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PULLBACK ATTRACTORS AND INVARIANT MEASURES FOR THE NON-AUTONOMOUS GLOBALLY MODIFIED NAVIER–STOKES EQUATIONS*

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Abstract. This paper studies the non-autonomous globally modified Navier–Stokes equations. The authors first prove that the associated process possesses a pullback attractor. Then they establish that there exists a unique family of Borel invariant probability measures on the pullback attractor.

Keywords. pullback attractor; invariant measures; non-autonomous globally modified Navier–Stokes equations.

AMS subject classifications. 35B41; 35D99; 76F20.

1. Introduction

This paper studies the following non-autonomous globally modified Navier–Stokes equations

$$\frac{\partial u}{\partial t} - \nu\Delta u + F_N(\|u\|_V)(u\cdot\nabla)u + \nabla p = g(t) \quad \text{in} \quad (\tau,+\infty)\times\Omega, \tag{1.1}$$

$$\nabla\cdot u = 0, \tag{1.2}$$

$$u = 0 \quad \text{on} \quad (\tau,+\infty)\times\partial\Omega, \tag{1.3}$$

$$u(x,\tau) = u_\tau(x), \quad x\in\Omega, \tag{1.4}$$

where u denotes the velocity field of the fluid, p is the pressure, and g is the external force. In equation (1.1), $\|u\|_V = \|\nabla u\|_{(L^2(\Omega))^3}$, and the function $F_N(\cdot):(0,\infty)\mapsto(0,1]$ is defined as

$$F_N(r) = \min\{1, N/r\}, \quad r\in(0,+\infty),$$

where $N\in(0,+\infty)$ is given. In addition, $\tau\in\mathbf{R}$ and $\Omega\subset\mathbf{R}^3$ is a suitable smooth domain satisfying the Poincaré inequality.

The globally modified Navier–Stokes equations were initiated in the papers [3] and [10]. The modifying factor $F_N(\|u\|_V)$ depends on the norm $\|u\|_V = \|\nabla u\|_{(L^2(\Omega))^3}$, which in turn depends on ∇u over the whole domain Ω and not just at or near the point $x\in\Omega$ under consideration. Essentially, it prevents large gradients from dominating the dynamics and leading to explosions ([5]).

The globally modified Navier–Stokes equations are interesting in themselves, but, more importantly, can be used to obtain useful information about the Navier–Stokes equations. For example, they were used as an intermediate step by Kloeden and Valero in [11] to prove that the attainability set of the weak solutions of the three-dimensional

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(3D) Navier–Stokes equations which satisfy an energy constraint is a compact and connected set in the weak topology. Nowadays, the globally modified Navier–Stokes equations have been widely studied by many researchers. For instance, the existence and uniqueness of solutions was investigated in [6, 20, 23]; the existence of attractors was studied in [3, 10, 21]; the existence of invariant measures and statistical solutions was proved in [5, 12, 17], etc.

The invariant measures and statistical solutions have proven to be very useful in the understanding of turbulence in the case of Navier–Stokes equations (see Foias *et al.* [7]). The main reason is that the measurements of several aspects of turbulent flows are actually measurements of time-average quantities. Very recently, Łukaszewicz and Robinson ([17]) used the techniques developed in the papers Łukaszewicz [15] and Łukaszewicz *et al.* [16], which were in turn based on works of Foias *et al.* [7] and Wang [25], to provide a construction of invariant measures for non-autonomous systems with minimal assumptions on the underlying dynamical process.

The results of Łukaszewicz and Robinson ([17, Thereom 3.1]) show that a continuous process $\{\mathcal{U}(t,\tau)\}_{t\geq\tau}$ on a complete metric space X possesses a family of Borel invariant probability measures in X if $\{\mathcal{U}(t,\tau)\}_{t\geq\tau}$ satisfies

(i) For every $v_0\in X$ and every $t\in\mathbf{R}$, the X-valued function $\tau\longmapsto U(t,\tau)v_0$ is continuous and bounded on $(-\infty,t]$.

(ii) The process $\{\mathcal{U}(t,\tau)\}_{t\geq\tau}$ possesses a pullback attractor in X.

In particular, in the end of the article [17], the authors pointed out that it would be interesting to apply the invariant measures theory there to a non-autonomous version of the globally modified Navier–Stokes equations studied in [5] and [12]. Following this clue, we will investigate the existence of the invariant measures for equations (1.1)–(1.4).

The first objective of this paper is to prove the existence of the pullback attractor for the process associated with the globally modified Navier–Stokes equations (1.1)–(1.4). We want to remark that Kloeden *et al.* established, via the flattening property, the existence and finite dimension of the pullback V-attractor for equations (1.1)–(1.4) in [10]. There they need the external force function $g(t)\in W^{1,2}_{\text{loc}}(\mathbf{R};(L^3(\Omega))^3)$ and

$$\int_{-\infty}^{t} e^{\sigma s}\|g(s)\|^2\mathrm{d}s<+\infty,\quad \forall t\in\mathbf{R}\quad\text{and}\quad \lim_{\tau\to-\infty} e^{\sigma\tau}\int_{\tau}^{\tau+1}\|g'(s)\|^2\mathrm{d}s=0.$$

Here we will use the approach of enstrophy (see e.g. [8]) to prove the existence of the pullback attractor for equations (1.1)–(1.4). The condition (see Assumption $(\mathbf{H}_1)$) on the external force function $g(t)$ here is weaker than that of [10].

The idea of energy equation (or enstrophy equation) was initially introduced by Ball ([1]) and developed later by Moise *et al.* [18, 19] in a systematic and abstract framework. In fact, such idea has been well extended and widely used in verifying the asymptotic compactness of the semigroup or process associated with the partial differential equations on unbounded domain (see e.g. [4, 9, 14, 22, 24, 26–28]). Recently, the approach of enstrophy equation was used in a concise form in [8] to investigate the existence and tempered behavior of the pullback attractor for the two-dimensional (2D) Navier–Stokes equations. This concise form was used in [29, 30] to investigate the existence of the pullback attractor for the 2D non-Newtonian fluid equations and 2D non-autonomous micropolar fluid flows with infinite delays.

The second goal of this paper is to establish the existence of a family of Borel invariant probability measures for the globally modified Navier–Stokes equations (1.1)–(1.4). By the results of Łukaszewicz and Robinson ([17, Thereom 3.1]), we shall check that the V-valued function $\tau \longmapsto U(t,\tau)u_0$ is continuous and bounded on $(-\infty,t]$. In fact, we prove these facts by estimating the difference $u(t)-v(t)$ between two solutions and the nonlinear term

$$F_N(\|u\|_V)(u\cdot\nabla)u - F_N(\|v\|_V)(v\cdot\nabla)v.$$

The rest of this paper is arranged as follows. The next section introduces notation, some operators, as well as a theorem concerning the existence and uniqueness of solutions to the globally modified Navier–Stokes equations. Section 3 is devoted to establishing the existence of pullback attractors for the associated process in space V via the approach of enstrophy equations. In the last section, we prove that there exists a unique family of Borel invariant probability measures on the pullback attractor.

