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GLOBAL SOLUTIONS TO ONE-DIMENSIONAL EQUATIONS FOR A SELF-GRAVITATING VISCOUS RADIATIVE AND REACTIVE GAS WITH DENSITY-DEPENDENT VISCOSITY*

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Abstract. In this paper we are concerned with the global existence of smooth solutions to two types of initial-boundary value problems to a system of equations describing one-dimensional motion of self-gravitating, radiative and chemically reactive gas whose viscosity coefficient depends on density. The main ingredient of the analysis is to derive the positive lower and upper bounds on both the specific volume and the absolute temperature.

Keywords. global solutions; self-gravitating viscous radiative and reactive gas; density-dependent viscosity; density- and temperature-dependent heat conductivity.

AMS subject classifications. 35Q35; 35D35; 76D05; 76V05.

1. Introduction and main results

We consider the one-dimensional motion of a compressible, viscous and heat-conducting gas which is self-gravitating, radiative and chemically reactive. Such a gaseous motion, especially in the processes of the unimolecular reactions whose kinetic order is one, is described by the following equations in the Lagragian mass coordinates

$$
\begin{aligned}
v_t - u_x &= 0,\\
u_t + p(v,\theta)_x &= \left(\mu(v)\frac{u_x}{v}\right)_x - G\left(x-\frac{1}{2}\right),\\
e_t + p u_x &= \frac{\mu(v)u_x^2}{v} + \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x + \lambda\phi z,\\
z_t &= \left(\frac{d}{v^2}z_x\right)_x - \phi z.
\end{aligned}
\tag{1.1}
$$

Here $x\in\Omega\subseteq\mathbb{R}$ is the Lagrangian space variable with Ω being some nonempty open set of $\mathbb{R}$, $t\in\mathbb{R}^+$ the time variable, and the primary dependent variables are the specific volume $v=v(t,x)$, the velocity $u=u(t,x)$, the absolute temperature $\theta=\theta(t,x)$ and the mass fraction of the reactant $z=z(t,x)$. $\mu(v)$ is the viscosity coefficient which satisfies $\mu(v)>0$ for all $v>0$ and the positive constants G, d and λ are the Newtonian gravitational constant, the species diffusion coefficient and the difference in the heat between the reactant and the product, respectively. The pressure p and the internal energy per unit mass e are defined by

$$
p(v,\theta) = \frac{R\theta}{v} + \frac{\alpha\theta^4}{3}, \quad e = c_v\theta + \alpha v\theta^4,
\tag{1.2}
$$

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where the positive constants R, c_v and α are the perfect gas constant, the specific heat capacity at constant volume and the radiation-density constant, respectively. The second terms on the right-hand side of both relations in the definition (1.2) stand for the effect of radiation phenomena, whose forms are given by the famous Stefan–Boltzmann law (see [29]). In the radiating regime, we naturally take into account the heat flux from the radiative contribution, not only from the heat-conductive contribution. We also assume that the bulk viscosity $\mu(v)$ is a positive function of specific volume $v(t,x)$ and the thermal conductivity $\kappa=\kappa(v,\theta)$ takes the form (see for example, [35,36])

$$\kappa(v,\theta)=\kappa_1+\kappa_2 v\theta^b \tag{1.3}$$

with positive constants κ_1, κ_2 and b.

Furthermore, as in [35], we assume that the reaction rate function $\phi=\phi(\theta)$ is defined, from the Arrhenius law, by

$$\phi(\theta)=K\theta^{\beta}\exp\left(-\frac{A}{\theta}\right), \tag{1.4}$$

where positive constants K and A are the coefficients of the rate of the reactant and the activation energy, respectively, and β is a non-negative number.

For mathematical theories on the equations (1.1), (1.2), (1.3), (1.4) with certain prescribed initial and/or boundary conditions, although some well-posedness theories have been obtained by many mathematicians recently, cf. [5, 11–16, 35, 36]) and the references cited therein, all these results are concerned with the case when the viscosity coefficient μ is a positive constant. We note, however, that since the energy producing process inside the medium is taken into account in the Equations (1.1), that is, the gas consists of a reacting mixture and the combustion process is current at the high temperature stage, and the experimental results for gases at high temperatures in [39] show that the viscosity coefficient μ may depend on the specific volume and temperature. Thus it is necessary and interesting to consider such a case and *the main purpose of this paper is concentrated on the case when the viscosity coefficient μ is a smooth function of the specific volume v.* For the case when the viscosity coefficient μ depends also on the temperature, since, as pointed out in [17] for one-dimensional compressible Navier–Stokes equations, temperature dependence of the viscosity μ has turned out to be especially problematic, we hope that we can have some contributions in such a problem in the near future.

Throughout the rest of this paper, we assume that the reference configuration is the unit interval $[0,1]$, i.e. $\Omega=(0,1)$ and our fist goal in this paper focuses on the outer pressure problem to the system (1.1), (1.2), (1.3), (1.4) with prescribed initial condition

$$(v(0,x),u(0,x),\theta(0,x),z(0,x))=(v_0(x),u_0(x),\theta_0(x),z_0(x)) \quad \text{for } x\in\overline{\Omega}=[0,1] \tag{1.5}$$

and boundary conditions

$$(\sigma(t,x),\theta_x(t,x),z_x(t,x))|_{x=0,1}=(-p_e,0,0) \quad \text{for } t>0, \tag{1.6}$$

which is studied in [31, 35, 36] with positive constant viscosity coefficient, where $\sigma=-p(v,\theta)+\mu(v)\frac{u_x}{v}$ stands for the stress and p_e (a positive constant) is the external pressure.

In this case, to illustrate our main ideas in deducing the desired global solvability result, as in [3,33], we assume that $\mu(v)$ takes the form

$$\mu(v)=v^{-a}. \tag{1.7}$$

Here $a\geq 0$ is some nonnegative constant. Moreover, without loss of generality, we may assume that, cf. [35, 36]

$$\int_0^1 u_0(x)dx=0. \tag{1.8}$$

For such a problem, our first result in this paper can be stated as in the following theorem

THEOREM 1.1. *Suppose that*

- *The viscosity coefficient μ satisfies the equation* (1.7)*;*
- *The parameters a,b and β are assumed to satisfy $0\leq a<\frac{1}{b+1}$, $b\geq 8$ and $0\leq\beta<b+9$;*
- *The initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$ satisfies the compatibility conditions, the condition (1.8) and*

$$(v_0(x),u_0(x),\theta_0(x),z_0(x))\in H^1(\Omega), \tag{1.9}$$

$$\inf_{x\in\Omega} v_0(x)>0, \quad \inf_{x\in\Omega}\theta_0(x)>0, \quad 0\leq z_0(x)\leq 1 \quad for \quad x\in\overline{\Omega}. \tag{1.10}$$

Then there exists a unique solution $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ of the initial-boundary value problem (1.1), (1.5, (1.6) with (1.2), (1.3), (1.4), (1.7), such that for any $T>0$

$$\begin{aligned}
(v(t,x),u(t,x),\theta(t,x),z(t,x))&\in C^0\left(0,T;H^1(\Omega)\right),\\
(u_x(t,x),\theta_x(t,x))&\in L^2\left(0,T;H^1(\Omega)\right),\\
\underline{V}_1\leq v(t,x)\leq\overline{V}_1, &\quad \forall(t,x)\in[0,T]\times\Omega\\
\underline{\Theta}_1\leq\theta(t,x)\leq\overline{\Theta}_1, &\quad \forall(t,x)\in[0,T]\times\Omega\\
0\leq z(t,x)\leq 1, &\quad \forall(t,x)\in[0,T]\times\Omega.
\end{aligned}$$

Here T is any given positive constant and $\underline{V}_1,\overline{V}_1,\underline{\Theta}_1,\overline{\Theta}_1$ are some positive constants which may depend on T and the initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$.

REMARK 1.1. Some remarks concerning Theorem 1.1 are listed below:

- From the proof of Theorem 1.1, it is easy to see that similar result still holds when the viscosity coefficient μ is a general function of v which is assumed to be sufficiently regular and satisfies the same asymptotic behaviors when $v\to 0^+$ and $v\to+\infty$ as the function (1.7);
- As demonstrated in [35], the initial-boundary value problem (1.1), (1.5, (1.6) with (1.2), (1.3), (1.4), (1.7) is equivalent to the free boundary problem of the equations (1.1) in Eulerian coordinates

$$\begin{aligned}
\rho_t+u\rho_y&=-\rho u_y,\\
\rho(u_t+uu_y)&=(-p+\mu u_y)_y+\rho f,\\
\rho(e_t+ue_y)&=(\kappa\theta_y)_y+(-p+\mu u_y)u_y+\lambda\rho\phi z,\\
\rho(z_t+uz_y)&=(d\rho z_y)_y-\rho\phi z
\end{aligned}$$

in $\cup_{t>0}(\{t\}\times\Omega_t)$, where y is the Eulerian space variable, $\rho=v^{-1}$ is the density, $\Omega_t:=\left\{y\in\mathbb{R}\,\middle|\,y_1(t)<y<y_2(t)\right\}$ and $y_i(t)$ for $i=1,2$ are fluctuating boundary

functions which satisfy the dynamical and kinematic boundary conditions for $i=1,2$

$$(-p+u_y)|_{y=y_i(t)}=-p_e \quad \text{for } t>0,$$
$$\frac{dy_i(t)}{dt}=v_i(t,y_i(t)) \quad \text{for } t>0$$

for some positive constant $p_e>0$ (the outer pressure) and the thermal and chemical boundary conditions for $i=1,2$

$$(\kappa\theta_y)|_{y=y_i(t)}=0 \quad \text{for } t>0,$$
$$(d\rho z_y)|_{y=y_i(t)}=0 \quad \text{for } t>0,$$

and the initial condition

$$(\rho(t,x),u(t,x),\theta(t,x),z(t,x))|_{t=0}=(\rho_0(x),u_0(x),\theta_0(x),z_0(x)) \quad \text{for } y\in\overline{\Omega}_0$$

under the standard normalization $\int_{\Omega_0}\rho_0(x)dx=1$. Here the external force per unit mass $f=f(t,y)$ is given by $f=-U_y$, where $U(t,y)$ is the solution of the boundary value problem

$$\frac{\partial^2 U(t,y)}{\partial y^2}=G\rho(t,y), \quad (t,y)\in\cup_{t>0}(\{t\}\times\Omega_t),$$
$$U(t,y)|_{y=y_1(t)}=U(t,y)|_{y=y_2(t)}=0, \quad t\in\mathbb{R}^+.$$

Our second goal in this paper is to deal with the system (1.1), (1.2), (1.3), (1.4) with prescribed initial condition (1.5) and homogeneous Dirichlet boundary condition with respect to the velocity $u(t,x)$ and homogeneous Neumann boundary conditions with respect to both the temperature $\theta(t,x)$ and the mass fraction of the reactant $z(t,x)$, i.e.

$$(u(t,x),\theta_x(t,x),z_x(t,x))|_{x=0,1}=0 \quad \text{for } t>0. \tag{1.11}$$

Although the case when the viscosity coefficient is a positive constant has been studied in [8], we consider in this paper the case when $\mu(v)$ is a smooth function of v for $v>0$. Due to the limit of our technique, we need to ask that $\mu(v)$ is non-degenerate in the sense that

$$\mu(v)\sim\begin{cases} v^{-l_1}, & v\to 0^+,\\ v^{l_2}, & v\to\infty,\end{cases} \tag{1.12}$$

where l_1,l_2 are positive constants and $f(x)\sim g(x)$ as $x\to x_0$ means that there exists a positive constant $C\geq 1$ such that $C^{-1}g(x)\leq f(x)\leq Cg(x)$ in a neighborhood of x_0.

For the global solvability of the initial-boundary value problem (1.1), (1.5), (1.11), we have

THEOREM 1.2. *Suppose that*

- $\mu(v)$ *is assumed to satisfy the condition* (1.12) *and the parameters* $l_1>1$, $l_2>0$, $0\leq\beta\leq b+4$, *and* b *satisfy one of the following two conditions:*
  - *(i)* $b\geq 8$,
  - *(ii)* $\frac{44l_1l_2+54l_1+32l_2+16}{6l_1l_2+7l_1+4l_2+2}<b<8$;

- *The viscosity coefficient* $\mu(v)$ *satisfies*

$$v|\mu'(v)|^2 \le \mu^3(v); \tag{1.13}$$

- *The initial data* $(v_0(x),u_0(x),\theta_0(x),z_0(x))$ *satisfies the compatibility conditions and*

$$(v_0(x),u_0(x),\theta_0(x),z_0(x)) \in H^1(\Omega), \tag{1.14}$$

$$v_0(x) > 0, \quad \theta_0(x) > 0, \quad 0 \le z_0(x) \le 1 \quad for \quad x \in [0,1]. \tag{1.15}$$

Then there exists a unique solution $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ *of the initial-boundary value problem (1.1), (1.5), (1.11) with (1.2), (1.3), (1.4), (1.12), such that for any* $T>0$

$$\begin{aligned}
(v(t,x),u(t,x),\theta(t,x),z(t,x)) &\in C^0\left(0,T;H^1(\Omega)\right),\\
(u_x(t,x),\theta_x(t,x)) &\in L^2\left(0,T;H^1(\Omega)\right),\\
\underline{V}_2 \le v(t,x) \le \overline{V}_2, &\quad \forall (t,x) \in [0,T]\times\Omega,\\
\underline{\Theta}_2 \le \theta(t,x) \le \overline{\Theta}_2, &\quad \forall (t,x) \in [0,T]\times\Omega,\\
0 \le z(t,x) \le 1, &\quad \forall (t,x) \in [0,T]\times\Omega.
\end{aligned}$$

Here T *is any given positive constant and* $\underline{V}_2,\overline{V}_2,\underline{\Theta}_2,\overline{\Theta}_2$ *are some positive constants which may depend on* T *and the initial data* (v_0,u_0,θ_0,z_0).

Remark 1.2. Some remarks concerning Theorem 1.2 are given below:

- The condition $b > \frac{44l_1l_2+54l_1+32l_2+16}{6l_1l_2+7l_1+4l_2+2}$ implies $b \ge 3$, which will be frequently used in Section 3.
- The Assumptions (1.12) we imposed on the viscosity coefficient $\mu(v)$ in Theorem 1.2 implies that it is non-degenerate and tends to infinity both for $v \to 0^+$ and for $v \to +\infty$. Such an assumption excludes the case when the viscosity coefficient is a positive constant. The problem to get a global solvability theory of the initial-boundary value problem (1.1), (1.5), (1.11) with (1.2), (1.3), (1.4) and more general, say for example degenerate, viscosity coefficient is under our current research.

