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GLOBAL WELL-POSEDNESS OF STRONG SOLUTIONS TO THE 2D DAMPED BOUSSINESQ AND MHD EQUATIONS WITH LARGE VELOCITY*

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Abstract. In this paper, we obtain global well-posedness for the 2D damped Boussinesq equations. Based on the estimate of the damped Euler equations leading to the uniform corresponding bound which does not grow in time, we can achieve this goal by using a new decomposition technique. Comparing with the previous works [D. Adhikar, C. Cao, J. Wu, and X. Xu, J. Diff. Eqs., 256:3594–3613, 2014] and [J. Wu, X. Xu, and Z. Ye, J. Nonlineal Sci., 25:157–192, 2015], we do not need any small assumptions of the initial velocity. As an application of our method, we obtain a similar result for the 2D damped MHD equations.

Keywords. Boussinesq equations, MHD equations, global well-posedness.

AMS subject classifications. 35Q35, 76B03.

1. Introduction

The 2D damped Boussinesq equations read as follows:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nu u + \nabla p = \theta e_2, \\ \partial_t \theta + u\cdot\nabla\theta + \eta\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where u and θ stand for the fluid velocity and the temperature in thermal convection or the density in geophysical flows, respectively, p represents the pressure, $e_2 = (0,1)$ is the unit vector in the vertical direction, and ν and η are positive parameters.

When $\nu = \eta = 0$, system (1.1) reduces to the inviscid model. Taking the operator "curl" on the velocity equation, by treating the initial data near nontrivial steady, [4] and [11] studied a new system

$$\begin{cases} \partial_t \omega + u\cdot\nabla\omega = \kappa\partial_1\rho, \\ \partial_t \rho + u\cdot\nabla\rho = \kappa u_2, \\ u = \nabla^{\perp}(-\Delta)^{-1}\omega, \\ \omega(0,x) = \omega_0(x), \quad \rho(0,x) = \rho_0(x), \end{cases} \tag{1.2}$$

where ρ is the temperature, like θ in system (1.1). When $\kappa = 1$, by using the decay estimate

$$\|e^{\mathcal{R}_1 t} f\|_{L^\infty} \le C t^{-\frac{1}{2}} \|f\|_{\dot{B}^2_{1,1}},$$

Elgindi and Widmayer [4] obtained the long time existence for equation (1.2). Specifically, when the initial norm is smaller than ϵ, they proved that the lifespan $T^\star$ of the local solution satisfies $T^\star > \epsilon^{-\frac{4}{3}}$, which is larger than the standard level ϵ^{-1}. When κ is large enough, by using some Strichartz-type estimates like

$$\|e^{\pm\mathcal{R}_1 t} f\|_{L^q(\mathbb{R};L^r(\mathbb{R}^2))} \le C\|f\|_{\dot{B}^{1-\frac{2}{r}}_{2,1}(\mathbb{R}^2)},$$

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Wan and Chen [11] obtained global well-posedness for equation (1.2), the main result of which is consistent with [4] if $\kappa=1$.

When $\nu>0$ and $\eta>0$, Adhikari et al. [1] proved global well-posedness for system (1.1) with the initial data satisfying

$$\|\nabla u_0\|_{\dot{B}^0_{\infty,1}}<\min\{\frac{\nu}{2C_0},\frac{\eta}{C_0}\},\ \|\nabla\theta_0\|_{\dot{B}^0_{\infty,1}}<\frac{\nu}{2C_0}\|\nabla u_0\|_{\dot{B}^0_{\infty,1}},$$

where $\dot{B}^0_{\infty,1}$ is the Besov space. The authors in [12] also obtained similar results for the n dimensional Boussinesq equations and other related models. We refer to the works [3,5–7,9,10,13] and the references therein for the global well-posedness of the 2D dissipative Boussinesq equations.