2. Preliminaries

In this section, we first introduce some notations and operators. Then we present the existence and uniqueness of solutions to the globally modified Navier–Stokes equations (1.1)–(1.4).

In this paper we use the following notations:

$\mathbf{R}$–the set of real numbers, $\mathbf{N}$–the set of positive integers;
$c(\cdot,\cdot)$–the generic constant that can take different values in different places;
$\mathbf{L}^p(\Omega)=(L^p(\Omega))^3$–the 3D Lebesgue space with norm $\|\cdot\|_{\mathbf{L}^p(\Omega)}$; $\|\cdot\|_{\mathbf{L}^2(\Omega)}=\|\cdot\|$;
$\mathbf{H}^m(\Omega)=$ the 3D Sobolev space $\{\phi\in\mathbf{L}^2(\Omega)|\nabla^k\phi\in\mathbf{L}^2(\Omega), k\le m\}$ with norm $\|\cdot\|_{\mathbf{H}^m(\Omega)}$;
$\mathbf{H}_0^1(\Omega)=$ closure of $\{\phi|\phi\in(\mathcal{C}_0^\infty(\Omega))^3\}$ in $\mathbf{H}^1(\Omega)$ with norm $\|\cdot\|_{\mathbf{H}^1(\Omega)}$;
$\mathcal{V}=\{\phi\in(\mathcal{C}_0^\infty(\Omega))^3|\nabla\cdot\phi=0\}$;
$H=$ closure of $\mathcal{V}$ in $\mathbf{L}^2(\Omega)$ with norm $\|\cdot\|$; $H'=$ dual space of H;
$V=$ closure of $\mathcal{V}$ in $\mathbf{H}^1(\Omega)$ with norm $\|\cdot\|_V=\|\cdot\|_{\mathbf{H}^1(\Omega)}$;
$V'=$ dual space of V with norm $\|\cdot\|_{V'}$;
$(\cdot,\cdot)$– the inner product in H, $\langle\cdot,\cdot\rangle$– the dual pairing between V and V';
$\mathrm{dist}_M(X,Y)$– the Hausdorff semidistance between $X\subseteq M$ and $Y\subseteq M$ defined by

$$\mathrm{dist}_M(X,Y)=\sup_{x\in X}\inf_{y\in Y}\|x-y\|_M.$$

"$\longrightarrow$" and "$\rightharpoonup$" denote the strong and weak convergence, respectively.

To write equations (1.1)–(1.4) in an abstract form, we next introduce some operators. Firstly, we consider the operator $A: V\mapsto V'$ defined as

$$\langle Au,v\rangle=(\nabla u,\nabla v),\ \ u,v\in V. \tag{2.1}$$

Denoting $D(A)=\mathbf{H}^2(\Omega)\cap V$, then $Au=-P\Delta u$, $\forall u\in D(A)$, is the Stokes operator, where P is the Leray–Helmholtz projection from $\mathbf{L}^2(\Omega)$ onto H. Secondly, we define a continuous trilinear form

$$b(u,v,w)=\sum_{i,j=1}^{3}\int_\Omega u_i\frac{\partial v_j}{\partial x_i}w_j\,\mathrm{d}x,\ \ u,v,w\in\mathbf{H}_0^1(\Omega).$$

Note that $V\subseteq\mathbf{H}_0^1(\Omega)$ is a closed subspace, $b(u,v,w)$ is continuous on $V\times V\times V$, and

$$b(u,v,w)=-b(u,w,v),\, b(u,v,v)=0,\ \ \forall u,v,w\in V.$$

For any $u,v\in V$,

$$\langle B(u,v),w\rangle=b(u,v,w),\ \ \forall w\in V,$$

defines a continuous function $B(u,v)$ on $V\times V$. We further set

$$b_N(u,v,w)=F_N(\|v\|_V)b(u,v,w),\ \ \forall u,v,w\in V. \tag{2.2}$$

$$\langle B_N(u,v),w\rangle=b_N(u,v,w),\ \ \forall u,v,w\in V. \tag{2.3}$$

Note that the form $b_N(u,v,w)$ is linear in u and w, but it is nonlinear in v. For above introduced operators, we select the following estimations.

Lemma 2.1 ([5,12]). *There exist two positive constants c_1 and c_2 depending only on Ω such that*

$$|b(u,v,w)|\le c_1\|u\|_V\|v\|_V\|w\|_V,\ \ \forall u,v,w\in V, \tag{2.4}$$

$$|b(u,v,w)|\le c_2\|Au\|\|v\|_V\|w\|,\ \ \forall u\in D(A),v\in V,w\in H, \tag{2.5}$$

$$|b(u,v,w)|\le c_2\|u\|^{1/4}\|Au\|^{3/4}\|v\|_V\|w\|,\ \ \forall u\in D(A),v\in V,w\in H. \tag{2.6}$$

We can easily obtain from (2.2) and (2.5) that

$$\|B_N(u,u)\|\le c_2N\|Au\|,\quad \forall u\in D(A). \tag{2.7}$$

With the above notation, excluding the pressure p, we can express the weak version of equations (1.1)–(1.4) in the solenoidal vector field as follows (see e.g. [5,12]):

$$u'(t)+\nu Au(t)+B_N(u(t),u(t))=g(t)\ \text{ in}\,\mathcal{D}'(\tau,+\infty;V'), \tag{2.8}$$

$$u(x,\tau)=u_\tau,\ \ x\in\Omega, \tag{2.9}$$

where $u'(t)=\dfrac{\partial u(t)}{\partial t}$.

We next specify the definition of solutions to problem (2.8)–(2.9).

Definition 2.1. *Let $\tau\in\boldsymbol{R}$ and any $T>\tau$ be given. Let $u_\tau\in H$ and $g\in L^2(\tau,T;H)$. A weak solution of problem (2.8)–(2.9) is any function $u(x,t)$ that belongs to $L^\infty(\tau,T;H)\cap L^2(\tau,T;V)$ for all $T>\tau$, with $u(x,t)|_{t=\tau}=u_\tau$, such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),\phi)+\nu\langle Au(t),\phi\rangle+\langle B_N(u(t),u(t)),\phi\rangle=(g(t),\phi),\quad \forall\phi\in V,$$

holds in the distribution sense of $\mathcal{D}'(\tau,+\infty)$. If $u(x,t)$ is a weak solution and $u(x,t)\in L^\infty(\tau,T;V)\cap L^2(\tau,T;D(A))$ for all $T>\tau$, then $u(x,t)$ is called a strong solution of problem (2.8)–(2.9).

For the existence and uniqueness of solutions to problem (2.8)–(2.9), we have the following result.

