COMMUN. MATH. SCI.
Vol. 15, No. 5, pp. 1403–1422

MEANFIELD GAMES AND MODEL PREDICTIVE CONTROL*

PIERRE DEGOND†, MICHAEL HERTY‡, AND JIAN-GUO LIU§

Abstract. Mean-Field Games are games with a continuum of players that incorporate the time-dimension through a control-theoretic approach. Recently, simpler approaches relying on the Best-Reply Strategy have been proposed. They assume that the agents navigate their strategies towards their goal by taking the direction of steepest descent of their cost function (i.e. the opposite of the utility function). In this paper, we explore the link between Mean-Field Games and the Best Reply Strategy approach. This is done by introducing a Model Predictive Control framework, which consists of setting the Mean-Field Game over a short time interval which recedes as time moves on. We show that the Model Predictive Control offers a compromise between a possibly unrealistic Mean-Field Game approach and the sub-optimal Best-Reply Strategy.

Keywords. Mean-Field Games; multi-agent systems.

AMS subject classifications. 90B30; 35L65.

1. Introduction

In the past years there has been a growing interest in particle or agent descriptions for sociological and economical processes [14,36]. Among the many examples are pedestrian flow dynamics [13,20], swarming problems [24,34], systemic risk problems in large scale economies [4,26,32,37,38], opinion formation models [1,2,42] or price formation models [11,12]. In particular, the study of the limit for infinitely many agents has been intensively studied in order to obtain for example qualitative results on pattern formation. Hierarchies of kinetic and continuum models are obtained using meanfield limits. Recently, the agent dynamics have been extended to include control-theoretic aspects. In order to pass to the limit in the number of agents or particles those control actions are typically closed-loop controls. The interplay of control action on the level of the agents as well as in the meanfield limit has been studied to various extends in different areas of applications. We give some examples.

In [1,2] control problems for opinion formation models in large agent populations [40] have been studied. Therein, suitable (feedback) control measures have been formulated in order to drive diverging opinions towards consensus.

In [13,20] control mechanism on the continuum level for pedestrian dynamics have been discussed and the corresponding agent based formulation introduced. The control measures should provide fast exit strategies in the case of pedestrian crowd dynamics.

In economics, a market may be modeled using a game theoretic setting, i.e. a set of agents endowed with strategies (and possibly other attributes) that they may play upon to maximize their utility function [26,36]. In a game, the utility function depends on the other agents' strategies. The proper functioning of a market is associated to a Nash equilibrium of this game, i.e. a set of strategies such that no agent can improve on his utility function by changing his own strategy, given that the other agents' strategies are fixed. At the market scale, the number and diversity of agents is huge and it is

*Received: April 10, 2016; accepted (in revised form): February 11, 2017. Communicated by Lorenzo Pareschi.

†Department of Mathematics, Imperial College London, London SW7 2AZ, United Kingdom (pdegond@imperial.ac.uk).

‡RWTH Aachen University, Department of Mathematics, RWTH Aachen University, D-52062 Aachen, Germany (herty@mathc.rwth-aachen.de).

§Department of Physics and Department of Mathematics, Duke University, Durham, NC 27708, USA (jliu@phy.duke.edu).

more effective to use games with a continuum of players. Games with a continuum of players or agents have been discussed for example in [4, 32, 37, 38]. A recent work [17], a meanfield limit is considered to model risk associated with inter-bank borrowing and lending strategies.

Here, we consider a control-theoretic approach for possibly infinitely many agents, where the optimal goal is not a simple Nash equilibrium, but a whole set of optimal trajectories of the agents in the strategy space. Those problems may have applications in economics, crowd dynamics, pricing models or opinion formation. Such an approach has been formalized in the seminal work of [31] and popularized under the name of 'Mean-Field Game (MFG)'. It has given rise to an abundant literature, among which (to cite only a few) [6–8, 16, 29].

However, the fact that the agents are able to optimize their trajectory over a large time horizon in the spirit of physical particles subjected to the least action principle can be seen as a bit unrealistic. A related but different approach has been proposed for example in [23] and builds on earlier work on pedestrian dynamics [20]. It consists in assuming that agents perform the so-called 'Best-Reply Strategy' (BRS): they determine a local (in time) direction of steepest descent (of the cost function, i.e. minus the utility function) and evolve their strategy variable in this direction. This approach has been applied to models describing the evolution of the wealth distribution among interacting economies, in the case of conservative [21] and nonconservative economies [22]. However, the link between MFG and BRS was still to be elaborated. This is the object of the present paper. We show that the BRS can be obtained as a MFG over a short interval of time which recedes as times evolves. This type of control is known as Model Predictive Control (MPC) or as Receding Horizon Control. The fact that the agents may be able to optimize the trajectories in the strategy space over a small but finite interval of time is certainly a reasonable assumption and this MPC strategy could be viewed as an improvement over the BRS and some kind of compromise between the BRS and a fully optimal but fairly unrealistic MFG strategy. In this paper though, we propose a general framework to connect BRS to MFG through MPC.

Recently, many contributions on meanfield games and control mechanisms for particle systems have been made. For more details on meanfield games we refer to [6–8, 16, 29, 31]. Among the many possible meanfield games to consider we are interested in differential (Nash) games of possibly infinitely many particles (also called players). Most of the literature in this respect treats theoretical and numerical approaches for solving the Hamilton–Jacobi Bellmann (HJB) equation for the value function of the underlying game, see e.g. [16] for an overview. Solving the HJB equation allows to determine the optimal control for the particle game. However, the associated HJB equation posses several theoretical and numerical difficulties among which the need to solve it backwards in time is the most severe one, at least from a numerical perspective. Therefore, recently model predictive control (MPC) concepts on the level particles or of the associated kinetic equation have been proposed [1, 15, 15, 19–22, 25]. While MPC has been well established in the case of finite-dimensional problems [28, 33, 39], and also in engineering literature under the term receding horizon control, contributions to systems of infinitely many interacting particles and/or game theoretic questions related to infinitely many particles are rather recent. It has been shown that MPC concepts applied to problems of infinitely many interacting particles have the advantage to allow for efficient computation [1, 21]. However, by construction MPC only leads to suboptimal solutions, see for example [30] for a comparison in the case of simple opinion formation model. Also, the existing approaches mostly for alignment models do not necessarily treat game theoretic concepts but focus on for example sparse global controls [15,19,25], time-scale separation and local mean-field controls [22] called best-reply strategy, or MPC on very short time-scales [1] called instantaneous control. Typically the MPC strategy is obtained solving an auxiliary problem (implicit or explicit) and the resulting expression for the control is substituted back into the original dynamics leading to a possibly modified and new dynamics. Then, a meanfield description is derived using Boltzmann or a macroscopic approximation. This requires the action of the control to be *local* in time and independent of future states of the system contrary to solutions of the HJB equation. Usually in MPC approaches independent optimal control problems are solved where particles do not anticipate the optimal control choices other particles contrary to meanfield games [31].

In this paper we contribute to the recent discussion by formal computations leading to a link between meanfield games and MPC concepts proposed on the level of particle games and associated kinetic equations. The relationship we plan to establish is highlighted in Figure 1.1. More precisely, we want to show that the MPC concept of the best-reply strategy [21] may be at least formally be derived from a meanfield games context.

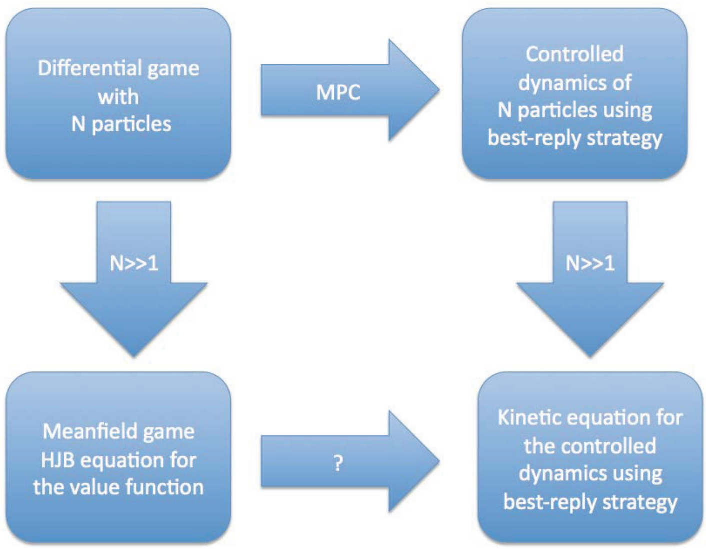

FIG. 1.1. *Relation between MPC concepts and meanfield games. The starting point are finite-dimensional differential games with N players in the top left part (Section 2). The connection for $N \to \infty$ of this games has been investigated for example in [16, 31] and leads to the HJB for meanfield games in the bottom left part of the figure (Section 3). If applying MPC concepts to the differential game as for example the best-reply strategy we obtain a controlled dynamics for N particles in the top right part [21] (Section 2.1). The meanfield limit for $N \to \infty$ leads to a kinetic equation in the bottom right part (Section 2.2). This paper also investigates the link between the meanfield game and the kinetic equation indicated by a question mark.*

2. Setting of the problem

In this section we introduce the basic notation of the studied problem. We consider nonlinear controlled particle dynamics. Each particle follows an ordinary differential equation and each particle has its own control. The control minimizes a suitable cost functional. This depends on the state of all other particles as well as their choice of control. The particle dynamics are particular in the sense that they are symmetric in order to allow for a suitable meanfield limit when the number of particles tend to infinity.

