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INVERSE EIGENVALUE PROBLEM FOR TENSORS*

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Abstract. Let $\mathbb{T}(\mathbb{C}^n, m+1)$ be the space of tensors of order $m+1$ and dimension n with complex entries. A tensor $\mathcal{T} \in \mathbb{T}(\mathbb{C}^n, m+1)$ has nm^{n-1} eigenvalues (counted with algebraic multiplicities). The inverse eigenvalue problem for tensors is a generalization of the inverse eigenvalue problem for matrices. Namely, given a multiset $S \in \mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ of total multiplicity nm^{n-1}, is there a tensor in $\mathbb{T}(\mathbb{C}^n, m+1)$ such that the set of eigenvalues of $\mathcal{T}$ is exactly S? The solvability of the inverse eigenvalue problem for tensors is studied in this article. With tools from algebraic geometry, it is proved that the necessary and sufficient condition for this inverse problem to be generically solvable is $m=1$, or $n=2$, or $(n,m)=(3,2)$, $(4,2)$, $(3,3)$.

Keywords. tensor; eigenvalue; inverse problem.

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1. Introduction

Eigenvalues of a tensor, as a natural generalized notion of eigenvalues of a square matrix, has attracted increasing attention in fields related to numerical multilinear algebra (see [2, 3, 7, 17, 18, 21, 23–25] and references therein). For a tensor of order $m+1$ and dimension n, its eigenvalues are the roots of the characteristic polynomial, which is a monic polynomial of degree nm^{n-1} [13, 24]. As a consequence, the number of eigenvalues, counted with multiplicities, is equal to nm^{n-1}. Equivalently, we can define eigenvalues as complex numbers such that a system of parametrized polynomial equations has a non-zero solution which recover eigenvalues of matrices when $m=1$ (cf. Definition 2.1). However, both computation and structures of the eigenvalues are very complicated, and tough to investigate [13, 14, 24]. The situation can be improved if the eigenvalues are shown to lie in a variety in $\mathbb{C}^{nm^{n-1}}$ with a much smaller dimension. Our main Theorem 1.1 shows that in most cases, eigenvalues of tensors indeed lie in a proper sub-variety of $\mathbb{C}^{nm^{n-1}}$. We hope that our results in this article may shed some light on how to simplify the computation of tensor eigenvalues.

Let $\mathbb{T}(\mathbb{C}^n, m+1)$ be the space of tensors of order $m+1$ and dimension n with entries in the field $\mathbb{C}$ of complex numbers. When $m=1$, we get the space of $n \times n$ matrices over $\mathbb{C}$. The eigenvalues of a matrix $A=(a_{ij}) \in \mathbb{T}(\mathbb{C}^n, 2)$, as roots of the characteristic polynomial

$$\det(\lambda I - A) = \lambda^n + c_{n-1}(A)\lambda^{n-1} + c_1(A)\lambda + c_0(A),$$

can be written as hypergeometric series in terms of the components a_{ij}'s (cf. [27]), since $c_i(A) \in \mathbb{C}[A]$ is a homogeneous polynomial of degree $n-i$ for $i=1,\ldots,n$. We can collect the n hypergeometric series to form a multiset-valued mapping $\phi: \mathbb{T}(\mathbb{C}^n, 2) \to \mathbb{C}^n/\mathfrak{S}(n)$, where $\mathfrak{S}(n)$ is the group of permutations on n elements. Thus, $\phi(A)$ is the multiset of eigenvalues of A. The set $\mathbb{C}^n/\mathfrak{S}(n)$ is the nth symmetric product of $\mathbb{C}$, and (cf. [11])

$$\dim(\mathbb{C}^n/\mathfrak{S}(n)) = n.$$

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A well-known result from linear algebra (cf. [12]) is that this mapping is surjective, i.e.,

$$\operatorname{image}(\phi) = \mathbb{C}^n/\mathfrak{S}(n). \tag{1.1}$$

This implies that any given n-tuple of complex numbers can be realized as the eigenvalues of an $n\times n$ matrix.

It is shown in [13] that the codegree i coefficient of the characteristic polynomial of a tensor is a homogeneous polynomial of degree i in terms of the tensor components. Therefore, we can define the multiset-valued eigenvalue mapping $\phi: \mathbb{T}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ in a similar way for the tensor eigenvalues. Here we use the same symbol ϕ for all positive integers m and n, when m,n are understood in the context. A basic question arises for tensors when one is trying to understand their eigenvalues:

what can the eigenvalues of a tensor be?

This question is of course very hard to answer in general. However, we can ask the question about *the existence of tensors for a given multiset of eigenvalues*, which has the nomenclature *inverse eigenvalue problem for tensors* in general. We refer to [4, 6] and references therein for the inverse eigenvalue problems for matrices.

In this article, we will study the counterpart of the equality (1.1) for tensors: when do the eigenvalues fulfill the whole quotient space? It turns out that this question is not easy to answer. However, with the help of concepts from algebraic geometry, we are able to answer a weaker version of the question: "when do the eigenvalues almost fulfill the whole quotient space?", or more precisely in mathematical language: "when does the image of the multiset-valued eigenvalue map contain an open dense subset of $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$?" Unless otherwise stated, we will always adopt the Zariski topology for the ambient space. The map ϕ is *dominant* if its image contains an open dense subset of $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ (cf. Definition 2.2). We have the following main theorem of this article.

THEOREM 1.1 (Dominant Theorem). *The eigenvalue map* $\phi: \mathbb{T}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ *is dominant if and only if*

$$m=1, \text{ or } n=2, \text{ or } (n,m) = (3,2),\ (4,2),\ (3,3).$$

Proof. The case $m=1$ is the trivial matrix counterpart. For the other cases, the necessity follows from Proposition 3.2, and the sufficiency follows from Propositions 4.1, 5.1 and 5.2. ∎

Since the topology on $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ is the Zariski topology, the fact that the image of ϕ contains an open dense subset of $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ implies that for almost every multiset S in $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$, there exists a tensor $\mathcal{T}$ in $\mathbb{T}(\mathbb{C}^n, m+1)$ such that the set of eigenvalues of $\mathcal{T}$ is exactly S. More precisely, the fact that the image of ϕ contains an open dense subset of $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ implies that the probability that a randomly picked multiset S can be realized as the set of eigenvalues of a tensor in $\mathbb{T}(\mathbb{C}^n, m+1)$ is one.

2. Preliminaries

2.1. Eigenvalues of tensors. There are two most popular definitions of tensor eigenvalues in the literature [18, 24]. Throughout this article, eigenvalues and eigenvectors of tensors are restricted to the next definition.

DEFINITION 2.1 (Eigenvalues and Eigenvectors [18, 24]). *Let tensor* $\mathcal{T}=(t_{ji_1\dots i_m})\in\mathbb{T}(\mathbb{C}^n,m+1)$*. A number* $\lambda\in\mathbb{C}$ *is called an eigenvalue of* $\mathcal{T}$*, if there exists a vector* $\mathbf{x}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$*, which is called an eigenvector, such that*

$$\mathcal{T}\mathbf{x}^m=\lambda\mathbf{x}^{[m]}, \tag{2.1}$$

where $\mathbf{x}^{[m]}\in\mathbb{C}^n$ *is an n-dimensional vector with its jth component being* x_j^m*, and* $\mathcal{T}\mathbf{x}^m\in\mathbb{C}^n$ *with*

$$\left(\mathcal{T}\mathbf{x}^m\right)_j:=\sum_{i_1,\dots,i_m=1}^{n} t_{ji_1\dots i_m}x_{i_1}\dots x_{i_m},\quad j=1,\dots,n.$$

We would like to remark that the system (2.1) is a system of n polynomial equations in n variables and it is parametrized by λ. Therefore, the resultant of the system (2.1) is a univariate polynomial

$$\chi(\lambda)=\det(\lambda\mathcal{I}-\mathcal{T}),$$

where $\mathcal{I}$ is the identity tensor of appropriate size, e.g., $\mathcal{I}=(i_{ji_1\dots i_m})$ with $i_{jj\dots j}=1$ for $j=1,\dots,n$ and zero otherwise. The polynomial $\chi(\lambda)$ is called the *characteristic polynomial* of $\mathcal{T}$ [24]. The degree of $\chi(\lambda)$ for $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m+1)$ is nm^{n-1}. According to the definition of the resultant of a polynomial system, we see that $\chi(\mu)=0$ if and only if the system (2.1) has a non-zero solution when $\lambda=\mu$. Let μ be an eigenvalue of $\mathcal{T}$ and let $a(\mu)$ be the multiplicity of μ as a root of $\chi(\lambda)$. We call $a(\mu)$ the *algebraic multiplicity* of μ as an eigenvalue of $\mathcal{T}$. The multiset of eigenvalues of a given tensor $\mathcal{T}$, which is denoted as $\sigma(\mathcal{T})$, is defined as (A,ψ) with A being the set of eigenvalues of $\mathcal{T}$ and the multiplicity map ψ being the algebraic multiplicity of the eigenvalue. The summation $\sum_{a\in A}\psi(a)$ is called the *total multiplicity* of the multiset $\sigma(\mathcal{T})$. Then, $\sigma(\mathcal{T})$ is always of total multiplicity nm^{n-1} [13, 24]. Thus, $\sigma(\mathcal{T})$ can be identified with an element in $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ for any $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m+1)$. We refer interested readers to [13, 14] for more details of algebraic multiplicity and the characteristic polynomial.

The multiset-valued eigenvalue map $\phi:\mathbb{T}(\mathbb{C}^n,m+1)\to\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ is defined as

$$\phi(\mathcal{T})=\sigma(\mathcal{T}).$$

As long as we are only concerned about eigenvalues of tensors, it is sufficient to consider the tensor space $\mathbb{TS}(\mathbb{C}^n,m+1):=\mathbb{C}^n\otimes\mathrm{S}^m(\mathbb{C}^n)$ (cf. [14, Section 5.2]). For any $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m+1)$, we can symmetrize its jth slice $\mathcal{T}_j:=(t_{ji_1\dots i_m})_{1\le i_1,\dots,i_m\le n}$ via

$$\left(\mathcal{T}\mathbf{x}^m\right)_j=\langle\mathrm{Sym}(\mathcal{T}_j),\mathbf{x}^{\otimes(m)}\rangle:=\sum_{i_1,\dots,i_m=1}^{n}\left(\mathrm{Sym}(\mathcal{T}_j)\right)_{i_1\dots i_m}x_{i_1}\dots x_{i_m}\ \text{for all } \mathbf{x}\in\mathbb{C}^n,$$

where $\mathrm{Sym}(\mathcal{T}_j)$ is the symmetrization of the tensor $\mathcal{T}_j$ as defined in the above equalities. We refer to [16] for basic concepts of tensors. Therefore, for every tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m+$

1), we associate it an element $\mathrm{eSym}(\mathcal{T})$ in $\mathbb{TS}(\mathbb{C}^n, m+1)$ by symmetrizing its slices. It is easy to see that

$$\mathcal{T}\mathbf{x}^m = \mathrm{eSym}(\mathcal{T})\mathbf{x}^m \text{ for all } \mathbf{x} \in \mathbb{C}^n.$$

We see that all tensors in the fibre of the surjective mapping $\mathrm{eSym}: \mathbb{T}(\mathbb{C}^n, m+1) \to \mathbb{TS}(\mathbb{C}^n, m+1)$ have the same defining equations for the eigenvalue problem. Therefore, we have

$$\phi(\mathbb{T}(\mathbb{C}^n, m+1)) = \phi(\mathbb{TS}(\mathbb{C}^n, m+1)).$$

2.2. Algebraic geometry. We list here some notions from algebraic geometry which we will use in this article. We refer to [5, 10, 11, 26] for basic algebro-geometric concepts.