Lemma 2.2 ([3]). *Suppose that $g(t)\in L^2(\tau,T;H)$ for all $T>\tau$ and let $u_\tau\in V$ be given. Then there exists a unique weak solution u of problem (2.8)–(2.9), which is in fact a strong solution and satisfies*

$$u\in\mathcal{C}([\tau,T];V),\quad for\ \ all\quad T>\tau. \tag{2.10}$$

Here we want to give a remark. Using the similar derivations as those as in [3], we can prove that if $u_\tau\in H\setminus V$ and $g(t)\in L^2(\tau,T;H)$, then problem (2.8)–(2.9) possesses a weak solution. However, we do not know if it is unique. This is the reason we consider the evolutionary process associated with problem (2.8)–(2.9) in space V. In fact, we can prove that the solutions of problem (2.10)–(2.11) depend continuously on the initial value in the topology of V. From this fact and Lemma 2.2, we see that the family of solution operators

$$U(t,\tau): u_\tau \in V \longmapsto U(t,\tau)u_\tau = u(t;\tau,u_\tau) \in V, \quad \forall t \geq \tau,$$

generates a continuous process $\{U(t,\tau)\}_{t\geq\tau}$ in space V, hereinafter $u(t;\tau,u_\tau)$ denotes the solution of problem (2.8)–(2.9) corresponding to the initial datum u_τ at initial time τ. Moreover, we can conclude from (2.10) that

$$\begin{cases} \text{for given } \tau \in \mathbf{R} \text{ and } u_\tau \in V, \text{ the function } \tau < t \longmapsto U(t,\tau)u_\tau \\ \text{is continuous with values in } V. \end{cases} \tag{2.11}$$

3. Existence of the pullback attractors in space V

In this section, we first introduce some definitions concerning the pullback attractors. Then we establish the existence of pullback attractors for the process in space V.

In the sequel, we use $\mathcal{P}(V)$ to denote the family of all nonempty subsets of V, and consider a family of nonempty sets $\widehat{D_0} = \{D_0(t) | t \in \mathbf{R}\} \subseteq \mathcal{P}(V)$. Let $\mathcal{D}$ be a given nonempty class of families parameterized in time $\widehat{D} = \{D(t) | t \in \mathbf{R}\} \subseteq \mathcal{P}(V)$. The class $\mathcal{D}$ will be called a universe in $\mathcal{P}(V)$.

Definition 3.1. *It is said that $\widehat{D_0} = \{D_0(t) | t \in \boldsymbol{R}\} \subseteq \mathcal{P}(V)$ is pullback $\mathcal{D}$-absorbing for the process $\{U(t,\tau)\}_{t\geq\tau}$ in V if for any $t \in \boldsymbol{R}$ and any $\widehat{D} = \{D(t) | t \in \boldsymbol{R}\} \in \mathcal{D}$, there exists a $\tau_0(t,\widehat{D}) \leq t$ such that $U(t,\tau)D(\tau) \subseteq D_0(t)$ for all $\tau \leq \tau_0(t,\widehat{D})$.*

Definition 3.2. *The process $\{U(t,\tau)\}_{t\geq\tau}$ is said to be pullback $\widehat{D_0}$-asymptotically compact if for any $t \in \boldsymbol{R}$ and any sequences $\{\tau_n\} \subseteq (-\infty,t]$ and $\{x_n\} \subseteq V$ satisfying $\tau_n \to -\infty$ and $x_n \in D_0(\tau_n)$ for all n, the sequence $\{U(t,\tau_n)x_n\}$ is relatively compact in V. $\{U(t,\tau)\}_{t\geq\tau}$ is said to be pullback $\mathcal{D}$-asymptotically compact if it is $\widehat{D}$-asymptotically compact for any $\widehat{D} \in \mathcal{D}$.*

Definition 3.3. *A family $\hat{A}_{\mathcal{D}} = \{\mathcal{A}_{\mathcal{D}}(t) | t \in \boldsymbol{R}\} \subseteq \mathcal{P}(V)$ is said to be a pullback $\mathcal{D}$-attractor for the process $\{U(t,\tau)\}_{t\geq\tau}$ in V if it has the following properties:*

(a) `Compactness`: *for any $t \in \boldsymbol{R}$, $\mathcal{A}_{\mathcal{D}}(t)$ is a nonempty compact subset of V;*

(b) `Pullback attracting`: *$\hat{\mathcal{A}}_{\mathcal{D}}$ is pullback $\mathcal{D}$-attracting in the following sense*

$$\lim_{\tau\to-\infty} \operatorname{dist}_V(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}}(t)) = 0, \ \forall \widehat{D} = \{D(t) | t \in \boldsymbol{R}\} \in \mathcal{D}, t \in \boldsymbol{R};$$

(c) `Invariance`: *$U(t,\tau)\mathcal{A}_{\mathcal{D}}(\tau) = \mathcal{A}_{\mathcal{D}}(t), \ \forall \tau \leq t.$*

References [4, 8] proved the general existence and minimality results of a pullback attractor and its property for general process. For example, García-Luengo, Marín-Rubio and Real in [8] pointed out that the family $\mathcal{A}_{\mathcal{D}}$ is minimal in the sense that if $\widehat{C} = \{C(t) | t \in \mathbf{R}\} \subseteq \mathcal{P}(V)$ is a family of closed sets such that for any $\widehat{D} = \{D(t) | t \in \mathbf{R}\} \in \mathcal{D}$,

$$\lim_{\tau\to-\infty} \operatorname{dist}_V(U(t,\tau)D(\tau), C(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}}(t) \subseteq C(t)$.

To prove the existence of the pullback attractors for the process $\{U(t,\tau)\}_{t\geq\tau}$ in space V, we need the following assumption on the external force function $g(t)$.

Assumption $(\mathbf{H}_1)$. Assume $g\in L^2_{\text{loc}}(\mathbf{R};H)$ and

$$\int_{-\infty}^{t} e^{\lambda_1\nu s}\|g(s)\|^2 \mathrm{d}s < +\infty, \quad \text{for all } t\in\mathbf{R}, \tag{3.1}$$

where λ_1 is the first eigenvalue of the operator A.

