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ON THE GLOBAL REGULARITY OF THE 2D CRITICAL BOUSSINESQ SYSTEM WITH $\alpha>2/3$*

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Abstract. This paper examines the question for global regularity for the Boussinesq equation with critical fractional dissipation $(\alpha,\beta):\alpha+\beta=1$. The main result states that the system admits global regular solutions for all (reasonably) smooth and decaying data, as long as $\alpha>2/3$. This improves upon some recent works [Q. Jiu, C. Miao, J. Wu and Z. Zhang, SIAM J. Math. Anal., 46:3426–3454, 2014] and [A. Stefanov and J. Wu, J. Anal. Math., 2015].

The main new idea is the introduction of a new, second generation Hmidi-Keraani-Rousset type, change of variables, which further improves the linear derivative in temperature term in the vorticity equation. This approach is then complemented by a new set of commutator estimates (in both negative and positive index Sobolev spaces!), which may be of independent interest.

Keywords. Boussinesq equations; fractional dissipation; global regularity.

AMS subject classifications. 35Q35; 35B65; 76B03.

1. Introduction

The two-dimensional (2D) Boussinesq equations with fractional dissipation is

$$\begin{cases} \partial_t u+u\cdot\nabla u+\nu\Lambda^\alpha u=-\nabla p+\theta\mathbf{e}_2, & x\in\mathbf{R}^2,\ t>0,\\ \nabla\cdot u=0, & x\in\mathbf{R}^2,\ t>0,\\ \partial_t\theta+u\cdot\nabla\theta+\kappa\Lambda^\beta\theta=0, & x\in\mathbf{R}^2,\ t>0,\\ u(x,0)=u_0(x),\ \theta(x,0)=\theta_0(x), & x\in\mathbf{R}^2, \end{cases} \tag{1.1}$$

where $u=u(x,t)=(u_1(x,t),u_2(x,t))$ denotes the velocity vector field, $p=p(x,t)$ is the scalar pressure, the scalar function $\theta=\theta(x,t)$ is the temperature, $\mathbf{e}_2$ the unit vector in the vertical direction, and $\nu\ge0$, $\kappa\ge0$, $0\le\alpha\le2$ and $0\le\beta\le2$ are real parameters. Here $\Lambda=\sqrt{-\Delta}$ is the Zygmund operator defined through the Fourier transform,

$$\widehat{\Lambda^\alpha f}(\xi)=|\xi|^\alpha\widehat{f}(\xi),$$

where the Fourier transform and its inverse are given by

$$\widehat{f}(\xi)=\int_{\mathbf{R}^2}e^{-ix\cdot\xi}f(x)\,dx,\quad f(x)=(2\pi)^{-2}\int_{\mathbf{R}^2}e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi.$$

This model is of importance in a number of studies on atmospheric turbulence, [18,21]. The standard model (where both dissipations are taken to be the classical Laplacian, $\alpha=\beta=2$) is a primary model for atmospheric fronts and oceanic circulation as well as in the study of Raleigh-Bernard convection, [3,8,17,21,24,25]. The fractional diffusion operators considered herein appear naturally in the study in hydrodynamics, [7] as well as anomalous diffusion in semiconductor growth, [20]. There are also other models

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in which the Boussinesq equations with fractional Laplacian naturally arise, namely in models where the kinematic and thermal diffusion is attenuated by the thinning of atmosphere, [8].

Mathematically, the problem for global regularity of ((1.1)) is an interesting and subtle one. Intuitively, the lower the values of α, β, the harder it is to prove that solutions emanating from sufficiently smooth and localized data persist globally. In particular, the problem with no dissipation (i.e. $\nu=\kappa=0$) remains open. This is very similar to the Euler equation in two and three spatial dimensions and in fact numerous studies explore the possibility of finite time blow up, [22].

Next, we take on the difficult task of reviewing the recent results regarding well-posedness issues for equation ((1.1)). Indeed, there has been tremendous interest in this problem in the last fifteen years. In the classical case, when the diffusion is given by the regular Laplacian (i.e. $\alpha=\beta=2$), the global regularity follows just as it does for the 2D Navier–Stokes model, [6, 18]. In the works [1, 12], global regularity was proved in the presence of one full Laplacian, that is in the cases $\alpha=2, \beta=0$ or $\alpha=0, \beta=2$. In more recent years, the full two parameter range of α, β was explored in detail. Based on the currently available results, it is natural to draw the conclusion that one expects global regularity in the cases $\alpha+\beta \geq 1$, while the case $\alpha+\beta<1$ generally remains open[1]. We thus adopt the notion of criticality - namely, we say that a pair (α, β) is subcritical if $\alpha+\beta>1$, critical if $\alpha+\beta=1$ and supercritical if $\alpha+\beta<1$.

As it was alluded above, in the supercritical regime the behavior of the solutions remains a mystery. Apart from some numerical simulations, the only rigorous result that we are aware of is the eventual regularity of the solutions, [27], for appropriate supercritical regime of the diffusivity parameters. To be sure, such statement does not, *per se* exclude a finite time blow up of some solutions. It remains to discuss the critical and subcritical cases. This is probably a good place to observe that if global regularity holds for critical pair $(\alpha_0, \beta_0): \alpha_0+\beta_0=1$, then it must hold for all subcritical pairs in the form[2] $(\alpha, \beta_0), \alpha>\alpha_0$ and $(\alpha_0, \beta): \beta>\beta_0$. Thus, clearly global regularity results on the critical line are superior, in the sense described above, to subcritical ones. That being said, the subcritical theory is far from obvious or well-understood. Many results have been put forward in the last ten or so years. The following (very incomplete and yet very long) list accounts for some recent accomplishments - [2, 4, 5, 15, 16, 19, 29–37].

Next, we give a full account of the global regularity results for diffusivity parameters on the critical line $\alpha+\beta=1$. First, in series of works, Hmidi-Keraani-Rousset, [10, 11] established global regularity in the two critical and endpoint cases $(\alpha, \beta)=(1,0),(0,1)$. In their work, they employed clever change of variables, thus introducing a new hybrid quantity, depending on both vorticity and temperature[3]. In a subsequent paper, by developing more sophisticated function spaces, Jiu-Miao-Wu-Zhang, [13] were able to extend the global regularity results to the case $\alpha+\beta=1$,

$$\alpha>\frac{23-\sqrt{145}}{12} \approx 0.9132...$$

Subsequently, the second name author, in collaboration with J. Wu, [23] significantly

[1]in some numerical simulations, there was a reason to believe that finite time blow up might occur, but this is at present still a conjecture

[2]We believe that this statement, while not a rigorous result, can be made an exact theorem on a case by case basis, by just reworking a proof for (α_0, β_0) to cover the higher dissipation cases (in either the u or the θ variables)

[3]which is better suited (and looses less derivatives than either vorticity and temperature separately)

extended the results in [13], by covering the critical line $\alpha+\beta=1$, up to

$$\alpha>\frac{\sqrt{1777}-23}{24}\approx 0.798103...$$

Quite recently[4], we have learned that in [28] Wu-Xu-Xue-Ye have managed to further lower the allowable α exponents to

$$\alpha>\frac{10}{13}\sim 0.7692.$$

These results were achieved thanks to more sophisticated commutator estimates, both in Sobolev and Besov spaces, by essentially working in the setup of Hmidi-Keraani-Rousset (HKR for short), [10]. It was our (informal) conclusion in [23] that tightening of the commutator estimates in the HKR variables has exhausted (or nearly exhausted) the possible improvements. In other words, one needs to introduce better, more sophisticated change of variables, which in conjunction with sharp commutator estimates yields wider range of critical indices (α,β), for which one has global regularity.

The purpose of this paper is to do just that. We aim at further improving upon the results in [23]. In particular, we still work in the regime[5] $\alpha>\frac{1}{2}>\beta$, but in order to obtain better range, we perform a second generation HKR change of variables, which positions us for a better result. As we mention above, this is complemented by very precise commutator estimates, see Section (2.2).

We note that we do not, at this point, have anything new to say in the regime $\beta>\frac{1}{2}>\alpha$, for which the only available global regularity result is for $\alpha=0,\beta=1$. We hope to be able to report on these cases in the near future.

1.1. Main result.

We are ready to state our main result.

THEOREM 1.1. *Consider the Boussinesq equation* ((1.1)) *with*

$$\frac{2}{3}<\alpha<1, \alpha+\beta=1.$$

Suppose also that

$$\|u_0\|_{H^{1+\rho}(\mathbf{R}^2)}<\infty, \quad \|\theta_0\|_{H^{1+\beta+\rho}(\mathbf{R}^2)}+\|\nabla\theta_0\|_{L^\infty}+\|\theta_0\|_{L^1(\mathbf{R}^2)}<\infty,$$

where $0<\rho\ll 1$. *Then, equation* ((1.1)) *has a unique global solution* (u,θ) *satisfying, for any* $T>0$ *and moreover*

$$(u,\theta)\in C([0,T];H^{1+\rho}(\mathbf{R}^2)\times H^{2+\rho}(\mathbf{R}^2)).$$

1.2. Some initial reductions.

It is well-known that for sufficiently smooth and decaying data, the problem has a local solution, say in some interval $[0,T]$. The global regularity problem then reduces to showing that $T=\infty$. One proceeds to establish that by a contradiction argument. That is, if one assumes that $T<\infty$, the contradiction will arise out of impossibility of blow up at time T. Thus, one seeks to prove *a priori* estimates on the solutions, which will prevent them from blowing up. Let us mention for now, that the problem allows for some elementary *a priori* estimates

$$\begin{cases} \|\theta(t)\|_{L^p}\le\|\theta_0\|_{L^p}, \text{for } \ p\in[1,\infty], \\ \|\theta(t)\|_{L^2}^2+2\kappa\int_0^t\|\Lambda^{\frac{\beta}{2}}\theta(\tau)\|_{L^2}^2d\tau=\|\theta_0\|_{L^2}^2, \\ \|u(t)\|_{L^2}^2+2\nu\int_0^t\|\Lambda^{\frac{\alpha}{2}}u(\tau)\|_{L^2}^2d\tau\le(\|u_0\|_{L^2}^2+t\|\theta_0\|_{L^2}^2). \end{cases} \tag{1.2}$$

[4]after major part of this paper was completed

[5]noting that the HKR framework takes a slightly different form in the case $\beta>\frac{1}{2}>\alpha$

which are valid, whenever $0<t<T$. These will be used repeatedly in the argument, but as such they will be inadequate to conclude global regularity, they are just too weak for that. From now on, due to the fact that the precise values of the physical constants $\kappa,\nu>0$ are unimportant in the arguments, we set them to one, $\kappa=\nu=1$.