(1) An *algebraic variety* in $\mathbb{C}^n$ is a set of common zeros of some polynomials in n variables. In particular, the linear space $\mathbb{C}^n$ is an algebraic variety.
(2) The coordinate ring $\mathbb{C}[X]$ of an algebraic variety X is defined to be the quotient ring $\mathbb{C}[x_1, \cdots, x_n]/I(X)$ where $I(X)$ is the ideal of all polynomials vanishing on X.
(3) A map $f: X \to Y$ between two algebraic varieties X and Y is said to be a *morphism* if f is induced by a homomorphism of coordinate rings $\psi: \mathbb{C}[Y] \to \mathbb{C}[X]$. In particular, polynomial maps between two linear spaces are morphisms.
(4) Let f be a morphism between X and Y. If its image is Zariski dense, i.e., $\overline{f(X)} = Y$, then f is called a *dominant morphism*.
(5) An algebraic variety X is *irreducible* if $X = X_1 \cup X_2$ for closed subvarieties X_1 and X_2 implies that $X_1 = X$ or $X_2 = X$.
(6) We say that a property P holds for a *generic point* in $\mathbb{C}^n$ if the set of points in $\mathbb{C}^n$ that do not satisfy P is contained in a proper subvariety of $\mathbb{C}^n$. For example, fix an algebraic variety $X \subset \mathbb{C}^n$, we say that a generic point in $\mathbb{C}^n$ is not in X.

REMARK 2.1. We remark here that if we put the outer Lebesgue measure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$ then a proper subvariety X of $\mathbb{C}^n$ has measure zero. Hence the fact that a property P holds for a generic point in $\mathbb{C}^n$ implies that the probability that a randomly picked point from $\mathbb{C}^n$ has property P is one.

The main result of this article will be proved based on the following algebraic version of the open mapping theorem.

PROPOSITION 2.1 ([28]). *If $f: X \to Y$ is a dominant morphism between two irreducible algebraic varieties then $f(X)$ contains an open dense subset of Y.*

The following two facts are obvious to those who are familiar with algebraic geometry, while we supply proofs here for completeness.

PROPOSITION 2.2. *If a morphism $f: X \to Y$ between two algebraic varieties X and Y is dominant, then it holds that*

$$\dim(X) \geq \dim(Y).$$

Proof. It is known that $\dim(X)$ is the same as the transcendence degree of the function field $\mathbb{C}(X)$ over $\mathbb{C}$ and f is dominant if and only if the ring map $\psi: \mathbb{C}[Y] \to \mathbb{C}[X]$ is an inclusion of rings. Since ψ is an inclusion of rings we obtain that ψ induces an inclusion of fields $\mathbb{C}(Y) \to \mathbb{C}(X)$. Therefore we have

$$\mathrm{tr.d}_{\mathbb{C}}(\mathbb{C}(X)) \geq \mathrm{tr.d}_{\mathbb{C}}(\mathbb{C}(Y)).$$

□

An algebraic variety $X\subseteq\mathbb{C}^n$ is *smooth* if the tangent space $T_{\mathbf{x}}(X)$ has constant dimension (i.e., $\dim(X)$) for every $\mathbf{x}\in X$. We would like to remind readers that when X and Y are smooth algebraic varieties, they can be regarded as smooth manifolds. Moreover, the differential of a morphism $f:X\to Y$ can be calculated as the differential of a smooth map between two manifolds. In particular, if $f:X\to Y$ is a polynomial map $f=(P_1,\dots,P_m)$, where $P_1,\dots,P_m$ are polynomials in n variables, then the differential $d_{\mathbf{x}}f$ is simply

$$d_{\mathbf{x}}f=(d_{\mathbf{x}}P_1,\dots,d_{\mathbf{x}}P_m).$$

Here $d_{\mathbf{x}}P_i$ is the differential of the polynomial P_i at the point $\mathbf{x}$, $i=1,\dots,m$.

Proposition 2.3. *Let $f:X\to Y$ be a morphism between two smooth algebraic varieties with* $\dim(X)\geq\dim(Y)$*. If there exists a point $\mathbf{x}\in X$ such that the rank of the differential of f at $\mathbf{x}$ is equal to* $\dim(Y)$*, then the morphism f is dominant.*

Proof. If f is not dominant, then $\overline{f(X)}$ is a proper subvariety of Y. Hence it factors as

$$X\xrightarrow{g}\overline{f(X)}\xrightarrow{i}Y,$$

where g is defined by $g(\mathbf{x})=f(\mathbf{x})$ and i is the inclusion of $\overline{f(X)}$ into Y. Then the differential $d_{\mathbf{x}}f$ factors as

$$T_{\mathbf{x}}X\xrightarrow{d_{\mathbf{x}}g}T_{f(\mathbf{x})}\overline{f(X)}\xrightarrow{d_{f(\mathbf{x})}i}T_{f(\mathbf{x})}Y,$$

where $T_{\mathbf{x}}X$ is the tangent space of the variety X at the point $\mathbf{x}$. Since $\overline{f(X)}$ is a proper subvariety of Y, it has strictly smaller dimension than $\dim(Y)$, which implies that $\operatorname{rank}(d_{\mathbf{x}}g)$ is at most $\dim T_{f(x)}\overline{f(X)}<\dim(Y)$. Therefore, we get a contradiction to the assumption that the rank of $d_{\mathbf{x}}f$ is $\dim(Y)$. □

For any positive integer $d>0$, the roots of the univariate polynomial equation

$$t^d+p_{d-1}t^{d-1}+\cdots+p_1t+p_0=0$$

depends continuously on the coefficient vector $\mathbf{p}:=(p_{d-1},\dots,p_0)^{\mathsf{T}}$ [27]. We define a multiset-valued map $q:\mathbb{C}^d\to\mathbb{C}^d/\mathfrak{S}(d)$ by

$$q(\mathbf{w}):=\{\text{roots (with mutliplicities) of } t^d+w_1t^{d-1}+\cdots+w_d=0\}. \tag{2.2}$$

Definition 2.2. *Let $p_i(\mathbf{y})\in\mathbb{C}[\mathbf{y}]$ be a polynomial for all $i=0,\dots,d-1$ with $\mathbf{y}=(y_1,\dots,y_k)^{\mathsf{T}}\in\mathbb{C}^k$, and let $\mathbf{p}:\mathbb{C}^k\to\mathbb{C}^d$ be the mapping defined by*

$$\mathbf{p}(\mathbf{y}):=(p_{d-1}(\mathbf{y}),\dots,p_0(\mathbf{y}))^{\mathsf{T}}.$$

The mapping $q\circ\mathbf{p}:\mathbb{C}^k\to\mathbb{C}^d/\mathfrak{S}(d)$ is called dominant if $\operatorname{image}(q\circ\mathbf{p})$ *contains a Zariski open dense subset of $\mathbb{C}^d/\mathfrak{S}(d)$.*

Definition 2.2 is an extension of dominant morphisms, since $q\circ\mathbf{p}$ is not a morphism.

Lemma 2.1. *For any positive integer $d>0$, let $\mathbf{p}:\mathbb{C}^k\to\mathbb{C}^d$ be a polynomial mapping as in Definition 2.2. Then,*

(1) the composite mapping $q\circ\mathbf{p}:\mathbb{C}^k\to\mathbb{C}^d/\mathfrak{S}(d)$ is surjective, i.e., $\operatorname{image}(q\circ\mathbf{p})=\mathbb{C}^d/\mathfrak{S}(d)$ *if and only if the mapping $\mathbf{p}$ is a surjective morphism, i.e.,* $\operatorname{image}(\mathbf{p})=\mathbb{C}^d$.

(2) *the composite mapping* $q\circ\mathbf{p}:\mathbb{C}^k\to\mathbb{C}^d/\mathfrak{S}(d)$ *is dominant, i.e.,* $\mathrm{image}(q\circ\mathbf{p})$ *contains an open dense subset of* $\mathbb{C}^d/\mathfrak{S}(d)$ *if and only if the mapping* $\mathbf{p}$ *is a dominant morphism, i.e.,* $\overline{\mathrm{image}(\mathbf{p})}=\mathbb{C}^d$.

Proof. Note that $q:\mathbb{C}^d\to\mathbb{C}^d/\mathfrak{S}(d)$ is bijective. In fact, for any given vector $\mathbf{w}$, the roots of the univariant polynomial determined by $\mathbf{w}$ (cf. the definition (2.2)) is uniquely defined. Thus, q is an injection from $\mathbb{C}^d$ to $\mathbb{C}^d/\mathfrak{S}(d)$. The surjection of q is clear, since for each given multiset of roots, there exists a $\mathbf{w}\in\mathbb{C}^d$ such that the roots of the corresponding polynomial are exactly them. Then, $q\circ\mathbf{p}$ is surjective if and only if $\mathbf{p}$ is surjective.

We consider the map $g:\mathbb{C}^d/\mathfrak{S}(d)\to\mathbb{C}^d$ defined by sending a multiset $\{\lambda_1,\dots,\lambda_d\}$ to the vector formed by coefficients (except the leading term) of the polynomial $(t-\lambda_1)\cdots(t-\lambda_d)$ in increasing codegree order. Then g is a morphism. It is easy to see that g is the inverse of the map $q:\mathbb{C}^d\to\mathbb{C}^d/\mathfrak{S}(d)$.

If $q\circ\mathbf{p}$ is dominant, then there is $V\subseteq\mathrm{image}(q\circ\mathbf{p})$ such that V is an open dense subset of $\mathbb{C}^d/\mathfrak{S}(d)$. V is also an Euclidean open subset. Since q is in addition continuous, $q^{-1}(V)=g(V)\subseteq\mathbf{p}(\mathbb{C}^k)$ is an Euclidean open subset. If $q^{-1}(V)$ is not dense, then there is small Euclidean open ball $\hat{V}\in\mathbb{C}^d$ such that $q^{-1}(V)\cap\hat{V}=\emptyset$. Since g is continuous, $g^{-1}(\hat{V})$ is an Euclidean open set in $\mathbb{C}^d/\mathfrak{S}(d)$ (whose Euclidean topology is the induced one from $\mathbb{C}^d$). We must have $g^{-1}(\hat{V})\cap V=\emptyset$, since g is bijective. Thus, we obtain a contradiction to the choice of V. Therefore, $\mathbf{p}(\mathbb{C}^k)$ should contains an Euclidean open dense subset of $\mathbb{C}^d$. On the other hand, it is also true that the Euclidean closure of $\mathbf{p}(\mathbb{C}^k)$ is contained in the Zariski closure of $\mathbf{p}(\mathbb{C}^k)$. Thus, $\overline{\mathbf{p}(\mathbb{C}^k)}=\mathbb{C}^d$, and hence $\mathbf{p}$ is a dominant morphism.