We consider N particles labeled by $i=1,\dots,N$ where each particle has a state $x_i \in \mathbb{R}$. We denote by $X=(x_i)_{i=1}^N$ the state of all particles and by $X_{-i}=(x_j)_{j=1,j\neq i}^N$ the states of all particles except i. Further, we assume that each particle's dynamics is governed by a smooth function $f_i:\mathbb{R}^N \to \mathbb{R}$ depending on the state X and we assume that each particle may control its dynamics by a control u_i. The dynamics for the particles $i=1,\dots,N$ is then given by

$$\frac{d}{dt}x_i(t)=f_i(X(t))+u_i(t), i=1,\dots,N, \tag{2.1}$$

and initial conditions

$$x_i(0)=\bar{x}_i. \tag{2.2}$$

We will drop the time-dependence of the variables whenever the intention is clear. Examples of models of the type (2.1) are alignment models in socio-ecological context, microscopic traffic flow models, production and many more, see e.g. the recent survey [34, 35, 41]. In recent contributions to control theory for equation (2.1) the case of a *single* control variable $u_i \equiv u$ for all i has been considered [1, 2, 15]. Here, we allow each particle to chose its own control strategy u_i. We suppose a control horizon of $T>0$ be given. As in [16] we suppose that particle i minimizes its own objective functional and determines therefore the optimal u_i^* by

$$u_i^*(\cdot)= \text{ argmin }_{u_i:[0,T]\to\mathbb{R}} \int_0^T \left(\frac{\alpha_i(s)}{2}u_i^2(s)+h_i(X(s))\right)ds, i=1,\dots,N. \tag{2.3}$$

Herein, $X(s)$ is the solution to equations (2.1) and (2.2). The optimal control and the corresponding optimal trajectory will from now on be denoted with superscript $*$. The minimization is performed on all sufficiently smooth functions $u_i:[0,T]\to\mathbb{R}$. There is no restriction on the control u_i similar to [31]. The objective $h_i:\mathbb{R}^N\to\mathbb{R}$ related to particle i is also supposed to be sufficiently smooth. The weights of the control $\alpha_i(t)>0, \forall i, t\geq 0$ and under additional conditions convexity of each optimization problem (2.3) is guaranteed. We will consider open-loop Nash equilibria [10, Definition 4.1]. For general cost functions and dynamics there is no guarantee that such points exists. However, there is a subclass of problems where a solution can be expected under additional regularity and convexity assumptions [10, Lemma 2.1]. The presented cost functional and dynamics belong to this class due to the linear dependence of the dynamics on the control and the decoupling of the control in the cost.

A challenge in solving the problem (2.3) relies on the fact that the associated HJB has to be solved backwards in time. Contrary to [1, 15] problem (2.3) are in fact N optimization problems that need to be solved *simultaneously* due to the dependence of X on $U=(u_i)_{i=1}^N$ through equation (2.1). This implies that each particle i *anticipates the optimal* strategy of all other particles U_{-i}^* when determining its optimal control u_i^*. Obviously, the problem (2.3) is simpler when each particle i *anticipates a fixed strategy* of all other particles U_{-i}. The optimization problems (2.3) decouple but the dynamics is still coupled. It has been argued that this is the case for reaction in pedestrian motions [22]. In fact, therein the following best-reply strategy has been proposed as a substitute for problem (2.1)

$$u_i(t)=-\partial_{x_i}h_i(X(t)), t\in[0,T]. \tag{2.4}$$

As in the meanfield theory presented in [16,31] we need to impose Assumption **(A)** on $f_i(X)$ and $h_i(X)$ before passing to the limit $N\to\infty$. The Assumption **(B)** will be used in Section 3.

(A) For all $i=1,\dots,N$ and any permutation $\sigma_i:\{1,\dots,N\}\backslash\{i\}\to\{1,\dots,N\}\backslash\{i\}$ we have

$$f_i(X)=f(x_i,X_{-i}) \text{ and } f(x_i,X_{-i})=f(x_i,X_{\sigma_i}),$$

for a smooth function $f:\mathbb{R}\times\mathbb{R}^{N-1}\to\mathbb{R}$ and where $X_{\sigma_i}=(x_{\sigma_i(j)})_{j=1,j\neq i}^N$. Further we assume that for each i the function $h_i(X)$ enjoys the same properties as stated for $f_i(X)$.

(B) We assume that $\alpha_i(t)=\alpha(t)$ for all $t\in[0,T]$ and all $i=1,\dots,N$.

Under additional growth conditions sequences of symmetric functions in many variables have a limit in the space of functions defined on probability measures, see e.g. [16, Theorem 2.1], [8, Theorem 4.1]. The corresponding result is recalled as Theorem 4.1 in the appendix for convenience.

To exemplify computations later on we will use a basic alignment or consensus model. This model has been extended to model wealth evolution by including additional factors like randomness and limited resources, see e.g. [9,18,22,23] The basic alignment problem is

$$f_i(X)=\frac{1}{N}\sum_{j=1}^N P(x_i,x_j)(x_j-x_i), \tag{2.5}$$

for some bounded, non-negative and smooth function $P(x,\tilde{x})$. Clearly, f fulfills Assumption **(A)**. As objective function we use a measure depending only on aggregated quantities as in [21]. An example fulfilling Assumption **(A)** is

$$h_i(X)=\frac{1}{N-1}\sum_{j=1,j\neq i}^N \phi(x_i,x_j) \tag{2.6}$$

for some smooth function $\phi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Finally, we introduce some additional notation. We denote by $\mathcal{P}(\mathbb{R})$ the space of Borel probability measures over $\mathbb{R}$. The empirical discrete probability measure $m^N\in\mathcal{P}(\mathbb{R})$ concentrated at a positions $X\in\mathbb{R}^N$ is denoted by

$$m_X^N=\frac{1}{N}\sum_{i=1}^N\delta(x-x_i).$$

We also use this notation if X is time dependent, i.e., $X=X(t)$, leading to the family of probability measures $m_X^N=m_X^N(t)=\frac{1}{N}\sum_{i=1}^N\delta(x-x_i(t))$. If the intention is clear we do not explicitly denote the dependence on x of the measure m_X^N (and on time t if $X=X(t)$ is time-dependent).

Based on the Assumption **(A)** we will frequently use Theorem [16, Theorem 2.1], see Theorem 4.1. In view of this theorem we will denote the limit of a family of functions $(f^N)_N:\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$ by $\mathbf{f}:\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$ and write

$$f(\xi,X_{-i})=f^N(\xi,m_{X_{-i}}^{N-1})\sim\mathbf{f}(\xi,m_X^N).$$

2.1. From differential games to controlled particle dynamics. In this section we derive a closed form of the optimal control for each particle using an approximation called MPC. By suitable discretization and approximation it is possible to break the complexity of the coupled optimization problems introduced in the previous section. The derived closed form allows to reformulate the particle dynamics in terms of the states of the remaining particles only.

The best-reply strategy (2.4) is obtained also from a MPC approach [33] applied to equations (2.1) and (2.3). In order to derive the best-reply strategy we consider the following problem: suppose we are given the state $X(t)$ of the system (2.1) at time $t>0$. Then, we consider a control horizon of the MPC of $\Delta t>0$ and supposedly small. We assume that the applied control $u_i(s)$ on $(t,t+\Delta t)$ is constant. For particle i we denote the unknown constant by $\tilde{u}_i$. Instead of solving the problem (2.3) on the full time interval (t,T), we consider the objective function only on the receding time horizon $(t,t+\Delta t)$. We also scale the cost by $\frac{1}{\Delta t}$ on this interval to have a meaningful integral for small Δt. Further, we discretize the dynamics (2.1) on $(t,t+\Delta t)$ using an explicit Euler discretization for the initial value $\bar{X}=X_i(t)$. We discretize the objective function by a Riemann sum. A naive discretization leads to a penalization of the control of the type $\frac{\alpha_i(t+\Delta t)}{2}\tilde{u}^2$. Since the explicit Euler discretization in equation (2.7) is only accurate up to order $O((\Delta t)^2)$ we additionally require to have $\tilde{u}_i=O(1)$ to obtain a meaningful control in the discretization (2.7) and also in the limit for $\Delta t\to 0$. A crucial point of the scaling is in α_i that is altered to $\Delta t\,\alpha_i$. This may be explained as follows: the term $\int_0^T \alpha_i(s)u^2(s)ds$ defines a global energy that is supposed to be minimized. In the MPC framework each time interval does not see the global energy and no scaling of α_i would give a contribution of u on a time interval $[t,t+\Delta t]$ towards the global energy of order $O(\Delta t)$ that will vanish for $\Delta t\to 0$. In order to compensate this the penalty α_i is scaled. Finally, this leads to a MPC problem associated with equation (2.3) and given by

$$x_i(t+\Delta t)=\bar{x}_i+\Delta t\left(f_i(\bar{X})+\tilde{u}_i\right), \qquad i=1,\dots,N, \tag{2.7}$$

$$\tilde{u}_i= \operatorname{argmin}_{\tilde{u}\in\mathbb{R}}\left(h_i\left(X(t+\Delta t)\right)+\Delta t\frac{\alpha_i(t+\Delta t)}{2}\tilde{u}^2\right), \qquad i=1,\dots,N. \tag{2.8}$$

Solving the minimization problem (2.8) leads to

$$\alpha_i(t+\Delta t)\,\tilde{u}_i=-\partial_{x_i}h_i(\bar{X}),\, i=1,\dots,N.$$

Now, we obtain a $\tilde{u}_i$ of order $O(1)$ by Taylor expansion of α_i at time t. Within the MPC approach the control for the time interval $(t,t+\Delta t)$ is therefore given by equation (2.9).

$$\tilde{u}_i=-\frac{1}{\alpha_i(t)}\partial_{x_i}h_i(\bar{X}),\, i=1,\dots,N. \tag{2.9}$$

Usually, the dynamics (2.7) is then computed with the computed control up to $t+\Delta t$. Then, the process is repeated using the new state $X(t+\Delta t)$. Substituting (2.9) into (2.7) and letting $\Delta t\to 0$ we obtain

$$\frac{d}{dt}x_i(t)=f_i(X(t))-\frac{1}{\alpha_i(t)}\partial_{x_i}h_i(X(t)),\, i=1,\dots,N, t\in[0,T]. \tag{2.10}$$

This dynamics coincide with the dynamics generated by the best-reply strategy (2.4) provided that $\alpha_i(t)\equiv 1$. Therefore, on a particle level the controlled dynamics (2.10) of the best-reply strategy [21] is equivalent to a MPC formulation of the problem (2.3). For the toy example we obtain

$$\frac{d}{dt}x_i = \frac{1}{N}\sum_{j=1}^{N} P(x_i,x_j)(x_j-x_i) - \frac{1}{(N-1)\alpha_i(t)}\sum_{j=1,j\neq i}^{N}\partial_{x_i}\phi(x_i,x_j). \tag{2.11}$$

2.2. From controlled particle dynamics (2.10) **to kinetic equation.** In this section we derive the meanfield limit of the previous dynamics. Those equations will be compared later with the meanfield limit of the full coupled optimization problem.