1.3. Change of variables: vorticity equation and beyond. It turns out that it is easier to work with the vorticity equation. A quick inspection shows that the vorticity $\omega=\nabla\times u$, a scalar quantity, satisfies

$$\begin{cases} \partial_t\omega+u.\nabla\omega+\Lambda^\alpha\omega=\partial_1\theta, \\ u=\nabla^\perp\psi, \quad \Delta\psi=\omega \quad \text{or} \quad u=\nabla^\perp\Delta^{-1}\omega. \end{cases} \tag{1.3}$$

Therefore the problem reduces to the problem of considering the regularity and existence of the classical solution of the equations

$$\begin{cases} \omega_t+\Lambda^\alpha\omega+u\cdot\nabla\theta=\partial_1\theta, \\ \theta_t+\Lambda^\beta\theta+u\cdot\nabla\theta=0. \end{cases} \tag{1.4}$$

One notices of course, that the right-hand side of the vorticity equation has a full derivative acting on θ, which is challenging to control. The strategy (first applied by Hmidi-Keraani-Rousset, [10]) is to consider a combined quantity of the vorticity and (a derivative of) the temperature θ, which one would eventually be able to control via energy estimates. More precisely, note that since we can write

$$\Lambda^\alpha\omega-\partial_1\theta=\Lambda^\alpha[\omega-\Lambda^{-\alpha}\partial_1\theta],$$

it is worth introducing the quantity $G=\omega-\Lambda^{-\alpha}\partial_1\theta$. For it, we have the equation,

$$G_t+u\cdot\nabla G+\Lambda^\alpha G=\Lambda^{\beta-\alpha}\partial_1\theta+[R_\alpha,u\cdot\nabla]\theta.$$

This is the evolution equation used in [10] and subsequent papers, [13, 23, 28]. It turns out however that the presence of the factor $\Lambda^{\beta-\alpha}\partial_1\theta$ is still too rough in the range of $\alpha>\frac{2}{3}$, thus preventing us from getting the desired bounds. In order to remove it as is done above in the G construction, we introduce a new variable $f=G-\Lambda^{\beta-2\alpha}\partial_1\theta$. This is the second generation HKR change of variables that we have alluded to above. We have

$$G_t+u\cdot\nabla G+\Lambda^\alpha(G-\Lambda^{\beta-2\alpha}\partial_1\theta)=[R_\alpha,u\cdot\nabla]\theta.$$

Again by adding and subtracting some terms and using the equation for θ, we get

$$\begin{aligned}(G-\Lambda^{\beta-2\alpha}\partial_1\theta)_t+u.\nabla(G-\Lambda^{\beta-2\alpha}\partial_1\theta)+\Lambda^\alpha(G-\Lambda^{\beta-2\alpha}\partial_1\theta)+\Lambda^{\beta-2\alpha}\partial_1\theta_t \\ +u\cdot\nabla\Lambda^{\beta-2\alpha}\partial_1\theta=[R_\alpha,u\cdot\nabla]\theta\end{aligned}$$

which gives

$$f_t+u.\nabla f+\Lambda^\alpha f+(-\Lambda^{2(\beta-\alpha)}\partial_1\theta-\Lambda^{\beta-2\alpha}\partial_1(u\cdot\nabla\theta))+u.\nabla\Lambda^{\beta-2\alpha}\partial_1\theta=[R_\alpha,u\cdot\nabla]\theta,$$

hence

$$f_t+u\cdot\nabla f+\Lambda^\alpha f=\Lambda^{2(\beta-\alpha)}\partial_1\theta+[R_\alpha,u\cdot\nabla]\theta+[\Lambda^{\beta-2\alpha}\partial_1,u\cdot\nabla]\theta. \tag{1.5}$$

Note that since $\beta-\alpha=1-2\alpha<0$, the term $[\Lambda^{\beta-2\alpha}\partial_1,u\cdot\nabla]\theta=[\Lambda^{\beta-\alpha}R_\alpha,u\cdot\nabla]\theta$ will always be easier to treat than the similar term $[R_\alpha,u\cdot\nabla]\theta$. For this reason, we will ignore this term in our discussion, with the understanding that a rigorous proof can always be produced by following the corresponding proof for the (harder to treat) commutator term $[R_\alpha, u\cdot\nabla]\theta$.

Based on the definition above

$$f = G - \Lambda^{\beta-2\alpha}\partial_1\theta = G - R_\alpha \Lambda^{\beta-\alpha}\theta = \omega - R_\alpha\theta - R_\alpha\Lambda^{\beta-\alpha}\theta = \omega - (R_\alpha + R_\alpha\Lambda^{\beta-\alpha})\theta,$$

therefore

$$u = \nabla^\perp\Delta^{-1}\omega = \nabla^\perp\Delta^{-1} f + \nabla^\perp\Delta^{-1}R_\alpha(I+\Lambda^{\beta-\alpha})\theta := u_f + u_\theta. \tag{1.6}$$

With this definition it is clear that, $u_f \sim \Lambda^{-1} f$ and $u_\theta \sim \Lambda^{-\alpha}\theta + \Lambda^{1-3\alpha}\theta$.

1.4. Regularity criteria for the Boussinesq system. The question for global regularity is reduced to a certain, so called regularity criteria, namely a quantity[6], which if controlled up to time T, will keep all higher Sobolev norms finite and non-blowing up to time T, hence the global regularity. This is a well-known problem in many quasilinear problems, for example in the standard Navier–Stokes posed on $\mathbf{R}^1_+ \times \mathbf{R}^d$, it suffices to control *a priori* $\sup_{0\le t\le T}\|u(t)\|_{\dot{H}^{d/2}}$ or $\sup_{0\le t\le T}\|u(t)\|_{L^d}$ or some mixed norm quantities of the form $\|u\|_{L^p_t(0,T)L^q(\mathbf{R}^d)}, \frac{2}{p}+\frac{d}{q}=1, 2\le p\le\infty$. These are all quantities, which of course scale nicely according to the natural scaling of the NLS problem. One difficulty with equation ((1.1)) is that the problem does not have scaling invariance, outside of the case $\alpha=\beta$. Nevertheless, there exists a regularity result for the Boussinesq system, namely Theorem 1.2 in [13]. Although, it is not quite stated in the clean form that we described above for NLS, it provides for a regularity result for the temperature equation[7] in equation ((1.1)). More precisely, we have

PROPOSITION 1.1 (Theorem 1.2 in [13]). *Let $\beta\in(0,1)$, $\tilde{u}:\nabla\cdot\tilde{u}=0$ with*

$$M = \|\tilde{u}\|_{L^\infty(0,T)L^2(\mathbf{R}^2)} + \|\nabla\tilde{u}\|_{L^\infty(0,T)L^\infty(\mathbf{R}^2)} < \infty. \tag{1.7}$$

Assume that $\theta:\theta\in L^2(\mathbf{R}^2), \nabla\theta\in L^2\cap L^\infty$ satisfies the generalized critical surface quasi-geostrophic equation

$$\begin{cases} \theta_t + \Lambda^\beta\theta + u\cdot\nabla\theta = 0 \\ u = \tilde{u}+v, v = -\nabla^\perp\Lambda^{-3+\beta}\partial_1\theta \\ \theta(0,x) = \theta_0(x). \end{cases} \tag{1.8}$$

Then, the equation ((1.8)) has an unique solution $\theta\in C([0,T), H^1(\mathbf{R}^2))$,

$$\|\nabla\theta\|_{L^\infty(0,T)L^\infty(\mathbf{R}^2)} \le C(T, M, \|\nabla_0\|_{H^1}, \|\nabla\theta_0\|_{L^\infty}).$$

for some continuous function C.

Having θ as smooth as guaranteed by Proposition (1.1) in turn allows us to conclude the regularity of u in the full Boussinesq system ((1.1)). Thus, the regularity criteria, which we need, is exactly

$$M_T = \sup_{0\le t\le T}[\|u_f\|_{L^\infty(0,t)L^\infty_x(\mathbf{R}^2)} + \|\nabla u_f\|_{L^\infty(0,t)L^\infty_x(\mathbf{R}^2)}] < \infty.$$

In order to extract an easy quantity to work with, we make use of the following result.

[6] usually a norm of the solution

[7] Given the form of the equation ((1.6)), the motivation for the form of u below becomes clear

Proposition 1.2 (Lemma 2.5, p. 1969, [28]). *Let* $\alpha,\beta:\alpha+\beta\leq 1, \frac{1}{2}<\alpha<1$ *and*

$$G_t+u\cdot\nabla G+\Lambda^\alpha G=[R_\alpha,u\cdot]\nabla+\Lambda^{\beta-\alpha}\partial_{x_1}\theta. \tag{1.9}$$

Then, if $\frac{2}{1-\alpha}>q>\frac{2}{\alpha}$ *and*

$$\sup_{0\leq t\leq T}\|G(t,\cdot)\|_{L^q(\mathbf{R}^2)}<\infty,$$

then for any $0\geq s<\max(3\alpha-2,0)$, *one has the bound*

$$\sup_{0\leq t\leq T}\|G(t,\cdot)\|_{B^s_{r,\infty}}<\infty,$$

where

$$\frac{2}{2\alpha-1}<r\leq\frac{2q}{2-(1-\alpha)q}.$$

Let us mention that the equation G displayed in equation ((1.9)) corresponds to the change of variables used in previous works (dubbed first generation Hmidi-Keraani-Rousset). On the other hand, we would like to apply Proposition (1.2) to the solution f of equation ((1.5)). Note however that the terms in (1.5) are either the same or more regular than the corresponding terms[8] in inequality ((1.10)). Thus, we can apply Proposition (1.2) to f. Using this result, we can reduce matters to verifying

$$\sup_{0\leq t\leq T}\|f(t,\cdot)\|_{L^6(\mathbf{R}^2)}<\infty. \tag{1.10}$$

Indeed, assuming that we have established the bound ((1.10)), we apply Proposition (1.2) with $q=6$ (which is exactly in the range $(\frac{2}{\alpha},\frac{2}{1-\alpha})$). We obtain the following bound for f

$$\sup_{0\leq t\leq T}\|f\|_{B^{3\alpha-2}_{\frac{6}{3\alpha-2},\infty}}<\infty.$$

But then, by elementary Sobolev embedding, we have for every small $\delta>0$,

$$\|\nabla u_f\|_{L^\infty_x}\leq C_\delta\|f\|_{W^{\frac{3\alpha-2-\delta}{3},\frac{6}{3\alpha-2}}}\leq C_\delta\|f\|_{B^{3\alpha-2}_{\frac{6}{3\alpha-2},\infty}}$$

which would have verified the bound ((1.7)). Thus, it remains to verify ((1.10)).