Suppose that $\mathbf{p}:\mathbb{C}^k\to\mathbb{C}^d$ is a dominant morphism. It follows from Proposition 2.1 that $\mathbf{p}(\mathbb{C}^k)$ contains an open dense subset U of $\mathbb{C}^d$. Since g is a morphism, $g^{-1}(U)$ is an open dense subset of $\mathbb{C}^d/\mathfrak{S}(d)$. Therefore, $g^{-1}(U)=q(U)\subseteq q(\mathbf{p}(\mathbb{C}^k))\subseteq\mathbb{C}^d/\mathfrak{S}(d)$ is an open dense subset. Thus, $q\circ\mathbf{p}$ is dominant by Definition 2.2. ∎

3. Necessary conditions

In this short section, we will apply tools from algebraic geometry we have reviewed in Section 2 to find a necessary condition for the eigenvalue mapping to be dominant. We can expand the characteristic polynomial $\chi(\lambda)$ of a tensor $\mathcal{T}\in\mathbb{TS}(\mathbb{C}^n,m+1)$ as

$$\chi(\lambda)=\lambda^{nm^{n-1}}+c_{nm^{n-1}-1}(\mathcal{T})\lambda^{nm^{n-1}-1}+\cdots+c_1(\mathcal{T})\lambda+c_0(\mathcal{T}).$$

According to [13], each $c_i(\mathcal{T})\in\mathbb{C}[\mathcal{T}]$ is a homogeneous polynomial in the variables $t_{ji_1\dots i_m}$'s of degree $nm^{n-1}-i$ for $i=0,\dots,nm^{n-1}-1$. We define the *coefficient map* $\mathbf{c}:\mathbb{TS}(\mathbb{C}^n,m+1)\to\mathbb{C}^{nm^{n-1}}$ as

$$\mathbf{c}(\mathcal{T}):=(c_{nm^{n-1}-1}(\mathcal{T}),\dots,c_0(\mathcal{T}))^\mathsf{T}\text{ for all }\mathcal{T}\in\mathbb{TS}(\mathbb{C}^n,m+1). \tag{3.1}$$

It is easy to see that $\mathbf{c}$ is a morphism between two smooth varieties. It is also easy to see that $\phi=q\circ\mathbf{c}$ (cf. Definition 2.2). These, together with Lemma 2.1, imply the next proposition.

PROPOSITION 3.1 (Equivalent Relation). *For any positive integers* m *and* n,

(1) *the multiset-valued eigenvalue map* $\phi:\mathbb{TS}(\mathbb{C}^n,m+1)\to\mathbb{C}^{nm^{n-1}}/\mathfrak{S}_{nm^{n-1}}$ *is surjective if and only if the coefficient map* $\mathbf{c}:\mathbb{TS}(\mathbb{C}^n,m+1)\to\mathbb{C}^{nm^{n-1}}$ *is a surjective morphism.*

(2) the multiset-valued eigenvalue map $\phi : \mathbb{TS}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}/\mathfrak{S}_{nm^{n-1}}$ *is dominant if and only if the coefficient map* $\mathbf{c} : \mathbb{TS}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}$ *is a dominant morphism.*

LEMMA 3.1. *For all positive integers* $m, n \geq 2$*, it holds that*

$$\binom{n+m-1}{m} < m^{n-1}, \tag{3.2}$$

unless $n=2$*, or*

$$(n,m) = (3,2),\ (4,2),\ (3,3).$$

Proof. First note that, for fixed $m \geq 2$, if the inequality (3.2) hods for some $n \geq 2$, then it also holds for $n+1$, since

$$\binom{n+m}{m} = \frac{n+m}{n}\binom{n+m-1}{m} < m\binom{n+m-1}{m}.$$

Second, note that for fixed $n \geq 2$, if the inequality (3.2) hods for some $m \geq 2$, then it also holds for $m+1$, since

$$\frac{\binom{n+m-1}{m}}{m^{n-1}} = \frac{(1+\frac{n-1}{m})\cdots(1+\frac{1}{m})}{(n-1)!}.$$

Last, it is then a direct calculation to see that the listed cases are the only exceptions to the inequality (3.2). ∎

The next proposition establishes the necessary condition under which the multiset-valued eigenvalue map is dominant. It says that in most situations, the eigenvalue map ϕ fails to be dominant.

PROPOSITION 3.2 (Necessary condition). *Let integers* $m, n \geq 2$*. A necessary condition for the map* $\phi : \mathbb{TS}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}/\mathfrak{S}(n)$ *being dominant is that either* $n=2$*, or*

$$(n,m) = (3,2),\ (4,2),\ (3,3).$$

Proof. Recall that Proposition 3.1 shows that a necessary condition for ϕ being dominant is that the coefficient map $\mathbf{c} : \mathbb{TS}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}$ is a dominant morphism. While, Proposition 2.2 indicates that a necessary condition for the mapping $\mathbf{c}$ being a dominant morphism is the dimension of the tensor space $\mathbb{TS}(\mathbb{C}^n, m+1)$ is not smaller than nm^{n-1}. Note that the dimension of the tensor space $\mathbb{TS}(\mathbb{C}^n, m+1)$ is

$$n\binom{n+m-1}{m}.$$

Last, Lemma 3.1 says that this can only happen for the listed cases. ∎

4. Tensors with dimension $n=2$

4.1. Basics. In this section, we consider tensors in $\mathbb{TS}(\mathbb{C}^2, m+1)$. The multiset-valued eigenvalue map is therefore

$$\phi : \mathbb{TS}(\mathbb{C}^2, m+1) \to \mathbb{C}^{2m}/\mathfrak{S}(2m).$$

The system of eigenvalue equations of a tensor $\mathcal{T}=(t_{i_0\ldots i_m})$ is (cf. the system (2.1))

$$\begin{cases} a_0x^m + a_1x^{m-1}y + \cdots + a_my^m = \lambda x^m, \\ b_0x^m + \cdots + b_{m-1}xy^{m-1} + b_my^m = \lambda y^m, \end{cases}$$

where we parameterize $\mathcal{T}$ as

$$a_0 := t_{1111\ldots 1},\ a_1 := mt_{1211\ldots 1}, a_2 = \frac{m(m-1)}{2}t_{1221\ldots 1}, \ldots, a_m = t_{1222\ldots 2},$$
$$b_0 = t_{2111\ldots 1}, \ldots, b_{m-2} = \frac{m(m-1)}{2}t_{2112\ldots 2}, b_{m-1} = mt_{2122\ldots 2},\ b_m = t_{2222\ldots 2}.$$

It follows from the Sylvester formula for the resultant of two homogeneous polynomials in two variable (cf. [8,27]) that the characteristic polynomial is $\det(M-\lambda I)$ with the identity matrix $I \in \mathbb{C}^{2m\times 2m}$ and the matrix $M \in \mathbb{C}^{2m\times 2m}$

$$M = \begin{bmatrix} a_0 & a_1 & a_2 & \ldots & a_m & 0 & 0 & \ldots \\ 0 & a_0 & a_1 & a_2 & \ldots & a_m & 0 & \ldots \\ 0 & 0 & a_0 & a_1 & a_2 & \ldots & a_m & \ldots \\ & & & & \ldots & & & \\ 0 & \ldots & 0 & a_0 & a_1 & a_2 & \ldots & a_m \\ b_0 & b_1 & b_2 & \ldots & b_m & 0 & 0 & \ldots \\ 0 & b_0 & b_1 & b_2 & \ldots & b_m & 0 & \ldots \\ 0 & 0 & b_0 & b_1 & b_2 & \ldots & b_m & \ldots \\ & & & & \ldots & & & \\ 0 & \ldots & 0 & b_0 & b_1 & b_2 & \ldots & b_m \end{bmatrix}. \tag{4.1}$$

For all $k=1,\ldots,2m$, denote by $M_k := \{A : A \text{ is a } k\times k \text{ principal submatrix of } M\}$ the set of all $k\times k$ principal submatrices of M. It is known that

$$\det(M-\lambda I) = \sum_{k=0}^{2m} (-1)^k \Bigg(\sum_{A\in M_{2m-k}} \det(A) \Bigg) \lambda^k,$$

where $M_0 := \emptyset$ by convenience, and the summation over an empty set is defined as 1. Denote by

$$c_k(\mathcal{T}) := (-1)^k \sum_{A\in M_{2m-k}} \det(A), \text{ for all } k = 0,\ldots,2m. \tag{4.2}$$

We have (cf. [13])

$$c_0(\mathcal{T}) = \det(\mathcal{T}) = \det(M),\ c_{2m-1}(\mathcal{T}) = -m(a_0+b_m), \text{ and } c_{2m}(\mathcal{T}) = 1.$$

It is easy to see that each $c_i \in \mathbb{C}[\mathcal{T}]$ is a homogeneous polynomial of degree $2m-i$ for $i=0,\ldots,2m$, and $c_{2m-1}(\mathcal{T}),\ldots,c_0(\mathcal{T})$ are the components of the coefficient map $\mathbf{c}$ (cf. the definition (3.1)). Denote by $H\in\mathbb{C}^{2m\times(2m+2)}$ the Jacobian matrix of the coefficient map $\mathbf{c}:=(c_{2m-1},\dots,c_0)^\mathsf{T}:\mathbb{C}^{2m+2}\to\mathbb{C}^{2m}$ with respect to variables $a_0,\dots,a_m,b_0,\dots,b_m$:

$$h_{ij}:=\begin{cases}\frac{\partial c_{2m-i}}{\partial a_{j-1}} & \text{if } j\le m+1,\\ \frac{\partial c_{2m-i}}{\partial b_{j-m-2}} & \text{otherwise.}\end{cases}$$

Denote by the submatrix $H_{:,1:2m}$ of H as K. Here we use the Matlab notation for submatrices: $A_{a:b,c:d}$ means the submatrix of $A\in\mathbb{C}^{p\times q}$ formed by the row index set $\{a,a+1,\dots,b\}$ and the column index set $\{c,c+1,\dots,d\}$, $A_{:,c:d}$ means the corresponding row index set being the entire $\{1,\dots,p\}$, etc. So, K is a $2m\times 2m$ matrix with entries in $\mathbb{C}[a_0,\dots,a_m,b_0,\dots,b_m]$. Moreover, it follows that the monomial of every term in each entry of the ith row of K is of the same degree $i-1$ with respect to $a_0,\dots,a_m,b_0,\dots,b_m$.

In order to show that the map ϕ is dominant for $\mathbb{TS}(\mathbb{C}^2,m+1)$, which is the same as the map $\mathbf{c}$ being dominant (cf. Proposition 3.1), our goal is to show that the matrix H is of full rank for some tensor $\mathcal{T}$ (cf. Proposition 2.3), which will be a consequence of the nonsingularity of K at that tensor point. Actually, we will show: the determinant of the matrix K is a nonzero polynomial in $\mathbb{C}[\mathcal{T}]$, which implies the nonsingularity generically. To achieve this, we only need to show that there is a term $\alpha a_1^{\frac{m(m-1)}{2}}a_m^{m-1}b_{m-1}^{\frac{m(m+1)}{2}+(m-1)^2}$ in the determinant $\det(K)$ for some nonzero scalar α.

To illustrate the proof of the general case we first work out the following example.