The considerations herein have essentially been studied for the best-reply strategy in the series of papers [21–23] and it is only repeated for convenience. In order to pass to the meanfield limit we assume that Assumptions **(A)** and **(B)** hold true. We also identify the measure with its density function without explicitly stating this in the following and subsequent sections. Then the particles are governed by

$$\frac{d}{dt}x_i(t) = f(x_i(t),X_{-i}(t)) - \frac{1}{\alpha(t)}\partial_{x_i}h(x_i(t),X_{-i}(t)),\ i=1,\dots,N. \tag{2.12}$$

Associated with the trajectories $X=X(t)$ the discrete probability measure m_X^N is given by $m_X^N = \frac{1}{N}\sum_{j=1}^{N}\delta(x-x_i(t))$. Using the weak formulation for a test function $\psi:\mathbb{R}\to\mathbb{R}$ we compute the dynamics of m_X^N over time as

$$\frac{d}{dt}\int \psi(x)m_X^N dx = \frac{1}{N}\sum_{i=1}^{N}\int \psi'(x)\left(f(x,X_{-i}) - \frac{1}{\alpha}\partial_x h(x,X_{-i})\right)\delta(x-x_i(t))dx.$$

Using [16, Theorem 2.1] and denoting by $m_{X_{-j}}^{N-1}(t) = \frac{1}{N-1}\sum_{k=1,k\neq j}\delta(x-x_k(t))$ a family of empirical measures on $\mathbb{R}$ we obtain from the previous equation

$$\frac{d}{dt}\int \psi(x)m_X^N dx = \frac{1}{N}\sum_{i=1}^{N}\int \psi'(x)\left(f^N(x,m_{X_{-i}}^{N-1}) - \frac{1}{\alpha}\partial_x h^N(x,m_{X_{-i}}^{N-1})\right)\delta(x-x_i(t))dx,$$

for some function $f^N,h^N:\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$. Assume that f and h fulfill the assertions of [8, Theorem 4.1]. Then, $\mathbf{f}(x,m_X^N)\sim f^N(x,m_{X_{-i}}^{N-1})$ and $\mathbf{h}(x,m_X^N)\sim h^N(x,X_{-i})$ and

$$\frac{d}{dt}\int \psi(x)m_X^N dx = \int \psi'(x)m_X^N\left(\mathbf{f}(x,m_X^N) - \frac{1}{\alpha}\partial_x\mathbf{h}(x,m_X^N)\right)dx.$$

This is the weak form of the kinetic equation for a probability measure $m=m(t,x)$

$$\partial_t m + \partial_x\left(m\left(\mathbf{f}(x,m) - \frac{1}{\alpha}\partial_x\mathbf{h}(x,m)\right)\right) = 0. \tag{2.13}$$

3. Results related to meanfield games

In this paragraph we consider the limit of the problem (2.3) for a large number of particles. This has been investigated for example in [31] and derivations (in a slightly different setting) have been detailed in [16, Section 7]. In order to show the links presented in Figure 1.1 we require to extend and further develop the computations in [16, Section 7].

A notion of solution to the competing N optimization problem (2.3) is the concept of Nash equilibrium, in fact, we study the so-called open loop Nash equilibrium. If it exists it may be computed for the differential games using the HJB equation. We briefly present computations leading to the HJB equation. Then, we discuss the large particle limit of the HJB equation and derive the best-reply strategy.

3.1. Derivation of the finite-dimensional HJB equation. In this section we solve the coupled optimization problem introduce in Section 2. In order to obtain a closed form of the solution we introduce the HJB equations and state the formal optimality conditions. An approximation to the dynamics of the HJB equations shows on the level of the particles already the similarity to the BRS or MPC.

The HJB equation describes the evolution of a objective function $V_i = V_i(t,Y)$ of the particle i defined as the future costs for a particle trajectory governed by equation (2.1) and starting at time $t \in (0,T)$, position Y and control $u_i : (t,T) \to \mathbb{R}, i = 1,\dots,N$,

$$V_i(t,Y) = \int_t^T \left(\frac{\alpha_i(s)}{2} u_i^2(s) + h_i(X(s)) \right) ds, \tag{3.1}$$

where $X(s) = (x_i(s))_{i=1}^N$ is the solution to equation (2.1) with control U and initial condition

$$X(t) = Y. \tag{3.2}$$

Among all possible controls u_i we denote by u_i^* the optimal control that minimizes $V_i(t,Y)$. We investigate the relation of V_i of particle i to the optimal control u_i^*. To this end assume that the coupled problem (2.3) has a unique solution denoted by $U^* = (u_i^*)_{i=1}^N$. Each $u_i^* : [t,T] \to \mathbb{R}$ for each $i = 1,\dots,N$, is hence a solution to

$$u_i^* = \text{ argmin }_{u_i(\cdot):[t,T]\to\mathbb{R}} \{V_i(t,Y) : X \text{ solves (2.1)}\}, \, i = 1,\dots,N.$$

The corresponding particle trajectories are denoted by $X^* = (x_i^*)_{i=1}^N$ and are obtained through (2.1) for an initial condition $X^*(0) = \bar{X}$.

Since $X^*(\cdot)$ depends on U^*, minimizing the objective function (3.1) leads to the computation of formal derivatives of V_i with respect to u_i. The optimal control u_i^* is then found as formal point (in function space) where the derivative of V_i with respect to u_i vanishes:

$$\frac{d}{du_i} V_i(t,Y)[v] = \int_t^T \left(\alpha_i(s) u_i^*(s) + \sum_{k=1}^N \partial_{x_k} h_i(X^*(s)) \partial_{u_i}(x_k^*(s)) \right) v(s) ds = 0. \tag{3.3}$$

The derivative is not easily computed due to the unknown derivative of each state x_k^* with respect to the acting control u_i^*. However, choosing a set of suitable co-states $\phi_j^i :$ $[0,T] \to \mathbb{R}$ for $i = 1,\dots,N$ and $j = 1,\dots,N$, we may simplify the previous equation (3.3): we test equation (2.1) by functions $\phi_j^i : [0,T] \to \mathbb{R}$ for $i,j = 1,\dots,N$ such that $\phi_j^i(T) = 0$, integrate on (t,T) with $0 \le t < T$, sum over all particles and use the initial data at $X^*(t) = Y$ to obtain

$$\sum_{j=1}^N \left\{ \int_t^T -\frac{d}{ds}(\phi_j^i(s)) x_j^*(s) - \phi_j^i(s) \left(f_j(X^*(s)) + u_j^*(s) \right) ds - \phi_j^i(t) y_j \right\} = 0, \, i = 1,\dots,N.$$

The derivative with respect to u_i in an arbitrary direction v is then

$$\int_t^T \left\{ \sum_{j=1}^N \left(-\frac{d}{ds}(\phi_j^i(s))\partial_{u_i}\left(x_j^*(s)\right) - \phi_j^i(s)\left(\sum_{k=1}^N \partial_{x_k}\left(f_j(X^*(s))\right)\partial_{u_i}(x_k^*(s))\right)\right) - \phi_i^i(s) \right\} \times v(s)ds = 0. \quad (3.4)$$

The previous equation can be equivalently rewritten as

$$\sum_{k=1}^N \left(-\frac{d}{ds}\phi_k^i(s) - \sum_{j=1}^N \phi_j^i(s)\partial_{x_k} f_j(X^*(s)) \right) \partial_{u_i}(x_k^*(s))v(s)ds = \int_t^T \phi_i^i(s)v(s)ds.$$

Let ϕ_j^i for $i,j=1,\ldots,N$ fulfill the coupled linear system of adjoint equations (or co-state equations), solved backwards in time,

$$-\frac{d}{dt}\phi_j^i(t) - \sum_{k=1}^N \phi_k^i(t)\partial_{x_j}(f_k(X^*(t)) = \partial_{x_j} h_i(X^*(t)),\ \phi_j^i(T) = 0. \quad (3.5)$$

Then, formally for every $s \in (t,T)$ we have

$$\begin{aligned}
&\sum_{j=1}^N \partial_{x_j} h_i(X^*(s))\partial_{u_i}(x_j^*(s)) \\
&= \sum_{k=1}^N -\frac{d}{dt}\phi_k^i(s)\partial_{u_i}(x_k^*(s)) - \sum_{j=1}^N\sum_{k=1}^N \phi_j^i(s)\partial_{x_k}(f_j(X^*(s))\partial_{u_i}(x_k^*(s)).
\end{aligned}$$

and it follows that

$$\sum_{j=1}^N \partial_{x_j} h_i(X^*(s))\partial_{u_i}\left(x_j^*(s)\right) = \phi_i^i(s), \forall s \in (t,T).$$

At optimality the necessary condition is for a.e. $s \in (t,T)$,

$$(\alpha_i(s)u_i^*(s)) + \sum_{j=1}^N \partial_{x_j} h_i(X^*(s))\partial_{u_i}\left(x_j^*(s)\right) = 0.$$

This leads to the following equation a.e. $s \in (t,T)$

$$\alpha_i(s)u_i^*(s) + \phi_i^i(s) = 0. \quad (3.6)$$

From now on we assume that the associated optimal controls u_i^* fulfill this system. The corresponding trajectories and co-state are denoted by $\mathcal{S}$. We formally derive the HJB based on the previous equations of PMP and refer to [27, Chapter 8] for a careful theoretical discussion.