Let us point out that both [1] and [12] require small velocity and temperature data. However, in this paper, we can get global well-posedness with large velocity. The main result reads as follows:

THEOREM 1.1. *Let $m>3$. Assume that*

$$u_0\in H^m(\mathbb{R}^2),\quad \theta_0\in H^{m-1}(\mathbb{R}^2),\ \nabla\cdot u_0=0.$$

If there exist some constants C and C_0 such that u_0 and θ_0 satisfy

$$\begin{aligned}&C_0\|\theta_0\|^2_{H^{m-1}}\exp\left\{C\frac{(1+\nu\eta)^2}{\nu^2\eta}\|u_0\|_{H^m}\exp\{\frac{C}{\nu}\|\|\Omega_0\|_{L^2\cap L^\infty}A(\nu,u_0,\Omega_0)\}\right\}\\&<\frac{\nu^2\eta^2}{(1+\nu\eta)^2}\min\{\nu^3\eta,\ \frac{\eta}{\nu}\},\end{aligned}\tag{1.3}$$

where $\Omega_0:=\nabla\times u_0$ and $A(\nu,u_0,\Omega_0):=\ln(e+\frac{\|u_0\|_{H^m}}{\nu})\exp\left\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\nu}\right\}$, then system (1.1) admits a unique global solution satisfying

$$(u,\theta)\in C(\mathbb{R}^+;H^{m-1}(\mathbb{R}^2)).$$

REMARK 1.1. In fact, this damped Boussinesq equations can be seen to be a special supercritical case (see e.g. Stefanov and Wu [10]). To the best of our knowledge, Theorem 1.1 is a first result on the global well-posedness with large velocity.

Now, let us give some comments on the proof.

1) By methods similar to those used in [1] and [12], we shall use the velocity equation and temperature equation simultaneously to bound the nonlinear terms and the special term θe_2. In this paper, we find a new method that splits system (1.1) into two new systems, one of which has a unique global solution. We call this system the damped Euler equations.
2) To obtain the global bound (independent of the growth of time) for the 2D damped Euler equations, we need different regularity of u_0 and θ_0, and we need to establish a new interpolation inequality— see Lemma 2.2.

Similarly, we can also get global well-posedness for the 2D damped MHD equations given by

$$\begin{cases}\partial_t u+u\cdot\nabla u+\kappa u+\nabla p=B\cdot\nabla B,\\ \partial_t B+u\cdot\nabla B-B\cdot\nabla u+\mu B=0,\\ \nabla\cdot u=\nabla\cdot B=0,\\ u(x,0)=u_0(x),\quad B(x,0)=B_0(x),\end{cases}\tag{1.4}$$

where u, B, and p are, respectively, the fluid velocity, the magnetic field of the fluid,and the pressure. The constants κ and μ are positive parameters.

THEOREM 1.2. *Let $m>3$. Assume that*

$$u_0\in H^m(\mathbb{R}^2),\quad B_0\in H^{m-1}(\mathbb{R}^2),\ \nabla\cdot u_0=\nabla\cdot B_0=0.$$

If there exist some constants C and C_0 such that u_0 and B_0 satisfy

$$C_0(\frac{1}{\kappa}+\frac{1}{\mu})\|B_0\|_{H^{m-1}}^2\exp(\frac{C(\kappa+\mu)^2}{\kappa^2\mu^2}\|u_0\|_{H^m}^2\exp\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\kappa}A_1(\kappa,u_0,\Omega_0)\})$$
$$\leq\min\{\kappa,\nu\},\tag{1.5}$$

where $\Omega_0:=\nabla\times u_0$ and $A_1(\kappa,u_0,\Omega_0):=\ln(e+\frac{\|u_0\|_{H^m}}{\kappa})\exp\Big\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\kappa}\Big\}$, then system (1.4) admits a unique global solution satisfying

$$(u,B)\in C(\mathbb{R}^+;H^{m-1}(\mathbb{R}^2)).$$

REMARK 1.2. Comparing with the small condition (1.5) of the MHD equations, it may be strange that the initial data of the Boussinesq equations is smaller than $\frac{1}{\nu}$. As a matter of fact, we shall need the damped term $\eta\theta$ to control the special term θe_2, but this situation does not occur in the MHD equations.