From now on, we set $\gamma=\lambda_1\nu$ and denote by $\mathcal{D}_\gamma$ the class of all families of nonempty subsets $\widehat{D}(t)=\{D(t)|t\in\mathbf{R}\}\subseteq\mathcal{P}(V)$ such that

$$\lim_{\tau\to-\infty}\big(e^{\gamma\tau}\sup_{u\in D(\tau)}\|u\|_V^2\big)=0. \tag{3.2}$$

Lemma 3.1. *Let Assumption* $(\mathbf{H}_1)$ *hold. Then for any* $t\in\boldsymbol{R}$ *and* $\widehat{D}=\{D(t)|t\in\boldsymbol{R}\}\in\mathcal{D}_\gamma$*, there exists some* $\tau_1(\widehat{D},t)<t-3$*, such that for any* $\tau\leq\tau_1(\widehat{D},t)$ *and any* $u_\tau\in D(\tau)$*, we have*

$$\|u(r;\tau,u_\tau)\|^2\leq\rho_1(t), \quad \forall r\in[t-3,t], \tag{3.3}$$

$$\|u(r;\tau,u_\tau)\|_V^2\leq\rho_2(t), \quad \forall r\in[t-2,t], \tag{3.4}$$

$$\int_{r-1}^{r}\|Au(\theta;\tau,u_\tau)\|^2\mathrm{d}\theta\leq\rho_3(t), \quad \forall r\in[t-1,t], \tag{3.5}$$

$$\int_{r-1}^{r}\|u'(\theta;\tau,u_\tau)\|^2\mathrm{d}\theta\leq\rho_4(t), \quad \forall r\in[t-1,t], \tag{3.6}$$

where

$$\rho_1(t)=1+\frac{e^{\gamma(3-t)}}{\gamma}\int_{-\infty}^{t}e^{\gamma\theta}\|g(\theta)\|^2\mathrm{d}\theta, \tag{3.7}$$

$$\rho_2(t)=c(c_2,\nu,\gamma,N)\max_{r\in[t-2,t]}\Big\{\rho_1(r)+\int_{r-1}^{r}\|g(\theta)\|^2\mathrm{d}\theta\Big\}, \tag{3.8}$$

$$\rho_3(t)=c(c_2,\nu,N)\rho_1(t)+\rho_2(t)/\nu+\frac{2}{\nu^2}\int_{t-2}^{t}\|g(\theta)\|^2\mathrm{d}\theta, \tag{3.9}$$

$$\rho_4(t)=c(c_2,\nu,N)\rho_3(t)+\int_{t-2}^{t}\|g(\theta)\|^2\mathrm{d}\theta. \tag{3.10}$$

Proof. We know that A is a positive self-adjoint linear and elliptic operator with compact inverse. By the classical spectral theory of elliptic operators (see e.g. [2]), there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfying

$$0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_n\leq\cdots, \ \ \lambda_n\to+\infty \ \text{ as } \ n\to\infty, \tag{3.11}$$

and a family of elements $\{w_n\}_{n=1}^{\infty} \subseteq D(A)$, which forms a Hilbert basis of V and is orthonormal in H, such that

$$Aw_n = \lambda_n w_n, \ \ \forall n \in \mathbf{N}. \tag{3.12}$$

We now consider some $u_\tau \in \mathcal{D}_\gamma$ and let $u(t;\tau,u_\tau)$ be the corresponding solution with initial datum u_τ. For each integer $n \geq 1$, we denote by $u_n(t)$ the Galerkin approximation of the solution $u(t;\tau,u_\tau)$ of problem (2.8)–(2.9), which is the weak solution of

$$u_n'(t) + \nu A u_n(t) + P_n B_N(u_n(t), u_n(t)) = P_n g(t), \quad t > \tau, \tag{3.13}$$
$$u_n(\tau) = u_{n,\tau} = P_n u_\tau,$$

where P_n is the projection onto the subspace H spanned by $\{w_1, w_2, \cdots, w_n\}$. From the proof of Theorem 7 in [3] and the uniqueness of the solution, we conclude that

$$u_n \longrightarrow u \ \text{ strongly in } \ L^2(\tau,t;V), \tag{3.14}$$
$$u_n \rightharpoonup u \ \text{ weakly in } \ L^2(\tau,t;D(A)), \tag{3.15}$$
$$u_n' \rightharpoonup u' \ \text{ weakly in } \ L^2(\tau,t;H). \tag{3.16}$$

For the Galerkin approximate solution u_n, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|^2 + \gamma \|u_n(t)\|^2 \leq \|g(t)\|^2/\gamma, \tag{3.17}$$

where we have used the Cauchy inequality and the following Poincaré inequality

$$\lambda_1 \|\psi\|^2 \leq \|\psi\|_V^2, \quad \forall \psi \in \mathbf{H}_0^1(\Omega). \tag{3.18}$$

Applying Gronwall inequality to estimate (3.17) yields

$$\|u_n(t)\|^2 \leq \|u_\tau\|^2 e^{-\gamma(t-\tau)} + \frac{e^{-\gamma t}}{\gamma} \int_{-\infty}^{t} e^{\gamma\theta} \|g(\theta)\|^2 \mathrm{d}\theta, \quad \forall t \geq \tau. \tag{3.19}$$

Therefore, if $\widehat{D} = \{D(t) | t \in \mathbf{R}\} \in \mathcal{D}_\gamma$ and $u_\tau \in D(\tau)$, then

$$\lim_{\tau \to -\infty} (e^{-\gamma(t-\tau)} \|u_\tau\|^2) \leq \frac{1}{\sqrt{\lambda_1}} \lim_{\tau \to -\infty} (e^{-\gamma(t-\tau)} \|u_\tau\|_V^2) = 0.$$

Hence, there exists some $\tau_1(\widehat{D}, t) < t - 3$ such that for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$, we have

$$\|u_n(t)\|^2 \leq 1 + \frac{e^{-\gamma t}}{\gamma} \int_{-\infty}^{t} e^{\gamma\theta} \|g(\theta)\|^2 \mathrm{d}\theta, \quad \forall t > \tau. \tag{3.20}$$

Therefore, for any $\tau \leq \tau_1(\widehat{D}, t)$ and any $u_\tau \in D(\tau)$,

$$\|u_n(r;\tau,u_\tau)\|^2 \leq \rho_1(t), \quad \forall r \in [t-3, t], \tag{3.21}$$

where $\rho_1(t)$ is given by formula (3.7).

To prove estimate (3.4), we take the inner product of equation (3.13) with $Au_n(t)$ and obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|_V^2+\nu\|Au_n(t)\|^2+\langle P_nB_N(u_n(t),u_n(t)),Au_n(t)\rangle\\&=(P_ng(t),Au_n(t))\le\frac{\nu}{4}\|Au_n(t)\|^2+\frac{\|g(t)\|^2}{\nu},\quad \forall t>\tau.\end{aligned}\tag{3.22}$$

By estimate (2.6) and the Young inequality, there is a positive constant $c(c_2,\nu,N)$ such that

$$\begin{aligned}|\langle P_nB_N(u_n(t),u_n(t)),Au_n(t)\rangle|&\le c_2N\|u_n(t)\|^{1/4}\|Au_n(t)\|^{7/4}\\&\le\frac{\nu}{4}\|Au_n(t)\|^2+c(c_2,\nu,N)\|u_n(t)\|^2.\end{aligned}\tag{3.23}$$

Inserting estimate (3.23) into inequality (3.22) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|_V^2+\nu\|Au_n(t)\|^2\le c(c_2,\nu,N)\|u_n(t)\|^2+\frac{2\|g(t)\|^2}{\nu},\quad \forall t>\tau.\tag{3.24}$$

