{\otimes m}$$

and

$$\mathcal{B} = \sum_{l=1}^{p} \left(\mathbf{v}^{(l)}\right)^{\otimes m}.$$

Then is easy to see that

$$\mathcal{A} \circ \mathcal{B} = \sum_{k=1}^{r} \sum_{l=1}^{p} \left(\mathbf{u}^{(k)} \circ \mathbf{v}^{(l)}\right)^{\otimes m},$$

i.e., $\mathcal{A}\circ\mathcal{B}$ is completely decomposable. □

This property is no longer true for regularly decomposable tensors. In this sense, completely decomposable tensors are similar to completely positive tensors studied in [20]: the Hadamard product of two completely positive tensors is still a completely positive tensor.

4.3. Duality between the PSD tensor cone and the completely decomposable tensor cone. Denote the set of all completely decomposable tensors in $S_{m,n+1}$ by $CD_{m,n+1}$, the set of all regularly decomposable tensors in $S_{m,n+1}$ by $RD_{m,n+1}$. Let m be even, denote the set of all PSD tensors in $S_{m,n+1}$ by $PSD_{m,n+1}$, the set of all SOS tensors in $S_{m,n+1}$ by $SOS_{m,n+1}$. Then $CD_{m,n+1}$, $RD_{m,n+1}$, $PSD_{m,n+1}$, and $SOS_{m,n+1}$ are cones.

Let C be a cone in $S_{m,n+1}$. Then its dual cone C^* is defined by

$$C^* := \{\mathcal{A}\in S_{m,n+1} : \mathcal{A}\bullet\mathcal{B}\geq 0, \text{ for all } \mathcal{B}\in C\}.$$

The dual cone C^* is a closed convex cone. The dual cone of C^* is the closure of the convex hull of C. If C is closed and convex, then C and C^* are dual cones to each other. Let $\mathcal{A}=(a_{i_1\ldots i_m})\in S_{m,n+1}$ and $\Re^{n+1}_+$ be the nonnegative orthant of $\Re^{n+1}$. If for any $\mathbf{x}\in\Re^{n+1}_+$, we have $\mathcal{A}\bullet\mathbf{x}^{\otimes m}\geq 0$, then we say that $\mathcal{A}$ is a **copositive** tensor. Copositive tensors have also applications in physics [9]. By [20], the completely positive tensor cone and copositive tensor cone are dual cones to each other.

By [33] and the definition of completely decomposable tensors, we have the following proposition. A part of this proposition is covered by Proposition 4.2 of [13].

PROPOSITION 4.2. *Suppose that m is even. Then $PSD_{m,n+1}$ and $CD_{m,n+1}$ are dual cones to each other. Thus, both are closed convex cones.*

4.4. Closedness and convexity of the regularly decomposable tensor cone. In the last subsection, we already knew that if m is even, then $PSD_{m,n+1}$ and $CD_{m,n+1}$ are closed convex cones. By [23], if m is odd, $CD_{m,n+1}$ is the linear space $S_{m,n+1}$. By [13], $SOS_{m,n+1}$ is also a closed convex cone. We now discuss closedness and convexity of $RD_{m,n+1}$.

PROPOSITION 4.3. *$RD_{m,n+1}$ is a closed convex cone.*

Proof. Suppose that $\{\mathcal{A}^{(l)} : l=1,2,\ldots,\}$ is a sequence of regularly decomposable tensors in $RD_{m,n+1}$ such that

$$\mathcal{A}=\lim_{l\to\infty}\mathcal{A}^{(l)}.$$

By Theorem 2.1, we may assume that

$$\mathcal{A}^{(l)}=\sum_{k=1}^{r_l}\alpha_{k,l}\left(\mathbf{v}^{(k,l)}\right)^{\otimes m},$$

where $\alpha_{k,l}\geq 0$, $\mathbf{v}^{(k,l)}=(1,v_1^{(k,l)},\ldots,v_1^{(k,l)})^\top$,

$$\left(v_1^{(k,l)}\right)^2+\ldots+\left(v_n^{(k,l)}\right)^2=1,$$

for $k=1,\ldots,r_l$, for $l=1,2,\ldots$. By the Carathéodory theorem, we may assume that

$$r_l\leq R\equiv\binom{n+m+2}{m}+1.$$

Thus, by taking a subsequence if necessary, without loss of generality, there is a $r \leq R$ such that $r_l = r$ for $l = 1,2,\ldots$. Then, we may conclude that there are $\alpha_k \geq 0$, $\mathbf{v}^{(k)} = (1, v_1^{(k)}, \ldots, v_1^{(k)})^\top$,

$$\left(v_1^{(k)}\right)^2 + \ldots + \left(v_n^{(k)}\right)^2 = 1,$$

for $k = 1,\ldots,r$. Thus, by Theorem 2.1, $\mathcal{A}$ is a regularly decomposable tensor. This shows that $RD_{m,n+1}$ is a closed cone. Following directly from Theorem 2.1, we see that $RD_{m,n+1}$ is also a convex cone. □

5. Concluding remarks

In this paper, we have introduced the concept of regularly decomposable tensors. We have shown that a spin state is classical if and only if its representing tensor is a regularly decomposable tensor. Thus, the problem for determining whether a spin state is classical or not is mathematically equivalent to the problem of determining whether a given regular symmetric tensor is a regularly decomposable tensor or not.

How can we construct an algorithm for determining a given regular symmetric tensor is a regularly decomposable tensor or not? We see that the properties of completely decomposable tensors and regularly decomposable tensors in some extent are similar to those of completely positive tensors [20, 34]. Recently, an algorithm for determining whether a given symmetric nonnegative tensor is completely positive or not was proposed [35]. Perhaps we may learn from that algorithm how to construct an algorithm determining whether a given regular symmetric tensor is regularly decomposable or not.

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