The present paper is structured as follows:
In the second section, we provide some definitions of spaces and an important lemma. In the third section, we prove Theorem 1.1. In the fourth section, we prove Theorem 1.2.

Let us complete this section by describing the notation we shall use in this paper.
Notation. For operators A and B, we denote by $[A,B]=AB-BA$ the commutator of A and B. The uniform constant C is different on different lines. In some places in this paper, we may use L^p and H^s to stand for $L^p(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2)$, respectively. We shall denote by $(a|b)$ the L^2 inner product of a and b, and $(a|b)_{\dot H^s}$ will stand for the standard $\dot H^s$ inner product of a and b. More precisely, $(a|b)_{\dot H^s}=(\Lambda^s a|\Lambda^s b)$, and furthermore

$$(a|b)_{H^s}:=(a|b)_{\dot H^s}+(a|b),\ s>0.$$

2. Preliminaries

In this section, we give some necessary definitions and propositions.

The fractional Laplacian operator $\Lambda^\alpha=(-\Delta)^{\frac{\alpha}{2}}$ is defined by the Fourier transform. Namely,

$$\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\widehat{f}(\xi),$$

where the Fourier transform is given by

$$\widehat{f}(\xi)=\int_{\mathbb{R}^2}e^{-ix\cdot\xi}f(x)dx.$$

The $\dot H^s(\mathbb{R}^2)$ and $H^s(\mathbb{R}^2)$ $(s>0)$ norm of f can be also defined as follows:

$$\|f\|_{\dot H^s(\mathbb{R}^2)}\overset{def}{:=}\|\Lambda^s f\|_{L^2(\mathbb{R}^2)}$$

and

$$\|f\|_{H^s(\mathbb{R}^2)} \stackrel{def}{:=} \|f\|_{L^2(\mathbb{R}^2)} + \|\Lambda^s f\|_{L^2(\mathbb{R}^2)}.$$

Let us recall a standard commutator estimate.

LEMMA 2.1. *[8] Let $s>0$, and $1<p<\infty$, then*

$$\|[\Lambda^s,f]g\|_{L^p(\mathbb{R}^2)} \le C\left\{\|\nabla f\|_{L^{p_1}(\mathbb{R}^2)}\|\Lambda^{s-1}g\|_{L^{p_2}(\mathbb{R}^2)} + \|\Lambda^s f\|_{L^{p_3}(\mathbb{R}^2)}\|g\|_{L^{p_4}(\mathbb{R}^2)}\right\}$$

where $1<p_2,p_3<\infty$ such that $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}$.

The following proposition provides Bernstein type inequalities.

PROPOSITION 2.1. *Let $1\le p\le q\le\infty$. Then for any $\beta,\gamma\in(\mathbb{N}\cup\{0\})^2$, there exists a constant C independent of f,j such that:*

1) If f satisfies

$$supp\,\widehat{f}\subset\{\xi\in\mathbb{R}^2 : |\xi|\le\mathcal{K}2^j\},$$

then

$$\|\partial^\gamma f\|_{L^q(\mathbb{R}^2)} \le C2^{j|\gamma|+jd(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p(\mathbb{R}^2)}.$$

2) If f satisfies

$$supp\,\widehat{f}\subset\{\xi\in\mathbb{R}^2 : \mathcal{K}_1 2^j\le|\xi|\le\mathcal{K}_2 2^j\}$$

then

$$\|f\|_{L^p(\mathbb{R}^2)} \le C2^{-j|\gamma|}\sup_{|\beta|=|\gamma|}\|\partial^\beta f\|_{L^p(\mathbb{R}^2)}.$$

The following lemma gives an interpolation inequality.