Remark 1.1. Originally, our proof proceeded via a Sobolev embedding control of the form $\|\nabla u_f\|_{L^\infty_x(\mathbf{R}^2)}\leq C(\|\Lambda^\delta\nabla f\|_{L^2(\mathbf{R}^2)}+\|f\|_{L^2})$ and then controlling this last Sobolev norm. We gratefully acknowledge Professor Ye's contribution, which lead us to this much shorter argument.

1.5. Strategy of the proof and the organization of the paper. As we have alluded to before, the strategy is to follow the standard approach for such models - namely one starts with a local solution[9]. Such solution may of course be defined for short time only and it may blow up at some finite time $T_0<\infty$. We henceforth do not worry about the existence and the regularity of the solution up to time T_0, but we need good *a*

[8] Thanks to Prof. Ye for pointing this out to us in a private communication

[9] which is immediately smooth for any time $t>0$

priori estimates. More precisely, in the discussion leading to ((1.10)), we explained that blow up is possible, only if $\limsup_{t\to T_0-}\|f\|_{L^\infty(0,t)L^6_x(\mathbf{R}^2)}=\infty$. Thus, a contradiction will be reached (whence $T_0=\infty$ and the solution is global), if one can provide *a priori* bound in the form $\sup_{0\le t\le T_0}\| \ f\|_{L^\infty(0,t)L^6_x(\mathbf{R}^2)}=M_0<\infty$. In practice, we construct $M=M(T;\|\theta_0\|_{L^1\cap H^{2+\rho}(\mathbf{R}^2)},\|u_0\|_{H^{1+\rho}(\mathbf{R}^2)})$ a continuous function in all arguments, so that $\sup_{0\le t\le T}\|f\|_{L^\infty(0,t)L^6_x(\mathbf{R}^2)}\le M(T)$.

Starting with the obvious *a priori* bounds ((1.2)), we gradually improve it to finally obtain ((1.10)). More precisely, in Section (3), we first establish an L^2 bound for f (see Proposition (3.1)), together with some Sobolev bounds for θ. Next, using the L^2 bounds from Proposition (3.1), we bootstrap Proposition (3.2), in order to establish L^4 bounds for f, together with the uniform in time Sobolev bounds for f,θ and some L^2 averaged in time Sobolev bounds. These are all (considerably) better than the one in Proposition (3.1). We finish Section (3) by bootstrapping Proposition (3.2) yet again to establish L^6 bounds for f, together with even better uniform and L^2 time averaged Sobolev bounds for f,θ. The uniform Sobolev bounds in time required for the global regularity in ((1.10)) do not come cheaply and by themselves - instead one seems to need to cook up energy functionals involving L^p (p larger) norms of f. In other words, for low α one faces not only the usual derivative difficulties as in previous works, but also integrability issues for f. Having Proposition (3.3) is enough, by the discussion in Section (1.4) below to conclude the global regularity claimed in Theorem (1.1).

2. Preliminaries

For the proof, we need a number of technical tools, which we now introduce. We start with the L^p spaces and Littlewood-Paley theory.

2.1. Function spaces. We use standard notation for L^p spaces and Sobolev spaces, namely for $s>0, p\in[1,\infty)$,

$$\|f\|_{L^p}=\left(\int |f(x)|^p dx\right)^{1/p}$$
$$\|f\|_{W^{s,p}}=\|\Lambda^s f\|_{L^p}+\|f\|_{L^p}$$

We need to quickly introduce some elementary Littlewood-Paley theory. To that end, let Υ be an even and smooth function on $\mathbf{R}^1$, so that *supp* $\Upsilon\subset[-2,2]$, so that $\Upsilon(\xi)=1, |\xi|<1$. Define $\zeta:\mathbf{R}^2\to\mathbf{R}^1$ via $\zeta(\xi)=\Upsilon(|\xi|)-\Upsilon(2|\xi|)$, so that $\zeta\in C^\infty(\mathbf{R}^2)$, with *supp* $\zeta\subset\{\xi:\frac{1}{2}<|\xi|<2\}$. In addition,

$$\sum_{k=-\infty}^{\infty}\zeta(2^{-k}\xi)=1,\quad \xi\neq 0.$$

This allows us to define the Littlewood-Paley operators $\widehat{\Delta_j f}(\xi):=\zeta(2^{-j}\xi)\hat{f}(\xi)$, restricting the Fourier transform of f to the annulus $\{\xi:|\xi|\sim 2^j\}$. We will often denote $f_k=\Delta_k f$, $f_{\sim k}=\sum_{j=k-10}^{k+10}\Delta_j f$ and $f_{<k}=\sum_{j<k}\Delta_j f$.

2.2. Commutator estimates. In this section, we present some commutator estimates, which will be useful in our arguments. Some of them, Lemma (2.2) and Lemma (2.3) appear to be new. We start with a lemma developed in [23] (see Lemma 2.5 there and Corollary 2.6).

Lemma 2.1. *Let $\nabla\cdot g=0$, $0<S<1$ and $1<p_2<\infty$, $1<p_1,p_3\le\infty$, so that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. For every $0\le S_1,S_2,S_3\le 1$ that satisfy $S_1+S_2+S_3>1+S$, there exists a*

$C=C(p_1,p_2,p_3,S_1,S_2)$, so that

$$|\int_{R^d} h[\Lambda^S, g\cdot\nabla]\psi dx| \leq C\|\Lambda^{S_1}\psi\|_{L^{p_1}}\|\Lambda^{S_2}h\|_{L^{p_2}}\|\Lambda^{S_3}g\|_{p_3}. \tag{2.1}$$

In particular if $p_3<\infty$ then

- *for $S_1=s_1$, $S_2=s_2$ and $S_3=1$ where $s_1+s_2>1-\alpha$*

$$|\int_{R^d} h[R_\alpha, V\cdot\nabla]\psi dx| \leq C\|\Lambda^{s_1}\psi\|_{L^{p_1}}\|\Lambda^{s_2}h\|_{L^{p_2}}\|\nabla V\|_{L^{p_3}}, \tag{2.2}$$

- *similarly, for every $0\leq s_2,s_3<1$, so that $s_2+s_3>1+S$ we have*

$$|\int_{R^d} h[\Lambda^S, V\cdot\nabla]\psi dx| \leq C\|\psi\|_{L^{p_1}}\|\Lambda^{s_2}h\|_{L^{p_2}}\|\Lambda^{s_3}V\|_{L^{p_3}}. \tag{2.3}$$

Note that in all statements, one could have replaced $R_\alpha=\partial_1\Lambda^{-\alpha}$ by any multiplier, which acts as differentiation of order $1-\alpha$, for example $\Lambda^{1-\alpha}$.

Note that in this lemma, one has to always allow for small derivative loss. Lemma (2.1) will be adequate for many terms, except when we need to account for all derivatives. In other words, we need a variant which is lossless in the derivative count (and/or endpoint estimates). We have two versions - Lemma (2.2) is for estimates in (homogeneous) Sobolev spaces of negative index, and the other one, Lemma (2.3), for estimates in (homogeneous) Sobolev spaces of positive index. We mostly need Lemma (2.2) throughout the paper, the need for Lemma (2.3) arises at the very end of our argument. Interestingly, in the proof (presented in the Appendix), we do not distinguish much between these two cases. Note that the results in Lemmas (2.2) and Lemma (2.3) hold under somewhat more general assumptions than the one that we displayed below, but we prefer to keep it simple and convenient for the applications.

Lemma 2.2. *Let s_1,s_2 be two real numbers so that $0\leq s_1$ and $0\leq s_2-s_1\leq 1$. Let p,q,r be related via the Hölder's $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$, where $2<q<\infty$, $1<p,r<\infty$. Finally, let $\nabla\cdot V=0$.*

Then for any $a\in[s_2-s_1,1]$

$$\|\Lambda^{-s_1}[\Lambda^{s_2}, V\cdot\nabla]\varphi\|_{L^p} \leq C\|\Lambda^a V\|_{L^q}\|\Lambda^{s_2-s_1+1-a}\varphi\|_{L^r}. \tag{2.4}$$

In addition, we have the following end-point estimate. For $s_1>0, s_2>0, s_3>0$ and $s_1<1, s_3<1, s_2<s_1+s_3$, there is[10]

$$\|\Lambda^{-s_1}[\Lambda^{s_2}, \Lambda^{-s_3}V\cdot\nabla]\varphi\|_{L^2} \leq C\|V\|_{L^\infty}\|\Lambda^{s_2-s_1+1-s_3}\varphi\|_{L^2}. \tag{2.5}$$

We have the following useful corollary of inequality ((2.4)).

Corollary 2.1. *Let $p_1,p_2,p_3: \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$ and $p_1>2$. Assume that $0\leq s\leq 1$. Then,*

$$|\langle[\Lambda^s, V\cdot\nabla]\varphi,\psi\rangle| \leq C\|\nabla V\|_{L^{p_1}}\|\Lambda^s\varphi\|_{L^{p_2}}\|\psi\|_{L^{p_3}} \tag{2.6}$$

$$|\langle[\Lambda^s, V\cdot\nabla]\varphi,\psi\rangle| \leq C\|\Lambda^a V\|_{L^{p_1}}\|\Lambda^{s+1-a}\varphi\|_{L^{p_2}}\|\psi\|_{L^{p_3}} \tag{2.7}$$

[10]Note that in the statement of inequality ((2.5)), one does not necessarily need precisely the form $\Lambda^{-s_3}V$. In fact, the estimate applies for any Fourier multiplier Q, with the property that $\|QV_k\|_{L^\infty}\sim 2^{-ks_3}\|V_k\|_{L^\infty}$

whenever $a\in[s,1]$.

The next lemma is basically identical to Lemma (2.2), except that s_1 has the opposite sign.

LEMMA 2.3. *Let* $0\le s_1$, $0<s_2$, $0\le s_1+s_2<1$, $s_2+s_1<a\le 1$, $2<q<\infty, 1<r<\infty$ *and* $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$. *Then*

$$\|\Lambda^{s_1}[\Lambda^{s_2},V\cdot\nabla]\varphi\|_{L^p}\le C\|\Lambda^a V\|_{L^q}\|\Lambda^{1+s_2+s_1-a}\varphi\|_{L^r}. \tag{2.8}$$

The following corollary is a direct result of the above lemma.

COROLLARY 2.2. *Let* $0\le s<\alpha$, $\beta+s<a\le 1$, $2<q,r<\infty$ *and* $\frac{1}{2}=\frac{1}{q}+\frac{1}{r}$. *Then*

$$\|\Lambda^{s}[R_\alpha,V\cdot\nabla]\varphi\|_{L^2}\le C\|\Lambda^a V\|_{L^q}\|\Lambda^{1+\beta+s-a}\varphi\|_{L^r}. \tag{2.9}$$

Next, we need to prepare a technical point, which will be useful in the sequel.