Consider the function $V_i(t,Y)$ evaluated along the optimal trajectory $\mathcal{S}$, i.e., let $\mathcal{V}_i(t) = V_i(t,X^*(t))$. Then, by definition of V_i and $\mathcal{S}$ we have

$$-\frac{\alpha_i(t)}{2}(u_i^*)^2(t) - h_i(X^*(t)) = \frac{d}{dt}\mathcal{V}_i(t)$$

$$=\partial_t V_i(t,X^*(t))+\sum_{k=1}^{N}\partial_{x_k}V_i(t,X^*(t))\left(f_k(X^*(t))+u_k^*(t)\right).$$

Using the necessary condition (3.6) we obtain

$$\begin{aligned}&-\frac{1}{2\alpha_i(t)}(\phi_i^i)^2(t)-h_i(X^*(t))\\&=\partial_t V_i(t,X^*(t))+\sum_{k=1}^{N}\partial_{x_k}V_i(t,X^*(t))\left(f_k(X^*(t))-\frac{1}{\alpha_k(t)}\phi_k^k(t)\right).\end{aligned}\tag{3.7}$$

The trajectory of $X^*(s)$ depends on the initial condition $Y=(y_i)_{i=1}^N$. The variation of $V_i(t,Y)$ with respect to y_o for $o\in\{1,\dots,N\}$ and evaluating at $\mathcal{S}$ can be explicitly computed using the weak form of the state equation. Using the co-state equation we obtain $\nabla_Y V_i(t,Y)=(\phi_k^i)_{k=1}^N$ provided that ϕ_k^i is a solution to equation (3.5).

Now, along $\mathcal{S}$ we may express in equation (3.7) the co-state by the derivative of V_i with respect to Y. Replacing $Y=X^*(t)$ we obtain

$$\begin{aligned}&-\frac{1}{2\alpha_i(t)}\left(\partial_{x_i}V_i(t,X^*(t))\right)^2-h_i(X^*(t))\\&=\partial_t V_i(t,X^*(t))+\sum_{k=1}^{N}\partial_{x_k}V_i(t,X^*(t))\left(f_k(X^*(t))-\frac{1}{\alpha_k(t)}\partial_{x_k}V_k(t,X^*(t))\right).\end{aligned}$$

By definition we have $V_i(T,X)=0$ for all X. Therefore, instead of solving the PMP equation we may ask to solve the N HJB for $V_i=V_i(t,X)$ on $[0,T]\times\mathbb{R}^N$ for $i=1,\dots,N$ given by the reformulation of the previous equation:

$$\begin{aligned}&\partial_t V_i(t,X)+\sum_{k=1,k\neq i}^{N}\partial_{x_k}V_i(t,X)\left(f_k(X)-\frac{1}{\alpha_k(t)}\partial_{x_k}V_k(t,X)\right)+\partial_{x_i}V_i(t,X)f_i(X)\\&=-h_i(X)+\frac{1}{2\alpha_i(t)}(\partial_{x_i}V_i(t,X))^2,\end{aligned}\tag{3.8}$$

with terminal condition

$$V_i(T,X)=0,\ i=1,\dots,N.\tag{3.9}$$

REMARK 3.1. There are shorter ways to derive the HJB equation (3.8), see [27]. We pursued the presented way in order to compare it with the limit $N\to\infty$ in the subsequent discussion.

The aspect of the game theoretic concept is seen in the HJB equation (3.8) in the mixed terms $\partial_{x_k}V_i$. If we model particles i that do not anticipate the optimal choice of the control of other particles $j\neq i$, then N minimization problems for V_i in equation (3.1) are independent. Therefore the corresponding HJB for V_i and V_j with $j\neq i$ decouple and all mixed terms vanish. In a different setting this situation has been studied in [1,2] where only a single control for all particles is present.

Assume that we have a (sufficiently regular) solution $(V_i)_{i=1}^N$ with $V_i:[0,T]\times\mathbb{R}^N\to\mathbb{R}$. Then, we obtain the optimal control $u_i^*(t)$ and the optimal trajectory $X^*(t)$ for minimizing V_i by

$$u_i^*(t)=-\frac{1}{\alpha_i(t)}\partial_{x_i}V_i(t,X^*(t)),\ i=1,\dots,N,\tag{3.10}$$

where X^* fulfills equation (2.1). Then, the associated controlled dynamics are given by

$$\frac{d}{dt}x_i(t) = f_i(X(t)) - \frac{1}{\alpha_i(t)}\partial_{x_i}V_i(t,X(t)),\, j=1,\dots,N, \tag{3.11}$$

and initial conditions (2.2). Comparing the HJB controlled dynamics with equation (2.10) we observe that in the best-reply strategy the full solution to the HJB is not required. Instead, $\partial_{x_i}V_i(t,X)$ is approximated by $\partial_{x_i}h_i(X(t))$. This approximation is also obtained using a discretization of equation (3.8) in a MPC framework. Since the equation for V_i is backwards in time we may use a semi discretization in time on the interval $(T-\Delta t, T)$ given by

$$\frac{V_i(T,X)-V_i(T-\Delta t,X)}{\Delta t} + \sum_{k=1,k\neq i}^{N} \partial_{x_k}V_i(T,X)\left(f_k(X) - \frac{1}{\alpha_k(t)}\partial_{x_k}V_k(T,X)\right)$$
$$+\partial_{x_i}V_i(T,X)f_i(X) = -h_i(X) + \frac{1}{2\alpha_i(t)}(\partial_{x_i}V_i(T,X))^2 + O(\Delta t),$$
$$V_i(T,X)=0.$$

Using the terminal condition we obtain that $V_i(T-\Delta t,X) = h_i(X)$ for all $X\in\mathbb{R}^N$.

The derivation of the equation for the HJB equation for $V_i(t,Y)$ allows for an arbitrary choice of $T>t$. Hence we may set the terminal time T also to $T:=t+\Delta t$. If we consider the function

$$V_i^{\Delta t}(t,Y) = \frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\frac{\Delta t\,\alpha_i(s)}{2}u_i^2(s) + h_i(X(s))\right)ds,$$

where $X(s), s\in(t,t+\Delta t)$ fulfills the dynamics and where we indicate the dependence on Δt by a superscript on V_i. Note that the scaling of the weight α_i is done as in the MPC approach for the discretized problem (2.7) and (2.8). Applying the explicit Euler discretization as shown before leads therefore to

$$V_i^{\Delta t}(t,Y) = h_i(Y),\, Y = X(t).$$

Hence, the best-reply strategy applied at time t for a finite-dimensional problem of N interacting particles coincides with an explicit Euler discretization of the HJB equation for a function given by $V_i^{\Delta t}(t,Y)$ where $Y=X(t)$ is the state of the particle system at time t.

3.2. Meanfield limit of the HJB equation (3.8)**.** Due to the similarity of the approximation to the dynamics of the HJB equations for the particles we expect a similar relation on the meanfield limit. To this end we require a meanfield limit of the HJB equations. Using the strong symmetry in the dynamics and the cost function we obtain the corresponding meanfield equation.

We turn to the meanfield limit of equation (3.8) for $i=1,\dots,N$. To this end we assume that Assumptions **(A)** and **(B)** hold. We further recall and introduce some notation;

$$X=(x_i)_{i=1}^N,\, Z=(z_i)_{i=1}^N,\, \mathbb{Z}=(\eta,z_1,\dots,z_{N-1}),\, \mathbb{Z}_k := (z_k,\eta,z_1,\dots,z_{k-1},z_{k+1},\dots,z_{N-1}).$$

We obtain the following set of equations for $V_i(t,X)$ and $i=1,\dots,N$,

$$\partial_t V_i(t,X) + \sum_{k=1,k\neq i}^{N}\partial_{x_k}V_i(t,X)\left(f(x_k,X_{-k}) - \frac{1}{\alpha(t)}\partial_{x_k}V_k(t,X)\right)$$

$$+\partial_{x_i} V_i(t,X) f(x_i,X_{-i}) = -h(x_i,X_{-i}) + \frac{1}{2\alpha(t)} \left(\partial_{x_i} V_i(t,X)\right)^2, \qquad V_i(t,X)=0. \tag{3.12}$$