LEMMA 2.2. *Let $s>2$, the vector function v satisfies $\nabla\cdot v=0$. There exists a constant C such that the following inequality holds for any constant M:*

$$\|\nabla v\|_{L^\infty} \le C\left\{(\|\nabla\times v\|_{L^2}+\|\nabla\times v\|_{L^\infty})\ln(e+M\|v\|_{H^s})+\frac{1}{M}\right\} \tag{2.1}$$

Proof. By the inhomogeneous Bony decomposition, $I=\sum_{j\ge-1}\Delta_j$, and Bernstein's inequality, (see e.g. the Chapter 2 in [2]), we have

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\le \|\nabla\Delta_{-1}v\|_{L^\infty}+\sum_{j=0}^{N}\|\nabla\Delta_j v\|_{L^\infty}+\sum_{j\ge N+1}2^{2jN}\|\Delta_j v\|_{L^2}\\ &\le C\|\nabla v\|_{L^2}+CN\|\nabla v\|_{\dot{B}^0_{\infty,\infty}}+C2^{-N(s-2)}\|v\|_{H^s}\\ &\le C\|\nabla\times v\|_{L^2}+CN\|\nabla\times v\|_{L^\infty}+C2^{-N(s-2)}\|v\|_{H^s},\end{aligned}$$

where $\dot{B}^0_{\infty,\infty}$ is the homogeneous Besov space, and we have used

$$\|\nabla v\|_{\dot{B}^0_{\infty,\infty}} \le C\|\nabla\times v\|_{\dot{B}^0_{\infty,\infty}} \le C\|\nabla\times v\|_{L^\infty},\ \|\nabla v\|_{L^p}\le C\|\nabla\times v\|_{L^p},\ 1<p<\infty$$

and

$$\|u\|_{H^s} \approx \Big(\sum_{j\geq -1} 2^{2js}\|\Delta_j u\|_{L^2}^2\Big)^{\frac{1}{2}}.$$

For details about Besov spaces, see Chapter 2 in [2]. By setting

$$N=\frac{2}{s-2}\{[\log_2(e+M\|v\|_{H^s})]+1\},$$

we get

$$2^{-N(s-2)}\|v\|_{H^s}\leq \frac{\|v\|_{H^s}}{(e+M\|v\|_{H^s})^2}\leq \frac{1}{M},$$

and then the desired inequality (2.1) can be proved. □

3. Proof of Theorem 1.1

As the comments in Section 1, we will split system (1.1) into two systems, that is, a system of 2D damped Euler equations

$$\begin{cases} \partial_t V+V\cdot\nabla V+\nu V+\nabla p_V=0,\\ \nabla\cdot V=0,\\ V(x,0)=u_0(x) \end{cases} \tag{3.1}$$

and

$$\begin{cases} \partial_t W+V\cdot\nabla W+W\cdot\nabla(V+W)+\nu W+\nabla p_W=\theta e_2,\\ \partial_t\theta+(V+W)\cdot\nabla\theta+\eta\theta=0,\\ \nabla\cdot W=0,\\ W(x,0)=0, \quad \theta(x,0)=\theta_0(x). \end{cases} \tag{3.2}$$

One can easily get the local well-posedness of equations (1.1), (3.1), and (3.2), so $u=W+V$ is the unique local solution to system (1.1). It therefore suffices to prove the global bound of the solutions to systems (3.1) and (3.2). The proof will be split into two steps.

Step 1 Global regularity for system (3.1). Denote $\Omega:=\nabla\times V$. By the standard commutator estimate and by equation (2.1), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|V\|_{H^m}^2+\nu\|V\|_{H^m}^2\leq C\|\nabla V\|_{L^\infty}\|V\|_{H^m}^2\\ \leq& C(\|\Omega\|_{L^2}+\|\Omega\|_{L^\infty})\ln(e+M\|V\|_{H^m})\|V\|_{H^m}^2+\frac{C}{M}\|V\|_{H^m}^2. \end{aligned}$$