2.3. The scaled variables. For technical reasons, we use the following scaled variables

$$\begin{cases} \theta(t,x)=\Theta(\epsilon_0^\beta t,\epsilon_0 x), & u(t,x)=U(\epsilon_0^\beta t,\epsilon_0 x)\\ f(t,x)=F(\epsilon_0^\beta t,\epsilon_0 x), & U=U_F+U_\Theta, \end{cases} \tag{2.10}$$

where ϵ_0 is a small parameter to be determined in each energy estimate later on separately. Clearly

$$\Theta_t+\epsilon_0^\alpha U\cdot\nabla\Theta+\Lambda^\beta\Theta=0.$$

The corresponding equation for F is

$$\begin{aligned}\epsilon_0^\beta F_t+\epsilon_0 U\cdot\nabla F+\epsilon_0^\alpha\Lambda^\alpha F=\epsilon_0^{1+2(\beta-\alpha)}\Lambda^{2(\beta-\alpha)}\partial_1\Theta+\epsilon_0^{3\beta}[\Lambda^{\beta-2\alpha}\partial_1,U\cdot\nabla]\Theta\\ +\epsilon_0^{1+\beta}[R_\alpha,U\cdot\nabla]\Theta.\end{aligned}$$

Thus, our new system now is in the form of

$$\begin{cases} F_t+\epsilon_0^\alpha U\cdot\nabla F+\epsilon_0^{\alpha-\beta}\Lambda^\alpha F=N(U,F,\Theta),\\ \Theta_t+\epsilon_0^\alpha U\cdot\nabla\Theta+\Lambda^\beta\Theta=0. \end{cases} \tag{2.11}$$

with $N(U,F,\Theta)=\epsilon_0^{2-3\alpha}\Lambda^{2(\beta-\alpha)}\partial_1\Theta+\epsilon_0[R_\alpha,U\cdot\nabla]\Theta+\epsilon_0^{2\beta}[\Lambda^{\beta-2\alpha}\partial_1,U.\nabla]\Theta$. Note that in this case $\|\theta\|_{L^p}=\epsilon_0^{-2/p}\|\Theta\|_{L^p}$, in particular $\|\Theta\|_{L^\infty}=\|\theta\|_{L^\infty}$ and similar for f,F.

2.4. Some basic energy inequalities. Now suppose $\kappa,s\ge 0$, and take Λ^s and Λ^κ derivatives, and then dot product with $\Lambda^s F$ and $\Lambda^\kappa\Theta$ in system ((2.11)), respectively, to get

$$\begin{aligned}\frac{1}{2}\partial_t\|\Lambda^s F\|_{L^2}^2+\epsilon_0^{\alpha-\beta}\|\Lambda^{s+\frac{\alpha}{2}}F\|_{L^2}^2\le \epsilon_0^\alpha\Big|\int(\Lambda^s[U\cdot\nabla F])\Lambda^s F dx\Big|\\ +\epsilon_0^{2-3\alpha}|\langle\Lambda^{2(\beta-\alpha)+s}\partial_1\Theta,\Lambda^s F\rangle|+\epsilon_0|\langle\Lambda^s[R_\alpha,U\cdot\nabla]\Theta,\Lambda^s F\rangle|\\ +\epsilon_0^{2\beta}|\langle\Lambda^s[\Lambda^{\beta-2\alpha}\partial_1,U\cdot\nabla]\Theta,\Lambda^s F\rangle|=I_1+I_2+I_3+I_4\end{aligned} \tag{2.12}$$

and,

$$\frac{1}{2}\partial_t\|\Lambda^\kappa\Theta\|_{L^2}^2+\|\Lambda^{\kappa+\frac{\beta}{2}}\Theta\|_{L^2}^2\le\epsilon_0^\alpha|\langle\Lambda^\kappa(U\cdot\nabla\Theta),\Lambda^\kappa\Theta\rangle|:=I_5. \tag{2.13}$$

In the case that $s<1$ or $\kappa<1$ we can easily rewrite I_1 and I_5 in the commutator forms:

$$I_1=\epsilon_0^\alpha|\langle[\Lambda^s,U\cdot\nabla]F,\Lambda^sF\rangle|,\qquad I_5=\epsilon_0^\alpha|\langle[\Lambda^\kappa,U\cdot\nabla]\Theta,\Lambda^\kappa\Theta\rangle|.$$

Now, take dot product with $F|F|^{p-2}$ in ((2.11)), and get

$$\begin{aligned}&\frac{1}{p}\partial_t\|F\|_{L^p}^p+\epsilon_0^{\alpha-\beta}|\int F|F|^{p-2}\Lambda^\alpha F dx|\le\epsilon_0^{2-3\alpha}\int F|F|^{p-2}\Lambda^{2(\beta-\alpha)}\partial_1\Theta dx\\&\quad+\epsilon_0|\langle[R_\alpha,U\cdot\nabla]\Theta,F|F|^{p-2}\rangle|+\epsilon_0^{2\beta}|\langle[\Lambda^{\beta-2\alpha}\partial_1,U\cdot\nabla]\Theta,F|F|^{p-2}\rangle|\\&:=K_1+K_2+K_3.\end{aligned}$$

By maximum principle

$$\begin{aligned}\epsilon_0^{\alpha-\beta}|\int F|F|^{p-2}\Lambda^\alpha F dx|&\ge C_0\epsilon_0^{2\alpha-1}\int|\Lambda^{\frac{\alpha}{2}}(F^{\frac{p}{2}})|^2dx\ge C_0\epsilon_0^{2\alpha-1}\|F^{\frac{p}{2}}\|^2_{L^{\frac{4}{2-\alpha}}}\\&=C_0\epsilon_0^{2\alpha-1}\|F\|^p_{L^{\frac{2p}{2-\alpha}}}.\end{aligned}$$

Therefore

$$\frac{1}{p}\partial_t\|F\|_{L^p}^p+C_0\epsilon_0^{\alpha-\beta}\|F\|^p_{\frac{2p}{2-\alpha}}\le\epsilon_0^{2-3\alpha}\int F^3\Lambda^{2(\beta-\alpha)}\partial_1\Theta dx=K_1+K_2+K_3.\quad(2.14)$$

In our proofs, we usually combine two or three relations of ((2.12)), ((2.13)) and ((2.14)), with different κ, s, and p, and try to find the proper estimate for the right hand side, and then use the Gronwall's inequality to close the arguments. In our discussion, we shall ignore the estimates for I_4 and K_3, as they are easier to deal with than the corresponding terms I_3 and K_2.

3. L^p bounds on f

In this section we prove L^2, L^4 and L^6 bound for f. We start with L^2 bound and then proceed with L^4 bound and finally we get the L^6 bound. During the discussion we also raise the derivative on both θ and f. This allows us to jump to higher derivatives in the next section.

3.1. L^2 Estimate.

PROPOSITION 3.1. *Let $0<\rho\ll 1$, $\gamma=\frac{\beta}{2}-2\rho$, $f_0\in H^{\frac{\alpha}{2}}$ and $\theta_0\in L^\infty\cap H^\gamma$ then*

$$\|f\|_{L^2}+\|\Lambda^\gamma\theta\|_{L^2}\le C_T\quad(3.1)$$

$$\int_0^T(\|\Lambda^{\frac{\alpha}{2}}f(.,t)\|^2_{L^2}+\|\Lambda^{\frac{\beta}{2}+\gamma}\theta(.,t)\|^2_{L^2})dt\le C_T\quad(3.2)$$

where $C_T=C(T,\|\theta_0\|_{L^\infty},\|f\|_{H^{\frac{\alpha}{2}}},\|\theta\|_{H^{\frac{\alpha}{2}}})$.

Proof. We start with the scaled variables. In each case, we specify how small ϵ needs to be in order to close the estimates. In the end, we choose and fix one such ϵ, say the half of the smallest upper bound. This argument will then imply the estimates ((3.1)) and ((3.2)).

In ((2.12)) and ((2.13)) take $\kappa=0$ and $s=\gamma$, then we want to bound the right hand side of the following relation

$$\frac{1}{2}\partial_t(\|F\|^2_{L^2}+\|\Lambda^\gamma\Theta\|^2_{L^2})+\epsilon_0^{\alpha-\beta}\|\Lambda^{\frac{\alpha}{2}}F\|^2_{L^2}+\|\Lambda^{\gamma+\frac{\beta}{2}}\Theta\|^2_{L^2}\le I_1+I_2+I_3+I_4+I_5.\quad(3.3)$$

Since $\langle U\cdot\nabla F,F\rangle=-\frac{1}{2}\langle\nabla\cdot U,F^2\rangle=0$, we have $I_1=0$.

3.1.1. Estimate for I_2.
Case $1, \alpha > \frac{3}{4}$:

$$I_2 \le \epsilon_0^{2-3\alpha} \|\Lambda^{3-4\alpha}\Theta\|_{L^2} \|F\|_{L^2} \le \epsilon_0^{2-3\alpha} \|\Theta\|_{L^{\frac{1}{2\alpha-1}}} \|F\|_{L^2} \le \frac{1}{100}\|F\|_{L^2}^2 + C_{\epsilon_0}.$$

Case $2, \alpha \le \frac{3}{4}$:
we have by Hölder's and Gagliardo-Nirenberg,

$$I_2 \le \epsilon_0^{2-3\alpha} \|\Lambda^{3-4\alpha} F\|_{L^2} \|\Theta\|_{L^2} \le C\epsilon_0^{2-3\alpha} \|\theta_0\|_{L^2} \|\Lambda^{\frac{\alpha}{2}} F\|_{L^2}^{\delta} \|F\|_{L^2}^{1-\delta}.$$

for $\delta = \frac{3-4\alpha}{\alpha/2}$. Note that $\delta \in (0,1)$, since $\alpha > 2/3$. Applying Young's inequality gives us

$$I_2 \le \frac{\epsilon_0^{2\alpha-1}}{100} \|\Lambda^{\frac{\alpha}{2}} F\|_{L^2}^2 + C_{\epsilon_0, \|\theta_0\|_{L^2}} (1 + \|F\|_{L^2})^2.$$

3.1.2. Estimate for I_3. In this case we are seeking bounds for two terms I_3^f and I_3^θ

$$I_3 \le \epsilon_0 \left| \int F[R_\alpha, U_\Theta.\nabla]\Theta dx \right| + \epsilon_0 \left| \int F[R_\alpha, U_F.\nabla]\Theta dx \right| := I_3^\theta + I_3^f.$$

Now, for I_3^θ, we apply inequality ((2.3)), with $p_1 = \infty, p_2 = p_3 = 2$ and $s_3 = \frac{\beta}{2} + \gamma + \alpha = 1 - 2\rho$ and $s_2 = \frac{\alpha}{2}$. Note that this is within the range of applicability of inequality ((2.3)), since $s_2 + s_3 = 1 + \frac{\alpha}{2} - 2\rho > 2 - \alpha$, whenever $\alpha > \frac{2}{3}$ and $0 < \rho \ll \alpha - \frac{2}{3}$. We get

$$I_3^\theta \le C\epsilon_0 \|\Theta\|_{L^\infty} \|\Lambda^{\frac{\alpha}{2}} F\|_{L^2} \|\Lambda^{\frac{\beta}{2}+\gamma+\alpha} U_\Theta\|_{L^2} \le C\epsilon_0 \|\theta_0\|_{L^\infty} \|\Lambda^{\frac{\alpha}{2}} F\|_{L^2} \|\Lambda^{\frac{\beta}{2}+\gamma}\Theta\|_{L^2}.$$

Thus,

$$I_3^\theta \le \frac{\epsilon_0^{2\alpha-1}}{100} \|\Lambda^{\frac{\alpha}{2}} F\|_{L^2}^2 + C\epsilon_0^{3-2\alpha} \|\theta_0\|_{L^\infty}^2 \|\Lambda^{\frac{\beta}{2}+\gamma}\Theta\|_{L^2}^2.$$

Taking $\epsilon_0 : C\epsilon_0^{3-2\alpha} \|\theta_0\|_{L^\infty}^2 \le \frac{1}{100}$ will ensure that we can absorb the second term above behind $\|\Lambda^{\frac{\beta}{2}+\gamma}\Theta\|_{L^2}^2$ on the left-hand side.