Now we outline the main ideas to deduce our main results Theorem 1.1 and Theorem 1.2. As is usual for the wellposedness theories of nonlinear partial differential equations, the main difficulty in deducing the global solvability results to the above two types of initial-boundary value problems is to control the possible growth of their solutions induced by the nonlinearities of the equations (1.1) suitably and the key point is to obtain the positive lower and upper bounds on the specific volume $v(t,x)$ and the temperature $\theta(t,x)$.

To explain our main ideas, we first recall the arguments used in [35] to deal with the outer pressure problem (1.1), (1.5), (1.6) with (1.2), (1.3), (1.4), (1.7) for the case of $a=0$ and in [8] to treat the initial-boundary value problem (1.1), (1.5), (1.11) with (1.2), (1.3), (1.4) and constant viscosity. In both cases, since the viscosity coefficient μ is a positive constant, the following useful explicit representation formula for the specific volume $v(t,x)$

$$v(t,x) = \frac{1}{B(t,x)Y(t,x)D(t,x)}\left(v_0 + \frac{R}{\mu}\int_0^t \theta(\tau,x)B(\tau,x)Y(\tau,x)D(\tau,x)\,d\tau\right) \tag{1.16}$$

is derived which is motivated by an argument developed by Kazhikhov and Shelukhin to study the wellposedness problem of an one-dimensional viscous and heat conducting ideal polytropic gas motion (cf. [23]). Here

$$B(t,x) := \exp\left[\frac{1}{\mu}\int_0^x (u_0(\xi) - u(t,\xi))\,d\xi\right], \quad Y(t,x) := \exp\left(\frac{1}{\mu} f(x) t\right), \tag{1.17}$$

$$f(x) := p_e + \frac{1}{2} G x (1-x), \quad D(t,x) := \exp\left(-\frac{\alpha}{3\mu}\int_0^t \theta(\tau,x)^4\,d\tau\right). \tag{1.18}$$

In fact, based on the above explicit representation formula for $v(t,x)$ and the basic energy type estimates obtained in Lemma 2.2 and Lemma 3.2, Umehara and Tani [35] and Ducomet [8] can derive the desired positive lower and upper bounds on $v(t,x)$ first. With such an estimate on $v(t,x)$ in hand, they can derive the positive lower bound on $\theta(t,x)$ as in [2–4, 22], while the upper bound on $\theta(t,x)$ can obtained by employing the argument used in [22].

But for the outer pressure problem (1.1), (1.5, (1.6) with (1.2), (1.3), (1.4), (1.7) for the case of $a > 0$ considered in this paper, since the viscosity coefficient coefficient $\mu(v)$ is degenerate, one cannot hope to derive an explicit representation formula for $v(t,x)$ similar to (1.17) which holds only for the case of $a = 0$, one cannot hope to deduce the positive lower and upper bounds on $v(t,x)$. To overcome such a difficulty, our main idea is to estimate $v(t,x)$ and $\theta(t,x)$ simultaneously and the key points in our analysis can be outlined as in the following:

(i) Motivated by [22, 28], our first step is to deduce the lower bound of the specific volume $v(t,x)$ based on the identity (2.10) for the auxiliary function $g(v(t,x)) = \int_1^v \mu(z)/z\,dz$. It is worth to pointing out that the boundary condition (1.6) plays an essential role in deducing the identity (2.10);

(ii) The second step is to control the lower bound of the absolute temperature in terms of the upper bound on $v(t,x)$, cf. the estimate (2.34) obtained in Lemma 2.6;

(iii) With the results obtained in the above two steps, we can then deduce an estimate on $\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx$ in terms of the upper bound on $v(t,x)$ which is motivated by an observation of Kanel' for a viscous isentropic gas motion [21], cf. the estimate obtained in Lemma 2.7, from which one can derive an upper bound on $v(t,x)$ provided that the parameters a and b involved satisfy certain conditions stated in Theorem 1.1. Having obtained the upper bound on $v(t,x)$, we can then combine the estimate obtained in the second step to obtain the lower bound of $\theta(t,x)$;

(iv) Having obtained the above bounds, the only thing left is to get the desired upper bound on $\theta(t,x)$. The argument here to deduce such a bound is similar to those used in [22, 31].

For the initial-boundary value problem (1.1), (1.5), (1.11) with (1.2), (1.3), (1.4), (1.12), in addition to fact that one cannot hope to deduce the desired positive lower and upper bounds on $v(t,x)$ via deriving an explicit representation formula similar to the formula (1.17) for $v(t,x)$ as in [8] for the case of constant viscosity, the story is a little bitter different. In fact since the outer pressure boundary condition (1.6) imposed on the stress $\sigma = -p(v,\theta) + \frac{\mu(v)u_x}{v}$ is now replaced by the homogeneous Dirichlet boundary condition (1.11) imposed on the velocity $u(t,x)$, the identity (2.10) for the auxiliary function $g(v(t,x)) = \int_1^v \mu(z)/z\,dz$, which holds for the initial-boundary value problem

(1.1), (1.5), (1.6) with (1.2), (1.3), (1.4), (1.7), does not hold any longer. Thus one cannot hope to use such an argument to yield the desired positive lower bound on $v(t,x)$. To overcome such a difficulty, we adopt Kanel's argument (cf. [21]) to achieve our goal in Section 3. The strategy to prove Theorem 1.2 can be outlined as follows:

(i) Our first observation is that from the structure of the equations (1.1), (1.2), (1.3), (1.4), (1.12) under our consideration, we can derive the positive lower bound of $\theta(t,x)$ provided that the viscosity coefficient $\mu(v)$ is assumed to satisfy the condition (1.12) with the corresponding parameters l_1 and l_2 satisfying $l_1 \geq 1, l_2 \geq 0$. It is worth to pointing out that the assumption on the non-degeneracy of the viscosity coefficient $\mu(v)$ at both $v \to 0^+$ and $v \to +\infty$ plays an important role in our analysis;

(ii) Secondly, we apply Kanel's method, cf. [21], to yield an estimate on the positive lower bound and the upper bound of $v(t,x)$ in terms of $\|\theta^{8-b}\|_\infty$, cf. the estimates obtained in Lemma 3.6;

(iii) Finally, we use the trick in [33] to bound $\theta(t,x)$ as in Lemma 3.7. Then we can deduce the desired lower bound of $v(t,x)$ and upper bounds of $v(t,x)$ and $\theta(t,x)$ if the involved parameters l_1, l_2, b, β satisfy the given conditions given in Theorem 1.2.

Before concluding this section, we now review some related results briefly as follows:

- Firstly for the compressible viscous and heat-conducting model in one dimension space, the global existence and /or large time behavior of smooth solutions have been established by many authors:
  - For polytropic ideal gas with positive constant transport coefficients, see Antontsev, Kazhikhov and Monakhov [1], Kazhikhov and Shelukhin [23], Jiang [18–20], Li and Liang [26] and the references cited therein;
  - For polytropic ideal gas but with density and/or temperature dependent viscosity and density and temperature dependent heat conductivity, see Chen, Zhao and Zou [3], Liu, Yang, Zhao and Zou [27], Tan, Yang, Zhao and Zou [33], Wang and Zhao [37] and the references cited therein;
  - For polytropic ideal gas but with constant viscosity and density and/or temperature dependent heat conductivity, see Jenssen and Karper [17], Pan and Zhang [30] and the references cited therein;
  - For general gas with density-dependent viscosity and density and temperature dependent heat conductivity, see Dafemos and Hsiao [4], Kawhohl [22], Luo [28] and the references cited therein.
- Secondly, there are some recent results on heat-conducting radiative viscous gas, cf. [5, 11–16, 35, 36] and the references cited therein. Among them, Ducomet [10], Ducomet and Zlotnik [15, 16] studied a one-dimensional gaseous model similar to this paper. Among these papers [10, 15, 16], they adopted as self-gravitation, a special form independent of the time variable explicitly in the Lagrangian mass coordinate, not the exact form. For the outer pressure problem (1.1), (1.5), (1.6) with (1.2), (1.3), (1.4), (1.7), Umehara and Tani [35] proved the global solvability of smooth solutions when the coefficient of viscosity is a positive constant, i.e. the function (1.7) with $a=0$. Later on, they improved their results in [36]. Moreover, Qin [31] further improved their results. On the other hand, Ducomet [8] proved the global existence and exponential decay in H^1 of solutions to the initial-boundary value problem (1.1), (1.5), (1.11) with

(1.2), (1.3), (1.4) and constant viscosity coefficient.

The rest of the paper is organized as follows. The proofs of Theorem 1.1 and Theorem 1.2 will be given in Sections 2 and 3, respectively.

Notations: Throughout this paper, $C>1$ is used to denote a generic positive constant which may dependent on $\inf_{x\in\Omega} v_0(x)$, $\inf_{x\in\Omega}\theta_0(x)$, T and $\|(v_0,u_0,\theta_0,z_0)\|_{H^1(\Omega)}$. Here $T>0$ is some given constant. Note that these constants may vary from line to line. $C(\cdot,\cdot)$ stand for some generic constants depending only on the quantities listed in the parenthesis. ϵ stand for some small positive constants. For function spaces, $L^q(\Omega)(1\le q\le\infty)$ denotes the usual Lebesgue space on Ω with norm $\|\cdot\|_{L^q(\Omega)}$, while $H^q(\Omega)$ denotes the usual Sobolev space in the L^2 sense with norm $\|\cdot\|_{H^q(\Omega)}$. For simplicity, we use $\|\cdot\|_\infty$ to denote the norm in $L^\infty([0,T]\times\Omega)$. For two functions $f(x)$ and $g(x)$, $f(x)\sim g(x)$ as $x\to a$ means that there exists a positive constant $C>0$ such that $C^{-1}f(x)\le g(x)\le Cf(x)$ in the neighborhood of a.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1 based on the continuation argument. Such an argument is a combination of the local existence result with certain energy estimates on the local solutions constructed.

Firstly, under the assumptions given in Theorem 1.1, we can get the following local existence result.

Lemma 2.1 (Local existence). *Under the assumptions listed in Theorem 1.1 or Theorem 1.2, there exists a sufficiently small positive constant t_1, which depends on $\|(v_0,u_0,\theta_0,z_0)\|_{H^1(\Omega)}$, $\inf_{x\in\Omega} v_0(x)$ and $\inf_{x\in\Omega}\theta_0(x)$, such that the initial-boundary value problem (1.1), (1.5), (1.6) (or (1.1), (1.5), (1.11)) admits a unique smooth solution $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ defined on $[0,t_1]\times\Omega$.*

Moreover, $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ satisfies

$$\begin{aligned}
(v(t,x),u(t,x),\theta(t,x),z(t,x)) &\in C^0\left(0,T;H^1(\Omega)\right),\\
(u_x(t,x),\theta_x(t,x)) &\in L^2\left(0,T;H^1(\Omega)\right),\\
\frac{1}{2}\inf_{x\in\Omega} v_0(x)\le v(t,x) &\le 2\sup_{x\in\Omega} v_0(x),\forall(t,x)\in[0,t_1]\times\Omega,\\
\frac{1}{2}\inf_{x\in\Omega}\theta_0(x)\le\theta(t,x) &\le 2\sup_{x\in\Omega}\theta_0(x),\forall(t,x)\in[0,t_1]\times\Omega,\\
0\le z(t,x)\le 1,&\forall(t,x)\in[0,t_1]\times\Omega
\end{aligned}$$

and

$$\sup_{x\in\Omega}\left\{\|(v,u,\theta,z)(t)\|_{H^1(\Omega)}\right\}\le 2\|(v_0,u_0,\theta_0,z_0)\|_{H^1(\Omega)}.$$

Lemma 2.1 can be obtained by using a similar approach as in [23] for the one-dimensional case or [34] for the three dimensional case. Hence we omit the details for brevity.

Suppose that the local solution $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ constructed in Lemma 2.1 has been extended to $t=T\ge t_1$ and satisfies the a priori assumption

$$(H_1)\qquad \underline{V}_1'\le v(t,x)\le\overline{V}_1',\ \underline{\Theta}_1'\le\theta(t,x)\le\overline{\Theta}_1',\ \forall(t,x)\in[0,T]\times\Omega.$$

Here $\underline{V}_1',\overline{V}_1',\underline{\Theta}_1'$ and $\overline{\Theta}_1'$ are some positive constants. To extend such a solution step by step to a global one, we only need to deduce certain a priori estimates of

$(v(t,x),u(t,x),\theta(t,x),z(t,x))$, which are independent of $\underline{V}_1', \overline{V}_1', \underline{\Theta}_1'$, and $\overline{\Theta}_1'$, but may depend on the initial data $\big(v_0(x),u_0(x),\theta_0(x),z_0(x)\big)$ and the constant T.

We now deduce certain a priori estimates on $(v(t,x),u(t,x),\theta(t,x),z(t,x))$. The first one is concerned with the basic energy estimate whose proof can be found in [35] which is based on the following identity

$$\partial_t\left(\frac{u^2}{2}\right)+\Big(up\Big)_x-u_xp=\left(\frac{\mu(v)uu_x}{v}\right)_x-\frac{\mu(v)u_x^2}{v}-uG\left(x-\frac{1}{2}\right) \tag{2.1}$$

and its consequence

$$\frac{d}{dt}\int_0^1\left\{\frac{1}{2}u^2+\left[p_e+\frac{1}{2}G(1-x)x\right]v\right\}dx+\int_0^1\frac{\mu(v)u_x^2}{v}dx=\int_0^1 pu_x dx. \tag{2.2}$$

Lemma 2.2 (Basic energy estimates). *Under the assumptions given in Theorem 1.1, for any $t\in[0,T]$, we have*

$$\int_0^1\left\{\frac{1}{2}u^2+C_v\theta+\alpha v\theta^4+\lambda z+\left[p_e+\frac{1}{2}G(1-x)x\right]v\right\}dx\le C, \tag{2.3}$$

$$\begin{aligned}&\int_0^1\left\{C_v(\theta-1-\log\theta)+R(v-1-\log v)\right\}dx\\&\quad+\int_0^t\int_0^1\left(\frac{\mu(v)u_x^2}{v\theta}+\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}+\frac{\lambda\phi z}{\theta}\right)dxd\tau\le C,\end{aligned} \tag{2.4}$$

$$\int_0^1 zdx+\int_0^t\int_0^1\phi zdxd\tau=\int_0^1 z_0dx, \tag{2.5}$$

$$\int_0^1\frac{1}{2}z^2dx+\int_0^t\int_0^1\left(\frac{d}{v^2}z_x^2+\phi z^2\right)dxd\tau=\int_0^1\frac{1}{2}z_0^2dx. \tag{2.6}$$

The next lemma is concerned with the estimate of $z(t,x)$. To this end, we can deduce by repeating the method used in [2] that