Taking $M=\frac{2C}{\nu}$, then

$$\frac{1}{2}\frac{d}{dt}\|V\|_{H^m}^2+\frac{\nu}{2}\|V\|_{H^m}^2\leq C(\|\Omega\|_{L^2}+\|\Omega\|_{L^\infty})\ln(e+\frac{1}{\nu}\|V\|_{H^m})\|V\|_{H^m}^2.$$

Dividing by $\|V\|_{H^m}$ on both sides leads to

$$\frac{d}{dt}\|V\|_{H^m}+\frac{\nu}{2}\|V\|_{H^m}\leq C(\|\Omega\|_{L^2}+\|\Omega\|_{L^\infty})\ln(e+\frac{1}{\nu}\|V\|_{H^m})\|V\|_{H^m}. \tag{3.3}$$

Multiplying by $\frac{1}{\nu}$ and applying Gronwall's lemma yields

$$\ln(e+\frac{\|V(t)\|_{H^m}}{\nu})\leq\ln(e+\frac{\|u_0\|_{H^m}}{\nu})\exp\left\{C\int_0^t\|\Omega(\tau)\|_{L^2\cap L^\infty}d\tau\right\}. \tag{3.4}$$

Applying the operator "$\nabla\times$" on system (3.1), one gets

$$\partial_t\Omega+V\cdot\nabla\Omega+\nu\Omega=0.$$

Taking the inner product with $\Omega|\Omega|^{p-2}$, using $\nabla\cdot V=0$, and integrating by parts, we obtain

$$\frac{1}{p}\frac{d}{dt}\|\Omega\|_{L^p}^p+\nu\|\Omega\|_{L^p}^p=0.$$

After dividing by $\|\Omega\|_{L^p}^{p-1}$, and taking $p=2$ and ∞, one has

$$\|\Omega(t)\|_{L^2\cap L^\infty}+\nu\int_0^t\|\Omega\|_{L^2\cap L^\infty}d\tau=\|\Omega_0\|_{L^2\cap L^\infty}.$$

Inserting the above estimate into equation (3.4) gives

$$\ln(e+\frac{\|V(t)\|_{H^m}}{\nu})\leq A(\nu,u_0,\Omega_0),$$

where

$$A(\nu,u_0,\Omega_0):=\ln(e+\frac{\|u_0\|_{H^m}}{\nu})\exp\left\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\nu}\right\}.$$

Plugging the above Log type estimate into equation (3.3), and applying Gronwall's lemma, we get

$$\begin{aligned}&\|V(t)\|_{H^m}+\frac{\nu}{2}\int_0^t\|V(\tau)\|_{H^m}d\tau\\ \leq&\|u_0\|_{H^m}\exp\left\{CA(\nu,u_0,\Omega_0)\int_0^t\|\Omega\|_{L^2\cap L^\infty}d\tau\right\}\\ \leq&\|u_0\|_{H^m}\exp\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\nu}A(\nu,u_0,\Omega_0)\}. \end{aligned}\tag{3.5}$$

REMARK 3.1. We can see the growth of the bound of V is independent of t, which is very important in our proof.

Step 2 Global well-posedness for system (1.1). Let $s=m-1$. Integrating by parts, and applying the cancelation property

$$(W\cdot\nabla\Lambda^sW|\Lambda^sW)=(V\cdot\nabla\Lambda^sW|\Lambda^sW)=0,$$

and the standard commutator estimate and Young's inequality, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|W\|_{H^s}^2+\nu\|W\|_{H^s}^2=&-(W\cdot\nabla(V+W)|W)_{H^s}-(V\cdot\nabla W|W)_{H^s}+(\theta e_2|W)_{H^s}\\ \leq&C\|\nabla V\|_{H^s}\|W\|_{H^s}^2+C\|\nabla W\|_{L^\infty}\|W\|_{H^s}^2\\ &+\|\nabla W\|_{L^\infty}\|W\|_{H^s}\|V\|_{H^s}+\|\theta\|_{H^s}\|W\|_{H^s}\\ \leq&C\|V\|_{H^m}\|W\|_{H^s}^2+C\|W\|_{H^s}^3+\frac{1}{2\nu}\|\theta\|_{H^s}^2+\frac{\nu}{2}\|W\|_{H^s}^2,\end{aligned}$$

which leads to

$$\frac{1}{2}\frac{d}{dt}\|W\|_{H^s}^2+\frac{\nu}{2}\|W\|_{H^s}^2\le C(\|V\|_{H^m}+\|W\|_{H^s})\|W\|_{H^s}^2+\frac{1}{2\nu}\|\theta\|_{H^s}^2. \tag{3.6}$$