Regarding I_3^f, we have by inequality ((2.2)) with $p_3 = 2$, $s_1 = 0$, $s_2 = 1 - \alpha + \rho$, $\frac{2}{p_1} = \frac{3\alpha}{2} - 1 - \rho, \frac{2}{p_2} = 2 - \frac{3\alpha}{2} + \rho$,

$$I_3^f \le \epsilon_0 \|\nabla U_F\|_{L^2} \|\Lambda^{1-\alpha+\rho} F\|_{L^{p_2}} \|\Theta\|_{L^{p_1}} \le \epsilon_0 C \|F\|_{L^2} \|\Lambda^{\alpha/2} F\|_{L^2} \|\theta_0\|_{L^{p_1}}.$$

where we have used the Sobolev embedding estimate $\|\Lambda^{1-\alpha+\rho} F\|_{L^{p_2}} \le C \|\Lambda^{\alpha/2} F\|_{L^2}$. Applying Cauchy-Schwarz yields

$$\begin{aligned} I_3^f &\le \frac{\epsilon_0^{\alpha-\beta}}{100} \|\Lambda^{\alpha/2} F\|_{L^2}^2 + \epsilon^{1+2\beta} C \|\theta_0\|_{L^{p_1}}^2 \|F\|_{L^2}^2 \\ &\le \frac{\epsilon_0^{\alpha-\beta}}{100} \|\Lambda^{\alpha/2} F\|_{L^2}^2 + \frac{1}{100} \|\theta_0\|_{L^{p_1}}^2 \|F\|_{L^2}^2, \end{aligned}$$

where we took ϵ so that $\epsilon^{1+2\beta} C \le \frac{1}{100}$.

3.1.3. Estimate for I_5. For I_5^f, take $s_3 = 1-\rho, s_2 = \gamma+2\rho$, $p: \frac{1}{p} = \frac{1}{2} - \frac{\rho}{2}$, $q: \frac{1}{q} = \frac{\rho}{2}$, then we have by inequality ((2.3)),

$$|\langle [\Lambda^\gamma, U_F.\nabla]\Theta, \Lambda^\gamma \Theta\rangle| \leq C\|\Theta\|_{L^q}\|\Lambda^{2\gamma+2\rho}\Theta\|_{L^2}\|\Lambda^{1-\rho}U_F\|_{L^p}.$$

Also, by Sobolev embedding $\|\Lambda^{1-\rho}U_F\|_{L^p} \leq C\|\Lambda^{-\rho}F\|_{L^p} \leq C\|F\|_{L^2}$. All in all, noting that $2(\gamma+\rho) = \gamma+\beta/2$,

$$\begin{aligned} I_5^f = \epsilon_0^\alpha |\langle [\Lambda^\gamma, U_F.\nabla]\Theta, \Lambda^\gamma\Theta\rangle| &\leq \frac{1}{100}\|F\|_{L^2}^2 + \epsilon_0^{2\alpha} C\|\Theta_0\|_{L^q}^2 \|\Lambda^{\beta/2+\gamma}\Theta\|_{L^2}^2 \\ &\leq \frac{1}{100}\|F\|_{L^2}^2 + \frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^{\beta/2+\gamma}\Theta\|_{L^2}^2. \end{aligned}$$

where we took ϵ_0 so that $\epsilon_0 C\|\theta_0\|_{L^q} \leq \frac{1}{100}$.
For the term containing U_Θ, we have by inequality ((2.5)), with $s_1 = \beta/2, s_2 = \gamma, s_3 = \alpha$,

$$I_5^\theta = \epsilon_0^\alpha |\langle [\Lambda^\gamma, U_\Theta \cdot \nabla]\Theta, \Lambda^\gamma\Theta\rangle| \leq C\epsilon_0^\alpha \|\theta_0\|_{L^\infty}\|\Lambda^{\gamma+\beta/2}\Theta\|_{L^2}^2 \leq \frac{1}{100}\|\Lambda^{\gamma+\beta/2}\Theta\|_{L^2}^2 \tag{3.4}$$

where we take $C\epsilon_0^\alpha\|\theta_0\|_{L^\infty} \leq \frac{1}{100}$. Introducing

$$J(t) = \|\Lambda^\gamma\Theta\|_{L^2}^2 + \|F\|_{L^2}^2,$$

and putting all the estimates together, we obtain the bound

$$J'(t) + \|\Lambda^{\frac{\alpha}{2}} f\|_{L^2}^2 + \|\Lambda^{\frac{\beta}{2}+\gamma}\theta\|_{L^2}^2 \leq C_{\epsilon_0, \|\theta_0\|_{L^2\cap L^\infty}} J(t).$$

An application of the Gronwall's inequality yields the bounds for the right hand side of ((3.3)). □

Now that we have the estimate for $\sup_{0\leq t\leq T}\|f\|_{L^2}$, we use it to obtain the estimates for $\sup_{0\leq t\leq T}\|f\|_{L^4}$.

3.2. L^4 Estimate. The precise result that we prove is the following.

PROPOSITION 3.2. *Let $1 > \alpha > \frac{2}{3}$, (u,θ) be the solution of equation ((1.1)) and (u_0,θ_0) be as specified in Theorem (1.1). Assume f satisfies equation ((1.5)), then for any $T > 0$ there exists $C_T = C(T)$, such that*

$$\sup_{0\leq t\leq T}\|f\|_{L^4} + \int_0^T \|f\|_{L^{\frac{8}{2-\alpha}}}^4 < C_T$$

$$\sup_{0\leq t\leq T}\|\Lambda^{\frac{\alpha}{2}} f\|_{L^2} + \int_0^T \|\Lambda^\alpha f\|_{L_x^2}^2 dt < C_T$$

$$\sup_{0\leq t\leq T}\|\Lambda^{\frac{3\beta}{2}}\theta\|_{L^2} + \int_0^T \|\Lambda^{2\beta}\theta\|_{L_x^2}^2 dt < C_T$$

Proof. We again use the scaled variables. In inequalities ((2.12)), ((2.13)) and ((2.14)) take $\kappa = \frac{\alpha}{2}$, $s = \frac{3\beta}{2}$ and $p = 4$ to get

$$\begin{aligned} &\partial_t\left(\frac{1}{4}\|F\|_{L^4}^4 + \frac{1}{2}\|\Lambda^{\frac{3\beta}{2}}\Theta\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{\frac{\alpha}{2}}F\|_{L^2}^2\right) + C_0\epsilon_0^{\alpha-\beta}\|F\|_{L^{\frac{8}{2-\alpha}}}^4 \\ &\quad + \epsilon_0^{\alpha-\beta}\|\Lambda^\alpha F\|_{L^2}^2 + \|\Lambda^{2\beta}\Theta\|_{L^2}^2 \leq K_1 + K_2 + K_3 + I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

We now proceed to establish proper bounds for each term in the right hand side.

3.2.1. Estimate for K_1.
Case $3-4\alpha<0$:
In this case we have,

$$|\int F^3\Lambda^{3-4\alpha}\Theta\ dx|\le\|F\|_{L^4}^3\|\Lambda^{3-4\alpha}\Theta\|_{L^4},$$

and by Sobolev embedding

$$\|\Lambda^{3-4\alpha}\Theta\|_{L^4}\le\|\Theta\|_{L^{\frac{1}{2\alpha-\frac{5}{4}}}}\le C_\epsilon\|\theta_0\|_{L^{\frac{1}{2\alpha-\frac{5}{4}}}}$$

hence

$$K_1\le\|F\|_{L^4}^4+C_\epsilon.$$

Case $3-4\alpha>0$: We have

$$|\int F^3\Lambda^{3-4\alpha}\Theta\ dx|\le\|F\|^3_{L^{\frac{8}{2-\alpha}}}\|\Lambda^{3-4\alpha}\Theta\|_{L^{\frac{8}{2+3\alpha}}}.$$

Furthermore,

$$\|\Lambda^{3-4\alpha}\Theta\|_{\frac{8}{2+3\alpha}}\le\|\Lambda^{2\beta}\Theta\|^a_{L^2}\|\Theta\|^{1-a}_{L^{q_0}}\le C_{\epsilon_0}\|\Lambda^{2\beta}\Theta\|^a_{L^2}\|\theta_0\|^{1-a}_{L^{q_0}}$$

where $a=\frac{3-4\alpha}{2\beta}$ and $q_0=\frac{4(2\alpha-1)}{3\alpha\beta+6\alpha-4}$. Note that for $\alpha>2/3$, $q_0\ge1$, therefore

$$K_1\le C\epsilon_0^{2-3\alpha}\|F\|^3_{L^{\frac{8}{2-\alpha}}}\|\Lambda^{2\beta}\Theta\|^a_{L^2}\le\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^4_{L^{\frac{8}{2-\alpha}}}+\epsilon_0^{\frac{9-14\alpha}{2}}C\|\Lambda^{2\beta}\Theta\|^{4a}_{L^2}.$$

Clearly $4a<2$, hence

$$K_1\le\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^4_{L^{\frac{8}{2-\alpha}}}+\frac{1}{100}\|\Lambda^{2\beta}\Theta\|^2_{L^2}+C_{\epsilon_0}.$$