We show that a solution $(V_i)_{i=1}^N$ to the previous set of equations is obtained by considering the equation (3.13) below. Suppose that a function $W = W(t,\mathbb{Z}) : [0,T] \times \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}$ fulfills

$$\partial_t W(t,\mathbb{Z}) + \sum_{k=1}^{N-1} \partial_{z_k} W(t,\mathbb{Z}) \left(f(\mathbb{Z}_k) - \frac{1}{\alpha(t)} \partial_\eta W(t,\mathbb{Z}) \right) + \partial_\eta W(t,\mathbb{Z}) f(\mathbb{Z})$$
$$= -h(\mathbb{Z}) + \frac{1}{2\alpha(t)} \left(\partial_\eta W(t,\mathbb{Z})\right)^2, \tag{3.13}$$

and terminal condition $W(T,\mathbb{Z}) = 0$. Suppose a solution W to equation (3.13) exists and fulfills the previous equation pointwise a.e. $(t,\mathbb{Z}) \in [0,T] \times \mathbb{R}^N$. Then, we define

$$V_i(t,X) := W(t,x_i,X_{-i}),\, i = 1,\dots,N. \tag{3.14}$$

By definition $W = W(t,\mathbb{Z})$, therefore the partial derivatives of V_i are computed as follows where

$$(x_i, X_{-i}) = (\eta, z_1, \dots, z_{N-1}):$$

$$\begin{aligned}
\partial_t V_i(t,X) &= \partial_t W(t,x_i,X_{-i}),\ \partial_{x_i} V_i(t,X) = \partial_\eta W(t,x_i,X_{-i}),\\
\partial_{x_k} V_k(t,X) &= \partial_{x_k} W(t,x_k,X_{-k}) = \partial_\eta W(t,x_k,X_{-k}),\\
\partial_{x_k} V_i(t,X) &= \partial_{z_k} W(t,\mathbb{Z}) \ \text{ for } k \in \{1,\dots,i-1\},\\
\partial_{x_k} V_i(t,X) &= \partial_{z_{k-1}} W(t,\mathbb{Z}) \ \text{ for } k \in \{i+1,\dots,N\}.
\end{aligned}$$

Due to Assumption **(A)** we have that

$$f(\mathbb{Z}_k) = f(z_k, z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_{i-1}, \eta, z_i, \dots, z_{N-1}),$$

for any $i \in \{1,\dots,N-1\}$. The same is true for the argument of h. Therefore,

$$f(x_k, X_{-k}) = f(\mathbb{Z}_k) \text{ and } h(x_i, X_{-i}) = h(\mathbb{Z}).$$

Therefore, $V_i(t,X) = W(t,x_i,X_{-i})$ fulfills equation (3.12). Hence, instead of studying equation (3.12) we may study the limit for $N \to \infty$ of equations (3.13). In view of Theorem 4.1 a limit exists provided W is symmetric (and fulfills uniform bound and uniform continuity estimates).

Note that, W as a solution to equation (3.13) is symmetric with respect to the argument $(z_1,\dots,z_{N-1})$. This holds true, since f and h are symmetric with respect to X_{-i} for any $i \in \{1,\dots,N\}$. Hence, in the following we assume to have a solution W to equation (3.13) with the property that for any permutation $\sigma : \{1,\dots,N-1\} \to \{1,\dots,N-1\}$ we have

$$W(t,\mathbb{Z}) = W(t,\eta,z_{\sigma_1},\dots,z_{\sigma_{N-1}}). \tag{3.15}$$

In view of Theorem 4.1 we expect $W(t,\mathbb{Z})$ to converge for for $N \to \infty$ to a limit function $\mathbf{W} : [0,T] \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ in the sense of Theorem 4.1, i.e., up to a subsequence and for $Z \in \mathbb{R}^N$

$$\lim_{N\to\infty} \sup_{|\eta| \le R, t \in [0,T], Z_{-N} \subset \mathbb{R}^{N-1}} |W(t,\mathbb{Z}) - \mathbf{W}(t,\eta,m_{Z_{-N}}^{N-1})| = 0.$$

We obtain that the limit $\mathbf{W}:[0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$ fulfills the convergence if the measure $m^{N-1}_{Z_{-N}}$ is replaced by the empirical measure m^N_Z for any $Z\in\mathbb{R}^N$. Using the introduced notation in Section 2 we may therefore write

$$W(t,\mathbb{Z})=W^N(t,\eta,m^{N-1}_{Z_{-N}})\sim\mathbf{W}(t,\eta,m^N_Z).$$

Similarly, we obtain the following meanfield limits for N sufficiently large (and provided the assumptions of Theorem 4.1 and [8, Theorem 4.1] are fulfilled.

$$\begin{aligned}
\partial_t V_i(t,X)=\partial_t W(t,x_i,X_{-i})=\partial_t W^N(t,x_i,m^{N-1}_{X_{-i}}) &\quad\sim\partial_t\mathbf{W}(t,x_i,m^N_X),\\
h_i(X)=h(x_i,X_{-i})=h^N(x_i,m^{N-1}_{X_{-i}}) &\quad\sim\mathbf{h}(x_i,m^N_X),\\
(\partial_{x_i}V_i(t,X))^2=(\partial_{x_i}W(t,x_i,X_{-i}))^2=(\partial_{x_i}W^N(t,x_i,m^{N-1}_{X_{-i}}))^2 &\quad\sim(\partial_{x_i}\mathbf{W}(t,x_i,m^N_X))^2.
\end{aligned}$$

It remains to discuss the limit of the mixed term in equations (3.12) and (3.13), respectively.

$$\sum_{k=1}^{N-1}\partial_{z_k}W(t,\mathbb{Z})\left(f(\mathbb{Z}_k)-\frac{1}{\alpha(t)}\partial_\eta W(t,\mathbb{Z})\right). \tag{3.16}$$

In order to derive the meanfield limit for equation (3.16) we require f to be symmetric in *all* variables, i.e.,

(C) We assume that $f(Z)=f((z_{\sigma_i})_{i=1}^N)$ for any permutation $\sigma:\{1,\dots,N\}\to\{1,\dots,N\}$ and for all $Z\in\mathbb{R}^N$.

Under Assumption **(C)** we have in particular for all $k\in\{1,\dots,N\}$ and a permutation $\sigma:\{1,\dots,N-1\}\to\{1,\dots,N-1\}$

$$f(\mathbb{Z}_k)=f(\eta,z_1,\dots,z_{N-1})=f(\eta,z_{\sigma_1},\dots,z_{\sigma_{N-1}}).$$

Therefore, $f(\mathbb{Z})=f_N(\eta,m^{N-1}_{Z_{-N})})$. We further obtain $f_N(\eta,m^{N-1}_{Z_{-N})})\sim\mathbf{f}(\eta,m^N_Z)$ for any (η,Z). However under Assumption **(C)** we also obtain $f(Z)=f_N(m^N_Z)\sim\mathbf{f}(m^N_Z)$. Assuming the limit in Theorem 4.1 is unique we obtain that $\mathbf{f}$ is therefore *independent* of η.

Now, consider the discrete measure $m^N_Z=\frac{1}{N}\sum_{j=1}^N m^N_{z_j}$ and $m_{z_j}=\delta(x-z_j)\in\mathcal{P}(\mathbb{R})$. For each j we denote by $m_{z_j}(\zeta)=\mathcal{Z}(\zeta)\#m_{z_j}$ the push forward of the discrete measure with the flow field $c:(t,t+a)\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$ and $m_{z_j}(t)=m_{z_j}$. Let the characteristic equations for $\mathcal{Z}$ for fixed η be given by the flow field

$$\frac{d}{d\zeta}\mathcal{Z}(\zeta)=c(\zeta,\eta,m^N_Z(\zeta)):=\mathbf{f}(m^N_Z(\zeta))-\frac{1}{\alpha(\zeta)}\partial_\eta\mathbf{W}(\zeta,\eta,m^N_Z(\zeta)). \tag{3.17}$$

Similarly to equation (4.8), we obtain the directional derivative of the measure of $\mathbf{W}(t,\eta,m^N_Z)$with respect to the measure m^N_Z in direction of the vectorfield c at $\zeta=t$ as

$$\begin{aligned}
&\sum_{k=1}^{N-1}\partial_{z_k}W(t,\mathbb{Z})\left(f(\mathbb{Z})-\frac{1}{\alpha(t)}\partial_\eta W(t,\mathbb{Z})\right)\sim\\
&\qquad\langle\partial_m\mathbf{W}(t,\eta,m^N_Z),\mathbf{f}(m^N_Z)-\frac{1}{\alpha(t)}\partial_\eta\mathbf{W}(t,\eta,m^N_Z)\rangle_{L^2_{m^N_Z}},
\end{aligned}$$

where $L^2_{m_Z^N}$ denotes the space of square integrable functions for the measure m_Z^N. Performing the limits for $N\to\infty$, replacing η by x, we obtain finally the meanfield equation for $\mathbf{W}=\mathbf{W}(t,x,m):[0,T]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$ given by

$$\partial_t\mathbf{W}(t,x,m)+\langle\partial_m\mathbf{W}(t,x,m),\mathbf{f}(m)-\frac{1}{\alpha(t)}\partial_x\mathbf{W}(t,x,m)\rangle_{L^2_m}+\partial_x\mathbf{W}(t,x,m)\mathbf{F}(x,m)$$
$$=-\mathbf{h}(x,m)+\frac{1}{2\alpha(t)}\left(\partial_x\mathbf{W}(t,x,m)\right)^2,\ \mathbf{W}(T,x,m)=0. \tag{3.18}$$