Notice that $\lambda_1\|u_n(t)\|_V^2\le\|Au_n(t)\|^2$. Estimate (3.24) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|_V^2+\gamma\|u_n(t)\|_V^2\le c(c_2,\nu,N)\|u_n(t)\|^2+\frac{2\|g(t)\|^2}{\nu},\quad \forall t>\tau.\tag{3.25}$$

Applying the Gronwall inequality to expression (3.25) on $\tau\le r-1\le s\le r$ yields

$$\begin{aligned}\|u_n(r)\|_V^2\le\|u_n(s)\|_V^2&+\frac{2}{\nu}\int_{r-1}^{r}e^{-\gamma(r-\theta)}\|g(\theta)\|^2\mathrm{d}\theta\\&+c(c_2,\nu,N)\int_{r-1}^{r}e^{-\gamma(r-\theta)}\|u_n(\theta)\|^2\mathrm{d}\theta.\end{aligned}\tag{3.26}$$

Then integrating inequality (3.26) with respect to s over $[r-1,r]$ gives

$$\begin{aligned}\|u_n(r)\|_V^2\le\int_{r-1}^{r}\|u_n(s)\|_V^2\mathrm{d}s&+\frac{2}{\nu}\int_{r-1}^{r}e^{-\gamma(r-\theta)}\|g(\theta)\|^2\mathrm{d}\theta\\&+c(c_2,\nu,N)\int_{r-1}^{r}e^{-\gamma(r-\theta)}\|u_n(\theta)\|^2\mathrm{d}\theta,\end{aligned}\tag{3.27}$$

for all $\tau\le r-1$. To estimate the term $\int_{r-1}^{r}\|u_n(s)\|_V^2\mathrm{d}s$, we take the inner product of equation (3.13) with $u_n(t)$ and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n(t)\|^2+\nu\|u_n(t)\|^2\le\|g(t)\|^2/\gamma.\tag{3.28}$$

Integrating inequality (3.28) over $[r-1,r]$ yields

$$\int_{r-1}^{r}\|u_n(s)\|_V^2\mathrm{d}s\le\|u_n(r-1)\|^2+\frac{1}{\gamma}\int_{r-1}^{r}\|g(\theta)\|^2\mathrm{d}\theta.\tag{3.29}$$

Inserting estimate (3.29) into expression (3.27) and using estimate (3.21), we obtain for any $\tau \le \tau_1(\widehat{D},t)$ and any $u_\tau \in D(\tau)$ that

$$\|u(r;\tau,u_\tau)\|_V^2 \le \rho_2(t), \quad \forall r \in [t-2,t], \tag{3.30}$$

where $\rho_2(t)$ is given by formula (3.8).

We now integrate expression (3.24) on $[r-1,r]$ to obtain

$$\int_{r-1}^{r} \|Au_n(\theta)\|^2 \mathrm{d}\theta \le \|u_n(r-1)\|_V^2/\nu + \frac{2}{\nu^2}\int_{r-1}^{r} \|g(\theta)\|^2 \mathrm{d}\theta + c(c_2,\nu,N)\int_{r-1}^{r} \|u_n(\theta)\|^2 \mathrm{d}\theta. \tag{3.31}$$

Hence, we get for any $r \in [t-1,t]$ that

$$\int_{r-1}^{r} \|Au_n(\theta)\|^2 \mathrm{d}\theta \le \frac{2}{\nu^2}\int_{r-1}^{r} \|g(\theta)\|^2 \mathrm{d}\theta + c(c_2,\nu,N)\rho_1(t) + \rho_2(t)/\nu. \tag{3.32}$$

Finally, taking estimations (2.7), (3.32) and equation (3.13) into account, we have

$$\int_{r-1}^{r} \|u_n'(\theta)\|^2 \mathrm{d}\theta \le c(c_2,\nu,N)\int_{r-1}^{r} \|Au_n(\theta)\|^2 \mathrm{d}\theta + \int_{r-1}^{r} \|g(\theta)\|^2 \mathrm{d}\theta. \tag{3.33}$$

At this stage, we can pass to the limit in inequalities (3.21), (3.30) and (3.32)–(3.33), using the convergent relations (3.14)–(3.16), to get the desired results of Lemma 3.1. The proof is completed. □

Let

$$\widehat{D}_0 = \{\bar{\mathcal{B}}(0,\rho_2^{1/2}(t)) | t \in \mathbf{R}\} \tag{3.34}$$

be the family of closed balls in space V centered at zero and with radius $\rho_2^{1/2}(t)$. By Assumption $(\mathbf{H}_1)$, we have

$$\lim_{\tau \to -\infty} e^{\gamma t}\rho_2(t) = 0.$$

By Lemma 3.1 and above fact, we have the following result.

LEMMA 3.2. *Let Assumption $(\mathbf{H}_1)$ hold, then for any $t \in \boldsymbol{R}$ and any $\widehat{D} = \{D(t) | t \in \boldsymbol{R}\} \in \mathcal{D}_\gamma$, there exists some $\tau(\widehat{D},t) < t$ such that*

$$U(t,\tau)D(\tau) \subseteq \bar{\mathcal{B}}(0,\rho_2^{1/2}(t)), \quad \forall \tau \le \tau(\widehat{D},t).$$

Particularly, the family of closed balls $\widehat{D}_0$ defined by formula (3.34) is pullback $\mathcal{D}_\gamma$-absorbing for $\{U(t,\tau)\}_{t\ge\tau}$ in space V.

We next use the enstrophy equation of the addressed equations to investigate the pullback asymptotic compactness of the process $\{U(t,\tau)\}_{t\ge\tau}$ in space V for the universe $\mathcal{D}_\gamma$.

LEMMA 3.3. *Let Assumption $(\mathbf{H}_1)$ hold. Then the process $\{U(t,\tau)\}_{t\ge\tau}$ is pullback $\mathcal{D}_\gamma$-asymptotically compact in space V.*

Proof. Let us fix $t\in\mathbf{R}$, a family $\widehat{D}=\{D(t)|t\in\mathbf{R}\}\in\mathcal{D}_\gamma$, a sequence $\{\tau_n\}\subseteq(-\infty,t]$ with $\tau_n\to-\infty$, and a sequence $\{u_{\tau_n}\}\subseteq V$, with $u_{\tau_n}\in D(\tau_n)$ for all n. We shall prove that the sequence $\{u(t;\tau_n,u_{\tau_n})\}$ is relatively compact in V. For brevity, we set $u^{(n)}=u^{(n)}(\cdot)=u(\cdot;\tau_n,u_{\tau_n})$.