Similarly, using

$$((V+W)\cdot\nabla\Lambda^s\theta|\Lambda^s\theta)=0,\ \forall\ s\ge 0,$$

we have the H^s estimate for θ:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\theta\|_{H^s}^2+\eta\|\theta\|_{H^s}^2&=-((V+W)\cdot\nabla\theta|\theta)_{H^s}\\ &\le C(\|V\|_{H^s}+\|W\|_{H^s})\|\theta\|_{H^s}^2.\end{aligned} \tag{3.7}$$

Multiplying estimate (3.7) by $\frac{1}{\eta\nu}$ and then adding the resulting inequality to estimate (3.6) we have

$$\begin{aligned}&\frac{d}{dt}\{\frac{1}{\nu\eta}\|\theta\|_{H^s}^2+\|W\|_{H^s}^2\}+\nu\|W\|_{H^s}^2+\frac{1}{\nu}\|\theta\|_{H^s}^2\\ \le&\frac{C(1+\nu\eta)}{\nu\eta}(\|V\|_{H^m}+\|W\|_{H^s})(\|W\|_{H^s}^2+\|\theta\|_{H^s}^2).\end{aligned}$$

Now, let us denote

$$\bar{T}=:\sup\left\{t\in(0,T^\star):\ \|W(t)\|_{H^s}\le\frac{\nu\eta}{2C(1+\nu\eta)}\min\{\nu,\ \frac{1}{\nu}\}\right\},$$

where $T^\star$ is the lifespan of the local solution. To achieve our goal, it suffices to obtain a contradiction if $\bar{T}<T^\star$. For all $t\in(0,\bar{T})$,

$$\begin{aligned}&\frac{d}{dt}\{\frac{1}{\nu\eta}\|\theta\|_{H^s}^2+\|W\|_{H^s}^2\}+\frac{1}{2}\min\{\nu,\frac{1}{\nu}\}(\|W\|_{H^s}^2+\|\theta\|_{H^s}^2)\\ \le&C\frac{(1+\nu\eta)^2}{\nu\eta}\|V\|_{H^m}(\|W\|_{H^s}^2+\frac{1}{\nu\eta}\|\theta\|_{H^s}^2).\end{aligned}$$

Applying Gronwall's lemma yields

$$\frac{1}{\nu\eta}\|\theta(t)\|_{H^s}^2+\|W(t)\|_{H^s}^2\le\frac{1}{\nu\eta}\|\theta_0\|_{H^s}^2\exp\{C\frac{(1+\nu\eta)^2}{\nu\eta}\int_0^t\|V\|_{H^m}d\tau\}.$$

Because of estimate (3.5), we get $\forall\ t\in(0,\bar{T})$,

$$\begin{aligned}&\frac{1}{\nu\eta}\|\theta(t)\|_{H^s}^2+\|W(t)\|_{H^s}^2\\ \le&\frac{1}{\nu\eta}\|\theta_0\|_{H^s}^2\exp\left\{C\frac{(1+\nu\eta)^2}{\nu^2\eta}\|u_0\|_{H^m}\exp\{\frac{C}{\nu}\|\|\Omega_0\|_{L^2\cap L^\infty}A(\nu,u_0,\Omega_0)\}\right\}.\end{aligned}$$