3.2.2. Estimate for K_2.
Estimate for K_2^f:
For $0<\delta\ll1$, to be determined later, by inequality ((2.6)) with $s=1-\alpha$, then

$$\begin{aligned}K_2^f=\epsilon_0|\langle[R_\alpha,U_F.\nabla]\Theta,F^3\rangle|&\le\epsilon_0C\|\Theta\|_{L^{\frac{8}{3\alpha-2}}}\|\Lambda^{1-\alpha}(F^3)\|_{L^{\frac{4}{4-\alpha}}}\|\nabla U_F\|_{L^{\frac{8}{2-\alpha}}}\\&\le\epsilon_0C\|\Lambda^{1-\alpha}F\|_{L^2}\|F\|^3_{L^{\frac{8}{2-\alpha}}}\|\Theta\|_{L^{\frac{8}{3\alpha-2}}}.\end{aligned}$$

Hence we conclude

$$K_2^f\le\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^4_{L^{\frac{8}{2-\alpha}}}+C_{\epsilon_0}\|\Lambda^{1-\alpha}F\|^4_{L^2}.$$

But

$$\|\Lambda^{1-\alpha}F\|^4_{L^2}\le\|\Lambda^\alpha F\|_{L^2}^{\frac{4(1-\alpha)}{\alpha}}\|F\|_{L^2}^{\frac{2\alpha-1}{\alpha}}.$$

Note however that $\frac{4(1-\alpha)}{\alpha}<2$, since $\alpha>\frac{2}{3}$, therefore

$$K_2^f\le\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^4_{L^{\frac{8}{2-\alpha}}}+\frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^\alpha F\|^2_{L^2}+C_{\epsilon_0}.$$

Estimate for K_2^θ:
Again for $0<\delta\ll 1$, apply inequality ((2.3)) with $s_3=\alpha+\delta$ and $s_2=2(1-\alpha)$

$$K_2^\theta=\epsilon_0|\langle[R_\alpha,U_\Theta.\nabla]\Theta,F^3\rangle|\le \epsilon_0\|\Lambda^{\alpha+\delta}U_\Theta\|_{L^{\frac{8}{\alpha}}}\|\Lambda^{2(1-\alpha)}(F^3)\|_{L^{\frac{4}{4-\alpha}}}\|\Theta\|_{L^{\frac{8}{\alpha}}}$$
$$\le \epsilon_0 C\|\Lambda^\delta\Theta\|_{L^{\frac{8}{\alpha}}}\|F\|^2_{L^{\frac{8}{2-\alpha}}}\|\Lambda^{2(1-\alpha)}F\|_{L^2}\|\theta_0\|_{L^{\frac{8}{\alpha}}}.$$

Note

$$\|\Lambda^\delta\Theta\|_{L^{\frac{8}{\alpha}}}\le\|\Lambda^{\frac{\beta}{2}-2\rho}\Theta\|^a_{L^2}\|\theta\|^{1-a}_{L^q}$$

where $a=\frac{\delta}{\frac{\beta}{2}-2\rho}$ and $q:\frac{1-a}{q}+\frac{a}{2}=\frac{\alpha}{8}$. Clearly $q\in(1,\infty)$, provided $\delta\ll 1$. We have obtained

$$K_2^\theta\le\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^4_{L^{\frac{8}{2-\alpha}}}+\epsilon_0^{1+2\beta}C\|\Lambda^{2(1-\alpha)}F\|^2_{L^2}$$

and

$$\|\Lambda^{2(1-\alpha)}F\|_{L^2}\le\|\Lambda^\alpha F\|^a_{L^2}\|F\|^{1-a}_{L^2}$$

where $a=\frac{2(1-\alpha)}{\alpha}<1$. Thus,

$$K_2^\theta\le\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^4_{L^{\frac{8}{2-\alpha}}}+\frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^\alpha F\|^2_{L^2}+C_\epsilon$$

hence

$$K_2\le\frac{\epsilon_0^{\alpha-\beta 1}}{50}\|F\|^4_{L^{\frac{8}{2-\alpha}}}+\frac{\epsilon_0^{\alpha-\beta}}{50}\|\Lambda^\alpha F\|^2_{L^2}+C_\epsilon.$$

3.2.3. Estimate for I_1.
Estimate for I_1^f:

$$I_1^f=\epsilon_0^\alpha|\langle[\Lambda^{\frac{\alpha}{2}},U_F.\nabla]F,\Lambda^{\frac{\alpha}{2}}F\rangle|\le\epsilon_0^\alpha\|\Lambda^\alpha F\|_{L^2}\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{\alpha}{2}},U_F.\nabla]F\|_{L^2}$$

now in inequality ((2.4)) take $s_1=s_2=\frac{\alpha}{2}$, $V=\Lambda^{-1}F$, $\varphi=F$, $a=1$ and $q=r=4$ to get

$$\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{\alpha}{2}},U_F.\nabla]F\|_{L^2}\le\|F\|^2_{L^4}$$

then

$$I_1^f\le\epsilon_0^\alpha C\|\Lambda^\alpha F\|_{L^2}\|F\|^2_{L^4}\le\frac{1}{100}\|F\|^4_{L^4}+C\frac{\epsilon_0^{2\alpha}}{100}\|\Lambda^\alpha F\|^2_{L^2}$$
$$\le\frac{1}{100}\|F\|^4_{L^4}+\frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^\alpha F\|^2_{L^2}$$

where we took $\epsilon_0\le\frac{1}{100C}$.
Estimate for I_1^θ:

$$I_1^\theta=\epsilon_0^\alpha|\langle[\Lambda^{\frac{\alpha}{2}},U_\Theta.\nabla]F,\Lambda^{\frac{\alpha}{2}}F\rangle|\le\epsilon_0^\alpha\|\Lambda^\alpha F\|_{L^2}\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{\alpha}{2}},U_\Theta.\nabla]F\|_{L^2}$$

if in inequality ((2.5)) we take $s_1=s_2=\frac{\alpha}{2}$, $s_3=\alpha$, $V=\Theta$, $\varphi=F$, $a=1$ then

$$\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{\alpha}{2}},U_\Theta.\nabla]F\|_{L^2}\le\|\Theta\|_{L^\infty}\|\Lambda^\beta F\|_{L^2}$$
$$\le\|\Theta\|_{L^\infty}\|\Lambda^\alpha F\|^{\frac{\beta}{\alpha}}_{L^2}\|F\|^{\frac{\alpha-\beta}{\alpha}}_{L^2}$$

therefore

$$I_1^\theta\le\epsilon_0^\alpha C\|\Lambda^\alpha F\|^{1+\frac{\beta}{\alpha}}_{L^2}\le\frac{\epsilon_o^{\alpha-\beta}}{100}\|\Lambda^\alpha F\|^2_{L^2}+C_{\epsilon_0}.$$

3.2.4. Estimate for I_2. If $3-4\alpha<0$, then we have $3(1-\alpha)<\alpha$ and

$$|\langle \Lambda^{2(\beta-\alpha)+\frac{\alpha}{2}}\partial_1\Theta, \Lambda^{\frac{\alpha}{2}}F\rangle| \le \|\Lambda^{3(1-\alpha)}F\|_{L^2}\|\Theta\|_{L^2}$$

and

$$\|\Lambda^{3(1-\alpha)}F\|_{L^2} \le \|\Lambda^{\alpha}F\|_{L^2}^{a}\|F\|_{L^2}^{1-a}$$

where $a=\frac{3(1-\alpha)}{\alpha}<1$, therefore

$$I_2 \le C\epsilon_0^{2-3\alpha}\|\Lambda^{\alpha}F\|_{L^2}^{a} \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^{\alpha}F\|_{L^2}^2 + C_{\epsilon_0}.$$

If $3-4\alpha>0$, then by Hölder

$$|\langle \Lambda^{2(\beta-\alpha)+\frac{\alpha}{2}}\partial_1\Theta, \Lambda^{\frac{\alpha}{2}}F\rangle| \le \|\Lambda^{\alpha}F\|_{L^2}\|\Lambda^{3-4\alpha}\Theta\|_{L^2}.$$

But

$$\|\Lambda^{3-4\alpha}\Theta\|_{L^2} \le \|\Lambda^{2\beta}\Theta\|_{L^2}^{a}\|\Theta\|_{L^2}^{1-a},$$

where $a=\frac{3-4\alpha}{2\beta}$ and therefore

$$I_2 \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^{\alpha}F\|_{L^2}^2 + C_{\epsilon_0}\|\Lambda^{2\beta}\Theta\|_{L^2}^{2a}.$$

Since $a<1$,

$$I_2 \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^{\alpha}F\|_{L^2}^2 + \frac{1}{100}\|\Lambda^{2\beta}\Theta\|_{L^2}^2 + C_{\epsilon_0}.$$

Considering the two sub-cases above, the last inequality is the proper estimate for I_2.

3.2.5. Estimate for I_3.
Estimate for I_3^{θ}:

$$I_3^{\theta} = \epsilon_0|\langle \Lambda^{\frac{\alpha}{2}}[R_\alpha, U_\Theta.\nabla]\Theta, \Lambda^{\frac{\alpha}{2}}F\rangle| \le \epsilon_0\|\Lambda^{\alpha}F\|_{L^2}\|[R_\alpha, U_\Theta.\nabla]\Theta\|_{L^2}.$$

Now if in inequality ((2.4)) we take $s_1=0$, $s_2=\beta$, $V=\Lambda^{-\alpha}\Theta$, $a=1$, $p=2$ and $q=r=4$ then

$$\|[R_\alpha, U_\Theta.\nabla]\Theta\|_{L^2} \le \|\Lambda^{\beta}\Theta\|_{L^4}^2 \le (\|\Lambda^{2\beta}\Theta\|_{L^2}^{\frac{1}{2}}\|\Theta\|_{L^\infty}^{\frac{1}{2}})^2$$

therefore

$$I_3^{\theta} \le \epsilon_0\|\Theta\|_{L^\infty}^{\frac{3}{2}}\|\Lambda^{\alpha}F\|_{L^2}\|\Lambda^{2\beta}\Theta\|_{L^2} \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^{\alpha}F\|_{L^2}^2 + \frac{1}{100}\|\Lambda^{2\beta}\Theta\|_{L^2}^2$$

where we took $\epsilon_0 \le (\frac{1}{10^4\|\theta_0\|_{L^\infty}})^{\frac{1}{3-2\alpha}}$.