Example 4.1. *Let $m=2$. Then we have the Sylvester matrix*

$$M=\begin{bmatrix}a_0&a_1&a_2&0\\0&a_0&a_1&a_2\\b_0&b_1&b_2&0\\0&b_0&b_1&b_2\end{bmatrix}$$

and hence the coefficients of the characteristic polynomial of M (the same as that for the tensor) are

$$\begin{aligned}
&c_4(M)=1,\\
&c_3(M)=-2(a_0+b_2),\\
&c_2(M)=\det\begin{bmatrix}a_0&a_1\\b_1&b_2\end{bmatrix}+\textit{ principal } 2\times 2 \textit{ minors which do not involve } b_1a_i\text{'s},\\
&c_1(M)=-\det\begin{bmatrix}a_0&a_1&a_2\\b_1&b_2&0\\b_0&b_1&b_2\end{bmatrix}-\textit{ principal } 3\times 3 \textit{ minors which do not involve both } b_1^2a_i\text{'s},\\
&c_0(M)=\det(M), \textit{in which only one term can involve } b_0b_1, \textit{ that is, } a_1a_2b_0b_1.
\end{aligned}$$

It is easy to compute H:

$$H=\begin{bmatrix}-2&0&0&0&0&2\\ \zeta&-b_1&\zeta&\zeta&\kappa&\kappa\\ \eta&\eta&-b_1^2+\eta&\eta&\kappa&\kappa\\ \delta&\mu&\mu&-a_1a_2b_1+\mu&\kappa&\kappa\end{bmatrix},$$

where ζ's contain terms without the variable b_1, η's contain terms with the degrees of b_1 being strictly smaller than 2, and μ contains terms violating either (1) having variable b_1 or (2) only having the variables a_1, a_2 and b_1.

By definition the submatrix K of H is

$$K=\begin{bmatrix}-2 & 0 & 0 & 0\\ \zeta & -b_1 & \zeta & \zeta\\ \eta & \eta & -b_1^2+\eta & \eta\\ \delta & \mu & \mu & -a_1a_2b_1+\mu\end{bmatrix}.$$

Thus, the only way to obtain $a_1a_2b_1^4$ in $\det(K)$ is taking the diagonal entries of the submatrix $K_{1:3,1:3}$ of K (cf. Lemma 4.1 [1]*) and the antidiagonal entry of the submatrix $K_{4,4}$ of K (cf. Lemma 4.2). It is obvious that the coefficient of $a_1a_2b_1^4$ is nonzero. Note that the degree 4 for the variable b_1 is the maximal possible (cf. Lemma 4.3).*

4.2. The proof. This section is devoted to the proof for the dominance of the mapping ϕ when $n=2$. The rationale is given before Example 4.1. Precisely, the existence of a nonzero term $\alpha a_1^{\frac{m(m-1)}{2}} a_m^{m-1} b_{m-1}^{\frac{m(m+1)}{2}+(m-1)^2}$ in the determinant $\det(K)$ will be shown. This monomial has the maximal degree with respect to b_{m-1} among all monomials involving only the variables a_1, a_m and b_{m-1}. The routine is as follows: The submatrix $K_{1:m+1;1:m+1}$ of K will contribute a factor with $b_{m-1}^{\frac{m(m+1)}{2}}$ of the claimed monomial (cf. Lemma 4.1), while the submatrix $K_{m+2:2m;m+2:2m}$ of K will contribute the rest of the factors (cf. Lemma 4.2). Lemma 4.3 will show that the monomial constructed from the entries chosen according to Lemmas 4.1 and 4.2 indeed has the maximal degree with respect to b_{m-1} among all monomials involving only the variables a_1, a_m and b_{m-1}. Proposition 4.1 will implement the lemmas to produce the claimed result.

Let us look at the diagonal elements of the submatrix $K_{1:m+1;1:m+1}$ of K.

LEMMA 4.1. *For each $i=1,\dots,m+1$, there is a nonzero term of the monomial b_{m-1}^{i-1} in the entry K_{ii}. Moreover, this is the unique entry in the ith row of K containing a term of the monomial b_{m-1}^{i-1}.*

Proof. Let us visualize the submatrix $M_{m-1:2m,m-1:2m}$ of M:

$$P=\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_m\\ b_{m-1} & b_m & 0 & 0 & \dots & 0\\ & b_{m-1} & b_m & & & \\ & & \ddots & \ddots & & \\ & & & b_{m-1} & b_m & 0\\ & & & & b_{m-1} & b_m\end{bmatrix}.$$

The case when $i=1$ is trivial. Let $i>1$. It is easy to check that there is a term

$$(-1)^{i-1}a_{i-1}b_{m-1}^{i-1}$$

in the $i\times i$ leading principal minor of P for all $i=2,\dots,m+1$. It is also easy to see that any other $i\times i$ principal minor chosen from the matrix M cannot have a term of the monomial $a_{i-1}b_{m-1}^{i-1}$ for $i>1$, since only the $i\times i$ principal submatrices of P can contain $i-1$ rows for the variable b_{m-1} and one row for a's, and only the minor we have seen can result in a nonzero term of the monomial $a_{i-1}b_{m-1}^{i-1}$. Henceforth, it follows from the formulae for the coefficients and the definition for the Jacobian matrix that a monomial b_{m-1}^{i-1} occurs in the entry K_{ii} for all $i=1,\dots,m+1$.

[1]The choice for the monomial is clear for this example, while the reference to this lemma here and in the sequel are for the purpose of illustrating how the lemmas in Section 4.2 work.

Next we show the uniqueness. By the homogeneity of the polynomials in each entry, the case $i=1$ is trivial. Actually, it follows from [13] that $c_{2m-1}(\mathcal{T})=-m(a_0+b_m)$, which implies $K_{1j}=0$ for $j=2,\dots,2m$.

Let us fix $i>1$. First, each entry K_{ij} cannot have a nonzero term of the monomial b_{m-1}^{i-1} for $j>m+1$. Suppose on the contrary that it has such a term. Then c_{2m-i} contains a nonzero term of the monomial $b_{j-m-2}b_{m-1}^{i-1}$. It follows from the structure of the matrix M that this term comes from an $i\times i$ principal minor of the submatrix $M_{m+1:2m,m+1:2m}$:

$$M_1 := \begin{bmatrix} b_m & 0 & 0 & \dots & 0 \\ b_{m-1} & b_m & 0 & \dots & 0 \\ b_{m-2} & b_{m-1} & b_m & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ b_1 & b_2 & b_3 & \dots & b_m \end{bmatrix}.$$

However, this cannot happen, since the monomial of every term in any principal minor of this matrix contains the variable b_m.

Second, we show that each entry K_{ij} cannot have a nonzero term of the monomial b_{m-1}^{i-1} for $j\neq i$ within $j\in\{1,\dots,m+1\}$. Again, suppose on the contrary that it has such a term. Then c_{2m-i} contains a nonzero term of the monomial $a_{j-1}b_{m-1}^{i-1}$. It comes from a principal minor of M. The corresponding principal submatrix is denoted by $T\in\mathbb{C}^{i\times i}$. By the hypothesis, we should have that T is such a principal submatrix with whose $(i-1)\times(i-1)$ lasting principal submatrix comes from an $(i-1)\times(i-1)$ principal submatrix of M_1, since we should have $i-1$'s b_{m-1}. Note that each principal submatrix of M_1 is lower triangular with the last diagonal entry being b_m. Therefore, in order to get the monomial $a_{j-1}b_{m-1}^{i-1}$, we must have that the $(1,i)$th entry of T is a_{j-1}, and there is b_{m-1} in each sth row of T for $s=2,\dots,i$ by Laplace's determinant formula (cf. [12]). However, this can only happen when $j=i$ and T is a leading principal submatrix of P. A contradiction is therefore obtained.

In conclusion, K_{ii} is the unique entry in the ith row of K possessing a nonzero term of the monomial b_{m-1}^{i-1}. ∎

Let us look at the antidiagonal elements of the submatrix $K_{m+2:2m;m+2:2m}$ of K.

LEMMA 4.2. *For each $i=m+2,\dots,2m$, there is a nonzero term of the monomial $a_1^{i-m-1}b_{m-1}^{m-1}a_m$ in the $(i,3m-i+2)$th entry of K. Moreover, it is the unique entry with the maximal degree $m-1$ for b_{m-1} among terms involving only a_1,b_{m-1},a_m in the (i,j)th entry of K for $j=2,\dots,2m$.*

Proof. Obviously, we cannot get a nonzero term with degree m for the variable b_{m-1} in the (i,j)th entry of K for all $j=m+2,\dots,2m$ and $i=m+2,\dots,2m$, since only m rows of M contain b's. It can be seen from the submatrix $M_{2m-i+1:2m,2m-i+1:2m}$ of M that a nonzero term of the monomial

$$b_{2m-i}a_1^{i-m-1}b_{m-1}^{m-1}a_m$$

occurs in its determinant, which is an $i\times i$ principal minor. We claim that the determinant of $M_{2m-i+1:2m,2m-i+1:2m}$ is the unique $i\times i$ minor in the definition of c_{2m-i} (cf. the notation (4.2)) which has a nonzero term of the monomial $b_{2m-i}a_1^{i-m-1}b_{m-1}^{m-1}a_m$. To obtain $b_{2m-i}b_{m-1}^{m-1}$, for any $i\times i$ principal submatrix T of M, the matrix P (defined in the proof of Lemma 4.1) should be its principal submatrix as well, since only m rows of M contain b's and we should take them all.

First, the last column of the matrix T contains only a_m, b_m and 0's, from which a_m should be chosen, since $2m-i<m$ for all the possible $i=m+2,\dots,2m$. Second, we cannot choose b_{2m-i} from the lower triangular parts of the submatrix P, since otherwise, we can at most get b_{m-1}^{m-2} according to Laplace's determinant formula. Third, by the second, we can only choose b_{2m-i} from the first $i-m-1$ columns of T. Also, since we pick principal submatrices from M, the (1,1)th entry of T is a_0, and the others in the first column are distinct b_t's. Therefore, we must choose b_{2m-i} from the first column. Moreover, it should be the first nonzero entry other than a_0 in the first column. It then follows from the structure of the matrix M that the only possible principal submatrix is the submatrix $M_{2m-i+1:2m,2m-i+1:2m}$.

Therefore, it follows from the formulae for the coefficients and the definition of the Jacobian matrix that a nonzero term of the monomial $a_1^{i-m-1}b_{m-1}^{m-1}a_m$ occurs in the $(i,3m-i+2)$th entry of K for all $i=m+2,\dots,2m$.

With almost the same argument, we can see that there does not exist a nonzero term of a monomial with the maximum degree $m-1$ for b_{m-1} and with only the variables a_1, b_{m-1} and a_m in the (i,j)th entry of K for all $j\in\{m+2,\dots,2m\}\setminus\{3m-i+2\}$ and $i=m+2,\dots,2m$.

Next we show that there does not exist a nonzero term of a monomial

$$a_r a_1^p b_{m-1}^{m-1} a_m^q$$

with some integers $p+q=i-m$ for any $r=1,\dots,m$ in any $i\times i$ principal minor of M for all $i=m+2,\dots,2m$. Note that the first column of any $i\times i$ principal submatrix of M is of the form

$$(a_0,0,\dots,0,b_t,b_{t-1},\dots)^{\mathsf{T}}$$

for some $t<m-1$, since $i\geq m+2$. Therefore, each term of the minor must contain either the variable a_0 or a variable b_s for some $s<m-1$. Neither case will result in a nonzero term of the monomial involving only a_1, a_r, a_m and b_{m-1} for some $r=1,\dots,m$. Thus, no term involving only a_1, a_m and b_{m-1} exists in the (i,j)-entry of K for $i=m+2,\dots,2m$ and $j=2,\dots,m+1$.