The previous equation is reformulated using the concept of directional derivatives of measures m outlined in the Appendix 4. Denote by $c(t,x,m)=\mathbf{f}(m)-\frac{1}{\alpha(t)}\partial_x\mathbf{W}(t,x,m)$ a field. If $m_{x_j}(t)\in\mathcal{P}(\mathbb{R})$ for each t is obtained as push forward with the vector field c, then, m_{x_j} fulfills in a weak sense the continuity equation (4.4). Therefore, $m_X^N=\frac{1}{N}\sum_{j=1}^{N}m_{x_j}$ fulfills

$$\partial_t m_X^N(t,x)+\partial_x\left(c(t,x,m_X^N)m_X^N(t,x)\right)=0. \tag{3.19}$$

As seen from the previous equations and the computations in equation (4.8) we therefore have

$$\partial_t\mathbf{W}(t,x,m_X^N(t,\cdot))+\langle\partial_m\mathbf{W}(t,x,m_X^N(t,\cdot)),\mathbf{f}(m_X^N(t,\cdot))-\frac{1}{\alpha(t)}\partial_x\mathbf{W}(t,x,m_X^N(t,\cdot))\rangle_{L^2_{m_X^N}}$$
$$=\frac{d}{dt}\mathbf{W}(t,x,m_X^N(t,\cdot)).$$

This motivates the following definition. For a family of measures $(m(t))_{t\in[0,T]}$ with $m(t,\cdot)\in\mathcal{P}(\mathbb{R})$, define $\mathbf{w}:[0,T]\times\mathbb{R}\to\mathbb{R}$ by

$$\mathbf{w}(t,x):=\mathbf{W}(t,x,m(t)). \tag{3.20}$$

Then, from equation (3.18) we obtain

$$\partial_t\mathbf{w}(t,x)+(\partial_x\mathbf{w}(t,x))\mathbf{f}(m)=-\mathbf{h}(x,m)+\frac{1}{2\alpha(t)}\left(\partial_x\mathbf{w}(t,x)\right)^2, \tag{3.21}$$

and from equation (3.19) we obtain using the definition (3.20)

$$\partial_t m(t,x)+\partial_x\left(\left(\mathbf{f}(m)-\frac{1}{\alpha(t)}\partial_x\mathbf{w}(t,x)\right)m(t,x)\right)=0. \tag{3.22}$$

Provided we may solve the meanfield equations (3.21) and (3.22) for $(\mathbf{w},m)$ we obtain a solution $\mathbf{W}$ along the characteristics in $m-$space by the implicit relation (3.20). In this sense and under the Assumptions (**A**) to (**C**) the meanfield limit of equation (3.12) or respectively equation (3.13) is given by the system of the following equations (3.23) and (3.24) for $\mathbf{w}:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $m(t)\in\mathcal{P}(\mathbb{R})$ for all $t\in[0,T]$. The terminal condition for $\mathbf{w}$ is given by $\mathbf{w}(T,x)=0$.

$$\partial_t\mathbf{w}(t,x)+\partial_x\left(\mathbf{w}(t,x)\right)\mathbf{f}(m(t,x))-\frac{1}{2\alpha(t)}(\partial_x\mathbf{w}(t,x))^2=-\mathbf{h}(x,m(t,x)), \tag{3.23}$$

$$\partial_t m(t,x)+\partial_x\left(\left(\mathbf{f}(m(t,x))-\frac{1}{\alpha(t)}\partial_x\mathbf{w}(t,x)\right)m(t,x)\right)=0. \tag{3.24}$$

REMARK 3.2. The control u_i^* given by equation (3.10) can be expressed in the mean field limit as follows.

$$u_i^*(t) = -\frac{1}{\alpha(t)} \partial_{x_i} V_i(t, x_i(t), X_{-i}(t)).$$

Under Assumption **(B)** and using equation (3.14) and equation (3.20) for any X we have

$$-\frac{1}{\alpha(t)} \partial_{x_i} V_i(t,X) = -\frac{1}{\alpha(t)} \partial_x W(t, x_i, X_{-i}) \sim$$
$$-\frac{1}{\alpha(t)} \partial_x \mathbf{W}(t, x, m_X^N(t,\cdot)) = -\frac{1}{\alpha(t)} \partial_x \mathbf{w}(t,x).$$

3.3. MPC and best-reply strategy for the meanfield equation (3.23)–(3.24)**.** Finally, we approximate on the meanfield level the HJB dynamics. The approximation is similar to the one conducted on the particle level. In the end, we obtain the best-reply strategy through a MPC approach.

First note, that the calculations leading to equation (3.23) are independent of the terminal time T. Now, let a time $\tau \in [0,T]$ be fixed and let $\Delta t > 0$ be sufficiently small. Consider the function on the receding horizon $(\tau, \tau + \Delta t)$ with initial conditions given at τ and where we, as before, add Δt as a superscript to indicate the dependence on the short time horizon. Further, we also scale the control weight by Δt as in equations (2.7) and (2.8).

$$V_i^{\Delta t}(\tau, Y) = \frac{1}{\Delta t} \int_\tau^{\tau+\Delta t} \left(\frac{\Delta t \alpha_i(s)}{2} u_i^2(s) + h_i(X(s)) ds \right) ds. \tag{3.25}$$

We obtain equation (3.23) defined only for $t \in [\tau, \tau + \Delta t]$ as

$$\partial_t \mathbf{w}(t,x) + \partial_x (\mathbf{w}(t,x)) \mathbf{f}(m(t,x)) - \frac{1}{2\alpha(t)} (\partial_x \mathbf{w}(t,x))^2 = -\mathbf{h}(x, m(t,x)),$$
$$\partial_t m(t,x) + \partial_x \left(\left(\mathbf{f}(m(t,x)) - \frac{1}{\alpha(t)} \partial_x \mathbf{w}(t,x) \right) m(t,x) \right) = 0,$$
$$\mathbf{w}(\tau + \Delta t, x) = 0.$$

Note that $\mathbf{w}$ corresponds to the non-weighted value function $V_i(t,X)$ whereas we are now interested in the scaled value function $V_i^{\Delta t}(\tau, Y)$. The difference between V_i and $V_i^{\Delta t}$ is simply a scaling of $\Delta t \alpha_i$ and h_i by $\frac{1}{\Delta t}$. Therefore, an Euler backwards in time discretization of the meanfield equation in $(\mathbf{w}, m)$ corresponding to $V_i^{\Delta t}$ reads

$$\mathbf{w}(\tau + \Delta t) - \mathbf{w}(\tau, x) + \Delta t \, \partial_x (\mathbf{w}(\tau + \Delta t, x)) \mathbf{f}(m(\tau + \Delta t, x))$$
$$- \frac{\Delta t}{2\alpha(\tau + \Delta t)} (\partial_x \mathbf{w}(\tau + \Delta t, x))^2 = -\Delta t \frac{\mathbf{h}(x, m(\tau + \Delta t, x))}{\Delta t}.$$

Hence, we obtain

$$\mathbf{w}(\tau, x) = \mathbf{h}(x, m(t,x)) + O(\Delta t). \tag{3.26}$$

Substituting this relation in the equation for m we obtain the MPC meanfield equation for the running cost $V_i^{\Delta t}$ as

$$\partial_t m(t,x) + \partial_x \left(\left(\mathbf{f}(m(t,x)) - \frac{1}{\alpha(t)} \partial_x \mathbf{h}(x, m(t,x)) \right) m(t,x) \right) = 0. \tag{3.27}$$

This equation is precisely the same as we had obtained for the controlled dynamics using the best-reply strategy derived in the previous section and given by equation (2.13).

REMARK 3.3. The best-reply strategy for a meanfield game corresponds therefore to considering at each time τ an objective function measuring only the costs for a small next time step. Those costs may depend on the optimal choices of the other agents. However, for a small time horizon the derivative of the running costs (i.e. $\mathbf{h}$) is a sufficient approximation to the otherwise intractable solution to the full system of meanfield equations (3.23)-(3.24).

We summarize the findings in the following Proposition.

PROPOSITION 3.1. *Assume that Assumptions* **(A)** *to* **(C)** *holds true and let* $\Delta t > 0$ *be given. Denote by* $\mathbf{f}(m)$ *and* $\mathbf{h}(x,m)$ *the meanfield limit for* $N \to \infty$ *of* $f(X)$ *and* $h(X)$*, respectively. Assume that* $m:[0,T]\times\mathbb{R}\to\mathbb{R}$ *be such that* $m(t,\cdot)\in\mathcal{P}(\mathbb{R})$ *and fulfill equation*

$$\partial_t m(t,x) + \partial_x\left(\left(\mathbf{f}(m(t,x)) - \frac{1}{\alpha(t)}\partial_x \mathbf{h}(x,m(t,x))\right)m(t,x)\right) = 0. \tag{3.28}$$

and let

$$\mathbf{w}(t,x) = \mathbf{h}(t,x).$$

Then, for any $t\in[0,T]$ *and up to an error of order* $O(\Delta t)$ *the function* $\mathbf{W}:[t,t+\Delta t]\times\mathbb{R}\times\mathcal{P}(\mathbb{R})\to\mathbb{R}$ *implicitly defined by*

$$\mathbf{W}(s,x,m(t,x)) = \mathbf{w}(s,x),\ x\in\mathbb{R}, s\in[t,t+\Delta t],$$

is a solution to the meanfield equation

$$\begin{aligned}
&\partial_s \mathbf{W}(s,x,m) + \langle \partial_m \mathbf{W}(s,x,m), \mathbf{f}(m) - \frac{1}{\alpha(s)}\partial_x \mathbf{W}(s,x,m)\rangle_{L^2_m} + \partial_x \mathbf{W}(s,x,m)\mathbf{f}(m)\\
&= -\mathbf{h}(x,m) + \frac{1}{2\alpha(s)}\left(\partial_x \mathbf{W}(s,x,m)\right)^2,\ \mathbf{W}(t+\Delta t,x,m) = 0.
\end{aligned}$$