Thus, by choosing C_0 sufficiently large in equation (1.3), we get

$$\begin{aligned}&\|\theta_0\|_{H^s}^2\exp\left\{C\frac{(1+\nu\eta)^2}{\nu^2\eta}\|u_0\|_{H^m}\exp\{\frac{C}{\nu}\|\|\Omega_0\|_{L^2\cap L^\infty}A(\nu,u_0,\Omega_0)\}\right\}\\ <&\frac{\nu^2\eta^2}{16C^2(1+\nu\eta)^2}\min\{\nu^3\eta,\ \frac{\eta}{\nu}\}.\end{aligned}$$

This ensures that we can get $\bar{T}=T^\star=\infty$ by a continuous argument, which contradicts the previous assumption that $\bar{T}<T^\star$. So we have shown global regularity for system (3.2). This concludes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Like the previous proof of Theorem 1.1, we split the system (1.4) into two systems, namely,

$$\begin{cases} \partial_t V + V\cdot\nabla V + \kappa V + \nabla p_V = 0, \\ \nabla\cdot V = 0, \\ V(x,0) = u_0(x) \end{cases} \tag{4.1}$$

and

$$\begin{cases} \partial_t W + V\cdot\nabla W + W\cdot\nabla(V+W) + \kappa W + \nabla p_W = B\cdot\nabla B, \\ \partial_t B + (V+W)\cdot\nabla B - B\cdot\nabla(V+W) + \mu B = 0, \\ \nabla\cdot W = \nabla\cdot B = 0, \\ W(x,0) = 0, \quad B(x,0) = B_0(x). \end{cases} \tag{4.2}$$

Step 1. Global regularity for system (4.1). Following the Step 1 in Section 3, we have

$$\|V(t)\|_{H^m} + \frac{\kappa}{2}\int_0^t \|V(\tau)\|_{H^m} d\tau \le \|u_0\|_{H^m} \exp\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\nu} A_1(\kappa,u_0,\Omega_0)\}, \tag{4.3}$$

where

$$A_1(\kappa,u_0,\Omega_0) := \ln(e + \frac{\|u_0\|_{H^m}}{\kappa}) \exp\left\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\kappa}\right\}.$$

Step 2. Global well-posedness for system (1.4). It suffices to get global regularity for system (4.2). Let $s = m-1$. By the energy estimate, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|(W,B)\|_{H^s}^2 + \kappa\|W\|_{H^s}^2 + \mu\|B\|_{H^s}^2 = & \underbrace{-(V\cdot\nabla W|W)_{H^s} - (W\cdot\nabla(V+W)|W)_{H^s}}_{\Xi_1} \\ & + \underbrace{(B\cdot\nabla B|W)_{H^s} + (B\cdot\nabla W|B)_{H^s}}_{\Xi_2} \\ & + \underbrace{(B\cdot\nabla V|B)_{H^s} - ((V+W)\cdot\nabla B|B)_{H^s}}_{\Xi_3}. \end{aligned}$$

Using the cancelation property

$$(V\cdot\nabla\Lambda^s W|\Lambda^s W) = 0$$

and the standard commutator estimates, we have

$$\begin{aligned} \Xi_1 \le & C(\|[\Lambda^s, V\cdot\nabla]W\|_{L^2} + \|[\Lambda^s, W\cdot\nabla]W\|_{L^2} + \|W\cdot\nabla V\|_{H^s})\|W\|_{H^s} \\ \le & C(\|\nabla V\|_{L^\infty}\|W\|_{H^s} + \|\nabla W\|_{L^\infty}\|V\|_{H^s} + \|\nabla V\|_{H^s}\|W\|_{H^s})\|W\|_{H^s} \\ \le & \frac{C}{\kappa}(\|V\|_{H^m}^2 + \|W\|_{H^s}^2)\|W\|_{H^s}^2 + \frac{\kappa}{8}\|W\|_{H^s}^2. \end{aligned}$$