Henceforth, a nonzero term of the monomial $a_1^{i-m-1}b_{m-1}^{m-1}a_m$ uniquely appears in the $(i,3m-i+2)$-entry of K for every $i=m+2,\dots,2m$. ∎

Lemma 4.3. *The submatrix K of the Jacobian matrix H is nonsingular generically over the tensor space $\mathbb{T}(\mathbb{C}^2,m+1)$ for all $m=2,3,\dots$.*

Proof. We know that the first row of K is

$$(-m,0,\dots,0)^{\mathsf{T}}\in\mathbb{C}^{2m},$$

which implies that $\det(K)=-m\det(K_{2:2m,2:2m})$.

We consider terms of $\det(K)$ of monomials with only the variables a_1, b_{m-1} and a_m. For $i=2,\dots,2m$, each entry of the ith row of the matrix K is a homogeneous polynomial of degree $i-1$ in the variables $a_0,\dots,a_m,b_0,\dots,b_m$, and there are m rows of M containing b_{m-1}. This, together with Lemmas 4.1 and 4.2, implies that the maximal possible degree for the variable b_{m-1} in such a term in the determinant of the matrix K is

$$1+\cdots+m+(m-1)(m-1)=\frac{m(m+1)}{2}+(m-1)^2.$$

It follows from Lemmas 4.1 and 4.2 again that such a term is unique and there is a unique way to consitute it: choosing the diagonal entries of the submatrix $K_{1:m+1,1:m+1}$ and then the anti-diagonal entries of the submatrix $K_{m+2:2m,m+2:2m}$. Moreover, by the same lemmas, the term of the monomial $a_1^{\frac{m(m-1)}{2}} b_{m-1}^{\frac{2m^2-m+1}{2}} a_m^{m-1}$ in the determinant of the K has nonzero coefficient.

Therefore, the determinant of the matrix K is a nonzero polynomial over the polynomial ring $\mathbb{C}[a_0,\dots,a_m,b_0,\dots,b_m]$. By Hilbert's Nullstellensatz (cf. [10, 11, 27]), we conclude that the submatrix K of the Jacobian matrix is nonsingular generically in the tensor space. ∎

Proposition 4.1. *For any positive $m \geq 1$, the multiset-valued eigenvalue map ϕ: $\mathbb{TS}(\mathbb{C}^2, m+1) \to \mathbb{C}^{2m}/\mathfrak{S}(2m)$ is dominant, i.e., for a generic multiset $S \in \mathbb{C}^{2m}/\mathfrak{S}(2m)$, there exists a tensor $\mathcal{T} \in \mathbb{TS}(\mathbb{C}^2, m+1)$ such that the set of eigenvalues (counting with multiplicities) of $\mathcal{T}$ is S.*

Proof. Recall that Lemma 4.3 indicates that the Jacobian matrix H of the coefficient map $\mathbf{c}: \mathbb{TS}(\mathbb{C}^2, m+1) \to \mathbb{C}^{2m}$ has full rank, since it has a nonsingular submatrix at a generic point. It then implies that the mapping $\mathbf{c}$ is a dominant morphism warranted by Proposition 2.3. Since the fact that $\mathbf{c}$ being a dominant morphism implies the mapping ϕ being dominant is proved in Proposition 3.1, the conclusion follows. ∎

4.3. Sylvester matrices. We make a remark here about Sylvester matrices as a short subsection. A Sylvester matrix is a matrix of the form as M (cf. the matrix (4.1)):

$$\begin{bmatrix} a_0 & \dots & a_p & 0 & 0 & \dots \\ & & & \dots & & \\ 0 & \dots & 0 & a_0 & \dots & a_p \\ b_0 & \dots & b_q & 0 & 0 & \dots \\ & & & \dots & & \\ 0 & \dots & 0 & b_0 & \dots & b_q \end{bmatrix},$$

while in general there are q rows of a's and p row of b's for possibly different p, q. Therefore, the matrix is in $\mathbb{C}^{(p+q)\times(p+q)}$. Up to permutation, we can assume without loss of generality that $q \geq p$. Then, with almost the same argument as the preceding analysis, we can obtain the following result on the inverse eigenvalue problem for Sylvester matrices.

Proposition 4.2 (Sylvester Matrix). *Let $n \geq 2$ be a positive integer. Given a generic multiset $S \in \mathbb{C}^n/\mathfrak{S}(n)$ there exists a Sylvester matrix $A \in \mathbb{C}^{n\times n}$ such that the set of eigenvalues (counting with multiplicities) of A is S.*

4.4. Extensions. In the following, we will get back to tensors. We know in the matrix case that for a given multiset of n numbers, we can easily construct a matrix (e.g., the diagonal matrix with the diagonal elements being the given numbers) such that whose multiset of eigenvalues is precisely the given one. For the tensor case when $n = 2$, Section 4.2 shows the dominance of the eigenvalue mapping ϕ, while it is unclear whether this mapping is surjective or not, as well as how to construct a tensor such that whose multiset of eigenvalues is the given one. These issues will be addressed slightly in this section.

Note that by Pieri's formula (cf. [20, page 73]), we have a decomposition of

$\mathbb{TS}(\mathbb{C}^n, m+1)$ as a $\mathrm{GL}_n(\mathbb{C})$ module:

$$\mathbb{TS}(\mathbb{C}^n, m+1) = \mathbb{C}^n \otimes \mathrm{S}^m(\mathbb{C}^n) = \mathrm{S}^{m+1}\mathbb{C}^n \oplus \mathrm{S}_{m,1}\mathbb{C}^n.$$

In particular, when $n=2$ we have

$$\mathbb{TS}(\mathbb{C}^2, m+1) = \mathrm{S}^{m+1}\mathbb{C}^2 \oplus \mathrm{S}_{m,1}\mathbb{C}^2 = \mathrm{S}^{m+1}\mathbb{C}^2 \oplus (\wedge^2\mathbb{C}^2 \otimes \mathrm{S}^{m-1}\mathbb{C}^2).$$

Tensors in $\mathrm{S}^{m+1}\mathbb{C}^2$ are just symmetric tensors, which can be represented by $m+2$ parameters. More precisely, for each symmetric tensor $\mathcal{T}$, the homogeneous polynomial $\mathbf{z}^\mathsf{T}(\mathcal{T}\mathbf{z}^m)$ with $\mathbf{z} = (x,y)^\mathsf{T}$ can be parameterized as $F(x,y) = a_{m+1}x^{m+1} + \cdots + a_0 y^{m+1} \in \mathbb{C}[x,y]$ for a's.

LEMMA 4.4. *The system of eigenvalue equations associated to $\mathcal{T}$ is*

$$\frac{\partial F(x,y)}{\partial x} = \lambda x^m,$$
$$\frac{\partial F(x,y)}{\partial y} = \lambda y^m.$$

We characterize eigenvalues of a nonzero $\mathcal{T} \in \mathrm{S}^{m+1}\mathbb{C}^2$ in the next proposition.

PROPOSITION 4.3. *Let $p_1, \cdots, p_k$ be distinct zeros of $F(x,y)$ in $\mathbb{P}^1$, with multiplicities $m_1, \cdots, m_k$ respectively. Let L_i be the linear form vanishing on p_i respectively. Eigenvalues of $\mathcal{T}$ are 0 with multiplicity $\sum_{i=1}^k (m_i - 1)$ and $\lambda_j = \frac{\partial F}{\partial x}(\alpha_j, \beta_j)/\alpha_j^m, j = 1, \ldots, m+k-1$ with multiplicity one where (α_j, β_j) is a solution of*

$$\frac{y^m \frac{\partial F}{\partial x} - x^m \frac{\partial F}{\partial y}}{\prod_{i=1}^k L_i^{m_i - 1}} = 0.$$

Proof. By the equations in Lemma 4.4 we obtain

$$\lambda(y^m \frac{\partial F}{\partial x} - x^m \frac{\partial F}{\partial y}) = 0.$$

Let $\lambda \neq 0$ be an eigenvalue of $\mathcal{T}$ and $(\alpha, \beta) \neq (0,0)$ be an eigenvector corresponding to λ. Then, either $\frac{\partial F}{\partial x}(\alpha, \beta) \neq 0$ or $\frac{\partial F}{\partial y}(\alpha, \beta) \neq 0$, i.e., (α, β) is a root of

$$\frac{y^m \frac{\partial F}{\partial x} - x^m \frac{\partial F}{\partial y}}{\prod_{i=1}^k L_i^{m_i - 1}} = 0.$$

Since $\mathcal{T}$ has $2m$ eigenvalues and $\sum_{i=1}^k m_i = m+1$, we see that they are either 0 or of the forms as claimed. ∎

To conclude this section, we consider eigenvalues of tensors in $\wedge^2\mathbb{C}^2 \otimes \mathrm{S}^{m-1}\mathbb{C}^2$.

LEMMA 4.5. *The system of eigenvalue equations associated to a tensor $\mathcal{T} \in \wedge^2\mathbb{C}^2 \otimes \mathrm{S}^{m-1}\mathbb{C}^2$ is of the form*

$$yf(x,y) = \lambda x^m,$$
$$-xf(x,y) = \lambda y^m,$$

where $f(x,y)$ is a homogeneous polynomial of degree $m-1$.

Proof. Let us fix the standard basis $\{\mathbf{e}_1,\mathbf{e}_2\}$ for $\mathbb{C}^2$. Then an element in $\wedge^2\mathbb{C}^2\otimes \mathrm{S}^{m-1}\mathbb{C}^2$ is

$$\mathcal{T}=(\mathbf{e}_1\wedge\mathbf{e}_2)\otimes f,$$

where $f\in\mathrm{S}^{m-1}\mathbb{C}^2$. We identify $\mathrm{S}^{m-1}\mathbb{C}^2$ with the space of homogeneous polynomials of degree $m-1$ in two variables with coefficients in the field of complex numbers. Expand $\mathbf{e}_1\wedge\mathbf{e}_2$ and write out the equation system corresponding to $\mathcal{T}$; the claim follows. □

Let $\mathcal{T}\in\wedge^2\mathbb{C}^2\otimes\mathrm{S}^{m-1}\mathbb{C}^2$. Then we describe eigenvalues of $\mathcal{T}$ in the next proposition.