Lemma 2.3. *Under the assumptions stated in Theorem 1.1, for any $(t,x)\in[0,T]\times\Omega$, we have*

$$0\le z(t,x)\le 1. \tag{2.7}$$

To derive bounds on the specific volume $v(x,t)$, if we define

$$g(v):=\int_1^v\frac{\mu(\xi)}{\xi}d\xi, \tag{2.8}$$

then we can get that

$$u_t+p_x+G\left(x-\frac{1}{2}\right)=\left(\mu(v)\frac{u_x}{v}\right)_x=\left(\mu(v)\frac{v_t}{v}\right)_x=[g(v)]_{xt}. \tag{2.9}$$

Integrating the equation (2.9) over $[0,t]\times[0,x]$ and using the boundary condition (1.6), one has

$$-g(v(t,x))+\int_0^t p(\tau,x)d\tau=\int_0^x(u_0-u)dx-g\big(v_0(x)\big)$$

$$+\int_0^t p_e d\tau - \int_0^t \int_0^x G\left(x-\frac{1}{2}\right) dx d\tau, \tag{2.10}$$

which implies

$$-g(v(t,x)) \leq C.$$

That is

$$g(v(t,x)) \geq -C. \tag{2.11}$$

On the other hand, it is easy to see

$$g(v) = \begin{cases} \frac{1-v^{-a}}{a}, & a>0, \\ \ln v, & a=0, \end{cases} \tag{2.12}$$

which together with the estimate (2.11) implies that

Lemma 2.4. *Under the assumptions stated in Theorem 1.1, there exists a positive constant* $\underline{V}_1$ *depending only on* T *and the initial data* $(v_0(x), u_0(x), \theta_0(x), z_0(x))$ *such that*

$$v(t,x) \geq \underline{V}_1, \quad \forall (t,x) \in [0,T] \times \Omega. \tag{2.13}$$

With Lemma 2.2 and Lemma 2.4 in hand, we immediately get

$$\int_0^1 \theta^r d\tau \leq C, \qquad 0 \leq r \leq 4. \tag{2.14}$$

For each $t \in [0,T]$, there exists $x^*(t) \in [0,1]$ such that

$$\theta(t, x^*(t)) = \int_0^1 \theta(t,x) dx, \tag{2.15}$$

and therefore, for any $r \geq 0$ and $(t,x) \in [0,T] \times \Omega$, we have

$$\begin{aligned} \theta(t,x)^{\frac{r}{2}} &= \left(\int_0^1 \theta(t,x) dx\right)^{\frac{r}{2}} + \frac{r}{2} \int_{x^*(t)}^x \theta(t,\xi)^{\frac{r}{2}-1} \theta_\xi(t,\xi) d\xi \\ &\leq C\left(1 + \int_0^1 \frac{\kappa^{\frac{1}{2}} |\theta_x|}{\theta v^{\frac{1}{2}}} \frac{v^{\frac{1}{2}} \theta^{\frac{r}{2}}}{\kappa^{\frac{1}{2}}} dx\right) \\ &\leq C\left(1 + \left(\int_0^1 \frac{v\theta^r}{1+v\theta^b} dx\right)^{\frac{1}{2}} \left(\int_0^1 \left(\frac{\mu(v) u_x^2}{v\theta} + \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} + \frac{\lambda\phi z}{\theta}\right) dx\right)^{\frac{1}{2}}\right). \end{aligned} \tag{2.16}$$

It is easy to see that

$$\theta^r \leq C\left(1+\theta^{b+4}\right) \tag{2.17}$$

holds for $0 \leq r \leq b+4$. We have from the bound (2.14) that

$$\int_0^1 \frac{v\theta^r}{1+v\theta^b} dx \leq C \int_0^1 \left(v+\theta^4\right) dx \leq C. \tag{2.18}$$

Combining the estimates (2.4), (2.16) and (2.18), one has

LEMMA 2.5. *Under the assumptions listed in Theorem 1.1, for any $t\in[0,T]$ and $b\geq 0$*

$$\int_0^t \max_{x\in\Omega}\{\theta^r(\tau,x)\}\,d\tau \leq C, \qquad 0\leq r\leq b+4. \tag{2.19}$$

Now we turn to estimate the term $\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}dxd\tau$, which will be used later. Using the identity (2.2) and Cauchy's inequality, one has

$$\begin{aligned}&\frac{d}{dt}\int_0^1\left\{\frac{1}{2}u^2+\left[p_e+\frac{1}{2}G(1-x)x\right]v\right\}dx+\int_0^1\frac{\mu(v)u_x^2}{v}dx\\&\leq \epsilon\int_0^1\frac{\mu(v)u_x^2}{v}dx+C(\epsilon)\int_0^1\frac{\theta^2}{v\mu(v)}dx+C(\epsilon)\int_0^1\frac{v\theta^8}{\mu(v)}dx.\end{aligned} \tag{2.20}$$

Integrating the inequality (2.20) with respect to t over$(0,t)$, we obtain

$$\begin{aligned}&\int_0^1\left\{\frac{1}{2}u^2+\left[p_e+\frac{1}{2}G(1-x)x\right]v\right\}dx+\int_0^t\int_0^1\frac{\mu(v)u_x^2}{v}dxd\tau\\&\leq C+C\int_0^t\int_0^1\frac{\theta^2}{v^{1-a}}dxd\tau+\int_0^t\int_0^1 v^{1+a}\theta^8dxd\tau,\end{aligned} \tag{2.21}$$

while Lemma 2.2 together with Lemma 2.5 tells us that

$$\int_0^t\int_0^1\frac{\theta^2}{v^{1-a}}dxd\tau\leq C\int_0^t\int_0^1\theta^2dxd\tau\leq C, \tag{2.22}$$

$$\begin{aligned}\int_0^t\int_0^1 v^{1+a}\theta^8dxd\tau&\leq\|v\|_\infty^a\int_0^t\int_0^1 v\theta^8dxd\tau\leq C\|v\|_\infty^a\\&\leq\|v\|_\infty^a\int_0^t\left(\max_{x\in[0,1]}\left\{\theta(\tau,x)^4\right\}\int_0^1 v(\tau,x)\theta^4(\tau,x)dx\right)d\tau\\&\leq C\|v\|_\infty^a.\end{aligned} \tag{2.23}$$

Combining the inequalities (2.21)-(2.23), we get

$$\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxd\tau\leq C+C\|v\|_\infty^a. \tag{2.24}$$

Now we are concerned with the lower-bound estimate on $\theta(t,x)$. For this purpose, setting $h=\frac{1}{\theta}$, we can deduce from the equation $(1.1)_3$ that

$$e_\theta h_t=\left(\frac{\kappa(\theta)h_x}{v}\right)_x+\frac{vp_\theta^2}{4\mu}-\left[\frac{2\kappa(v,\theta)h_x^2}{vh}+\frac{\mu(v)h^2}{v}\left(u_x-\frac{vp_\theta}{2\mu h}\right)^2+\lambda h^2\phi z\right], \tag{2.25}$$

where

$$e_\theta=C_v+4\alpha v\theta^3,\quad p_\theta=\frac{R}{v}+\frac{4}{3}\alpha\theta^3. \tag{2.26}$$

It is easy to see that

$$e_\theta h_t\leq\left(\frac{\kappa(v,\theta)h_x}{v}\right)_x+\frac{vp_\theta^2}{4\mu}. \tag{2.27}$$

Using the identity (2.26) and Lemma 2.4, one has

$$h_t \le \frac{1}{e_\theta}\left(\frac{\kappa(v,\theta)h_x}{v}\right)_x + C\left(1+v^a\theta^3\right). \tag{2.28}$$

Multiplying the inequality (2.28) by h^{2p-1} and integrating the result with respect to x over Ω, one has

$$\|h\|_{L^{2p}}^{2p-1}\left(\|h\|_{L^{2p}}\right)_t \le C\int_0^1 \left(1+v^a\theta^3\right)h^{2p-1}dx + C\int_0^1 \frac{h^{2p-1}}{e_\theta}\left(\frac{\kappa(v,\theta)h_x}{v}\right)_x dx. \tag{2.29}$$

Noticing that

$$\begin{aligned}\frac{h^{2p-1}}{e_\theta}\left(\frac{\kappa(v,\theta)h_x}{v}\right)_x &= \left(\frac{\kappa(v,\theta)h^{2p-1}h_x}{ve_\theta}\right)_x - \frac{(2p-1)h^{2p-2}h_x^2}{ve_\theta}\\ &\quad + \frac{\kappa(v,\theta)h^{2p-1}h_x(e_\theta)_x}{ve_\theta^2}\end{aligned} \tag{2.30}$$

and

$$\frac{\kappa(v,\theta)h^{2p-1}h_x(e_\theta)_x}{e_\theta^2} \le (2p-1)\frac{h^{2p-2}h_x^2}{ve_\theta} + \frac{1}{2p-1}\frac{\kappa^2(v,\theta)h^{2p}\left((e_\theta)_x\right)^2}{ve_\theta^3}, \tag{2.31}$$

we obtain

$$\|h\|_{L^{2p}}^{2p-1}\left(\|h\|_{L^{2p}}\right)_t \le C\int_0^1 \left(1+v^a\theta^3\right)h^{2p-1}dx + \frac{1}{2p-1}\int_0^1 \frac{\kappa^2(v,\theta)h^{2p}\left((e_\theta)_x\right)^2}{ve_\theta^3}dx. \tag{2.32}$$

Integrating the inequality (2.32) with respect to t over $(0,t)$, one can get, by using Hölder's inequality and by letting $p \to +\infty$, that

$$\|h\|_{L^\infty(\Omega)} \le C + C\int_0^t \max_{x\in\Omega}\left\{v^a(\tau,x)\theta^3(\tau,x)\right\}d\tau \le C + C\|v\|_\infty^a. \tag{2.33}$$

Hence, we have

Lemma 2.6. *Under the assumption in Theorem 1.1, we have*

$$\frac{1}{\theta(t,x)} \le C + C\|v\|_\infty^a, \quad \forall (t,x)\in[0,T]\times\Omega. \tag{2.34}$$

Now we turn to estimate the upper bound of $v(t,x)$. To do so, since

$$\begin{aligned}v(t,x) &\le \int_0^1 v(t,x)\,dx + \int_0^1 |v_x(t,x)|\,dx\\ &\le C + C\|v\|_\infty^{a+\frac12}\left(\int_0^1 v\,dx\right)^{\frac12}\left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx\right)^{\frac12}\\ &\le C + C\|v\|_\infty^{a+\frac12}\left(\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx\right)^{\frac12},\end{aligned} \tag{2.35}$$

we need to estimate the term $\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx$ first. For this purpose, noticing that

$$p_x = \left(\frac{R\theta}{v} + \frac{\alpha\theta^4}{3}\right)_x = \frac{R\theta_x}{v} - \frac{R\theta v_x}{v^2} + \frac{4}{3}\alpha\theta^3\theta_x \tag{2.36}$$

and

$$\begin{aligned}\frac{u_t v_x}{v^{1+a}} &= \left(\frac{u v_x}{v^{1+a}}\right)_t - u\left(\frac{v_x}{v^{1+a}}\right)_t \\ &= \left(\frac{u v_x}{v^{1+a}}\right)_t - u\left(\frac{u_x}{v^{1+a}}\right)_x,\end{aligned} \tag{2.37}$$

we can get by multiplying the equation $(1.1)_2$ by $\frac{\mu(v)v_x}{v}$ and by using the above identities that

$$\left(\frac{\mu^2(v)v_x^2}{2v^2}\right)_t = \left(\frac{u v_x}{v^{1+a}}\right)_t + \frac{v_x}{v^{1+a}}p_x + G\left(x-\frac{1}{2}\right)\frac{v_x}{v^{1+a}} + \frac{u_x^2}{v^{1+a}} - \Big(u\sigma\Big)_x - \Big(up\Big)_x. \tag{2.38}$$

Integrating the identity (2.38) with respect to t and x over $(0,t)\times\Omega$, one has

$$\begin{aligned}\frac{1}{2}\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx \le C &+ \underbrace{\int_0^1 \frac{u v_x}{v^{1+a}}dx}_{I_1} + \underbrace{\int_0^t\int_0^1 \frac{u_x^2}{v^{1+a}}dxd\tau}_{I_2} - \underbrace{\int_0^t\int_0^1 (u\sigma)_x dxd\tau}_{I_3} \\ &+ \underbrace{\int_0^t\int_0^1 \frac{v_x}{v^{1+a}}p_x dxd\tau}_{I_4} - \underbrace{\int_0^t\int_0^1 (up)_x dxd\tau}_{I_5} \\ &+ \int_0^t\int_0^1 G\left(x-\frac{1}{2}\right)\frac{v_x}{v^{1+a}}dxd\tau.\end{aligned} \tag{2.39}$$

Now we deal with $I_j (j=1,2,3,4,5)$ term by term. Since the estimate (2.24) gives the desired estimate on I_2, we only need to control the other four terms. In fact for I_1, we can get by using Cauchy's inequality and the bound (2.3) that

$$\begin{aligned}I_1 &\le C(\epsilon)\int_0^1 u^2 dx + \epsilon\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx \\ &\le C(\epsilon) + \epsilon\int_0^1 \frac{v_x^2}{v^{2(1+a)}}dx\end{aligned} \tag{2.40}$$

holds for any $\epsilon > 0$.