Estimate for I_3^{f}:

$$\begin{aligned} I_3^{f} = \epsilon_0|\langle \Lambda^{\frac{\alpha}{2}}[R_\alpha, U_F.\nabla]\Theta, \Lambda^{\frac{\alpha}{2}}F\rangle| &\le \epsilon_0\|\Lambda^{\alpha}F\|_{L^2}\|[R_\alpha, U_F.\nabla]\Theta\|_{L^2} \\ &\le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^{\alpha}F\|_{L^2}^2 + C\epsilon_0^{3-2\alpha}\|[R_\alpha, U_F.\nabla]\Theta\|_{L^2}^2. \end{aligned}$$

Now by applying inequality ((2.4)) with $s_1=0, p=2, q=r=4$, $s_2=1-\alpha$, $a=1$, we have

$$\epsilon_0^{3-2\alpha}\|[R_\alpha, U_F.\nabla]\Theta\|_{L^2}^2 \le \epsilon_0^{3-2\alpha}\|F\|_{L^4}^2\|\Lambda^{1-\alpha}\Theta\|_{L^4}^2 \le \frac{1}{100}\|F\|_{L^4}^4 + C\epsilon_0^{6-4\alpha}\|\Lambda^\beta\Theta\|_{L^4}^4.$$

If we take ϵ_0 so that $\epsilon_0 \le (\frac{1}{100C\|\theta_0\|_{L^\infty}})^{\frac{1}{6-4\alpha}}$ then

$$\epsilon_0^{3-2\alpha}\|[R_\alpha, U_f.\nabla]\Theta\|_{L^2}^2 \le \frac{1}{100}\|F\|_{L^4}^4 + \frac{1}{100}\|\Lambda^{2\beta}\Theta\|_{L^2}^2,$$

therefore

$$I_3^f \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\Lambda^\alpha F\|_{L^2}^2 + \frac{1}{100}\|F\|_{L^4}^4 + \frac{1}{100}\|\Lambda^{2\beta}\Theta\|_{L^2}^2.$$

3.2.6. Estimate for I_5.

Estimate for I_5^θ: Apply ((2.5)) with $s_1=\beta/2, s_2=3\beta/2$ and $s_3=\alpha$,

$$I_5^\theta = \epsilon_0^\alpha|\langle[\Lambda^{\frac{3\beta}{2}}, U_\Theta\cdot\nabla]\Theta, \Lambda^{\frac{3\beta}{2}}\Theta\rangle| \le \epsilon_0^\alpha C\|\theta_0\|_{L^\infty}\|\Lambda^{2\beta}\Theta\|_{L^2}^2 \le \frac{1}{100}\|\Lambda^{2\beta}\Theta\|_{L^2}^2 \tag{3.5}$$

where we took $\epsilon_0 \le (\frac{1}{100C\|\Theta_0\|_{L^\infty}})^{\frac{1}{\alpha}}$.

Estimate for I_5^f:

$$\begin{aligned} I_5^f &= \epsilon_0^\alpha|\langle[\Lambda^{\frac{3\beta}{2}}, U_F\cdot\nabla]\Theta, \Lambda^{\frac{3\beta}{2}}\Theta\rangle| = \epsilon_0^\alpha|\langle\Lambda^{\frac{-\beta}{2}}[\Lambda^{\frac{3\beta}{2}}, U_F\cdot\nabla]\Theta, \Lambda^{2\beta}\Theta\rangle| \\ &\le \frac{1}{100}\|\Lambda^{2\beta}\Theta\|_{L^2}^2 + C\epsilon_0^{2\alpha}\|\Lambda^{\frac{-\beta}{2}}[\Lambda^{\frac{3\beta}{2}}, U_F\cdot\nabla]\Theta\|_{L^2}^2. \end{aligned}$$

We apply ((2.4)) with $s_1=\frac{\beta}{2}$, $s_2=\frac{3\beta}{2}$, $V=U_F$, $\phi=\Theta$ and $q=r=4$ then

$$\epsilon_0^{2\alpha}C\|\Lambda^{\frac{-\beta}{2}}[\Lambda^{\frac{3\beta}{2}}, U_F.\nabla]\Theta\|_{L^2}^2 \le \epsilon_0^{2\alpha}C\|F\|_{L^4}^2\|\Lambda^\beta\Theta\|_{L^4}^2 \le \frac{1}{100}\|F\|_{L^4}^4 + \epsilon_0^{4\alpha}CC\|\Lambda^\beta\Theta\|_{L^4}^4.$$

Applying again the Gagliardo-Nirenberg inequality $\|\Lambda^\beta\Theta\|_{L^4} \le \|\Lambda^{2\beta}\Theta\|_{L^2}^{\frac{1}{2}}\|\Theta\|_{L^\infty}^{\frac{1}{2}}$, we obtain

$$\epsilon_0^{2\alpha}\|\Lambda^{-\frac{\beta}{2}}[\Lambda^{\frac{3\beta}{2}}, U_F.\nabla]\Theta\|_{L^2}^2 \le \frac{1}{100}\|F\|_{L^4}^4 + C\epsilon_0^{2\alpha}\|\Theta_0\|_{L^\infty}^2\|\Lambda^{2\beta}\Theta\|_{L^2}^2.$$

From here we take ϵ_0 so small that $\epsilon_0 \le (\frac{1}{100C\|\Theta_0\|_{L^\infty}^2})^{\frac{1}{4\alpha}}$. We get

$$I_5^f \le \frac{1}{100}\|F\|_{L^4}^4 + \frac{1}{50}\|\Lambda^{2\beta}\Theta\|_{L^2}^2.$$

Now putting all the above estimates together along with a using of Gronwall's inequality finishes the proof for L^4. ∎

3.3. L^6 Estimate. Now we have enough information of θ and f to get the L^6 estimate

PROPOSITION 3.3. *Let $\alpha > \frac{2}{3}$, then for any $T>0$ there exists a C_T such that*

$$\sup_{0\le t\le T}\|F\|_{L^6}^6 + \int_0^T \|F\|_{L^{\frac{12}{2-\alpha}}}^6\,dt \le C_T,$$

$$\sup_{0\leq t\leq T}\|\Lambda^{\frac{1+\beta}{2}}F\|_{L^2}^2+\int_0^T\|\partial F\|_{L^2}^2dt\leq C_T,$$

and

$$\sup_{0\leq t\leq T}\|\Lambda^{\frac{5\beta}{2}}\Theta\|_{L^2}^2+\int_0^T\|\Lambda^{3\beta}\Theta\|_{L^2}^2dt\leq C_T.$$

Proof. In inequalities ((2.14)), ((2.12)) and ((2.13)) take $p=6$, $s=\frac{1+\beta}{2}$ and $\kappa=\frac{5\beta}{2}$ to get

$$\begin{aligned}&\partial_t(\frac{1}{6}\|F\|_{L^6}^6+\frac{1}{2}\|\Lambda^{\frac{1+\beta}{2}}F\|_{L^2}^2+\frac{1}{2}\|\Lambda^{\frac{5\beta}{2}}\Theta\|_{L^2}^2)\\&\quad+\epsilon_0^{\alpha-\beta}\|F\|_{L^{\frac{12}{2-\alpha}}}^6+\epsilon_0^{\alpha-\beta}\|\partial F\|_{L^2}^2+\|\Lambda^{3\beta}\Theta\|_{L^2}^2\\&\leq K_1+K_2+K_3+I_1+I_2+I_3+I_4+I_5.\end{aligned}$$

3.3.1. Estimate for K_1. If $3-4\alpha\geq 0$, by Hölder inequality

$$K_1\leq\epsilon_0^{2-3\alpha}|\int F^5\Lambda^{3-4\alpha}\Theta dx|\leq\epsilon_0^{2-3\alpha}\|F\|_{L^{\frac{12}{2-\alpha}}}^5\|\Lambda^{3-4\alpha}\Theta\|_{L^{\frac{12}{5\alpha+2}}}.$$

By Sobolev inequality

$$\|\Lambda^{3-4\alpha}\Theta\|_{L^{\frac{12}{5\alpha+2}}}\leq\|\Lambda^{\frac{22-29\alpha}{6}}\Theta\|_{L^2}.$$

Now since for $\frac{2}{3}<\alpha\leq\frac{3}{4}$, $0<\frac{22-29\alpha}{6}<\frac{3\beta}{2}$, there is a $0<\gamma<1$ such that

$$\|\Lambda^{\frac{22-29\alpha}{6}}\Theta\|_{L^2}\leq\|\Lambda^{\frac{3\beta}{2}}\Theta\|_{L^2}^{\gamma}\|\Theta\|_{L^2}^{1-\gamma}=C,$$

therefore

$$K_1\leq\epsilon_0^{2-3\alpha}C\|F\|_{L^{\frac{12}{2-\alpha}}}^5\leq\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|_{L^{\frac{12}{2-\alpha}}}^6+C_{\epsilon_0}.$$

If $3-4\alpha<0$, then we use Hölder and Sobolev inequalities to get

$$K_1\leq\epsilon_0^{2-3\alpha}\|F\|_{L^6}^5\|\Lambda^{3-4\alpha}\Theta\|_{L^6}\leq\epsilon_0^{2-3\alpha}\|F\|_{L^6}^5\|\Theta\|_{L^{\frac{6}{12\alpha-8}}}\leq\frac{1}{100}\|F\|_{L^6}^6+C_{\epsilon_0}.$$

Now we put both cases together to get

$$K_1\leq\frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|_{L^{\frac{12}{2-\alpha}}}^6+\frac{1}{100}\|F\|_{L^6}^6+C_{\epsilon_0}.$$

3.3.2. Estimate for K_2.
Estimate for K_2^f:
In inequality ((2.4)) take $s_1=0$, $s_2=\beta$, $a=1$, $V=\Lambda^{-1}F$, $\phi=\Theta$, $q=\frac{12}{2-\alpha}$ and $r=\frac{2}{\alpha}$ to get

$$\begin{aligned}K_2^f=\epsilon_0|\langle[R_\alpha,U_F\cdot\nabla]\Theta,F^5\rangle|&\leq\epsilon_0\|F\|_{L^{\frac{12}{2-\alpha}}}^5\|[R_\alpha,U_F.\nabla]\Theta\|_{L^{\frac{12}{5\alpha+2}}}\\&\leq\epsilon_0\|F\|_{L^{\frac{12}{2-\alpha}}}^6\|\Lambda^\beta\Theta\|_{L^{\frac{2}{\alpha}}},\end{aligned}$$

then

$$\|\Lambda^\beta\Theta\|_{L^{\frac{2}{\alpha}}} \le \|\Lambda^{\frac{3\beta}{2}}\Theta\|_{L^2}^{\frac{2}{3}} \|\Theta\|_{L^{\frac{2}{3\alpha-2}}}^{\frac{1}{3}} = C,$$

therefore

$$K_2^f \le \epsilon_0 C\|F\|^6_{L^{\frac{12}{2-\alpha}}} \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^6_{L^{\frac{12}{2-\alpha}}},$$

where we took ϵ_0 such that $\epsilon_0 \le [\frac{1}{100C}]^{\frac{1}{2\beta}}$.