PROPOSITION 4.4. *Eigenvalues of $\mathcal{T}$ are 0 with multiplicity $m-1$ and $\omega_i f(1,\omega_i), i=0,\dots,m$, where ω_i is an $(m+1)$-th root of -1.*

Proof. If $\lambda\neq 0$ is an eigenvalue of $\mathcal{T}$, then

$$yf(x,y)=\lambda x^m \text{ and } -xf(x,y)=\lambda y^m.$$

Thus

$$\lambda(x^{m+1}+y^{m+1})=0.$$

Since $\lambda\neq 0$ we can derive

$$y=\omega_i x, i=0,\dots,m.$$

It is easy to obtain

$$\lambda=yf(x,y)/x^m=\omega_i f(1,\omega_i).$$

Lastly, every homogeneous polynomial $f(x,y)$ definitely has a nontrivial solution in $\mathbb{C}^2$ by Hilbert's Nullstellensatz. We conclude that 0 is also an eigenvalue of $\mathcal{T}$. Since the total number of eigenvalues is $2m$, we see that 0 gets the rest multiplicity $m-1$. □

REMARK 4.1. We notice that Proposition 4.4 gives an algorithm to reconstruct a tensor $\mathcal{T}\in\wedge^2\mathbb{C}^2\otimes\mathrm{S}^{m-1}\mathbb{C}^2$ from given $m+1$ numbers $\lambda_0,\dots,\lambda_m$ such that eigenvalues of $\mathcal{T}$ are $\lambda_0,\dots,\lambda_m$ and zero by solving a linear system. Namely, we consider the following linear system

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ \omega_1 & \omega_1^2 & \cdots & \omega_1^{m-1} & \omega_1^m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \omega_m & \omega_m^2 & \cdots & \omega_m^{m-1} & \omega_m^m \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix},$$

where ω_i's are the $m+1$-th roots of -1.

(1) If this overdetermined linear system has no solution, then no tensor $\mathcal{T}\in\wedge^2\mathbb{C}^2\otimes\mathrm{S}^{m-1}\mathbb{C}^2$ can have $\{\lambda_0,\dots,\lambda_m,0,\dots,0\}$ as the multiset of eigenvalues.
(2) If this overdetermined linear system has a solution, then the solution gives a homogeneous polynomial $f(x,y)=\sum_{i=0}^{m-1}a_i x^{m-1-i}y_i$ which gives the desired tensor $\mathcal{T}$ (cf. Lemma 4.5).

5. The exceptional cases

In this section, we show that the eigenvalue map $\phi: \mathbb{TS}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ is dominant for the *exceptional cases*

$$(n,m) = (3,2),\ (4,2),\ (3,3).$$

We use Propositions 2.3 and 3.1 to prove the results. The basic idea is the same as Section 4: finding a point in $\mathbb{TS}(\mathbb{C}^n, m+1)$ such that the differential of the coefficient map $\mathbf{c}: \mathbb{TS}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}$ at this point has the maximal rank nm^{n-1}. The difference is instead of proving a generic property on the Jacobian matrix, we find a concrete point at which the Jacobian matrix is of full rank.

5.1. Macaulay's formula of characteristic polynomials. We refer to [8, Chapter 3] for the theory and computation of resultants and hyperdeterminants. More resources for resultants and hyperdeterminants are available in [5, 10, 27]. The determinant of a tensor is actually the resultant of a specially constructed system of homogeneous polynomials of the same degree [13].

Let $d = nm - n + 1$ and let $S = \{x_1^d, x_1^{d-1}x_2, \ldots, x_n^d\}$ be the set of monomials in $x_1, \ldots, x_n$ of degree d in lexicographic order. A monomial of degree d is written as $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$ with $\alpha \in \mathbb{N}^n$ and $\alpha_1 + \cdots + \alpha_n = d$. The set S divides into n subsets as follows:

$$S_i := \{\mathbf{x}^\alpha \in S : \alpha_i \geq m,\ \alpha_j < m \text{ for all } j = 1, \ldots, i-1\}, \text{ for all } i = 1, \ldots, n.$$

It is easy to see that $\{S_1, \ldots, S_n\}$ are mutually disjoint and $\cup_{i=1}^n S_i = S$. Note that the cardinality of S is

$$w = |S| = \binom{d+n-1}{d}.$$

Let $\mathcal{T} \in \mathbb{TS}(\mathbb{C}^n, m+1)$. We write

$$f_i(\mathbf{x}) := (\mathcal{T}\mathbf{x}^m - \lambda \mathbf{x}^{[m]})_i$$

as the ith defining equation for the eigenvalue problem for $i = 1, \ldots, n$. For the n homogeneous polynomials $f_1(\mathbf{x}), \ldots, f_n(\mathbf{x})$ in n variables $\mathbf{x} = (x_1, \ldots, x_n)$, parameterized by $\mathcal{T}$ and λ, we can formulate a system of w homogeneous polynomials

$$\mathbf{x}^{\alpha - m\mathbf{e}_i} \cdot f_i(\mathbf{x}), \text{ for all } \mathbf{x}^\alpha \in S_i, \text{ for all } i = 1, \ldots, n, \tag{5.1}$$

where $\mathbf{e}_i \in \mathbb{R}^n$ is the ith standard basis vector. This system of polynomials is naturally indexed by monomials $\mathbf{x}^\alpha \in S$. With respect to the basis S, we can represent the system (5.1) as a matrix $R \in \mathbb{C}[\mathcal{T}, \lambda]^{w \times w}$. A monomial $\mathbf{x}^\alpha$ is reduced, if there exists exactly one $i \in \{1, \ldots, n\}$ such that $\alpha_i \geq m$. The submatrix of R obtained by deleting all rows and columns of reduced monomials is denoted by R'. Note that the entries of both R and R' are linear forms of the variables $t_{ji_1\ldots i_m}$ and λ.

It follows from Macaulay's formula for resultant (cf. [19]) that the characteristic polynomial of $\mathcal{T}$ is

$$\det(\mathcal{T} - \lambda \mathcal{I}) = \pm \frac{\det(R)}{\det(R')}. \tag{5.2}$$

With the characteristic polynomial (5.2), we can compute out the coefficient map $\mathbf{c}: \mathbb{TS}(\mathbb{C}^n, m+1) \to \mathbb{C}^{nm^{n-1}}$ and its Jacobian matrix H. Note that, we may restrict our map $\mathbf{c}$ to a subspace $V \subseteq \mathbb{TS}(\mathbb{C}^n, m+1)$ as long as the dimension of V is larger than nm^{n-1} (cf. Proposition 2.2) to reduce the computational cost.

5.2. Third order three-dimensional tensors. In this section, we present the detailed computation for third order three-dimensional tensors, i.e., $(n,m)=(3,2)$. The details serve as an example to illustrate the method in Section 5.1. The computation is majorly conducted by Macaulay2 [9] together with Matlab.

For any tensor $\mathcal{T}\in\mathbb{TS}(\mathbb{C}^3,3)$, its system of eigenvalue equations is

$$\begin{cases}\sum_{j,k=1}^{3} t_{1jk}x_jx_k=\lambda x_1^2,\\ \sum_{j,k=1}^{3} t_{2jk}x_jx_k=\lambda x_2^2,\\ \sum_{j,k=1}^{3} t_{3jk}x_jx_k=\lambda x_3^2,\end{cases}$$

which can be equivalently parameterized as

$$\begin{cases} f_1(A,\lambda,\mathbf{x}):=a_{11}x_1^2+a_{12}x_1x_2+a_{13}x_2^2+a_{14}x_1x_3+a_{15}x_2x_3+a_{16}x_3^2-\lambda x_1^2=0,\\ f_2(A,\lambda,\mathbf{x}):=a_{21}x_1^2+a_{22}x_1x_2+a_{23}x_2^2+a_{24}x_1x_3+a_{25}x_2x_3+a_{26}x_3^2-\lambda x_2^2=0,\\ f_3(A,\lambda,\mathbf{x}):=a_{31}x_1^2+a_{32}x_1x_2+a_{33}x_2^2+a_{34}x_1x_3+a_{35}x_2x_3+a_{36}x_3^2-\lambda x_3^2=0.\end{cases}$$

Let

$$S=\{x_1^4,x_1^3x_2,\dots,x_2^4,\dots,x_3^4\}$$

be the set of all monomials of x_1,x_2,x_3 with total degree 4 in lexicographic order, and

$$T_1=\{x_1^2,x_1x_2,x_1x_3,x_2^2,x_2x_3,x_3^2\},\ T_2=\{x_1x_2,x_1x_3,x_2^2,x_2x_3,x_3^2\},$$

and

$$T_3=\{x_1x_2,x_1x_3,x_2x_3,x_3^2\}.$$

It follows that the cardinality of S is

$$|S|=\binom{3+4-1}{4}=15.$$

We generate a system of 15 polynomial equations via

$$f_ig_j^i=0,\ \text{for all } g_j^i\in T_i,\ \text{for all } i=1,2,3.$$

Regarding $f_ig_j^i\in\mathbb{C}[A,\lambda][\mathbf{x}]$, we can get a square matrix $M\in\big(\mathbb{C}[A,\lambda]\big)^{15\times 15}$ as the coefficient matrix of the polynomial equations

$$f_1g_1^1=0,\dots,f_1g_6^1=0,f_2g_1^2=0,\dots,f_2g_5^2=0,f_3g_1^3=0,\dots,f_3g_4^3=0$$

in the canonical basis S. It follows from Section 5.1 that the characteristic polynomial of $\mathcal{T}$ is

$$\det(\mathcal{T}-\lambda\mathcal{I})=\pm\frac{\det(M)}{(a_{11}-\lambda)^2(a_{23}-\lambda)}.$$

The matrix M is

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{11} & a_{12} & 0 & a_{14} & 0 & a_{13} & a_{15} & 0 & 0 & 0 & a_{16} & 0 & 0 & 0 \\
0 & 0 & a_{11} & 0 & 0 & 0 & a_{12} & a_{14} & a_{13} & a_{15} & a_{16} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{11} & a_{12} & a_{14} & 0 & a_{13} & 0 & 0 & 0 & a_{15} & a_{16} & 0 & 0 \\
0 & 0 & 0 & 0 & a_{11} & 0 & 0 & a_{12} & 0 & a_{13} & a_{15} & a_{14} & 0 & a_{16} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{11} & 0 & 0 & 0 & 0 & a_{13} & a_{12} & a_{14} & a_{15} & a_{16} \\
0 & a_{21} & a_{22} & 0 & a_{24} & 0 & a_{23} & a_{25} & 0 & 0 & 0 & a_{26} & 0 & 0 & 0 \\
0 & 0 & 0 & a_{21} & a_{22} & a_{24} & 0 & a_{23} & 0 & 0 & 0 & a_{25} & a_{26} & 0 & 0 \\
0 & 0 & a_{21} & 0 & 0 & 0 & a_{22} & a_{24} & a_{23} & a_{25} & a_{26} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{21} & 0 & 0 & a_{22} & 0 & a_{23} & a_{25} & a_{24} & 0 & a_{26} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{21} & 0 & 0 & 0 & 0 & a_{23} & a_{22} & a_{24} & a_{25} & a_{26} \\
0 & a_{31} & a_{32} & 0 & a_{34} & 0 & a_{33} & a_{35} & 0 & 0 & 0 & a_{36} & 0 & 0 & 0 \\
0 & 0 & 0 & a_{31} & a_{32} & a_{34} & 0 & a_{33} & 0 & 0 & 0 & a_{35} & a_{36} & 0 & 0 \\
0 & 0 & 0 & 0 & a_{31} & 0 & 0 & a_{32} & 0 & a_{33} & a_{35} & a_{34} & 0 & a_{36} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{31} & 0 & 0 & 0 & 0 & a_{33} & a_{32} & a_{34} & a_{35} & a_{36}
\end{bmatrix} - \lambda I,$$

where I is the identity matrix of appropriate size.