As to I_3, one can conclude from the boundary condition (1.6) that

$$\begin{aligned}I_3 &= \int_0^t \Big\{u(1,t)(-p_e) - u(0,t)(-p_e)\Big\}d\tau \\ &= -p_e\int_0^t\int_0^1 u_x dxd\tau \\ &= -p_e\int_0^t\int_0^1 v_t dxd\tau \\ &= -p_e\int_0^1 (v-v_0)dx \le C.\end{aligned} \tag{2.41}$$

Now for I_4, by using the identity (2.36) and Cauchy's inequality, we can get for each $\epsilon > 0$ that

$$I_4 \le C\int_0^t\int_0^1 \left(\frac{|\theta_x||v_x|}{v^{2+a}} + \frac{\theta^3|v_x||\theta_x|}{v^{1+a}}\right)dxd\tau - R\int_0^t\int_0^1 \frac{\theta v_x^2}{v^{3+a}}dxd\tau$$

$$\leq C\int_0^t\int_0^1\left(\frac{\theta_x^2}{v^{1+a}\theta}+\frac{\theta^3|v_x||\theta_x|}{v^{1+a}}\right)dxd\tau-(R-\epsilon)\int_0^t\int_0^1\frac{\theta v_x^2}{v^{3+a}}dxd\tau. \tag{2.42}$$

Finally for I_7, due to

$$I_5=\underbrace{\int_0^t\int_0^1 up_x dxd\tau}_{K_1}+\underbrace{\int_0^t\int_0^1 u_x p dxd\tau}_{K_2},$$

we can get from the equations (1.2), Lemma 2.4, the estimates (2.24) and (2.19), and Cauchy's inequality that

$$\begin{aligned}K_1&\leq C\int_0^t\int_0^1\left(\frac{|\theta_x||u|}{v}+\frac{\theta|u||v_x|}{v^2}+|u|\theta^3|\theta_x|\right)dxd\tau\\&\leq\epsilon\int_0^t\int_0^1\frac{\theta v_x^2}{v^{3+a}}dxd\tau+C\int_0^t\int_0^1\left(\frac{\theta u^2}{v^{1-a}}+\frac{\theta_x^2}{\theta v^{1+a}}+|u|\theta^3|\theta_x|\right)dxd\tau\end{aligned}$$

and

$$\begin{aligned}K_2&\leq C\int_0^t\int_0^1\left(\frac{\theta|u_x|}{v}+\theta^4|u_x|\right)dxd\tau\\&\leq C\int_0^t\int_0^1\frac{\theta^2}{v^{1-a}}dxd\tau+C\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxd\tau+C\int_0^t\int_0^1\theta^4|u_x|dxd\tau\\&\leq C+C\int_0^t\int_0^1\theta^4|u_x|dxd\tau+C\|v\|_\infty^a.\end{aligned}$$

Consequently

$$\begin{aligned}I_5\leq{}&C(1+\|v\|_\infty^a)+\epsilon\int_0^t\int_0^1\frac{\theta v_x^2}{v^{3+a}}dxd\tau\\&+C\int_0^t\int_0^1\left(\frac{\theta u^2}{v^{1-a}}+\frac{\theta_x^2}{\theta v^{1+a}}+|u|\theta^3|\theta_x|+\theta^4|u_x|\right)dxd\tau.\end{aligned} \tag{2.43}$$

Combining the estimates (2.39)-(2.43) and by choosing ϵ small enough, we have

$$\begin{aligned}&\int_0^1\frac{v_x^2}{v^{2(1+a)}}dx+\int_0^t\int_0^1\frac{\theta v_x^2}{v^{3+a}}dxd\tau\\&\leq C+C\int_0^t\int_0^1\left(\frac{\theta_x^2}{v^{1+a}\theta}+\frac{\theta^3|v_x||\theta_x|}{v^{1+a}}+|u|\theta^3|\theta_x|+\theta^4|u_x|\right)dxd\tau\\&\quad+C\int_0^t\int_0^1 G\left|x-\frac{1}{2}\right|\frac{|v_x|}{v^{1+a}}dxd\tau+C\|v\|_\infty^a.\end{aligned} \tag{2.44}$$

To bound the terms on the right hand side of the estimate (2.44), noticing that the estimate (2.4) tells us that

$$\int_0^t\int_0^1\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dxd\tau\leq C, \tag{2.45}$$

we thus get from the assumption $b\geq 8$ that

$$\int_0^t\int_0^1\frac{\theta_x^2}{v^{1+a}\theta}dxd\tau=\int_0^t\int_0^1\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\cdot\frac{\theta}{\kappa(v,\theta)v^a}dxd\tau$$

$$\leq C\left\|\frac{1}{\theta}\right\|_{\infty}^{b-1}, \tag{2.46}$$

$$\begin{aligned}\int_0^t\int_0^1\frac{\theta^3|v_x||\theta_x|}{v^{1+a}}dxd\tau &\leq C\int_0^t\int_0^1\frac{v_x^2}{v^{2(1+a)}}dxd\tau+C\int_0^t\int_0^1\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}\cdot\frac{v\theta^8}{\kappa(v,\theta)}dxd\tau\\ &\leq C\int_0^t\int_0^1\frac{v_x^2}{v^{2(1+a)}}dxd\tau+C\left\|\frac{1}{\theta}\right\|_{\infty}^{b-8}, \end{aligned}\tag{2.47}$$

$$\begin{aligned}\int_0^t\int_0^1|u|\theta^3|\theta_x|dxd\tau &\leq C\int_0^t\int_0^1\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}dxd\tau+C\int_0^t\int_0^1\frac{u^2\theta^8 v}{\kappa(v,\theta)}dxd\tau\\ &\leq C+C\int_0^t\int_0^1\frac{u^2}{\theta^{b-8}}dxd\tau\\ &\leq C+C\left\|\frac{1}{\theta}\right\|_{\infty}^{b-8}, \end{aligned}\tag{2.48}$$

and

$$\begin{aligned}\int_0^t\int_0^1\theta^4|u_x|dxdt &\leq C\int_0^t\int_0^1\frac{u_x^2}{v^{1+a}}dxd\tau+C\int_0^t\int_0^1\theta^8v^{1+a}dxd\tau\\ &\leq C+C\|v\|_\infty^a. \end{aligned}\tag{2.49}$$

Combining the estimates (2.44)-(2.49) and using Lemma 2.6, we arrive at

$$\begin{aligned}\int_0^1\frac{v_x^2}{v^{2(1+a)}}dx+\int_0^t\int_0^1\frac{\theta v_x^2}{v^{3+a}}dxd\tau &\leq C+C\|v\|_\infty^a+C\left\|\frac{1}{\theta}\right\|_{\infty}^{b-1}+C\left\|\frac{1}{\theta}\right\|_{\infty}^{b-8}\\ &\leq C+C\|v\|_\infty^a+C\|v\|_\infty^{a(b-1)}+C\|v\|_\infty^{a(b-8)}\\ &\leq C+C\|v\|_\infty^{a(b-1)}. \end{aligned}\tag{2.50}$$

Hence, we have the following lemma:

Lemma 2.7. *Under the assumptions listed in Theorem 1.1, for any $t\in[0,T]$, we have*

$$\int_0^1\frac{v_x^2}{v^{2(1+a)}}dx\leq C+C\|v\|_\infty^{a(b-1)}. \tag{2.51}$$

Combining the estimates (2.3), (2.35) and (2.51), we can get that

$$\begin{aligned}v(t,x) &\leq C+C\|v\|_\infty^{a+\frac{1}{2}}\left(1+\|v\|_\infty^{a(b-1)}\right)^{\frac{1}{2}}\\ &\leq C+C\|v\|_\infty^{a+\frac{1}{2}+\frac{a(b-1)}{2}}. \end{aligned}\tag{2.52}$$

Having obtained the estimate (2.52), if the parameters a and b are suitably chosen such that $0\leq a<\frac{1}{b+1}$ and by employing the estimate (2.34), we can obtain the following result

Lemma 2.8. *Under the assumptions given in Theorem 1.1, there exist positive constants $\overline{V}_1$ and $\underline{\Theta}_1$ depending only on T and the initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$ such that*

$$v(t,x)\leq\overline{V}_1,\quad \theta(t,x)\geq\underline{\Theta}_1\quad \forall(t,x)\in[0,T]\times\Omega. \tag{2.53}$$

Since up to now we have obtained the desired positive lower bound and upper bound of $v(t,x)$, the estimates (2.24) and (2.50) can be rewritten as

$$\int_0^t\int_0^1 u_x^2 dxd\tau \le C \tag{2.54}$$

and

$$\int_0^1 v_x^2 dx + \int_0^t\int_0^1 \theta v_x^2 dxd\tau \le C. \tag{2.55}$$

To continue our analysis, we need to estimate $\int_0^t\int_0^1 u_x^4 dxd\tau$, which will be used later. To this end, by employing the argument used in [4, 22, 25], we can get the following lemma.

LEMMA 2.9. *Under the assumptions stated in Theorem 1.1, for any $t\in[0,T]$, we have*

$$\int_0^t\int_0^1 u_x^4 dxd\tau \le C. \tag{2.56}$$

Proof. Setting

$$U(t,x) = \int_0^x u(t,y)dy, \tag{2.57}$$

under the boundary conditions (1.6), we can get by integrating the equation $(1.1)_2$ with respect to x over $(0,x)$ that

$$\begin{aligned} U_t - \frac{\mu(v)}{v}U_{xx} &= p_e - G\left(\frac{1}{2}x^2 - \frac{1}{2}x\right) - p(x,t),\\ U(0,x) &= \int_0^x u_0(y)dy,\\ U(t,0) &= 0,\\ U(t,1) &= \int_0^1 u(t,x)dx \le C. \end{aligned} \tag{2.58}$$

Hence the standard L^p-estimates for solutions to the linear problem (2.58), cf. [25], yields

$$\begin{aligned} \int_0^t\int_0^1 u_x^4 dxd\tau &= \int_0^t\int_0^1 U_{xx}^4 dxd\tau\\ &\le C + C\int_0^t\int_0^1 \left(p_e - G\left(\frac{1}{2}x^2 - \frac{1}{2}x\right) - p(x,t)\right)^4 dxd\tau\\ &\le C + C\int_0^t\int_0^1 \left(\theta^4 + \theta^{16}\right) dxd\tau\\ &\le C + C\int_0^t \max_{x\in\Omega}\left\{\theta^{12}(\tau,x)\right\} d\tau. \end{aligned} \tag{2.59}$$

Since $b \ge 8$, Lemma 2.5 implies that

$$\int_0^t \max_{x\in\Omega}\left\{\theta^{12}(\tau,x)\right\} d\tau \le C. \tag{2.60}$$

Thus the proof of Lemma 2.9 is complete. ∎

Finally, we derive the upper bound on $\theta(t,x)$. For this purpose, we set

$$\begin{aligned} X &:= \int_0^t \int_0^1 \left(1+\theta^{b+3}\right)\theta_t^2 dx d\tau, \\ Y &:= \max_t \int_0^1 \left(1+\theta^{2b}\right)\theta_x^2 dx, \\ Z &:= \max_t \int_0^1 u_{xx}^2 dx. \end{aligned} \tag{2.61}$$

Observe first that

$$\begin{aligned} \theta^{2b+6} &\le C + C\int_0^1 \theta^{2b+5}|\theta_x| dx \\ &\le C + C\|\theta\|_{L^\infty(\Omega)}^{b+3}\left(\int_0^1 \theta^4 dx\right)^{\frac{1}{2}}\left(\int_0^1 \theta^{2b}\theta_x^2 dx\right)^{\frac{1}{2}} \\ &\le C + C\|\theta\|_{L^\infty(\Omega)}^{b+3} Y^{\frac{1}{2}}, \end{aligned} \tag{2.62}$$

which implies

$$\|\theta\|_{L^\infty(\Omega)} \le C + CY^{\frac{1}{2b+6}}. \tag{2.63}$$

Secondly, noticing that

$$\int_0^1 u_x^2 dx \le C\int_0^1 u^2 dx + C\left(\int_0^1 u^2 dx\right)^{\frac{1}{2}}\left(\int_0^1 u_{xx}^2 dx\right)^{\frac{1}{2}}, \tag{2.64}$$

from which and the estimate (2.3), we have

$$\max_t \int_0^1 u_x^2 dx \le C + CZ^{\frac{1}{2}}. \tag{2.65}$$

Moreover, by using the inequality

$$u_x^2(t,x) \le \int_0^1 u_x^2(t,x) dx + 2\int_0^1 |u_x(t,x)||u_{xx}(t,x)| dx, \tag{2.66}$$

one has

$$\|u_x\|_{L^\infty(\Omega)} \le C + CZ^{\frac{3}{8}}. \tag{2.67}$$

With the above preparations in hand, our next result is to show that X and Y can be controlled by Z.

Lemma 2.10. *Under the assumptions listed in Theorem 1.1, we have*

$$X + Y \le C + CZ^{\frac{7}{8}}. \tag{2.68}$$

Proof. In the same manner as in [35] and [22], if we set

$$K(v,\theta) = \int_0^\theta \frac{\kappa(v,\xi)}{v} d\xi, \tag{2.69}$$

then it is easy to verify that

$$K_t = K_v u_x + \frac{\kappa(v,\theta)\theta_t}{v}, \tag{2.70}$$

$$K_{xt} = \left[\frac{\kappa\theta_x}{v}\right]_t + K_v u_{xx} + K_{vv} v_x u_x + \left(\frac{\kappa}{v}\right)_v v_x \theta_t, \tag{2.71}$$

$$|K_v(v,\theta)| + |K_{vv}(v,\theta)| \le C\theta. \tag{2.72}$$

Multiplying the equation $(1.1)_3$ by K_t and integrating the resulting identity over $(0,t)\times(0,1)$, we arrive at

$$\begin{aligned}&\int_0^t\int_0^1 \left(e_\theta\theta_t + \theta p_\theta u_x - \frac{\mu(v)u_x^2}{v}\right) K_t dx d\tau + \int_0^t\int_0^1 \frac{\kappa(v,\theta)}{v}\theta_x K_{tx} dx d\tau \\ &= \int_0^t\int_0^1 \lambda\phi z K_t dx d\tau.\end{aligned} \tag{2.73}$$

Combining the results (2.70)-(2.73), we have

$$\int_0^t\int_0^1 \frac{e_\theta\kappa(v,\theta)\theta_t^2}{v} dx d\tau + \int_0^t\int_0^1 \frac{\kappa(v,\theta)\theta_x}{v}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_t dx d\tau \le C + \sum_{k=6}^{15} I_k, \tag{2.74}$$

where the definition of $I_k(6\le k\le 15)$ will be given below.