From Lemma 3.1 we see that there exists a $\tau_1(\widehat{D},t)<t-3$, such that the sequence $\{u^{(n)}|\tau_n\le\tau_1(\widehat{D},t)\}$ is uniformly bounded in $L^\infty(t-2,t;V)\bigcap L^2(t-2,t;D(A))$, and $\{(u^{(n)})'\}$ is also uniformly bounded in $L^2(t-2,t;H)$. By the Aubin–Lions compactness lemma (see [13]) there exists an element $u\in L^\infty(t-2,t;V)\bigcap L^2(t-2,t;D(A))$ with $u'\in L^2(t-2,t;H)$, such that for a subsequence the following convergent relations hold:

$$u^n \rightharpoonup^* u \ \text{weak}-\text{star in}\ L^\infty(t-2,t;V), \tag{3.35}$$

$$u^n \rightharpoonup u \ \text{weakly in}\ L^2(t-2,t;D(A)), \tag{3.36}$$

$$(u^n)' \rightharpoonup u' \ \text{weakly in}\ L^2(t-2,t;H), \tag{3.37}$$

$$u^n \longrightarrow u \ \text{strongly in}\ L^2(t-2,t;V), \tag{3.38}$$

$$u^n(s) \longrightarrow u(s) \ \text{strongly in}\ V, \text{a.e.}\, s\in(t-2,t). \tag{3.39}$$

By relations (3.35)–(3.39) and the fact that $u\in\mathcal{C}([t-2,t];V)$, we have in the interval $(t-2,t)$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t),v)+\nu\langle Au(t),v\rangle+\langle B_N(u(t),u(t)),v\rangle=(g(t),v), \forall v\in V. \tag{3.40}$$

From relations (3.35)–(3.39), we also conclude that $\{u^{(n)}\}$ is equiv-continuous in H on the interval $[t-2,t]$. Since the sequence $\{u^{(n)}\}$ is uniformly bounded in $\mathcal{C}([t-2,];V)$ and the injection $V\hookrightarrow H$ is compact, we obtain by the Ascoli–Arzelá Theorem that

$$u^{(n)} \longrightarrow u \ \text{strongly in}\ \mathcal{C}([t-2,t];H), \tag{3.41}$$

by which and also the uniform boundedness of $\{u^{(n)}\}$ in $\mathcal{C}([t-2,t];V)$, we have

$$u^{(n)}(s_n) \rightharpoonup u(s_*) \ \text{weakly in}\ V,\ \forall\{s_n\}\subseteq[t-2,t] \text{ with } \ s_n\to s_*. \tag{3.42}$$

We next prove

$$u^{(n)} \longrightarrow u \ \text{strongly in}\ \mathcal{C}([t-2,t];V), \tag{3.43}$$

which will implies the desired relative compactness. To prove the limit (3.43), we argue by contradiction. If limit (3.43) does not hold, then there exist an $\epsilon_0>0$ and a sequence $\{t_n\}\subseteq[t-1,t]$ (without loss of generality converging to some t_*) such that

$$\|u^{(n)}(t_n)-u(t_*)\|_V\ge\epsilon_0,\ \forall n\ge1. \tag{3.44}$$

By the result (3.42) and the lower semi-continuity of the norm, we get

$$\|u(t_*)\|_V\le\liminf_{n\to\infty}\|u^{(n)}(t_n)\|_V. \tag{3.45}$$

Since V is a Hilbert space, inequality (3.44) will contradict the inequality (3.45) and the following inequality

$$\|u(t_*)\|_V\ge\limsup_{n\to\infty}\|u^{(n)}(t_n)\|_V. \tag{3.46}$$

Thus we only need prove inequality (3.46).

Now, using the enstrophy inequality (3.24) for u and all $u^{(n)}$, we have for all $t-2 \le s_1 \le s_2 \le t$ that

$$\|u^{(n)}(s_2)\|_V^2 + \nu \int_{s_1}^{s_2} \|Au^{(n)}(\theta)\|^2 \mathrm{d}\theta \le \frac{2}{\nu} \int_{s_1}^{s_2} \|g(\theta)\|^2 \mathrm{d}\theta + c(c_2,\nu,N) \int_{s_1}^{s_2} \|u^{(n)}(\theta)\|^2 \mathrm{d}\theta \quad (3.47)$$

and

$$\|u(s_2)\|_V^2 + \nu \int_{s_1}^{s_2} \|Au(\theta)\|^2 \mathrm{d}\theta \le \frac{2}{\nu} \int_{s_1}^{s_2} \|g(\theta)\|^2 \mathrm{d}\theta + c(c_2,\nu,N) \int_{s_1}^{s_2} \|u(\theta)\|^2 \mathrm{d}\theta. \quad (3.48)$$

Write

$$J_n(s) = \|u^{(n)}(s)\|_V^2 - \frac{2}{\nu} \int_{t-2}^{s} \|g(\theta)\|^2 \mathrm{d}\theta - c(c_2,\nu,N) \int_{t-2}^{s} \|u^{(n)}(\theta)\|^2 \mathrm{d}\theta,$$

$$J(s) = \|u(s)\|_V^2 - \frac{2}{\nu} \int_{t-2}^{s} \|g(\theta)\|^2 \mathrm{d}\theta - c(c_2,\nu,N) \int_{t-2}^{s} \|u(\theta)\|^2 \mathrm{d}\theta.$$

Then

$$\begin{aligned} &J_n(s_2) - J_n(s_1) \\ &= \|u^{(n)}(s_2)\|_V^2 - \|u^{(n)}(s_1)\|_V^2 - \frac{2}{\nu} \int_{s_1}^{s_2} \|g(\theta)\|^2 \mathrm{d}\theta - c(c_2,\nu,N) \int_{s_1}^{s_2} \|u^{(n)}(\theta)\|^2 \mathrm{d}\theta \\ &\le -\nu \int_{s_1}^{s_2} \|Au^{(n)}(\theta)\|^2 \mathrm{d}\theta \le 0, \ \text{ for } \ t-2 \le s_1 \le s_2 \le t. \end{aligned}$$

Hence, for each n, $J_n(\cdot)$ is a non-increasing function in $[t-2,t]$. Similarly, $J(\cdot)$ is a non-increasing function in $[t-2,t]$. Now, by relations (3.39) and (3.41), we see

$$J_n(s) \to J(s), \ \text{ a.e.}\, s \in (t-2,t).$$

Therefore, there exists a sequence $\{t_k^*\} \subseteq (t-2,t_*)$ such that $t_k^* \to t_*$ when $k \to \infty$, and

$$\lim_{n\to\infty} J_n(t_k^*) = J(t_k^*), \ \text{ for all } \ k.$$

Consider an arbitrary $\delta > 0$. By the continuity of J, there exists some k_δ such that

$$|J(t_k^*) - J(t_*)| < \delta/2, \ \forall k \ge k_\delta. \quad (3.49)$$

Now consider $n(k_\delta)$ such that for all $n \ge n(k_\delta)$, it holds

$$t_n \ge t_{k_\delta}^* \text{ and } \ |J_n(t_{k_\delta}^*) - J(t_{k_\delta}^*)| < \delta/2. \quad (3.50)$$