The meanfield equation is the formal limit for $N\to\infty$ *of an* N *particle game on the time interval* $(t,t+\Delta t)$ *and described by equation* (2.1) *for* $i=1,\dots,N$*, i.e.,*

$$\begin{aligned}
&\frac{d}{ds}x_i(s) = f_i(X(s)) + u_i(s),\\
&\quad u_i(s) = \ argmin\ _{u:[t,t+\Delta t]\to\mathbb{R}} \frac{1}{\Delta t}\int_t^{t+\Delta t}\left(\frac{\Delta t \alpha_i(s)}{2}u^2(r) + h_i(X(r))\right)dr.
\end{aligned}$$

A solution to the associated ith HJB equations for $V_i:[t,t+\Delta t]\times\mathbb{R}^N\to\mathbb{R}$ *are given by* $V_i(t,X) := \mathbf{W}(s,x_i,m^N_{X_{-i}})$ *for* $i=1,\dots,N$*, and the optimal control is* $u_i^*(s) = -\frac{1}{\alpha_i(s)}\partial_{x_i}V_i(s,X(s))$*.*

The meanfield equation (3.28) *coincides with the formal meanfield equation obtained using the best-reply strategy* (2.13)*.*

4. Technical details

This section includes results related to the meanfield limit of particle systems and shows the necessity of the requirements Assumptions **(A)** and **(C)**. We collect some results of [16] for convenience. The Kantorowich–Rubenstein distance $\mathbf{d}_1(\mu,\nu)$ for measures $\mu,\nu\in\mathcal{P}(Q)$ is given defined by

$$\mathbf{d}_1(\mu,\nu):=\sup\{\int\phi d(\mu-\nu):\phi:Q\to\mathbb{R},\phi\ 1\text{ - Lipschitz }\}. \tag{4.1}$$

THEOREM 4.1 (Theorem 2.1 [16]). *Let Q^N be a compact subset of $\mathbb{R}^N$. Consider a sequence of functions $(u_N)_{N=1}^{\infty}$ with $u_N:Q^N\to\mathbb{R}$. Assume that each $u_N(X)=u_N(x_1,\dots,x_N)$ is a symmetric function in all variables, i.e.,*

$$u_N(X)=u_N(x_{\sigma(1)},\dots,x_{\sigma(N)}),$$

for any permutation σ on $\{1,\dots,N\}$. Denote by $\mathbf{d}_1$ the Kantorowich–Rubenstein distance on the space of probability measures $\mathcal{P}(Q)$ and let ω be a modulus of continuity independent of N. Assume that the sequence is uniformly bounded $\|u_N\|_{L^\infty(Q^N)}\le C$. Further assume that for all $X,Y\in Q^N$ and all N we have

$$|u_N(X)-u_N(Y)|\le\omega(\mathbf{d}_1(m_X^N,m_Y^N)),$$

where $m_\xi^N\in\mathcal{P}(Q)$ is defined by $m_\xi^N(x)=\frac{1}{N}\sum_{i=1}^{N}\delta(x-\xi_i)$.

Then there exists a subsequence $(u_{N_k})_k$ of $(u_N)_N$ and a continuous map $U:\mathcal{P}(Q)\to\mathbb{R}$ such that

$$\lim_{k\to\infty}\sup_{X\in\mathbb{R}^N}|u_{N_k}(X)-U(m_X^{N_k})|=0. \tag{4.2}$$

An extension is found in [8, Theorem 4.1]. As toy example consider $u_N(X)=\frac{1}{N}\sum_{i=1}^{N}\phi(x_i)$. If $\phi:\mathbb{R}\to\mathbb{R}$ is compactly supported, bounded and $|\phi'(\xi)|\le C$ for all $\xi\in\mathbb{R}$, then the assumptions of the previous theorem are fulfilled. Note that the assumption on ϕ implies that for each i we have $|\partial_{x_i}u_N(X)|\le\frac{C}{N}$ for all X and all N. This condition implies the estimate on u_N. The corresponding limit is given by the function $U:\mathcal{P}(\mathbb{R})\to\mathbb{R}$ defined by $U(m)=\int\phi dm$. We have $U(m_X^N)=u_N(X)$.

Derivatives in the space of measures are described for example in [3]. They may be motivated by the following formal computation. Let ψ be a smooth function on $\mathbb{R}$ and let $y'(t)=c$ for $t\in(a,b)$ and $y(a)=x$. We denote by a subindex $t=a$ the evaluation at $t=a$ of the corresponding expression and by a prime the derivative of ψ. Then,

$$c\psi'(x)=\int\psi'(z)c\delta(x-z)dz=\left(\int\psi'(z)c\delta(y(t)-z)dz\right)|_{t=a}=\left(\int\psi\partial_z(c\delta(y(t)-z))\right)|_{t=a},$$
$$c\psi'(x)=\left(\frac{d}{dt}\psi(y(t))\right)|_{t=a}=\left(\frac{d}{dt}\int\psi(z)\delta(y(t)-z)dz\right)|_{t=a}.$$

Therefore, we may write

$$\partial_t\delta(y(t)-z)+\partial_z(c\delta(y(t)-z))=0,$$

provided that $y'(t)=c$. Further, $\delta(y(t)-z)=y(t)\#\delta(x-z)$ where $\#$ is the push forward operator, see below. Hence, for the family of measures $\delta(y(t)-z)$ the previous computation lead to a notion of derivatives. This can be formalized to a calculus for derivatives in measure space and we summarize in the following more general results from [3, Chapter II.8]. We consider the space of probability measures $\mathcal{P}_p(\mathbb{R})$ [3, eequa-tionquation (5.1.22)]:

$$\mathcal{P}_p(\mathbb{R}) = \left\{ \mu \in \mathcal{P}(\mathbb{R}) : \int |x-\bar{x}|^p d\mu(x) < \infty \text{ for some } \bar{x} \in \mathbb{R} \right\}.$$

Let $\mathcal{P}_p(\mathbb{R})$ be equipped with the Wasserstein distance $W_p(\mu,\nu)$ [3, Chapter 7.1.1]. In the case $p=1$ and for bounded measures μ,ν this distance is equivalent to $\mathbf{d}_1(\nu,\mu)$ defined in equation (4.1). For the case $p=2$ we refer to [5] for a different characterization.

We consider absolutely continuous curves $m:(a,b) \to \mathcal{P}_p(\mathbb{R})$. The curve m is called absolutely continuous if there exists a function $M \in L^1(a,b)$ such that for all $a \leq s < t \leq b$ we have

$$W_p(m(s),m(t)) \leq \int_s^t M(\xi) d\xi, \tag{4.3}$$

see [3, Definition 1.1.1]. For an absolutely continuous curve $m:(a,b) \to \mathcal{P}_p(\mathbb{R})$, i.e., $m(t) \in \mathcal{P}_p(\mathbb{R})$, and $p>1$ there exists a vector field $v:(a,b)\times\mathbb{R} \to \mathbb{R}$ with $v(t) \in L^p(\mathbb{R};m(t))$ a.e. $t \in (a,b)$ such that the continuity equation

$$\partial_t m(t,x) + \partial_x (v(t,x) m(t,x)) = 0, \tag{4.4}$$

holds in a distributional sense. Further, $\|v(t)\|_{L^p(\mathbb{R};m(t))} \leq |m'|(t)$ a.e. in t. Here, $t \to |m'|(t)$ for $t \in (a,b)$ is the metric derivative of the curve m. The precise statement is given in [3, Theorem 8.3.1] and the metric derivative is given in [3, Theorem 1.1.2] by

$$|m'|(t) := \lim_{s\to t} \frac{W_p(m(s),m(t))}{|s-t|}. \tag{4.5}$$

The limit exists a.e. in t, provided that m is absolutely continuous (4.3). We have $|m'|(t) \leq M(t)$ a.e. for each function M fulfilling equation (4.3), see [3, Chapter 1]. Also, the converse result holds true: If m fulfills in a weak sense equation (4.4) for some $v \in L^1(a,b;L^p(\mathbb{R};m(\cdot))$, then m is absolutely continuous. Furthermore, solutions to equation (4.4) can be represented using the methods of characteristics, see [3, Lemma 8.1.6, Proposition 8.1.8]. Under suitable assumptions on m and v we have that a weak solution to equation (4.4) is

$$m(t,\cdot) = X(t;a,\cdot)\#m(a,\cdot)\, \forall t \in [a,b], \tag{4.6}$$

provided that $X(t)$ solves characteristic system for every $x \in \mathbb{R}$ and every $s \in [a,b]$:

$$X(s;s,x) = x \text{ and } \partial_t X(t;s,x) = v(t, X(t;s,x)). \tag{4.7}$$

Here, (s,x) is the initial position of the characteristic in phase space and # is the push forward operator, i.e., if applied to the set $\{x\}$ we have $m(t,\{X(t;a,x)\}) = m(a,\{X(a;a,x)\}) = m(a,\{x\})$. equation (4.4) may also be viewed as the directional derivative of the family of measures $m(t,\cdot)$ in direction v.