We now turn to control $I_k(k=6,7,\cdots,15)$ term by term. To do so, we have first that

$$\begin{aligned}\int_0^t\int_0^1 \frac{e_\theta\kappa(v,\theta)\theta_t^2}{v} dx d\tau &\ge C\int_0^t\int_0^1 \left(1+\theta^3\right)\left(1+\theta^b\right)\theta_t^2 dx d\tau \\ &\ge CX\end{aligned} \tag{2.75}$$

and

$$\begin{aligned}&\int_0^t\int_0^1 \frac{\kappa(v,\theta)\theta_x}{v}\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_t dx d\tau \\ &= \frac{1}{2}\int_0^1 \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)^2 dx - \frac{1}{2}\int_0^1 \left(\frac{\kappa(v,\theta)\theta_x}{v}\right)^2 (0,x) dx \\ &\ge CY - C.\end{aligned} \tag{2.76}$$

With the above two estimates in hand, $I_k(k=6,7,\cdots,14)$ can be estimated term by term by employing Cauchy's inequality, Hölder's inequality and Young's inequality as follows

$$\begin{aligned}|I_6| &= \left|\int_0^t\int_0^1 e_\theta\theta_t K_v u_x dx d\tau\right| \\ &\le C\int_0^t\int_0^1 (1+\theta)^4 |\theta_t u_x| dx d\tau \\ &\le \epsilon X + C(\epsilon)\left(1+Z^{\frac{3}{4}}\right),\end{aligned} \tag{2.77}$$

$$|I_7| = \left|\int_0^t\int_0^1 \theta p_\theta u_x K_v u_x dx d\tau\right|$$

$$
\begin{aligned}
&\leq C\|1+\theta\|_\infty^5 \int_0^t \|u_x\|^2 d\tau \\
&\leq \epsilon Y + C(\epsilon),
\end{aligned}
\tag{2.78}
$$

where we have used the estimate (2.54),

$$
\begin{aligned}
|I_8| &= \left| \int_0^t \int_0^1 \frac{\mu(v) u_x^2}{v} K_v u_x dx d\tau \right| \\
&\leq C \int_0^t \int_0^1 \theta u_x^2 |u_x| dx d\tau \\
&\leq C \|u_x\|_\infty \max_{t\in[0,T]} \|u_x\|^2 \int_0^t \max_{x\in\Omega} \theta d\tau \\
&\leq C\left(1+Z^{\frac{7}{8}}\right),
\end{aligned}
\tag{2.79}
$$

$$
\begin{aligned}
|I_9| &= \left| \int_0^t \int_0^1 \frac{\theta p_\theta \kappa(v,\theta) u_x \theta_t}{v} dx d\tau \right| \\
&\leq C \int_0^t \int_0^1 \left(1+\theta^3\right) \theta \left(1+\theta^b\right) |u_x \theta_t| dx d\tau \\
&\leq \epsilon X + C(\epsilon) \|1+\theta\|_\infty^{b+5} \int_0^t \|u_x\|^2 d\tau \\
&\leq \epsilon (X+Y) + C(\epsilon),
\end{aligned}
\tag{2.80}
$$

$$
\begin{aligned}
|I_{10}| &= \left| \int_0^t \int_0^1 \frac{\mu(v) u_x^2 \kappa(v,\theta) \theta_t}{v^2} dx d\tau \right| \\
&\leq C \int_0^t \int_0^1 \left(1+\theta^b\right) |\theta_t| u_x^2 dx d\tau \\
&\leq C(\epsilon) + \epsilon X + C(\epsilon) \int_0^t \int_0^1 \theta^{b-3} u_x^4 dx d\tau \\
&\leq C(\epsilon) + \epsilon (X+Y),
\end{aligned}
\tag{2.81}
$$

where we have used Lemma 2.9,

$$
\begin{aligned}
|I_{11}| &= \left| \int_0^t \int_0^1 \frac{\kappa(v,\theta) \theta_x K_v u_{xx}}{v} dx d\tau \right| \\
&\leq C \int_0^t \left(\int_0^1 \frac{\kappa \theta_x^2}{\theta^2} dx \right)^{\frac{1}{2}} \left(\int_0^1 \kappa \theta^4 u_{xx}^2 dx \right)^{\frac{1}{2}} d\tau \\
&\leq C + C \left\| \kappa^{\frac{1}{2}} \theta^2 \right\|_\infty Z^{\frac{1}{2}} \\
&\leq C + \epsilon Y + C Z^{\frac{3}{4}},
\end{aligned}
\tag{2.82}
$$

$$
\begin{aligned}
|I_{12}| &= \left| \int_0^t \int_0^1 \frac{\lambda \phi z \kappa(v,\theta) \theta_t}{v} dx d\tau \right| \\
&\leq C \int_0^t \int_0^1 \left(1+\theta^b\right) z\phi |\theta_t| dx d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \epsilon X + C(\epsilon)\left(1+\theta^{b+\beta-3}\right)\int_0^t\int_0^1 \phi z^2 dx d\tau \\
&\leq C(\epsilon) + \epsilon X + CY^{\frac{b+\beta-3}{2b+6}} \\
&\leq C(\epsilon) + \epsilon (X+Y),
\end{aligned} \tag{2.83}
$$

where we have used the assumption $0 \leq \beta < b+9$ given in Theorem 1.1,

$$
\begin{aligned}
|I_{13}| &= \left| \int_0^t \int_0^1 \frac{\kappa(v,\theta)\theta_x K_{vv} v_x u_x}{v} dx d\tau \right| \\
&\leq C \int_0^t \int_0^1 \left(1+\theta^{b+1}\right)|\theta_x v_x u_x| dx d\tau \\
&\leq C \|u_x\|_\infty Y^{\frac{1}{2}} \left(\int_0^t \max_{x\in\Omega} \left(1+\theta^{b+2}\right)\|v_x\|^2 d\tau \right)^{\frac{1}{2}} \\
&\leq C + \epsilon Y + C(\epsilon) Z^{\frac{3}{4}},
\end{aligned} \tag{2.84}
$$

where we have used the inequality (2.55),

$$
\begin{aligned}
|I_{14}| &= \left| \int_0^t \int_0^1 \lambda \phi z K_v u_x dx d\tau \right| \\
&\leq C \|\theta\|_\infty \|u_x\|_\infty \int_0^t \int_0^1 \phi z dx d\tau \\
&\leq \epsilon Y + C(\epsilon)\left(1+Z^{\frac{3}{4}}\right).
\end{aligned} \tag{2.85}
$$

For I_{15}, similarly we can get that

$$
\begin{aligned}
|I_{15}| &= \left| \int_0^t \int_0^1 \frac{\kappa(v,\theta)\theta_x}{v} \left(\frac{\kappa}{v}\right)_v v_x \theta_t dx d\tau \right| \\
&\leq C \int_0^t \int_0^1 \left(1+\theta^b\right)|\theta_x v_x \theta_t| dx d\tau \\
&\leq \epsilon \int_0^t \int_0^1 \left(1+\theta^{b+3}\right)\theta_t^2 dx d\tau + C(\epsilon) + C(\epsilon)\int_0^t \left\| \frac{\kappa\theta_x}{v} \right\|_{L^\infty(\Omega)}^2 \|v_x\|^2 d\tau \\
&\leq \epsilon X + C(\epsilon) + C(\epsilon)\left(\int_0^t \left\| \frac{\kappa\theta_x}{v} \right\|^2 d\tau \right)^{\frac{1}{2}} \left(\int_0^t \left\| \left(\frac{\kappa\theta_x}{v}\right)_x \right\|^2 d\tau \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.86}
$$

Here we have used the Sobolev inequality and the inequality (2.55).

Since Lemma 2.9 tells us that

$$
\begin{aligned}
\int_0^t \left\| \frac{\kappa\theta_x}{v} \right\|^2 d\tau &\leq C\left(1+\|\theta\|_\infty^{b+2}\right) \\
&\leq C\left(1+Y^{\frac{b+2}{2b+6}}\right)
\end{aligned} \tag{2.87}
$$

and

$$
\int_0^t \left\| \left(\frac{\kappa\theta_x}{v}\right)_x \right\|^2 d\tau \leq C \int_0^t \int_0^1 \left(e_\theta^2 \theta_t^2 + \theta^2 p_\theta^2 u_x^2 + u_x^4 + \phi^2 z^2\right) dx d\tau
$$

$$
\begin{aligned}
&\le C\int_0^t\int_0^1\left[\left(1+\theta^6\right)\theta_t^2+\left(1+\theta^8\right)u_x^2+u_x^4+\theta^\beta\phi z\right]dxd\tau\\
&\le C\left(1+\left\|\frac{1}{\theta}\right\|_\infty^{b-3}\right)X+C\|u_x\|_\infty^2\int_0^t\left(1+\max_{x\in\Omega}\theta^4\right)\int_0^1\left(1+\theta^4\right)dxd\tau\\
&\quad +C+C\|\theta\|_\infty^\beta\int_0^t\int_0^1\phi z dxd\tau\\
&\le C+CX+CZ^{\frac{3}{4}}+CY^{\frac{\beta}{2q+6}},
\end{aligned}\tag{2.88}
$$

we can thus deduce from the estimates (2.86)-(2.88) that

$$
\begin{aligned}
I_{15}&\le C+\epsilon X+C\left(1+Y^{\frac{b+2}{2b+6}}\right)^{\frac{1}{2}}\left(1+X+Z^{\frac{3}{4}}+Y^{\frac{\beta}{2b+6}}\right)^{\frac{1}{2}}\\
&\le C+\epsilon X+C\left(1+X^{\frac{1}{2}}+Z^{\frac{3}{8}}+Y^{\frac{\beta}{4b+12}}+Y^{\frac{b+2}{4b+12}}+X^{\frac{1}{2}}Y^{\frac{b+2}{4b+12}}+Y^{\frac{b+2}{4b+12}}Z^{\frac{3}{8}}+Y^{\frac{b+2+\beta}{4b+12}}\right)\\
&\le C(\epsilon)+3\epsilon X+5\epsilon Y+CZ^{\frac{3}{8}}+C(\epsilon)Z^{\frac{3b+9}{6b+20}}\\
&\le C(\epsilon)+3\epsilon X+5\epsilon Y+\epsilon Z^{\frac{7}{8}}.
\end{aligned}\tag{2.89}
$$

Here we have used the assumption $0\le\beta<b+9$ given in Theorem 1.1 again.

Finally, combining the estimates (2.74)-(2.89) and choosing $\epsilon>0$ small enough, we can get the inequality (2.68). This completes the proof of Lemma 2.10. □

Our last result in this section is to show that Z can be bounded by X and Y.

LEMMA 2.11. *Under the assumptions listed in Theorem 1.1, we have*

$$
Z\le C+CX+CY+CZ^{\frac{3}{4}}.\tag{2.90}
$$

Proof. Differentiating the equation $(1.1)_2$ with respect to t, multiplying it by u_t, then integrating the resulting equation with respect to x over Ω and by using the boundary conditions (1.6), we have

$$
\frac{1}{2}\frac{d}{dt}\|u_t\|^2+\int_0^1\frac{u_{tx}^2}{v^{1+a}}dx=\int_0^1\left(p_t+\frac{(1+a)u_x^2u_{xt}}{v^{2+a}}\right)dx.\tag{2.91}
$$

Since

$$
p_t=\left(\frac{R}{v}+\frac{4}{3}\alpha\theta^3\right)\theta_t-\frac{R\theta u_x}{v^2},
$$

we can deduce from the identity (2.91) that

$$
\begin{aligned}
\|u_t(t)\|^2+\int_0^t\|u_{xt}(\tau)\|d\tau&\le C\left(1+\int_0^t\int_0^1\left(p_t^2+u_x^4\right)dxd\tau\right)\\
&\le C+C\int_0^t\int_0^1\left(1+\theta^6\right)\theta_t^2dxd\tau+C\int_0^t\int_0^1\theta^2u_x^2dxd\tau\\
&\le C+CX+CY^{\frac{2}{2b+6}}\\
&\le C+CX+CY.
\end{aligned}\tag{2.92}
$$

Moreover, we can conclude from the equation $(1.1)_2$ that

$$
u_{xx}=v^{1+a}\left[u_t+p_x+G\left(x-\frac{1}{2}\right)+\frac{(1+a)u_xv_x}{v^{2+a}}\right].\tag{2.93}
$$

Combining the inequality (2.92) and the identity (2.93), one has

$$\begin{aligned}\|u_{xx}(t)\|^2 &\le C\left(1+\int_0^1 \left(u_t^2+p_x^2+u_x^2v_x^2\right)(t,x)dx\right)\\ &\le C\bigg(\|u_t(t)\|^2+\int_0^1 \left(1+\theta^6(t,x)\right)\theta_x^2(t,x)dx\\ &\quad+\int_0^1 \left(\theta^2(t,x)+u_x^2(t,x)\right)v_x^2(t,x)dx\bigg)\\ &\le C+CX+CY+C\|\theta\|_\infty^2\|v_x(t)\|^2+C\|u_x\|_\infty^2\\ &\le C+CX+CY+CZ^{\frac{3}{4}},\end{aligned} \tag{2.94}$$

which gives the inequality (2.90) immediately. This completes the proof of Lemma 2.11. □

Combining Lemma 2.10 with Lemma 2.11, we can deduce that $Y\le C$. Thus we can get the desired upper bounds on $\theta(t,x)$ from the estimate (2.63). Since we have obtained the desired positive lower and upper bounds on $v(t,x)$ and $\theta(t,x)$, then Theorem 1.1 can be proved by employing the standard continuation argument and we omit the details for brevity.

3. Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Similar to the proof of Theorem 1.1, the argument here is also a combination of the local existence result with certain energy estimates on the local solutions constructed.

Suppose that the local solution $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ constructed in Lemma 2.1 has been extended to $t=T\ge t_1$ and satisfies the a priori assumption

$$(H_2)\qquad \underline{V}_2'\le v(t,x)\le \overline{V}_2',\ \underline{\Theta}_2'\le\theta(t,x)\le\overline{\Theta}_2',\ \forall (t,x)\in[0,T]\times\Omega.$$

Here $\underline{V}_2',\overline{V}_2',\underline{\Theta}_2'$ and $\overline{\Theta}_2'$ are some positive constants. Similar to that of Section 2, we only need to deduce certain a priori estimates on $(v(t,x),u(t,x),\theta(t,x),z(t,x))$ which are independent of $\underline{V}_2',\overline{V}_2',\underline{\Theta}_2'$, and $\overline{\Theta}_2'$ but may depend on the initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$ and the constant T.

Firstly, similar to that of Lemma 2.2 and Lemma 2.3, we have

Lemma 3.1 (Basic energy estimates). *Under the assumptions given in Lemma 2.1, for any $t\in[0,T]$, we have*

$$\int_0^1\left[\frac{1}{2}u^2+C_v\theta+\alpha v\theta^4+\lambda z+\frac{1}{2}G(1-x)xv\right]dx\le C, \tag{3.1}$$

$$\int_0^1\{C_v(\theta-1-\log\theta)+R(v-1-\log v)\}dx+\int_0^t\int_0^1\left(\frac{\mu(v)u_x^2}{v\theta}+\frac{\kappa(v,\theta)\theta_x^2}{v\theta^2}+\frac{\lambda\phi z}{\theta}\right)dxd\tau\le C, \tag{3.2}$$

$$\int_0^1 zdx+\int_0^t\int_0^1\phi zdxd\tau=\int_0^1 z_0dx, \tag{3.3}$$

$$\int_0^1 \frac{1}{2} z^2 dx + \int_0^t \int_0^1 \left(\frac{d}{v^2} z_x^2 + \phi z^2 \right) dx d\tau = \int_0^1 \frac{1}{2} z_0^2 dx. \tag{3.4}$$

$$0 \le z(t,x) \le 1. \tag{3.5}$$

Based on Lemma 3.1, especially the estimates (3.1) and (3.2), we can get the following lemma, which will be frequently used later on.