Then, since all J_n are non-increasing, we deduce from estimates (3.49) and (3.50) that for all $n \ge n(k_\eta)$

$$J_n(t_n) - J(t_*) \le J_n(t_{k_\delta}^*) - J(t_*) \le |J_n(t_{k_\delta}^*) - J(t_*)|$$

$$\leq |J_n(t^*_{k_\delta}) - J(t^*_{k_\delta})| + |J(t^*_{k_\delta}) - J(t_*)| < \delta, \tag{3.51}$$

which implies that

$$\limsup_{n\to\infty} J_n(t_n) \leq J(t_*). \tag{3.52}$$

From relations (3.35)–(3.39) and (3.52), we get inequality (3.46). The proof is completed. □

By combining Lemma 3.2, Lemma 3.3, and Garcia-Luengo *et al.* [8, Theorem 3.11, Corollary 3.13], we get the main theorem of this section as follows.

THEOREM 3.1. *Let Assumption* ($\mathbf{H}_1$) *hold. Then there exists the minimal pullback* $\mathcal{D}_\gamma$*-attractor* $\hat{\mathcal{A}}_{\mathcal{D}_\gamma} = \{\mathcal{A}_{\mathcal{D}_\gamma} | t \in \boldsymbol{R}\}$ *for the process* $\{U(t,\tau)\}_{t\geq\tau}$ *in* V*, which satisfies:*

(a) `Compactness`: *for any* $t \in \boldsymbol{R}$, $\mathcal{A}_{\mathcal{D}_\gamma}(t)$ *is a nonempty compact subset of* V*;*

(b) `Pullback attracting`: $\hat{\mathcal{A}}_{\mathcal{D}_\gamma}$ *is pullback* $\mathcal{D}_\gamma$*-attracting in the following sense*

$$\lim_{\tau\to-\infty} \mathrm{dist}_V(U(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\gamma}(t)) = 0, \ \forall \widehat{D} = \{D(t) | t \in \boldsymbol{R}\} \in \mathcal{D}_\gamma, t \in \boldsymbol{R};$$

(c) `Invariance`: $U(t,\tau)\mathcal{A}_{\mathcal{D}_\gamma}(\tau) = \mathcal{A}_{\mathcal{D}_\gamma}(t), \ \forall \tau \leq t.$

4. Invariant measure on the pullback attractor

The aim of this section is to employ the theory of Łukaszewicz and Robinson [17] to prove the unique existence of invariant measure on the pullback $\mathcal{D}_\gamma$- attractor $\hat{\mathcal{A}}_{\mathcal{D}_\gamma}$.

We first cite two definitions.

DEFINITION 4.1 ([7,17]). *A generalized Banach limit is any linear functional, which we denote by* $\mathrm{LIM}_{T\to\infty}$*, defined on the space of all bounded real-valued functions on* $[0,+\infty)$ *that satisfies*

(i) $\mathrm{LIM}_{T\to\infty} f(T) \geq 0$ *for nonnegative functions* f*;*

(ii) $\mathrm{LIM}_{T\to\infty} f(T) = \lim_{T\to\infty} f(T)$ *if the usual limit* $\lim_{T\to\infty} f(T)$ *exists.*

DEFINITION 4.2 ([17]). *A process* $\{\mathcal{U}(t,\tau)\}_{t\geq\tau}$ *is said to be* τ*- continuous on a metric space* X *if for every* $v_0 \in X$ *and every* $t \in \boldsymbol{R}$*, the* X*-valued function* $\tau \longmapsto U(t,\tau)v_0$ *is continuous and bounded on* $(-\infty, t]$.

The following result was proved by Łukaszewicz and Robinson in [17].

LEMMA 4.1 ([17]). *Let* $\{\mathcal{U}(t,\tau)\}_{t\geq\tau}$ *be a* τ*-continuous evolutionary process in a complete metric space* X *that has a pullback* $\mathcal{D}$*-attractor* $\mathcal{A}(\cdot)$*. Fix a generalized Banach limit* $\mathrm{LIM}_{T\to\infty}$ *and let* $\psi : \boldsymbol{R} \longmapsto X$ *be a continuous map such that* $\psi(\cdot) \in \mathcal{D}$*. Then there exists a unique family of Borel probability measures* $\{\mu_t\}_{t\in\boldsymbol{R}}$ *in* X *such the support of the measure* μ_t *is contained in* $\mathcal{A}(t)$ *and*

$$\mathrm{LIM}_{\tau\to-\infty} \frac{1}{t-\tau} \int_\tau^t \phi(\mathcal{U}(t,s)\psi(s))\mathrm{d}s = \int_{\mathcal{A}(t)} \phi(v)\mathrm{d}\mu_t(v)$$

for any real-valued continuous functional ϕ *on* X*. In addition,* μ_t *is invariant in the sense that*

$$\int_{\mathcal{A}(t)} \phi(v)\mathrm{d}\mu_t(v) = \int_{\mathcal{A}(\tau)} \phi(\mathcal{U}(t,\tau)v)\mathrm{d}\mu_\tau(v), \quad t \geq \tau.$$

In order to employ the above result to the pullback $\mathcal{D}_\gamma$-attractor $\hat{\mathcal{A}}_{\mathcal{D}_\gamma}$ obtained in Theorem 3.1, we need check the τ-continuous property of the process $\{U(t,\tau)\}_{t\geq\tau}$ in space V. We begin with the following estimation.

LEMMA 4.2. *There exists some positive constant $c(c_1,c_2,N)$ such that*

$$\|B_N(v,v)-B_N(u,u)\|\leq c(c_1,c_2,N)(\|Au\|+\|Av\|)\|v-u\|,\quad \forall u,v\in D(A). \tag{4.1}$$

Proof. For any $w\in H$, we have

$$\begin{aligned}|(B_N(v,v)-B_N(u,u),w)|&=|b_N(v,v,w)-b_N(u,u,w)|\\&=|F_N(\|v\|_V)b(v,v,w)-F_N(\|u\|_V)b(u,u,w)|\\&\leq I_1+I_2+I_3,\quad \forall u,v\in D(A),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}I_1&=|(F_N(\|v\|_V)-F_N(\|u\|_V))b(v,v,w)|,\\I_2&=|F_N(\|v\|_V)b(v-u,v,w)|,\\I_3&=|F_N(\|u\|_V)b(u,v-u,w)|.\end{aligned}$$

We next estimate the terms I_1, I_2 and I_3. Firstly, we have (see e.g. [5])

$$r|F_N(r)-F_N(s)|\leq|r-s|,\quad \forall r,s\geq 0. \tag{4.3}$$

It follows from estimates (2.5) and (4.3) that

$$\begin{aligned}I_1&\leq c_2|F_N(\|v\|_V)-F_N(\|u\|_V)|\,\|Av\|\|v\|_V\|w\|\\&\leq c_2\|v-u\|_V\|Av\|\|w\|.\end{aligned} \tag{4.4}$$