If we restrict the tensor space to be with $a_{21}=a_{31}=a_{13}=a_{33}=0$, then we have

$$\det(\mathcal{T}-\lambda\mathcal{I}) = \frac{\det(M)}{(a_{11}-\lambda)^2(a_{23}-\lambda)} = \det(M')$$

with

$$M' = \begin{bmatrix}
a_{11} & 0 & 0 & 0 & a_{12} & a_{14} & a_{15} & a_{16} & 0 & 0 & 0 & 0 \\
0 & a_{11} & a_{12} & a_{14} & 0 & 0 & 0 & 0 & a_{15} & a_{16} & 0 & 0 \\
0 & 0 & a_{11} & 0 & 0 & a_{12} & 0 & a_{15} & a_{14} & 0 & a_{16} & 0 \\
0 & 0 & 0 & a_{11} & 0 & 0 & 0 & 0 & a_{12} & a_{14} & a_{15} & a_{16} \\
a_{22} & 0 & a_{24} & 0 & a_{23} & a_{25} & 0 & 0 & a_{26} & 0 & 0 & 0 \\
0 & 0 & a_{22} & a_{24} & 0 & a_{23} & 0 & 0 & a_{25} & a_{26} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{25} & a_{24} & 0 & a_{26} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{23} & a_{22} & a_{24} & a_{25} & a_{26} \\
a_{32} & 0 & a_{34} & 0 & 0 & a_{35} & 0 & 0 & a_{36} & 0 & 0 & 0 \\
0 & 0 & a_{32} & a_{34} & 0 & 0 & 0 & 0 & a_{35} & a_{36} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{32} & 0 & a_{35} & a_{34} & 0 & a_{36} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{32} & a_{34} & a_{35} & a_{36}
\end{bmatrix} - \lambda I.$$

Note that we restricted our coefficient map $\mathbf{c}$ to a linear subspace V of dimension 14. We use Macaulay2 to compute the 12×14 Jacobian matrix. The evaluation of this matrix at the point

$$a_{11}=1, a_{12}=2, a_{14}=3, a_{15}=4, a_{16}=5,$$
$$a_{22}=6, a_{23}=7, a_{24}=8, a_{25}=9, a_{26}=10,$$
$$a_{32}=11, a_{34}=12, a_{35}=13, a_{36}=14$$

is

$$\left[\begin{array}{ccccccc}
-4 & 0 & 0 & 0 & 0 & 0 & -4 \\
348 & -12 & -24 & 0 & 0 & -4 & 324 \\
-11948 & 528 & 1575 & -336 & -420 & 123 & -10190 \\
229449 & -6573 & -42450 & 9606 & 14460 & -435 & 178549 \\
-2841839 & 8007 & 669288 & -123924 & -254385 & -40559 & -1983021 \\
24015886 & 693225 & -6820251 & 716979 & 2947131 & 838271 & 14685692 \\
-141005226 & -8897502 & 46520475 & -55500 & -23636976 & -7517499 & -74200394 \\
577067743 & 52779339 & -214721160 & -19191762 & 128734014 & 37178105 & 258147039 \\
-1615274021 & -168212115 & 650003046 & 85305210 & -443297901 & -110199791 & -603940123 \\
2874026450 & 286931673 & -1165235823 & -145117011 & 852500403 & 194375103 & 876534196 \\
-2794768018 & -259796358 & 1046384496 & 111968790 & -778096974 & -179135234 & -672256468 \\
1066887388 & 96499788 & -356759172 & -33512052 & 261090648 & 64501920 & 202844400
\end{array}\right.$$

$$
\left.\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & -4 \\
0 & -26 & 0 & 0 & -6 & -18 & 296 \\
-158 & 1685 & -382 & -223 & 156 & 652 & -8362 \\
4943 & -43158 & 16014 & 5650 & -1511 & -6084 & 129887 \\
-54113 & 607211 & -282327 & -46378 & -27079 & -23151 & -1258172 \\
150466 & -5348868 & 2845660 & -150997 & 963371 & 1047467 & 7850991 \\
170837 & 31218785 & -18669471 & 5370313 & -11890233 & -11634444 & -30119950 \\
-16953189 & -122285430 & 82279634 & -38993472 & 78057193 & 68998306 & 58177003 \\
64320415 & 313612725 & -232274827 & 133470656 & -288207851 & -225232919 & -1425840 \\
-125125090 & -487990298 & 384765126 & -232073969 & 569148965 & 386696721 & -187322475 \\
121312760 & 392623914 & -320537057 & 201649792 & -531624753 & -319797178 & 252532328 \\
-45364410 & -122396540 & 101857630 & -69231372 & 183581748 & 99950648 & -98555702
\end{array}\right]
$$

Using either Matlab or Macaulay2, we can check that the above matrix has full rank 12. Therefore, we arrive at the next proposition.

PROPOSITION 5.1. *The eigenvalue map* $\phi\colon \mathbb{TS}(\mathbb{C}^3,3)\to\mathbb{C}^{12}/\mathfrak{S}(12)$ *is dominant.*

5.3. Fourth order three-dimensional and third order four-dimensional tensors. In this section, we show that the eigenvalue maps $\phi\colon\mathbb{TS}(\mathbb{C}^3,4)\to\mathbb{C}^{27}/\mathfrak{S}(27)$ and $\phi\colon\mathbb{TS}(\mathbb{C}^4,3)\to\mathbb{C}^{32}/\mathfrak{S}(32)$ are both dominant. Note that, symbolically, the determinants of tensors of both formats $\mathbb{TS}(\mathbb{C}^3,4)$ and $\mathbb{TS}(\mathbb{C}^4,3)$ are most likely to have millions of terms, regarding the relationship between determinants and hyperdeterminants (cf. [22]) and already the approximately 3 million terms for the hyperdeterminant of tensors in $\mathbb{T}(\mathbb{C}^2,4)$ (cf. [15]). It is impossible to use Macaulay2 to compute out the characteristic polynomial $\det(\mathcal{T}-\lambda\mathcal{I})$ of a symbolic tensor in these two cases.

For a map $\mathbf{f}\colon V\to W$ between two vector spaces V and W, whenever the differential $d_{\mathbf{x}}\mathbf{f}$ exists at a point $\mathbf{x}$, we have that the directional derivative of $\mathbf{f}$ at direction $\mathbf{y}\in V$ is

$$\mathbf{f}'(\mathbf{x};\mathbf{y})=(d_{\mathbf{x}}\mathbf{f})\mathbf{y}. \tag{5.3}$$

Let $\dim(V)\geq\dim(W)$. As a linear map from V to W, $d_{\mathbf{x}}\mathbf{f}$ is of maximal rank $\dim(W)$ if we can find a set of directions $\{\mathbf{y}_1,\ldots,\mathbf{y}_k\}\subset V$ with $k\geq\dim(W)$ such that

$$\operatorname{rank}\big([(d_{\mathbf{x}}\mathbf{f})\mathbf{y}_1,\ \ldots,\ (d_{\mathbf{x}}\mathbf{f})\mathbf{y}_k]\big)=\dim(W).$$

Note that here our map is the coefficient map $\mathbf{c}$ (cf. the definition (3.1)). V is either $\mathbb{TS}(\mathbb{C}^3,4)$ or $\mathbb{TS}(\mathbb{C}^4,3)$, and W is respectively either $\mathbb{C}^{27}$ or $\mathbb{C}^{32}$. In both cases, we choose $k=\dim(V)$. We use formula (5.3) to compute $(d_{\mathbf{x}}\mathbf{f})\mathbf{y}_i$ for each $i=1,\ldots,k$. We first choose a point $\mathcal{T}\in V$ and a set of directions $\{\mathcal{T}_1,\ldots,\mathcal{T}_k\}$, which will be chosen as the set of standard basis of the space V. Then, we compute the characteristic polynomial of $\mathcal{T}+t\mathcal{T}_i$ with parameter t

$$\det(\mathcal{T}+t\mathcal{T}_i-\lambda\mathcal{I})$$

for all $i=1,\ldots,k$. Note that we have only two symbolic variables λ and t now. Write $\det(\mathcal{T}+t\mathcal{T}_i-\lambda\mathcal{I})$ as

$$\det(\mathcal{T}+t\mathcal{T}_i-\lambda\mathcal{I})=\sum_{s=0}^{N}c_s(t)\lambda^s,$$

for appropriate N, which is either 27 or 32. Note that $c_N(t)=\pm1$. It follows that

$$\mathbf{c}'(\mathcal{T};\mathcal{T}_i)=(d_{\mathcal{T}}\mathbf{c})\mathcal{T}_i=(c'_{N-1}(0),\ldots,c'_0(0))^{\mathsf{T}}.$$

In this way, we can try to find a tensor $\mathcal{T}\in V$ such that the resulting matrix

$$[(d_{\mathcal{T}}\mathbf{c})\mathcal{T}_1,\ \ldots,(d_{\mathcal{T}}\mathbf{c})\mathcal{T}_k]$$

has full rank. In fact, if such a tensor $\mathcal{T}$ exists, then a generic tensor will work. For $V=\mathbb{TS}(\mathbb{C}^3,4)$, the differential of the coefficient map at the tensor point (only independent entries are listed)

$$t_{1111}=1,\ t_{1112}=-1/3,\ t_{1122}=2/3,\ t_{1222}=-2,\ t_{1113}=1,\ t_{1123}=-1/2,\ t_{1223}=4/3,$$
$$t_{1133}=-4/3,\ t_{1233}=5/3,\ t_{1333}=-5, t_{2111}=6,\ t_{2112}=-2,\ t_{2122}=7/3,\ t_{2222}=-7,$$
$$t_{2113}=8/3, t_{2123}=-4/3,\ t_{2223}=3,\ t_{2133}=-3,\ t_{2233}=1/3,\ t_{2333}=2,\ t_{3111}=3,$$
$$t_{3112}=4/3,\ t_{3122}=5/3, t_{3222}=6,\ t_{3113}=0, t_{3123}=-1/6,$$
$$t_{3223}=-2/3,\ t_{3133}=-1,\ t_{3233}=-4/3,\ t_{3333}-5$$

is of full rank 27; and for $V=\mathbb{TS}(\mathbb{C}^4,3)$, the differential of the coefficient map at the tensor point (again, only independent entries are listed)

$$t_{111}=1,\ t_{112}=-1/2,\ t_{122}=2,\ t_{113}=-1,\ t_{123}=3/2,\ t_{133}=-3, t_{114}=2,\ t_{124}=-2,$$
$$t_{134}=5/2,\ t_{144}=-5, t_{211}=6,\ t_{212}=-3,\ t_{222}=7,\ t_{213}=-7/2,\ t_{223}=4,$$
$$t_{233}=-8, t_{214}=9/2,\ t_{224}=-9/2,\ t_{234}=1/2,\ t_{244}=2, t_{311}=3,\ t_{312}=2,$$
$$t_{322}=5,\ t_{313}=3,\ t_{323}=0,\ t_{333}=-1, t_{314}=-1,\ t_{324}=-3/2,\ t_{334}=-2,\ t_{344}=-5,$$
$$t_{411}=1,\ t_{412}=1,\ t_{422}=3/2,\ t_{413}=2,\ t_{423}=5/2,\ t_{433}=6,$$
$$t_{414}=7/2,\ t_{424}=4,\ t_{434}=9/2,\ t_{444}=10$$

is of full rank 32.