Lemma 3.2. *Under the assumption in Lemma 3.1, for any $t \in [0,T]$, we have*

$$\int_0^t \max_{x \in \Omega} \theta^r(\tau,x) d\tau \le C, \qquad 0 \le r \le b+1. \tag{3.6}$$

Proof. Using the same method in Lemma 2.5, for any $r \ge 0$ and $(t,x) \in [0,T] \times \Omega$, we have

$$\theta(t,x)^{\frac{r}{2}} \le C + C \left(\int_0^1 \frac{v\theta^r}{1+v\theta^b} dx \right)^{\frac{1}{2}} \left(\int_0^1 \left(\frac{\mu(v) u_x^2}{v\theta} + \frac{\kappa(v,\theta)\theta_x^2}{v\theta^2} + \frac{\lambda\phi z}{\theta} \right) dx \right)^{\frac{1}{2}}. \tag{3.7}$$

It is easy to see that

$$\theta^r \le C\left(1 + \theta^{b+1}\right), \tag{3.8}$$

holds for $0 \le r \le b+1$, thus

$$\int_0^1 \frac{v\theta^r}{1+v\theta^b} dx \le C \int_0^1 (v+\theta) dx \le C. \tag{3.9}$$

Combining the estimates (3.2), (3.7) and (3.9), we can complete the proof of this lemma. □

Remark 3.1. One can find the difference between Lemma 3.2 and Lemma 2.5. The main reason to cause such a difference is that we have already obtained the positive lower bound of $v(t,x)$ before proving Lemma 2.5, while we have not obtained the positive lower bound of $v(t,x)$ here, and consequently we can only make use of the estimate (3.1) to prove Lemma 3.2.

Now we turn to derive a positive lower bound estimate on the temperature $\theta(t,x)$ in the following lemma, and it is worth to pointing out that the nondegerate assumption (1.12) plays an essential role in our analysis.

Lemma 3.3. *Under the assumptions stated in Lemma 3.1, there exists a positive constant $\underline{\Theta}_2$ depending only on T and the initial data $(v_0(x), u_0(x), \theta_0(x), z_0(x))$ such that*

$$\theta(t,x) \ge \underline{\Theta}_2, \quad \forall (t,x) \in [0,T] \times \Omega. \tag{3.10}$$

Proof. Setting $h = \frac{1}{\theta}$, similar to the proof of Lemma 2.6, one can deduce that

$$h_t \le \frac{1}{e_\theta} \left(\frac{\kappa(v,\theta) h_x}{v} \right)_x + C\left(\frac{1}{v\mu(v)} + \frac{\theta^3}{\mu(v)} \right). \tag{3.11}$$

Multiplying the inequality (3.11) by h^{2p-1} and integrating the result with respect to x over Ω, one has

$$\|h\|_{L^{2p}}^{2p-1}(\|h\|_{L^{2p}})_t \leq C\int_0^1\left(\frac{1}{v\mu(v)}+\frac{\theta^3}{\mu(v)}\right)h^{2p-1}dx+C\int_0^1\frac{h^{2p-1}}{e_\theta}\left(\frac{\kappa(v,\theta)h_x}{v}\right)_x dx. \tag{3.12}$$

Using the identity (2.30) and the inequality (2.31), we arrive at

$$\|h\|_{L^{2p}}^{2p-1}(\|h\|_{L^{2p}})_t \leq C\int_0^1\left(\frac{1}{v\mu(v)}+\frac{\theta^3}{\mu(v)}\right)h^{2p-1}dx+\frac{1}{2p-1}\int_0^1\frac{\kappa^2(v,\theta)h^{2p}((e_\theta)_x)^2}{ve_\theta^3}dx. \tag{3.13}$$

Integrating the estimate (3.13) with respect to t over $(0,t)$, one can get, by using Hölder's inequality and by letting $p\to+\infty$, that

$$\begin{aligned}\|h\|_{L^\infty(\Omega)} &\leq C+C\int_0^t\left(\left\|\frac{1}{v\mu(v)}\right\|_{L^\infty(\Omega)}+\left\|\frac{\theta^3}{\mu(v)}\right\|_{L^\infty(\Omega)}\right)d\tau\\ &\leq C+C\left\|\frac{1}{v\mu(v)}\right\|_\infty+C\left\|\frac{1}{\mu(v)}\right\|_\infty\int_0^t\max_{x\in\Omega}\{\theta^3(\tau,x)\}d\tau\\ &\leq C+C\left\|\frac{1}{v\mu(v)}\right\|_\infty+C\left\|\frac{1}{\mu(v)}\right\|_\infty,\end{aligned} \tag{3.14}$$

where we have used Lemma 3.2 and the fact $b\geq 3$.

Since $l_1>1, l_2>0$, we can deduce from the condition (1.12) that

$$\left\|\frac{1}{v\mu(v)}\right\|_\infty+\left\|\frac{1}{\mu(v)}\right\|_\infty\leq C, \tag{3.15}$$

and the estimate (3.10) follows immediately from the estimates (3.14) and (3.15). This completes the proof of Lemma 3.3. ∎

Now we turn to deduce the desired positive lower and upper bound on $v(t,x)$. To this end, we first derive the following lemma, which will be used later.

Lemma 3.4. *Under the assumptions listed in Lemma 3.1, for any $t\in[0,T]$, we have*

$$\int_0^1\left[\frac{1}{2}u^2+\frac{1}{2}G(1-x)xv\right]dx+\int_0^t\int_0^1\frac{\mu(v)u_x^2}{v}dxd\tau\leq C. \tag{3.16}$$

Proof. Multiplying the equation $(1.1)_2$ by u and integrating the result with respect to t and x over $(0,t)\times(0,1)$, we arrive at

$$\begin{aligned}&\int_0^1\left[\frac{1}{2}u^2+\frac{1}{2}G(1-x)xv\right]dx+\int_0^t\int_0^1\frac{\mu(v)u_x^2}{v}dxd\tau\\ &\leq C+\int_0^t\int_0^1\left(\frac{R\theta u_x}{v}+\frac{\alpha}{3}\theta^4u_x\right)dxd\tau.\end{aligned} \tag{3.17}$$

Since Lemma 3.2 together with the fact $b\geq 3$ tells us that

$$\int_0^t\int_0^1\frac{R\theta u_x}{v}dxd\tau\leq\epsilon\int_0^t\int_0^1\frac{\mu(v)u_x^2}{v}dxd\tau+C(\epsilon)\int_0^t\int_0^1\frac{\theta^2}{v^2}\cdot\frac{v}{\mu(v)}dxd\tau$$

$$
\begin{aligned}
&\le \epsilon \int_0^t \int_0^1 \frac{\mu(v)u_x^2}{v} dx d\tau + C(\epsilon) \left\| \frac{1}{v\mu(v)} \right\|_\infty \\
&\le \epsilon \int_0^t \int_0^1 \frac{\mu(v)u_x^2}{v} dx d\tau + C(\epsilon) \qquad (3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^t \int_0^1 \frac{\alpha}{3}\theta^4 u_x dx d\tau &\le \epsilon \int_0^t \int_0^1 \frac{\mu(v)u_x^2}{v} dx d\tau + C(\epsilon) \int_0^t \int_0^1 \theta^8 \cdot \frac{v}{\mu(v)} dx d\tau \\
&\le \epsilon \int_0^t \int_0^1 \frac{\mu(v)u_x^2}{v} dx d\tau + C(\epsilon) \left\| \frac{1}{\mu(v)} \right\|_\infty \\
&\le \epsilon \int_0^t \int_0^1 \frac{\mu(v)u_x^2}{v} dx d\tau + C(\epsilon), \qquad (3.19)
\end{aligned}
$$

we can get the inequality (3.16) immediately by putting the estimates (3.17), (3.18) and (3.19) together and by choosing $\epsilon > 0$ small enough. This completes the proof of Lemma 3.4. □

Now we try to exploit Kanel's argument (see [21]) to deduce the desired positive lower bound and the upper bound on $v(t,x)$ in terms of $\|\theta^{8-b}\|_\infty$. To this end, set

$$
\Phi(v) = \int_1^v \frac{\sqrt{\phi(z)}}{z} \mu(z) dz, \quad where \quad \phi(x) = x - lnx - 1. \qquad (3.20)
$$

On one hand, it is easy to see that there exist positive constants C_1 and C_2 such that

$$
|\Phi(v)| \ge C_1 \left(v^{-l_1} + v^{l_2 + \frac{1}{2}} \right) - C_2. \qquad (3.21)
$$

On the other hand,

$$
\begin{aligned}
\Phi(v) &\le C + \int_0^1 \left| \Phi(v(y,t))_y \right| dy \\
&\le C + \int_0^1 \left| \frac{\sqrt{\phi(v)}}{v} \mu(v) v_x \right| dx \\
&\le C + C \left(\int_0^1 \phi(v) dx \right)^{\frac{1}{2}} \left(\int_0^1 \frac{\mu^2(v)v_x^2}{v^2} dx \right)^{\frac{1}{2}} \\
&\le C + C \left(\int_0^1 \frac{\mu^2(v)v_x^2}{v^2} dx \right)^{\frac{1}{2}}. \qquad (3.22)
\end{aligned}
$$

Thus to yield an estimate on the positive lower and upper bounds on $v(t,x)$, we need to estimate the term $\int_0^1 \frac{\mu^2(v)v_x^2}{v^2} dx$ suitably first. For this purpose, multiplying the equation $(1.1)_2$ by $\frac{\mu(v)v_x}{v}$, one has

$$
\begin{aligned}
\left(\frac{\mu^2(v)v_x^2}{2v^2} \right)_t = \left(\frac{\mu(v)uv_x}{v} \right)_t - \left(u \frac{\mu(v)u_x}{v} \right)_x + \frac{\mu(v)u_x^2}{v} \\
+ \frac{\mu(v)v_x p_x}{v} + G\left(x - \frac{1}{2} \right) \frac{\mu(v)v_x}{v}. \qquad (3.23)
\end{aligned}
$$

Integrating the identity (3.23) over $(0,t)\times\Omega$, one can deduce that

$$\frac{1}{2}\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dx \le C+\underbrace{\int_0^1 \frac{\mu(v)uv_x}{v}dx}_{I_{16}}+\underbrace{\int_0^t\int_0^1 \frac{\mu(v)u_x^2}{v}dxd\tau}_{I_{17}}+\underbrace{\int_0^t\int_0^1 \frac{\mu(v)v_xp_x}{v}dxd\tau}_{I_{18}}$$
$$+\int_0^t\int_0^1 G\left(x-\frac{1}{2}\right)\frac{\mu(v)v_x}{v}dxd\tau. \tag{3.24}$$

Now we control $I_i(i=16,17,18)$ suitably. To this end, since Lemma 3.4 gives a nice estimate on I_{17}, we only need to deal with the terms I_{16} and I_{18}. Firstly, for I_{16}, we can get by using Cauchy's inequality that

$$\begin{aligned} I_{16} &\le \epsilon\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dx + C(\epsilon)\int_0^1 u^2dx \\ &\le \epsilon\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dx + C(\epsilon). \end{aligned} \tag{3.25}$$

As for I_{18}, by making use of the identity (2.36), we have

$$I_{18}=\underbrace{\int_0^t\int_0^1 \frac{R\mu(v)v_x\theta_x}{v^2}dxd\tau}_{K_3}+\underbrace{\int_0^t\int_0^1 \frac{4\alpha\mu(v)v_x\theta^3\theta_x}{3v}dxd\tau}_{K_4}-\int_0^t\int_0^1 \frac{R\mu(v)v_x^2\theta}{v^3}dxd\tau,$$

while K_3 and K_4 can be estimated by exploiting Cauchy's inequality as follows

$$\begin{aligned} K_3 &\le C\int_0^t\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dxd\tau + C\int_0^t\int_0^1 \frac{\kappa\theta_x^2}{v\theta^2}\cdot\frac{\theta^2}{v\kappa}dxd\tau \\ &\le C\int_0^t\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dxd\tau + C\left\|\frac{\theta^2}{v\kappa}\right\|_\infty \\ &\le C\int_0^t\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dxd\tau + C\left\|\frac{1}{v}\right\|_\infty^2 \end{aligned}$$

and

$$\begin{aligned} K_4 &\le C\int_0^t\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dxd\tau + C\int_0^t\int_0^1 \frac{\kappa\theta_x^2}{v\theta^2}\cdot\frac{v\theta^8}{\kappa}dxd\tau \\ &\le C\int_0^t\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dxd\tau + C\left\|\frac{v\theta^8}{\kappa}\right\|_\infty \\ &\le C\int_0^t\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dxd\tau + C\left\|\theta^{8-b}\right\|_\infty. \end{aligned}$$

Consequently

$$\begin{aligned} I_{18} \le{}& C\int_0^t\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dxd\tau + C\left(\left\|\theta^{8-b}\right\|_\infty + \left\|\frac{1}{v}\right\|_\infty^2\right) \\ &-\int_0^t\int_0^1 \frac{R\mu(v)v_x^2\theta}{v^3}dxd\tau. \end{aligned} \tag{3.26}$$

Combining the estimates (3.24)-(3.26), the estimate (3.16) obtained in Lemma 3.4 and by using Gronwall's inequality, we have the following lemma:

LEMMA 3.5. *Under the assumptions given in Lemma 3.1, for any $t\in[0,T]$, we have*

$$\int_0^1 \frac{\mu^2(v)v_x^2}{v^2}dx+\int_0^t\int_0^1 \frac{\mu(v)\theta v_x^2}{v^3}dxd\tau\leq C+C\left\|\frac{1}{v}\right\|_\infty^2+C\left\|\theta^{8-b}\right\|_\infty. \tag{3.27}$$

Having obtained Lemma 3.5, we can deduce from the inequalities (3.21) and (3.22) and Lemma 3.5 that

$$v^{-l_1}+v^{l_2+\frac{1}{2}}\leq C+C\left\|\frac{1}{v}\right\|_\infty+C\left\|\theta^{8-b}\right\|_\infty^{\frac{1}{2}}. \tag{3.28}$$

Since $l_1>1$, with the help of the Young inequality, we can obtain that

$$v^{-l_1}+v^{l_2+\frac{1}{2}}\leq C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{1}{2}}. \tag{3.29}$$

Thus we have the following lemma:

LEMMA 3.6. *Under the assumptions listed in Lemma 3.1, for any $t\in[0,T]$, we have*

$$\frac{1}{v}\leq C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{1}{2l_1}}, \tag{3.30}$$

$$v\leq C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{1}{2l_2+1}}. \tag{3.31}$$

Now we turn to deduce the upper bound on $\theta(t,x)$. To this end, we first give the following lemma.