Secondly, employing the Sobolev embedding $\mathbf{H}_0^1(\Omega)\hookrightarrow\mathbf{L}^6(\Omega)$ and the fact that $|F_N(\cdot)|\leq 1$, we see that there exists some positive constant $c(\Omega)$ such that

$$I_2\leq c(\Omega)\|v-u\|_{\mathbf{L}^6(\Omega)}\|\nabla v\|_{\mathbf{L}^3(\Omega)}\|w\|\leq c(\Omega)\|v-u\|_V\|Av\|\|w\|. \tag{4.5}$$

Finally, we get directly from estimate (2.5) that

$$I_3\leq c_2\|v-u\|_V\|Au\|\|w\|. \tag{4.6}$$

We obtain the desired inequality (4.1) from estimates (4.4)–(4.6). ∎

LEMMA 4.3. *Let u_τ, $v_\tau\in V$, and $u(t)=u(t;\tau,u_\tau)$, $v(t)=u(t;\tau,v_\tau)$ be the corresponding solutions to problem (2.8)–(2.9). Then there exists some positive constant $c(c_1,c_2,N)$ such that for any $t>\tau$*

$$\|u(t)-v(t)\|_V^2\leq\|u_\tau-v_\tau\|_V^2\exp\Big\{c(c_1,c_2,N)\int_\tau^t(\|Au(\theta)\|^2+\|Av(\theta)\|^2)\mathrm{d}\theta\Big\}. \tag{4.7}$$

Proof. Since $u(t)$ and $v(t)$ are two solutions of problem (2.8)–(2.9) corresponding to the initial data u_τ and v_τ, respectively, we have

$$(u-v)'(t)+\nu A(u(t)-v(t))+B_N(u(t),u(t))-B_N(v(t),v(t))=0. \tag{4.8}$$

Taking the inner product of expression (4.8) with $A(u(t)-v(t))$ in H and using Lemma 4.2 and the Cauchy inequality yields

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-v(t)\|_V^2+\nu\|A(u(t)-v(t))\|^2\\ &\le\frac{\nu}{2}\|A(u(t)-v(t))\|^2+c(c_1,c_2,N)(\|Au(t)\|+\|Av(t)\|)^2\|u(t)-v(t)\|_V^2,\end{aligned}$$

that is

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)-v(t)\|_V^2+\nu\|A(u(t)-v(t))\|^2\\ &\le c(c_1,c_2,N)(\|Au(t)\|^2+\|Av(t)\|^2)\|u(t)-v(t)\|_V^2.\end{aligned}\tag{4.9}$$

Applying the Gronwall inequality, we obtain inequality (4.7). □

Lemma 4.4. *Let Assumption* ($\mathbf{H}_1$) *hold. Then the process* $\{U(t,\tau)\}_{t\ge\tau}$ *is* τ*-continuous in space* V*.*

Proof. Consider any $u_0\in V$ and $t\in\mathbf{R}$. We shall prove that for any $\epsilon>0$ there exists some $\delta=\delta(\epsilon)>0$, such that if $r<t$, $s<t$ and $|r-s|<\delta$, then $\|U(t,r)u_0-U(t,s)u_0\|_V<\epsilon$. We assume that $r<s$ without loss of generality. Then employing Lemma 4.3 and the property of the continuous process, we have

$$\begin{aligned}&\|U(t,r)u_0-U(t,s)u_0\|_V^2\\ &=\|U(t,s)U(s,r)u_0-U(t,s)U(r,r)u_0\|_V^2\\ &\le\|U(s,r)u_0-U(r,r)u_0\|_V^2\\ &\quad\bullet\exp\Big\{c(c_1,c_2,N)\int_s^t(\|AU(\theta,r)u_0\|^2+\|AU(\theta,s)u_0\|^2)\mathrm{d}\theta\Big\}.\end{aligned}\tag{4.10}$$

Now we can use the similar derivations as estimates (3.19) and (3.24) to see that

$$\exp\Big\{c(c_1,c_2,N)\int_s^t(\|AU(\theta,r)u_0\|^2+\|AU(\theta,s)u_0\|^2)\mathrm{d}\theta\Big\}$$

is bounded by a constant depending only on c_1, c_2, N, u_0, t and g, but being independent of s. Hence, From statement (2.11) we conclude that the right hand side of inequality (4.10) is as small as needed if $|r-s|$ is small enough. Therefore, the V-valued function $\tau\longmapsto U(t,\tau)u_0$ is continuous in space V. Finally, using the similar derivations as relation (3.25), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_V^2+\gamma\|u(t)\|_V^2\le c(c_2,\nu,N)\|u(t)\|^2+\frac{2\|g(t)\|^2}{\nu},\quad\forall t>\tau.\tag{4.11}$$

Integrating inequality (4.11) gives the boundedness of the V-valued function $\tau\longmapsto U(t,\tau)u_0$ on $(-\infty,t]$. The proof is completed. □

At this stage, we take Theorem 3.1, Lemma 4.1 and Lemma 4.4 into account and have the following result.

Theorem 4.1. *Suppose Assumption* ($\mathbf{H}_1$) *hold. Let* $\{U(t,\tau)\}_{t\ge\tau}$ *be the process associated to the solution operators of equation (2.8) and* $\hat{\mathcal{A}}_{\mathcal{D}_\gamma}$ *be the pullback* $\mathcal{D}_\gamma$*-attractor* *obtained in Theorem 3.1. Fix a generalized Banach limit* $\mathrm{LIM}_{T\to\infty}$ *and let* $\psi:\boldsymbol{R}\mapsto V$ *be a continuous map such that* $\psi(\cdot)\in\mathcal{D}_\gamma$. *Then there exists a unique family of Borel probability measures* $\{\mu_t\}_{t\in\boldsymbol{R}}$ *in space* V *such that the support of the measure* μ_t *is contained in* $\hat{\mathcal{A}}_{\mathcal{D}_\gamma}$ *and*

$$\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \phi(U(t,s)\psi(s))\mathrm{d}s=\int_{\hat{\mathcal{A}}_{\mathcal{D}_\gamma}(t)}\phi(u)\mathrm{d}\mu_t(u)$$

for any real-valued continuous functional ϕ *on* V. *In addition,* μ_t *is invariant in the sense that*

$$\int_{\hat{\mathcal{A}}_{\mathcal{D}_\gamma}(t)}\phi(u)\mathrm{d}\mu_t(u)=\int_{\hat{\mathcal{A}}_{\mathcal{D}_\gamma}(\tau)}\phi(U(t,\tau)u)\mathrm{d}\mu_\tau(u),\quad t\geq\tau.$$

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