Therefore, we have the next result from Propositions 2.3 and 3.1.

PROPOSITION 5.2. *The eigenvalue maps* $\phi:\mathbb{TS}(\mathbb{C}^3,4)\to\mathbb{C}^{27}/\mathfrak{S}(27)$ *and* $\phi:\mathbb{TS}(\mathbb{C}^4,3)\to\mathbb{C}^{32}/\mathfrak{S}(32)$ *are both dominant.*

6. Final remarks

The dominance of the eigenvalue mapping in the tensor space is investigated in this article. A tensor of order $m+1$ and dimension n possesses nm^{n-1} eigenvalues (with multiplicities). The underlying interconnections between this vast number of eigenvalues are rarely studied in the literature and therefore unknown. We showed in this work that the multisets of eigenvalues of tensors in $\mathbb{T}(\mathbb{C}^n,m+1)$ usually form a proper subset of $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ with a much smaller dimension. Thus, there are intrinsic structures underlying eigenvalues of tensors, and the algebraic relations between them are rich. This should be interesting and important for understanding eigenvalues of tensors, as well as warranting them for applications. There are several immediate issues (e.g., how to reconstruct a tensor given the eigenvalues, the dimension of the image of the eigenvalue mapping and its closedness, and the situation for structured tensors, such as symmetric tensors) worth investigating, which will be gently discussed in the following to conclude this article.

6.1. Reconstruction of a tensor.

The understanding of the ranges of the multiset-valued maps is an essential step to understand the configurations of the eigenvalues of general tensors.

PROPOSITION 6.1. *For any integers* $m,n\geq 2$*, there is a set* $W\subset\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ *whose closure has dimension* $2\lfloor\frac{n}{2}\rfloor m$ *contained in the closure of the image* $\phi(\mathbb{T}(\mathbb{C}^n,m+1))$.

Proof. For any $n\geq 2$, we can take $\lfloor\frac{n}{2}\rfloor$ tensors $\mathcal{A}_i\in\mathbb{T}(\mathbb{C}^2,m+1)$, and possibly a scalar α (when n is odd) as subtensors to form a diagonal block tensor $\mathcal{T}\in\mathbb{T}(\mathbb{C}^n,m+1)$.

It follows from [13] that

$$\det(\mathcal{T}-\lambda\mathcal{I})=(\alpha-\lambda)^q\prod_{i=1}^{\lfloor\frac{n}{2}\rfloor}\left[\det(\mathcal{A}_i-\lambda\mathcal{I})\right]^p$$

for some nonnegative integers p,q. So, it is clear that the set $\phi(\mathbb{T}(\mathbb{C}^2,m+1))\times\cdots\times\phi(\mathbb{T}(\mathbb{C}^2,m+1))$ with $\lfloor\frac{n}{2}\rfloor$ copies can be embedded into $\phi(\mathbb{T}(\mathbb{C}^n,m+1))$. It then follows from Proposition 4.1 that the closure of this set has dimension $2\lfloor\frac{n}{2}\rfloor m$. □

Similarly, we can use Propositions 5.1 and 5.2 to refine the blocks to get a variety with larger dimension in some cases. However, in general it is far away from the following expected dimension (6.1).

6.2. The dimension of the image of ϕ. It follows from Theorem 1.1 that for most tensor spaces $\mathbb{T}(\mathbb{C}^n,m+1)$, the multiset-valued eigenvalue map is not dominant. Thus, it is reasonable to expect that the dimension of $\overline{\phi(\mathbb{T}(\mathbb{C}^n,m+1))}$ is

$$\min\left\{n\binom{n+m-1}{m},nm^{n-1}\right\},\tag{6.1}$$

since $\phi(\mathbb{T}(\mathbb{C}^n,m+1))=\phi(\mathbb{TS}(\mathbb{C}^n,m+1))$ and $\dim(\mathbb{TS}(\mathbb{C}^n,m+1))=n\binom{n+m-1}{m}$. We also want to point out that the value (6.1) may not hold for all $m,n\geq 2$. We tested, by a similar method as Section 5.3, the case $(n,m)=(3,4)$. Note that the tensor space is of dimension 45 while the number of eigenvalues is 48. However, the ranks of the resulting Jacobian matrices for both the following two points (only independent elements are listed) are of the same value 43.

$t_{11111}=0,\ t_{11112}=3/4,\ t_{11122}=5/6,\ t_{11222}=1/4,\ t_{12222}=0,\ t_{11113}=-5/4,$
$t_{11123}=-1/6,\ t_{11223}=-1/6,\ t_{12223}=5/4,\ t_{11133}=-1/2,\ t_{11233}=-1/3,\ t_{12233}=-1/6,$
$t_{11333}=-1,\ t_{12333}=1/2, t_{13333}=4,\ t_{21111}=3,\ t_{21112}=5/4,\ t_{21122}=-1/2,\ t_{21222}=1,$
$t_{22222}=-2,\ t_{21113}=1,\ t_{21123}=1/6,\ t_{21223}=-5/12,\ t_{22223}=-1,\ t_{21133}=-1/6,$
$t_{21233}=0,\ t_{22233}=-1/3,\ t_{21333}=1,\ t_{22333}=-5/4,\ t_{23333}=-1, t_{31111}=-3,$
$t_{31112}=-5/4,\ t_{31122}=-1/3,\ t_{31222}=-1/2,\ t_{32222}=0,\ t_{31113}=3/4,\ t_{31123}=1/6,$
$t_{31223}=1/3,\ t_{32223}=2/3,\ t_{31133}=-2/3,\ t_{31233}=-1/6,$
$t_{32233}=1/3,\ t_{31333}=1,\ t_{32333}=-1,\ t_{33333}=0,$

and

$t_{11111}=7,\ t_{11112}=-3/2,\ t_{11122}=-4/3,\ t_{11222}=-9/4,\ t_{12222}=8,\ t_{11113}=7/4,$
$t_{11123}=3/4,\ t_{11223}=7/12,\ t_{12223}=-7/6,\ t_{11133}=5/6,\ t_{11233}=-1/12,\ t_{12233}=5/6,$
$t_{11333}=1,\ t_{12333}=9/4,\ t_{13333}=0,\ t_{21111}=10,\ t_{21112}=-1,\ t_{21122}=0,\ t_{21222}=5/4,$
$t_{22222}=-1,\ t_{21113}=7/4,\ t_{21123}=7/12,\ t_{21223}=0,\ t_{22223}=-7/6,\ t_{21133}=-7/6,$
$t_{21233}=-1/4,\ t_{22233}=5/6,\ t_{21333}=-5/2,\ t_{22333}=1,\ t_{23333}=6,\ t_{31111}=8,$
$t_{31112}=-3/2,\ t_{31122}=-1/3,\ t_{31222}=-5/4,\ t_{32222}=4,\ t_{31113}=9/4,\ t_{31123}=3/4,$
$t_{31223}=-1/3,\ t_{32223}=-4/3,\ t_{31133}=1/6,\ t_{31233}=5/6,\ t_{32233}=-1,$
$t_{31333}=3/2,\ t_{32333}=5/2,\ t_{33333}=6.$

Nevertheless, we have confidence to believe that the value (6.1) is true for all but a finite number of exceptions, as such phenomena happen in tensor problems, e.g., the famous Alexander–Hirschowitz theorem for symmetric tensor rank [1].

6.3. The closedness of the image of ϕ. We proved in Proposition 4.1 that for a generic multiset of total multiplicity $2m$ of complex numbers, the inverse eigenvalue problem is solvable. A natural question to ask is: is the inverse eigenvalue problem solvable for any multiset of total multiplicity $2m$ of complex numbers, i.e., is ϕ surjective? It is easy to show that when $m=2$, the answer is affirmative. We will need the following result.

PROPOSITION 6.2 ([6]). *Let $f=(f_1,\dots,f_n):\mathbb{C}^n\to\mathbb{C}^n$ be a polynomial map where each f_i is homogeneous. If $f(x_1,\dots,x_n)=0$ has only a trivial solution $(0,\dots,0)$, then $f(x_1,\dots,x_n)=\omega$ is always solvable for any $\omega\in\mathbb{C}^n$.*

PROPOSITION 6.3. *Given any multiset S of total multiplicity four of complex numbers, there exists a tensor $\mathcal{T}$ in $\mathbb{T}(\mathbb{C}^2,3)$ such that the set of eigenvalues of $\mathcal{T}$ is S.*

Proof. As before, we use a_0,a_1,a_2,b_0,b_1,b_2 to parametrize $\mathbb{TS}(\mathbb{C}^2,3)$ which is isomorphic to $\mathbb{C}^6$. Let $\mathbf{c}:\mathbb{C}^6\to\mathbb{C}^4$ be the map sending the vector $(a_0,a_1,a_2,b_0,b_1,b_2)$ to (c_1,c_2,c_3,c_4), where c_i is the codegree i coefficient of the characteristic polynomial of the tensor determined by $(a_0,a_1,a_2,b_0,b_1,b_2)$, $i=1,\dots,4$. Hence, the c_i's are homogeneous polynomials of degree $(4-i)$, respectively. By Proposition 6.2 and Proposition 3.1, it is sufficient to find a four-dimensional linear subspace $L\subset\mathbb{C}^6$ such that $\mathbf{c}^{-1}((0,0,0,0))\cap L$ is $(0,0,0,0)$. We consider the linear subspace L defined by equations

$$a_1+b_1+b_2=0, a_2+b_0=0.$$

Then it is easy to verify that $L\cap\mathbf{c}^{-1}((0,0,0,0))$ is $(0,0,0,0)$. □

REMARK 6.1. We use Macaulay2 to see that $L\cap\mathbf{c}^{-1}((0,0,0,0))$ is $(0,0,0,0)$. Since n generic homogeneous polynomials only have a trivial solution, the existence of L in the proof of Proposition 6.3 implies that a generic four-dimensional subspace of $\mathbb{C}^6$ should work.

REMARK 6.2. It is tempting to extend the proof of Proposition 6.3 to show that $\mathbf{c}$ is surjective in general. However, on one hand it is difficult to compute the intersection of $\mathbf{c}^{-1}(\mathbf{0})$ with a generic linear space of dimension $2m$ in general. On the other hand, when $m=3$ the dimension of $\mathbf{c}^{-1}(\mathbf{0})$ is three which is larger than the expected dimension two, hence the method used for $m=2$ does not work for $m=3$.

6.4. Eigenvalues of structured tensors. Let $\mathbb{V}$ be a subspace of $\mathbb{T}(\mathbb{C}^n,m+1)$. Theorem 1.1 implies the following proposition:

PROPOSITION 6.4. *If the eigenvalue map $\phi:\mathbb{V}\to\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ is dominant then (n,m) must be one of the following:*

$$m=1, \text{ or } n=2, \text{or } (n,m)=(3,2),\ (4,2),\ (3,3).$$

Therefore, if (n,m) is not equal to any one of the five exceptional cases described in Proposition 6.4 a generic element in $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$ cannot be the multiset of eigenvalues of a tensor in $\mathbb{V}$. In other words, the spectrum of most structured tensors (e.g., symmetric tensors, partially symmetric tensors), cannot fill the whole ambient space $\mathbb{C}^{nm^{n-1}}/\mathfrak{S}(nm^{n-1})$.

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