LEMMA 3.7. *Under the assumption in Lemma 3.1, for any $t\in[0,T]$, we have*

$$\|\theta\|_{L^\infty(\Omega)}\leq C+C\int_0^t\left(\left\|\frac{\mu(v)u_x^2}{v^2\theta^3}\right\|_{L^\infty(\Omega)}+\left\|\frac{u_x}{v^2\theta^2}\right\|_{L^\infty(\Omega)}+\left\|\frac{u_x\theta}{v}\right\|_{L^\infty(\Omega)}+\left\|\frac{\theta^\beta}{v\theta^3}\right\|_{L^\infty(\Omega)}\right)d\tau. \tag{3.32}$$

Proof. Firstly, the equation $(1.1)_3$ can be rewritten as

$$\left(C_v+4\alpha v\theta^3\right)\theta_t+\left(\frac{R\theta}{v}+\frac{4}{3}\alpha\theta^4\right)u_x=\frac{\mu(v)u_x^2}{v}+\left(\frac{\kappa(v,\theta)\theta_x}{v}\right)_x+\lambda\phi z. \tag{3.33}$$

Multiplying the above identity by $2p\theta^{2p-1}$, we arrive at

$$\begin{aligned}&\left(\theta^{2p}\right)_t+\frac{2p(2p-1)\theta^{2p-2}\kappa(v,\theta)\theta_x^2}{v(C_v+4\alpha v\theta^3)}\\&=\left\{\frac{2p\theta^{2p-1}\kappa(v,\theta)\theta_x}{v(C_v+4\alpha v\theta^3)}\right\}_x+\frac{2p\lambda\theta^{2p-1}\phi z}{C_v+4\alpha v\theta^3}+\frac{2p\theta^{2p-1}\mu(v)u_x^2}{v(C_v+4\alpha v\theta^3)}-\frac{2p\theta^{2p-1}u_x}{C_v+4\alpha v\theta^3}\left(\frac{R\theta}{v}+\frac{4}{3}\alpha\theta^4\right)\\&\quad+\frac{8p\alpha\theta^{2p-1}\kappa(v,\theta)\theta_x\left(v_x\theta^3+3v\theta^2\theta_x\right)}{v\left(C_v+4\alpha v\theta^3\right)^2}.\end{aligned} \tag{3.34}$$

Noticing that

$$\frac{8p\alpha\theta^{2p-1}\kappa\theta_x\left(v_x\theta^3+3v\theta^2\theta_x\right)}{v\left(C_v+4\alpha v\theta^3\right)^2}$$

$$
\begin{aligned}
&= \frac{\sqrt{2p(2p-1)}\theta^{p-1}\theta_x\sqrt{\kappa}}{\sqrt{v(C_v+4\alpha v\theta^3)}} \cdot \frac{8p\alpha\theta^p\sqrt{\kappa}\left(v_x\theta^3+3v\theta^2\theta_x\right)}{\sqrt{2p(2p-1)}\sqrt{v(C_v+4\alpha v\theta^3)^3}} \\
&\le \frac{1}{2}\frac{2p(2p-1)\theta^{2p-2}\kappa\theta_x^2}{v(C_v+4\alpha v\theta^3)} + \frac{64p^2\alpha^2\theta^{2p}\kappa\left(v_x\theta^3+3v\theta^2\theta_x\right)^2}{(2p^2-p)v(C_v+4\alpha v\theta^3)^3},
\end{aligned}
\tag{3.35}
$$

we can deduce from the identity (3.34) and the above inequality that

$$
\begin{aligned}
&\left(\theta^{2p}\right)_t + \frac{2p(2p-1)\theta^{2p-2}\kappa(v,\theta)\theta_x^2}{2v(C_v+4\alpha v\theta^3)} \\
&\le \left\{\frac{2p\theta^{2p-1}\kappa(v,\theta)\theta_x}{v(C_v+4\alpha v\theta^3)}\right\}_x + \frac{64p^2\alpha^2\theta^{2p-1}\kappa\theta\left(v_x\theta^3+3v\theta^2\theta_x\right)^2}{(2p^2-p)v(C_v+4\alpha v\theta^3)^3} \\
&\quad + \frac{2p\theta^{2p-1}\mu(v)u_x^2}{v(C_v+4\alpha v\theta^3)} - \frac{2p\theta^{2p-1}u_x}{C_v+4\alpha v\theta^3}\left(\frac{R\theta}{v}+\frac{4}{3}\alpha\theta^4\right) \\
&\quad + \frac{2p\lambda\theta^{2p-1}\phi z}{C_v+4\alpha v\theta^3}.
\end{aligned}
\tag{3.36}
$$

Integrating the above inequality with respect to x over Ω and using Hölder's inequality, we obtain

$$
\begin{aligned}
\left(\|\theta\|_{L^{2p}}\right)_t \le{}& \frac{2pC}{2p^2-p}\left\|\frac{\kappa\theta\left(v_x\theta^3+3v\theta^2\theta_x\right)^2}{v(C_v+4\alpha v\theta^3)^3}\right\|_{L^{2p}} + C\left\|\frac{\mu(v)u_x^2}{v(C_v+4\alpha v\theta^3)}\right\|_{L^{2p}} \\
&+ C\left\|\frac{|u_x|}{C_v+4\alpha v\theta^3}\left(\frac{\theta}{v}+\theta^4\right)\right\|_{L^{2p}} + C\left\|\frac{\theta^\beta}{C_v+4\alpha v\theta^3}\right\|_{L^{2p}}.
\end{aligned}
\tag{3.37}
$$

Integrating the above inequality with respect to t over $(0,t)$ and letting $p\to+\infty$, we can obtain the inequality (3.32). This completes the proof of Lemma 3.7. ∎

From Lemma 3.7, if $0\le\beta\le b+4$, one can deduce that

$$
\begin{aligned}
\|\theta\|_{L^\infty(\Omega)} \le{}& C + C\left\|\frac{\mu(v)}{v^2\theta^3}\right\|_\infty \int_0^t \|u_x(\tau)\|^2_{L^\infty(\Omega)}d\tau + C\int_0^t \|u_x(\tau)\|^2_{L^\infty(\Omega)}d\tau \\
&+ C\left\|\frac{1}{v^4\theta^4}\right\|_\infty + C\left\|\frac{1}{v}\right\|_\infty^2 \\
\le{}& C + C\left(1+\left\|\frac{\mu(v)}{v^2}\right\|_\infty\right)\int_0^t \|u_x(\tau)\|^2_{L^\infty(\Omega)}d\tau + C\left\|\frac{1}{v}\right\|_\infty^4.
\end{aligned}
\tag{3.38}
$$

Thus to deduce the desired upper bound on $\theta(t,x)$, we had to deal with the term $\int_0^t \|u_x(\tau)\|^2_{L^\infty(\Omega)}d\tau$. For this purpose, noticing that we can get from Lemma 3.4 that

$$
\begin{aligned}
\int_0^t \|u_x(\tau)\|^2_{L^\infty(\Omega)}d\tau \le{}& \int_0^t \|u_x(\tau)\|\|u_{xx}(\tau)\|d\tau \\
\le{}& \left(\int_0^t \|u_x(\tau)\|^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t \|u_{xx}(\tau)\|^2 d\tau\right)^{\frac{1}{2}} \\
\le{}& \left\|\frac{v}{\mu(v)}\right\|_\infty \left(\int_0^t\int_0^1 \frac{\mu(v)u_x^2}{v}dxd\tau\right)^{\frac{1}{2}}\left(\int_0^t\int_0^1 \frac{\mu(v)u_{xx}^2}{v}dxd\tau\right)^{\frac{1}{2}}
\end{aligned}
$$

$$\leq C\left\|\frac{v}{\mu(v)}\right\|_\infty \left(\int_0^t\int_0^1 \frac{\mu(v)u_{xx}^2}{v}dxd\tau\right)^{\frac{1}{2}}, \tag{3.39}$$

thus we need to estimate the term $\int_0^t\int_0^1 \frac{\mu(v)u_{xx}^2}{v}dxd\tau$. To this end, differentiating the equation $(1.1)_2$ with respect to x, multiplying the resulting identity by u_x, we have

$$\partial_t\left(\frac{u_x^2}{2}\right)+\left(\frac{\mu(v)u_x}{v}\right)_x u_{xx}=u_{xx}p_x+(u_xu_t)_x+G\left(x-\frac{1}{2}\right)u_{xx}. \tag{3.40}$$

Integrating the above identity with respect to t and x over $(0,t)\times(0,1)$ and using the boundary condition (1.11), we arrive at

$$\begin{aligned}
&\|u_x(t)\|^2+\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau\\
\leq{}& C+C\underbrace{\int_0^t\int_0^1\frac{|\mu'(v)v_xu_xu_{xx}|}{v}dxd\tau}_{I_{19}}+C\underbrace{\int_0^t\int_0^1\frac{|\theta_xu_{xx}|}{v}dxd\tau}_{I_{20}}\\
&+C\underbrace{\int_0^t\int_0^1\frac{\mu(v)|v_xu_xu_{xx}|}{v^2}dxd\tau}_{I_{21}}+C\underbrace{\int_0^t\int_0^1\frac{\theta|v_xu_{xx}|}{v^2}dxd\tau}_{I_{22}}\\
&+C\underbrace{\int_0^t\int_0^1\theta^3|\theta_xu_{xx}|dxd\tau}_{I_{23}}+C\underbrace{\int_0^t\int_0^1\left|x-\frac{1}{2}\right||u_{xx}|dxd\tau}_{I_{24}},
\end{aligned} \tag{3.41}$$

and the terms $I_j(j=19,\cdots,24)$ can be estimated as in the following

$$\begin{aligned}
I_{19}\leq{}&\epsilon\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau+C(\epsilon)\int_0^t\int_0^1\frac{|\mu'(v)|^2u_x^2v_x^2}{\mu(v)v}dxd\tau\\
\leq{}&\epsilon\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau+C(\epsilon)\int_0^t\left\|\frac{|\mu'(v)|^2u_x^2v}{\mu^3(v)}\right\|_{L^\infty(\Omega)}\int_0^1\frac{\mu^2(v)v_x^2}{v^2}dxd\tau\\
\leq{}&\epsilon\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau\\
&+C(\epsilon)\left\|\frac{|\mu'(v)|^2v}{\mu^3(v)}\right\|_\infty\left(1+\left\|\frac{1}{v}\right\|_\infty^2+\|\theta^{8-b}\|_\infty\right)\left\|\frac{v}{\mu(v)}\right\|_\infty\left(\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau\right)^{\frac{1}{2}}\\
\leq{}&2\epsilon\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau+C(\epsilon)\left\|\frac{|\mu'(v)|^2v}{\mu^3(v)}\right\|_\infty^2\left(1+\left\|\frac{1}{v}\right\|_\infty^2+\|\theta^{8-b}\|_\infty\right)^2\left\|\frac{v}{\mu(v)}\right\|_\infty^2,
\end{aligned} \tag{3.42}$$

where we have used the inequality (3.39) and Lemma 3.5,

$$\begin{aligned}
I_{20}\leq{}&\epsilon\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau+C(\epsilon)\int_0^t\int_0^1\frac{\kappa\theta_x^2}{v\theta^2}\cdot\frac{\theta^2}{\mu(v)\kappa}dxd\tau\\
\leq{}&\epsilon\int_0^t\int_0^1\frac{\mu(v)u_{xx}^2}{v}dxd\tau+C(\epsilon)\left\|\frac{1}{\mu(v)v\theta^{b-2}}\right\|_\infty
\end{aligned}$$

$$\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon), \tag{3.43}$$

$$\begin{aligned}
I_{21} &\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \int_0^1 \frac{\mu(v) u_x^2 v_x^2}{v^3} dx d\tau \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \left\| \frac{u_x^2}{\mu(v) v} \right\|_{L^\infty(\Omega)} \int_0^1 \frac{\mu^2(v) v_x^2}{v^2} dx d\tau \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau \\
&\quad + C(\epsilon) \left(1 + \left\| \frac{1}{v} \right\|_\infty^2 + \|\theta^{8-b}\|_\infty \right) \left\| \frac{v}{\mu(v)} \right\|_\infty \left(\int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau \right)^{\frac{1}{2}} \\
&\leq 2\epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \left\| \frac{v}{\mu(v)} \right\|_\infty^2 \left(1 + \left\| \frac{1}{v} \right\|_\infty^2 + \|\theta^{8-b}\|_\infty \right)^2,
\end{aligned} \tag{3.44}$$

where we have used the estimates (3.15) and (3.39) and Lemma 3.5,

$$\begin{aligned}
I_{22} &\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \int_0^1 \frac{\theta^2 v_x^2}{\mu(v) v^3} dx d\tau \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \left\| \frac{\theta^2}{\mu^3(v) v} \right\|_{L_x^\infty} \int_0^1 \frac{\mu^2(v) v_x^2}{v^2} dx d\tau \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \left\| \frac{1}{\mu^3(v) v} \right\|_\infty \left(1 + \left\| \frac{1}{v} \right\|_\infty^2 + C\|\theta^{8-b}\|_\infty \right) \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \left(1 + \left\| \frac{1}{v} \right\|_\infty^2 + C\|\theta^{8-b}\|_\infty \right).
\end{aligned} \tag{3.45}$$

Here we have used the fact $l_1 > 1, l_2 > 0$ again,

$$\begin{aligned}
I_{23} &\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \int_0^1 \frac{\kappa \theta_x^2}{v \theta^2} \cdot \frac{v^2 \theta^8}{\mu(v) \kappa} dx d\tau \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \left\| \frac{v^2 \theta^8}{\mu(v) \kappa} \right\|_\infty \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \left\| \frac{v \theta^{8-b}}{\mu(v)} \right\|_\infty \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \left\| \frac{v}{\mu(v)} \right\|_\infty \|\theta^{8-b}\|_\infty,
\end{aligned} \tag{3.46}$$

where we have used the condition (1.3) and the inequality (3.2), and

$$\begin{aligned}
I_{24} &= \int_0^t \int_0^1 \frac{\sqrt{\mu(v)} u_{xx}}{\sqrt{v}} \cdot \frac{\sqrt{v} \left| x - \frac{1}{2} \right|}{\sqrt{\mu(v)}} dx d\tau \\
&\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v) u_{xx}^2}{v} dx d\tau + C(\epsilon) \int_0^t \int_0^1 \frac{v \left(x - \frac{1}{2} \right)^2}{\mu(v)} dx d\tau
\end{aligned}$$

$$\leq \epsilon \int_0^t \int_0^1 \frac{\mu(v)u_{xx}^2}{v} dxd\tau + C(\epsilon)\left\|\frac{v}{\mu(v)}\right\|_\infty. \tag{3.47}$$

Here we have used the fact that $\Omega := (0,1)$ is a bounded domain and Cauchy's inequality.