Since

$$(B\cdot\nabla\Lambda^s B|\Lambda^s W) + (B\cdot\nabla\Lambda^s W|\Lambda^s B) = 0, \ s \ge 0,$$

we get

$$\begin{aligned}\Xi_2 \le& C(\|\nabla B\|_{L^\infty}\|B\|_{H^s}+\|\nabla W\|_{L^\infty}\|B\|_{H^s})\|B\|_{H^s}\\ \le& \frac{C}{\mu}(\|B\|_{H^s}^2+\|W\|_{H^s}^2)\|B\|_{H^s}^2+\frac{\mu}{8}\|B\|_{H^s}^2.\end{aligned}$$

Because

$$(f\cdot\nabla\Lambda^s B|\Lambda^s B)=0,\ \forall\ \nabla\cdot f=0,$$

we obtain

$$\begin{aligned}\Xi_3 \le& C(\|\nabla V\|_{H^s}\|B\|_{H^s}+\|[\Lambda^s,(V+W)\cdot\nabla]B\|_{L^2})\|B\|_{H^s}\\ \le& C(\|V\|_{H^m}\|B\|_{H^s}+\|\nabla(V+W)\|_{L^\infty}\|B\|_{H^s}\\ &+\|\nabla B\|_{L^\infty}\|V+W\|_{H^s})\|B\|_{H^s}\\ \le& C(\|V\|_{H^m}\|B\|_{H^s}+\|W\|_{H^s}\|B\|_{H^s})\|B\|_{H^s}\\ \le& \frac{C}{\mu}(\|V\|_{H^m}^2+\|W\|_{H^s}^2)\|B\|_{H^s}^2+\frac{\mu}{8}\|B\|_{H^s}^2.\end{aligned}$$

Collecting the above estimates, we have

$$\begin{aligned}\frac{d}{dt}\|(W,B)\|_{H^s}^2+\kappa\|W\|_{H^s}^2+\mu\|B\|_{H^s}^2 \le& \frac{C(\kappa+\mu)}{\kappa\mu}(\|V\|_{H^m}^2+\|W\|_{H^s}^2)\|(W,B)\|_{H^s}^2\\ &+\frac{C}{\mu}\|(B,W)\|_{H^s}^2\|B\|_{H^s}^2.\end{aligned}$$

Denote by

$$\bar{T}=:\left\{t\in(0,T^\star):\ \|(W,B)(t)\|_{H^s}^2\le\frac{\kappa\mu}{4C(\kappa+\mu)}\min\{\kappa,\mu\}\right\},$$

where $T^\star$ is the lifespan of the local solution. Assume $\bar{T}<T^\star$. For all $t\in(0,\bar{T})$, we have.

$$\frac{d}{dt}\|(W,B)\|_{H^s}^2+\kappa\|W\|_{H^s}^2+\mu\|B\|_{H^s}^2\le\frac{C(\kappa+\mu)}{\kappa\mu}\|V\|_{H^m}^2\|(W,B)\|_{H^s}^2.$$

Applying Gronwall's lemma, and using estimate (4.3) in Step 1 yields

$$\begin{aligned}\|(W,B)(t)\|_{H^s}^2 \le& \|B_0\|_{H^s}^2\exp\{\frac{C(\kappa+\mu)^2}{\kappa^2\mu^2}\int_0^t\|V\|_{H^m}^2d\tau\}\\ \le& \|B_0\|_{H^{m-1}}^2\exp(\frac{C(\kappa+\mu)^2}{\kappa^2\mu^2}\|u_0\|_{H^m}^2\exp\{\frac{C\|\Omega_0\|_{L^2\cap L^\infty}}{\kappa}A_1(\kappa,u_0,\Omega_0)\}).\end{aligned}$$

One can easily check that if we take C_0 large enough in equation (1.5), then

$$\|(W,B)(t)\|_{H^s}\le\frac{1}{16C}\frac{\kappa\mu}{\kappa+\mu}\min\{\kappa,\mu\}.$$

By a continuous argument, we can get a contradiction. So we have proven that system (4.2) admits a unique global solution. Therefore, we conclude the proof of Theorem 1.2.

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