In Section 3 we need to discuss a term of the type $\sum_{j=1}^{N} c(x_j)\partial_{x_j} f(x_1,\ldots,x_N)$ for a symmetric function f. Now, consider a family of paths $m_j:(a,b) \to \mathcal{P}(\mathbb{R})$ generated by $m_j(t,z) = y_j(t)\#\delta(x_j - z)$ where y_j solves the characteristic equation $y_j'(t) = c(y_j(t))$ and $y_j(a) = x_j$. Let $m_Y^N(t,z) := \frac{1}{N}\sum_{j=1}^{N} \delta(y_j(t)-z)$. We have then

$$\partial_t m_Y^N(t,x) = -\partial_x \left(c(x) m_Y^N(t,x) \right).$$

If we assume that f fulfills the assumption of Theorem 4.1, then, there exists: $\mathcal{P}(\mathbb{R}) \to \mathbb{R}$ and then $f(y_1(t),\dots,y_N(t)) = f_N(m_Y^N) \sim \mathbf{f}(m_Y^N)$. The following computation similar to the motivation shows the expression of the unknown term for large N:

$$\sum_{j=1}^{N} c(x_j)\partial_{x_j} f(x_1,\dots,x_N) = \frac{d}{dt} f(y_1(t),\dots,y_j(t),\dots,y_N(t))|_{t=a} \tag{4.8}$$

$$= \frac{d}{dt} f_N(m_Y^N(t))|_{t=a} \sim \frac{d}{dt}\mathbf{f}(m_{Y(t)}^N)|_{t=a}. \tag{4.9}$$

In order to make the link with the theory developed in [16], we note that the last derivative at $m = m_Y^N$ can be interpreted as

$$\frac{d}{dt}\mathbf{f}(m_{Y(t)}^N)|_{t=a} = \langle \partial_m \mathbf{f}(m), c\rangle_{L_m^2},$$

with L_m^2 the space of square integrable functions with respect to the measure m. This formula can either be seen as the definition of $\partial_m \mathbf{f}(m)$ if one follows the approach of [3] (which is the route taken here) or as a consequence of its definition if one follows the approach of [16].

Acknowledgments. This work has been supported by KI-Net NSF RNMS grant No. 1107444, DFG HE5386/13,14,15-1 and CNRS PEPS-HuMaI grant 'DEESSes'. JGL acknowledges support by the National Science Foundation under grant number DMS-1514826. MH and JGL are grateful for the opportunity to stay and work at Imperial College, London, UK, in the Department of Mathematics. PD acknowledges support by the Royal Society and the Wolfson Foundation through a Royal Society Wolfson Research Merit Award no. WM130048. PD is on leave from CNRS, Institut de Mathématiques de Toulouse, France.

REFERENCES

[1] G. Albi, M. Herty, and L Pareschi, *Kinetic description of optimal control problems and applications to opinion consensus*, Commun. Math. Sci., 13(6):1407–1429, 2015.

[2] Giacomo Albi, Lorenzo Pareschi, and Mattia Zanella, *Boltzmann type control of opinion consensus through leaders*, Philos. Trans. A, 372:20140138, 2014.

[3] L. Ambrosio, N. Gigli, and G. Savare, *Gradient Flows in Metric Spaces of Probability Measures*, Lectures in Mathematics. Birkhäuser Verlag, Basel, Boston, Berlin, 2005.

[4] Robert J. Aumann, *Existence of competitive equilibria in markets with a continuum of traders*, Econometrica, 34(1):1–17, 1966.

[5] Jean-David Benamou and Yann Brenier, *A computational fluid mechanics solution to the monge-kantorovich mass transfer problem*, Numer. Math., 84(3):375–393, 2000.

[6] A. Bensoussan, J. Frehse, and P. Yam, *Mean Field Games and Mean Field Type Control Theory*, Series: SpringerBriefs in Mathematics, New York, 2013.

[7] Alain Bensoussan, Jens Frehse, and Sheung Chi Phillip Yam, *The master equation in mean field theory*, J. Math. Pures Appl., 103(6):1441–1474, 2015.

[8] Adrien Blanchet and Guillaume Carlier, *From nash to cournot-nash equilibria via the monge-kantorovich problem*, Philos. Trans. A, 372:20130398, 2014.

[9] Jean-Philippe Bouchaud and Marc Mézard, *Wealth condensation in a simple model of economy*, Phys. A, 282(3):536–545, 2000.

[10] Alberto Bressan, *Noncooperative differential games*, Milan J. Math., 79(2):357–427, 2011.

[11] Martin Burger, Luis Caffarelli, Peter A. Markowich, and Marie-Therese Wolfram, *On a Boltzmann-type price formation model*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 469(2157):20130126, 20, 2013.

[12] Martin Burger, Luis Caffarelli, Peter A. Markowich, and Marie-Therese Wolfram, *On the asymptotic behavior of a Boltzmann-type price formation model*, Commun. Math. Sci., 12(7):1353–1361, 2014.

[13] Martin Burger, Marco Di Francesco, Peter A. Markowich, and Marie-Therese Wolfram, *Mean field games with nonlinear mobilities in pedestrian dynamics*, Discrete Contin. Dyn. Syst. Ser. B, 19(5):1311–1333, 2014.

[14] Martin Burger, Jan Ha **w**skovec, and Marie-Therese Wolfram, *Individual based and mean-field modeling of direct aggregation*, Phys. D, 260:145–158, 2013.

[15] Marco Caponigro, Massimo Fornasier, Benedetto Piccoli, and Emmanuel Trélat, *Sparse stabilization and optimal control of the Cucker-Smale model*, Math. Control Relat. Fields, 3(4):447–466, 2013.

[16] Pierre Cardaliaguet, *Notes on mean field games*, from P.-L. Lions lectures at College de France, 2010.

[17] Rene Carmona, Jean-Pierre Fouque, and Li-Hsien Sun, *Mean field games and systemic risk*, arXiv preprint, arXiv:1308.2172, 2013.

[18] Stephane Cordier, Lorenzo Pareschi, and Giuseppe Toscani, *On a kinetic model for a simple market economy*, J. Stat. Phys., 120(1-2):253–277, 2005.

[19] I.D. Couzin, J. Krause, N.R. Franks, and S.A. Levin, *Effective leadership and decision-making in animal groups on the move*, Nature, 433:513–516, 2005.

[20] Pierre Degond, Cécile Appert-Rolland, Mehdi Moussaid, Julien Pettré, and Guy Theraulaz, *A hierarchy of heuristic-based models of crowd dynamics*, J. Stat. Phys., 152(6):1033–1068, 2013.

[21] Pierre Degond, Jian-Guo Liu, and Christian Ringhofer, *Evolution of the distribution of wealth in an economic environment driven by local nash equilibria*, J. Stat. Phys., 154(3):751–780, 2014.

[22] Pierre Degond, Jian-Guo Liu, and Christian Ringhofer, *Evolution of wealth in a nonconservative economy driven by local nash equilibria*, Philos. Trans. A, 372, 2014.

[23] Pierre Degond, Jian-Guo Liu, and Christian Ringhofer, *Large-scale dynamics of mean-field games driven by local nash equilibria*, J. Nonlinear Sci., 24:93–115, 2014.

[24] Pierre Degond and Sébastien Motsch, *Continuum limit of self-driven particles with orientation interaction*, Math. Models Methods Appl. Sci., 18(1):1193–1215, 2008.

[25] Massimo Fornasier and Francesco Solombrino, *Mean-field optimal control*, ESAIM Control Optim. Calc. Var., 20:1123–1152, 2014.

[26] Jean-Pierre Fouque and Joseph A Langsam, *Handbook on Systemic Risk*, Cambridge University Press, 2013.

[27] Avner Friedman, *Differential games*, American Mathematical Society, Providence, R.I., 1974.

[28] Lars Grüne and Jürgen Pannek, *Nonlinear Model Predictive Control*, Communications and Control Engineering Series. Springer, London, 2011.

[29] Olivier Guéant, Jean-Michel Lasry, and Pierre-Louis Lions, *Mean field games and applications*, in Paris-Princeton Lectures on Mathematical Finance 2010, Springer, 205–266, 2011.

[30] M. Herty, S. Steffensen, and L. Pareschi, *Mean-field control and Riccati equations*, Netw. Heterog. Media, 10:699–715, 2015.

[31] Jean-Michel Lasry and Pierre-Louis Lions, *Mean field games*, Jpn. J. Math., 2(1):229–260, 2007.

[32] Andreu Mas-Colell, *On a theorem of schmeidler*, J. Math. Econom., 13(3):201–206, 1984.

[33] David Q. Mayne and Hannah Michalska, *Receding horizon control of nonlinear systems*, IEEE Trans. Automat. Control, 35(7):814–824, 1990.

[34] Sebastien Motsch and Eitan Tadmor, *Heterophilious dynamics enhances consensus*, SIAM Rev., 56:577–621, 2014.

[35] G. Naldi, L. Pareschi, and G. Toscani (eds.), *Mathematical Modeling of Collective Behavior in Socio-Economic and Life Sciences*, Series: Modeling and Simulation in Science, Engineering and Technology. Birkhauser, Boston, 2010.

[36] Lorenzo Pareschi and Giuseppe Toscani, *Interacting Multiagent Systems: Kinetic Equations and Monte Carlo Methods*, Oxford University Press, 2013.

[37] David Schmeidler, *Equilibrium points of nonatomic games*, J. Stat. Phys., 7(4):295–300, 1973.

[38] Norman Zalmon Shapiro and Lloyd S Shapley, *Values of large games, i: a limit theorem*, Math. Oper. Res., 3(1):1–9, 1978.

[39] Eduardo D. Sontag, *Mathematical Control Theory*, Texts in Applied Mathematics, Springer-Verlag, New York, Second Edition, 6, 1998.

[40] Giuseppe Toscani, *Kinetic models of opinion formation*, Commun. Math. Sci., 4(3):481–496, 2006.

[41] T. Vicsek and A. Zafeiris, *Collective motion*, Phys. Rep., 517:71–140, 2012.

[42] Marie-Therese Wolfram, *Opinion formation in a heterogenous society*, in Econophysics of order-driven markets, New Econ. Windows, Springer, Milan, 277–288, 2011.