Combining the estimates (3.41)-(3.47), by choosing $\epsilon > 0$ small enough and if we further assume that

$$v|\mu'(v)|^2 \leq \mu^3(v), \tag{3.48}$$

then we have the following lemma:

LEMMA 3.8. *Under the assumptions listed in Lemma 3.1, for any $t \in [0,T]$, we have*

$$\|u_x(t)\|^2 + \int_0^t \int_0^1 \frac{\mu(v)u_{xx}^2}{v} dxd\tau \leq C + C\left\|\frac{v}{\mu(v)}\right\|_\infty^2 \left(1 + \left\|\frac{1}{v}\right\|_\infty^2 + \|\theta^{8-b}\|_\infty\right)^2. \tag{3.49}$$

Having obtained Lemma 3.8, one can deduce from the estimate (3.39) that

$$\begin{aligned}
\int_0^t \|u_x(\tau)\|_{L^\infty(\Omega)}^2 d\tau &\leq C\left\|\frac{v}{\mu(v)}\right\|_\infty \left[1 + \left\|\frac{v}{\mu(v)}\right\|_\infty^2 \left(1 + \left\|\frac{1}{v}\right\|_\infty^2 + \|\theta^{8-b}\|_\infty\right)^2\right]^{\frac{1}{2}} \\
&\leq C + C\left\|\frac{v}{\mu(v)}\right\|_\infty + C\left\|\frac{v}{\mu(v)}\right\|_\infty^2 \left(1 + \left\|\frac{1}{v}\right\|_\infty^2 + \|\theta^{8-b}\|_\infty\right) \\
&\leq C + C\left\|\frac{v}{\mu(v)}\right\|_\infty^2 + C\left\|\frac{v}{\mu(v)}\right\|_\infty^2 \left\|\frac{1}{v}\right\|_\infty^2 + C\left\|\frac{v}{\mu(v)}\right\|_\infty^2 \|\theta^{8-b}\|_\infty.
\end{aligned} \tag{3.50}$$

Combining the above inequality with the estimate (3.38), we arrive at

$$\|\theta\|_{L^\infty(\Omega)} \leq C + \sum_{k=25}^{29} I_k \tag{3.51}$$

and the definition of $I_k (27 \leq k \leq 31)$ will be given below.

Now we estimate $I_{25} - I_{29}$ term by term. First of all, we can deduce from the condition (1.12) that

$$\mu(v) \leq C\left(1 + \left\|\frac{1}{v}\right\|_\infty^{l_1} + \|v\|_\infty^{l_2}\right). \tag{3.52}$$

Thus one can get from Lemma 3.6 that

$$\begin{aligned}
I_{25} &= C\left(1 + \left\|\frac{\mu(v)}{v^2}\right\|_\infty\right) \\
&\leq C + C\left\|\frac{1}{v}\right\|_\infty^2 \left(1 + \left\|\frac{1}{v}\right\|_\infty^{l_1} + \|v\|_\infty^{l_2}\right) \\
&\leq C + C\left\|\frac{1}{v}\right\|_\infty^{l_1+2} + C\left\|\frac{1}{v}\right\|_\infty^2 \|v\|_\infty^{l_2} \\
&\leq C + C\|\theta^{8-b}\|_\infty^{\frac{l_1+2}{2l_1}} + C\|\theta^{8-b}\|_\infty^{\frac{2}{2l_1}+\frac{l_2}{2l_2+1}},
\end{aligned} \tag{3.53}$$

$$\begin{aligned}
I_{26}=&C\left(1+\left\|\frac{\mu(v)}{v^2}\right\|_\infty\right)\left\|\frac{v}{\mu(v)}\right\|_\infty^2\\
\leq& C\left(1+\left\|\theta^{8-b}\right\|_\infty^{\frac{l_1+2}{2l_1}}+\left\|\theta^{8-b}\right\|_\infty^{\frac{2}{2l_1}+\frac{l_2}{2l_2+1}}\right)\left(1+\left\|\theta^{8-b}\right\|_\infty^{\frac{2}{2l_2+1}}\right)\\
\leq& C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}},
\end{aligned} \tag{3.54}$$

$$\begin{aligned}
I_{27}=&C\left(1+\left\|\frac{\mu(v)}{v^2}\right\|_\infty\right)\left\|\frac{v}{\mu(v)}\right\|_\infty^2\left\|\frac{1}{v}\right\|_\infty^2\\
\leq& C\left(1+\left\|\theta^{8-b}\right\|_\infty^{\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}}\right)\left(1+\left\|\theta^{8-b}\right\|_\infty^{\frac{2}{2l_1}}\right)\\
\leq& C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{l_1+4}{2l_1}+\frac{2}{2l_2+1}},
\end{aligned} \tag{3.55}$$

$$\begin{aligned}
I_{28}=&C\left(1+\left\|\frac{\mu(v)}{v^2}\right\|_\infty\right)\left\|\frac{v}{\mu(v)}\right\|_\infty^2\left\|\theta^{8-b}\right\|_\infty\\
\leq& C\left(1+\left\|\theta^{8-b}\right\|_\infty^{\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}}\right)\left\|\theta^{8-b}\right\|_\infty\\
\leq& C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}+1},
\end{aligned} \tag{3.56}$$

and

$$\begin{aligned}
I_{29}=&C\left\|\frac{1}{v}\right\|_\infty^4\\
\leq& C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{2}{l_1}}.
\end{aligned} \tag{3.57}$$

Combining the estimates (3.51)-(3.57), we finally arrive at

$$\|\theta\|_{L^\infty(\Omega)}\leq C+C\left\|\theta^{8-b}\right\|_\infty^{\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}+1}. \tag{3.58}$$

With the above presentation in hand, we now turn to deduce the desired lower and upper bounds on $v(t,x)$ and $\theta(t,x)$. In fact, we have

Corollary 3.1. *Under the conditions listed in Lemma 3.1, if we further assume that the parameters l_1, l_2 and b satisfy one of the following two conditions*

(i) $b\geq 8$;

(ii) $\frac{44l_1l_2+54l_1+32l_2+16}{6l_1l_2+7l_1+4l_2+2}<b<8$.

Then there exist positive constants $\underline{V}_2$, $\overline{V}_2$, $\underline{\Theta}_2$ and $\overline{\Theta}_2$ which depend only on the initial data $(v_0(x),u_0(x),\theta_0(x),z_0(x))$ and T, such that

$$\underline{V}_2\leq v(t,x)\leq\overline{V}_2,\quad \underline{\Theta}_2\leq\theta(t,x)\leq\overline{\Theta}_2,\quad \forall(t,x)\in[0,t]\times\Omega. \tag{3.59}$$

Proof. We first consider the case $b\geq 8$. In this case, the inequalities (3.30), (3.31) and (3.58) can be rewritten as

$$\frac{1}{v}\leq C+C\left\|\frac{1}{\theta}\right\|_\infty^{\frac{b-8}{2l_1}}, \tag{3.60}$$

$$v \le C + C\left\|\frac{1}{\theta}\right\|_{\infty}^{\frac{b-8}{2l_2+1}}, \tag{3.61}$$

and

$$\|\theta\|_{L^\infty(\Omega)} \le C + C\left\|\frac{1}{\theta}\right\|_{\infty}^{\left(\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}+1\right)(b-8)}. \tag{3.62}$$

With the above inequalities and Lemma 3.3 in hand, we can obtain the lower bound of $v(t,x)$ and the upper bound of $v(t,x), \theta(t,x)$ immediately.

For the case $b < 8$, (3.58) can be rewritten as

$$\|\theta\|_{L^\infty(\Omega)} \le C + C\|\theta\|_{\infty}^{\left(\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}+1\right)(8-b)}, \tag{3.63}$$

since $\frac{44l_1l_2+54l_1+32l_2+16}{6l_1l_2+7l_1+4l_2+2} < b < 8$, one has

$$0 < \left(\frac{l_1+2}{2l_1}+\frac{2}{2l_2+1}+1\right)(8-b) < 1. \tag{3.64}$$

With the inequalities (3.63) and (3.64) in hand, we can deduce the upper bound of $\theta(t,x)$ by using the Young inequality.

Having obtained the upper bound of $\theta(t,x)$, the lower bound and the upper bound of $v(t,x)$ can be obtained from Lemma 3.6. This completes the proof of the corollary.

□

With Corollary 3.9 in hand, Theorem 1.2 follows immediately from the standard continuation argument and we omit the details for brevity.

REFERENCES

[1] S.N. Antontsev, A.V. Kazhikov, and V.N. Monakhov, *Boundary Value Problems in Mechanics of Nonhomogeneous Fluids*, Amsterdam, New York, 1990.

[2] G.-Q. Chen, *Global solutions to the compressible Navier–Stokes equations for a reacting mixture*, SIAM J. Math. Anal., 23(3):609–634, 1992.

[3] Q. Chen, H.-J. Zhao, and Q.-Y. Zou, *Initial-boundary value problems to the one-dimensional compressible Navier–Stokes equations with degenerate transport coefficients*, Proc. Roy. Soc. Edinb. Sect. A, DOI:10.1017/S0308210516000342.

[4] C.M. Dafermos and L. Hsiao, *Global smooth thermomechanical processes in one-dimensional nonlinear thermoviscoelasticity*, Nonlinear Anal., 6(5):435–454, 1982.

[5] D. Donatelli and K. Trivisa, *On the motion of a viscous compressible radiative-reacting gas*, Commun. Math. Phys., 265(2):463–491, 2006.

[6] B. Ducomet, *On the stability of a stellar structure in one dimension*, Math. Models Methods Appl. Sci., 6(3):365–383, 1996.

[7] B. Ducomet, *On the stability of a stellar structure in one dimension II: The reactive case*, Math. Model. Number. Anal., 31(3):381–407, 1997.

[8] B. Ducomet, *A model of thermal dissipation for a one-dimensional viscous reactive and radiative gas*, Math. Methods Appl. Sci., 22(15):1323–1349, 1999.

[9] B. Ducomet, *Some stability results for reactive Navier–Stokes–Poisson systems*, Evolution Equations: Existence, Regularity and Singularities (Warsaw), 83–118, 1998.

[10] B. Ducomet, *Hydrodynamical models of gaseous stars*, Rev. Math. Phys., 7(8):957–1000, 1996.

[11] B. Ducomet and E. Feireisl, *The equations of magnetothydrodynamics: on the interaction between matter and radiation in the evolution of gaseous stars*, Commun. Math. Phys., 266(3):595–629, 2006.

[12] B. Ducomet and A. Zlotnik, *Stabilization for viscous compressible heat-conducting media equations with nonmonotone state functions*, C.R. Math. Acad. Sci. (Paris), 334(2):119–124, 2002.

[13] B. Ducomet and A. Zlotnik, *Stabilization for equations of one-dimensional viscous compressible heat-conducting media with nonmonotone equation of state*, J. Diff. Eqs., 194(1):51–81, 2003.

[14] B. Ducomet and A. Zlotnik, *Stabilization for 1D radiative and reactive viscous gas flow*, C.R. Acad. Sci.(Paris), 338(2):127–132, 2004.

[15] B. Ducomet and A. Zlotnik, *Lyapunov functional method for 1D radiative and reactive viscous gas dynamics*, Arch. Ration. Mech. Anal., 177(2):185–229, 2005.

[16] B. Ducomet and A. Zlotnik, *On the large-time behavior of 1D radiative and reactive viscous flows for higher-order kinetics*, Nonlinear Anal., 63(8):1011–1033, 2005.

[17] H.K. Jenssen and T.K. Karper, *One-dimensional compressible flow with temperature dependent transport coefficients*, SIAM J. Math. Anal., 42:904–930, 2010.

[18] S. Jiang, *Large-time behavior of solutions to the equations of a viscous polytropic ideal gas*, Ann. Mat. Pura Appl., 175:253–275, 1998.

[19] S. Jiang, *Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains*, Commun. Math. Phys., 200:181–193, 1999.

[20] S. Jiang, *Remarks on the asymptotic behaviour of solutions to the compressible Navier–Stokes equations in the half-line*, Proc. Roy. Soc. Edinb. A, 132:627–638, 2002.

[21] Y. Kanel', *On a model system of equations of one-dimensional gas motion*, Diff. Uravn., 4:374–380, 1968.

[22] B. Kawhohl, *Global existence of large solutions to initial boundary value problems for a viscous, heat-conducting, one-dimensional real gas*, J.Diff. Eqs., 58(1):76–103, 1985.

[23] A.V. Kazhikhov and V.V. Shelukhin, *Unique global solution with respect to time of initial boundary value problems for one-dimensinal equations of a viscous gas*, J. Appl. Math. Mech., 41(2):273–282, 1977.

[24] R. Kippenhahn and A. Weingert, *Stellar Structure and Evolution*, Springer-Verlag, Berlin, 1994.

[25] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N.Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, Amer. Math. Sot., Providence, RI, 1968.

[26] J. Li and Z. Liang, *Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier–Stokes system in unbounded domains with large data*, Arch. Ration. Mech. Anal., 220:1195–1208, 2016.

[27] H.-X. Liu, T. Yang, H.-J. Zhao, and Q.-Y. Zou, *One-dimensional compressible Navier–Stokes equations with temperature dependent transport coefficients and large data*, SIAM J. Math. Anal., 46(3):2185–2228, 2014.

[28] T. Luo, *On the outer pressure problem of a viscous heat-conductive one-dimensional real gas*, Acta Math. Appl. Sin. Engl. Ser., 13(3):251–264, 1997.

[29] D. Mihalas and B.W. Mihalas, *Foundations of Radiation Hydrodynamics*, Oxford Univ. Press, New York, 1984.

[30] R. Pan and W. Zhang, *Compressible Navier–Stokes equations with temperature dependent heat conductivity*, Commun. Math. Sci., 13:401–425, 2015.

[31] Y.-M. Qin, G.-L. Hu, T.-G. Wang, L. Huang, and Z.-Y. Ma, *Remarks on global smooth solutions to a 1D self-gravitating viscous radiative and reactive gas*, J. Math. Anal. Appl., 408(1):19–26, 2013.

[32] S.F. Shandarin and Y.B. Zel'dovichi, *The large-scale structure of the universe: Turbulence, intermittency, structures in a self-gravitating medium*, Rev. Modern Phys., 61(2):185–220, 1989.

[33] Z. Tan, T. Yang, H.-J. Zhao, and Q.-Y. Zou, *Global solutions to the one-dimensional compressible Navier–Stokes–Poisson equations with large data*, SIAM J. Math. Anal., 45(2):547–571, 2013.

[34] A. Tani, *On the first initial-boundary value problem for compressible viscous fluid motion*, Publ. Res. Inst. Math. Sci., 13:193–253, 1977.

[35] M. Umehara and A. Tani, *Global solution to one-dimensional equations for a self-gravitating viscous radiative and reactive gas*, J. Diff. Eqs., 234(2):439–463, 2007.

[36] M. Umehara and A. Tani, *Global solvability of the free-boundary problem for one-dimensinal motion of a self-gravitating viscous radiative and reactive gas*, Proc. Japan Acad. Ser. A Math. Sci., 84(7):123–128, 2008.

[37] T. Wang and H.-J. Zhao, *One-dimensional compressible heat-conducting gas with temperature-dependent viscosity*, Math. Models Methods Appl. Sci., 26(12):2237–2275, 2016.

[38] G.J.V. Wylen and R.E. Sonntag, *Fundamentals of Classical Thermodynamics*, Wiley, New York, 1985.

[39] Y.B. Zel'dovich and Y.P. Raizer, *Physics of Shock Waves and High-temperature Hydrodynamic Phenomena, vol. II*, Academic Press, New York, 1967.