Estimate for K_2^θ:

$$K_2^\theta = \epsilon_0|\langle[R_\alpha, U_\Theta.\nabla]\Theta, F^5\rangle| \le \epsilon_0\|\Lambda^\beta(F^5)\|_{L^{\frac{12}{7(2-\alpha)}}}\|\Lambda^{-\beta}[R_\alpha, U_\Theta.\nabla]\Theta\|_{L^{\frac{12}{7\alpha-2}}},$$

whence by Kato-Ponce,

$$\|\Lambda^\beta(F^5)\|_{L^{\frac{12}{7(2-\alpha)}}} \le C\|\Lambda^\beta F\|_{L^{\frac{4}{2-\alpha}}}\|F\|^4_{L^{\frac{12}{2-\alpha}}}$$

then

$$\|\Lambda^\beta F\|_{L^{\frac{4}{2-\alpha}}} \le C\|\nabla F\|^\beta_{L^2}\|F\|^\alpha_{L^4},$$

and if inequality ((2.4)) we take $s_1 = s_2 = \beta$, $V = \Lambda^{-\alpha}\Theta$, $\phi = \Theta$, $a = \alpha + \frac{\beta}{2}$ and $q = r = \frac{24}{7\alpha-2}$ we have

$$\|\Lambda^{-\beta}[R_\alpha, U_\Theta.\nabla]\Theta\|_{L^{\frac{12}{7\alpha-2}}} \le C\|\Lambda^{\frac{\beta}{2}}\Theta\|^2_{L^{\frac{24}{7\alpha-2}}} \le C(\|\Lambda^{3\beta}\Theta\|^{\frac{1}{6}}_{L^2}\|\Theta\|^{\frac{5}{6}}_{L^{\frac{20}{7\alpha-4}}})^2,$$

therefore

$$\begin{aligned} K_2^\theta &\le \epsilon_0 C\|F\|^4_{L^{\frac{12}{2-\alpha}}}\|\nabla F\|^\beta_{L^2}\|\Lambda^{3\beta}\Theta\|^{\frac{1}{3}}_{L^2} \\ &\le \frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^6_{L^{\frac{12}{2-\alpha}}} + (\epsilon_0^{\frac{1+4\beta}{3}} C\|\nabla F\|^\beta_{L^2}\|\Lambda^{3\beta}\Theta\|^{\frac{1}{3}}_{L^2})^3, \end{aligned}$$

now since $3(\beta + \frac{1}{3}) < 2$,

$$K_2^\theta \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|^6_{L^{\frac{12}{2-\alpha}}} + \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|^2_{L^2} + \frac{1}{100}\|\Lambda^{3\beta}\Theta\|^2_{L^2} + C_{\epsilon_0}.$$

3.3.3. Estimate for I_1.

Estimate for I_1^f:

$$\begin{aligned} I_1^f &= \epsilon_0^\alpha|\langle[\Lambda^{\frac{1+\beta}{2}}, U_F.\nabla]F, \Lambda^{\frac{1+\beta}{2}}F\rangle| \le \epsilon_0^\alpha\|\nabla F\|_{L^2}\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{1+\beta}{2}}, U_F.\nabla]F\|_{L^2} \\ &\le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|^2_{L^2} + \epsilon_0 C\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{1+\beta}{2}}, U_F.\nabla]F\|^2_{L^2}, \end{aligned}$$

then if in inequality ((2.4)) we take $s_1 = \frac{\alpha}{2}$, $s_2 = \frac{1+\beta}{2}$, $V = \Lambda^{-1}F$, $\phi = F$, $a = 1$ and $q = r = 4$, then a using of ((2.4)), Sobolev inequality and Gagliardo-Nirenberg gives

$$\begin{aligned} \|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{1+\beta}{2}}, U_F.\nabla]F\|_{L^2} &\le \|F\|_{L^4}\|\Lambda^\beta F\|_{L^4} = C\|\Lambda^\beta F\|_{L^4} \\ &\le C\|\Lambda^{\frac{1+2\beta}{2}}F\|_{L^2} \le C\|\nabla F\|^{\frac{1+2\beta}{2}}_{L^2}\|F\|^{\frac{1-2\beta}{2}}_{L^2}. \end{aligned}$$

Therefore

$$I_1^f \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + \epsilon_0 C\|\nabla F\|_{L^2}^{1+\beta} \le \frac{\epsilon_0^{\alpha-\beta}}{50}\|\nabla F\|_{L^2}^2 + C_{\epsilon_0}.$$

Estimate for I_1^θ:

$$\begin{aligned} I_1^\theta &= \epsilon_0^\alpha |\langle [\Lambda^{\frac{1+\beta}{2}}, U_\Theta.\nabla]F, \Lambda^{\frac{1+\beta}{2}}F\rangle| \le \epsilon_0^\alpha \|\nabla F\|_{L^2}\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{1+\beta}{2}}, U_\Theta.\nabla]F\|_{L^2} \\ &\le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + \epsilon_0 C\|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{1+\beta}{2}}, U_\Theta.\nabla]F\|_{L^2}^2, \end{aligned}$$

now if in inequality ((2.5)) we take $s_1 = \frac{\alpha}{2}$, $s_2 = \frac{1+\beta}{2}$, $s_3 = \alpha$, $V = \Theta$, $\phi = F$ then Gagliardo-Nirenberg yields

$$\begin{aligned} \|\Lambda^{-\frac{\alpha}{2}}[\Lambda^{\frac{1+\beta}{2}}, U_\Theta.\nabla]F\|_{L^2} &\le C\|\Theta\|_{L^\infty}\|\Lambda^{2\beta}F\|_{L^2} = C\|\Lambda^{2\beta}F\|_{L^2} \\ &\le C\|\nabla F\|_{L^2}^{2\beta}\|F\|_{L^2}^{1-2\beta}. \end{aligned}$$

therefore

$$I_1^\theta \le \frac{\epsilon_0^{\alpha-\beta}}{50}\|\nabla F\|_{L^2}^2 + C_{\epsilon_0}.$$

3.3.4. Estimate for I_2.

$$I_2 = \epsilon_0^{2-3\alpha}|\langle \Lambda^{2(\beta-\alpha)+\frac{1+\beta}{2}}\partial_1\Theta, \Lambda^{\frac{1+\beta}{2}}F\rangle| \le C\epsilon_0^{2-3\alpha}\|\nabla F\|_{L^2}\|\Lambda^{4-5\alpha}\Theta\|_{L^2}.$$

To find the bound for the right hand side,we consider two cases.
If $4-5\alpha \ge 0$, we have that $4-5\alpha \le 3\beta$, whence there is a $0<\gamma<1$, such that

$$\|\Lambda^{4-5\alpha}\Theta\|_{L^2} \le C\|\Lambda^{3\beta}\Theta\|_{L^2}^\gamma \|\Theta\|_{L^2}^{1-\gamma},$$

therefore in this case

$$I_2 \le \epsilon_0^{2-3\alpha}C\|\nabla F\|_{L^2}\|\Lambda^{3\beta}\Theta\|_{L^2}^\gamma \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2 + C_{\epsilon_0}.$$

If $4-5\alpha<0$, then by Sobolev inequality we have $\|\Lambda^{4-5\alpha}\Theta\|_{L^2} \le C\|\Theta\|_{L^{\frac{2}{5\alpha-3}}}$. Note that $\frac{2}{5\alpha-3} \ge 1$, whence

$$I_2 \le \epsilon_0^{2-3\alpha}C\|\nabla F\|_{L^2} \le \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + C_{\epsilon_0}.$$

3.3.5. Estimate for I_3.

Estimate for I_3^f:

$$\begin{aligned} I_3^f &= \epsilon_0|\langle \Lambda^{\frac{1+\beta}{2}}[R_\alpha, U_F.\nabla]\Theta, \Lambda^{\frac{1+\beta}{2}}F\rangle| \le \epsilon_0\|\nabla F\|_{L^2}\|\Lambda^\beta[R_\alpha, U_F.\nabla]\Theta\|_{L^2} \\ &\le C\frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + C\epsilon_0^{1+2\beta}\|\Lambda^\beta[R_\alpha, U_F.\nabla]\Theta\|_{L^2}^2. \end{aligned}$$

Now in inequality ((2.9)) take $s=\beta$, $V=U_F$, $\varphi=\Theta$, $a=1$, $q=6$, and $r=3$ to get

$$\|\Lambda^\beta[R_\alpha, U_F.\nabla]\Theta\|_{L^2} \le C\|F\|_{L^6}\|\Lambda^{2\beta}\Theta\|_{L^3}.$$

So

$$\epsilon_0^{1+2\beta} C\|\Lambda^\beta[R_\alpha, U_F.\nabla]\Theta\|_{L^2}^2 \leq \frac{1}{100}\|F\|_{L^6}^6 + \epsilon_0^{\frac{3(1+2\beta)}{2}}\|\Lambda^{2\beta}\Theta\|_{L^3}^3.$$

Now $\|\Lambda^{2\beta}\Theta\|_{L^3} \leq \|\Lambda^{3\beta}\Theta\|_{L^2}^{\frac{2}{3}}\|\Theta\|_{L^\infty}^{\frac{1}{3}}$, so we take $\epsilon_0 \leq (\frac{1}{100C\|\Theta_0\|_{L^\infty}})^{\frac{2}{3(1+2\beta)}}$ to get

$$I_3^f \leq \frac{1}{100}\|F\|_{L^6}^6 + \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2.$$

Estimate for I_3^θ:

$$\begin{aligned} I_3^\theta &= \epsilon_0|\langle\Lambda^{\frac{1+\beta}{2}}[R_\alpha, U_\Theta.\nabla]\Theta, \Lambda^{\frac{1+\beta}{2}}F\rangle| \leq \epsilon_0\|\nabla F\|_{L^2}\|\Lambda^\beta[R_\alpha, U_\Theta.\nabla]\Theta\|_{L^2} \\ &\leq \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + C\epsilon_0^{1+2\beta}\|\Lambda^\beta[R_\alpha, U_\Theta.\nabla]\Theta\|_{L^2}^2. \end{aligned}$$

Now in inequality ((2.9)) take $s=\beta$, $V=U_\Theta$, $\varphi=\Theta$, $a=1$, $q=6$, and $r=3$ to get

$$\begin{aligned} \|\Lambda^\beta[R_\alpha, U_\Theta.\nabla]\Theta\|_{L^2} &\leq C\|\Lambda^\beta\Theta\|_{L^6}\|\Lambda^{2\beta}\Theta\|_{L^3} \\ &\leq C(\|\Lambda^{3\beta}\Theta\|_{L^2}^{\frac{1}{3}}\|\Theta\|_{L^\infty}^{\frac{2}{3}})\,(\|\Lambda^{3\beta}\Theta\|_{L^2}^{\frac{2}{3}}\|\Theta\|_{L^\infty}^{\frac{1}{3}}) \\ &= C\|\Lambda^{3\beta}\Theta\|_{L^2}\|\Theta\|_{L^\infty}. \end{aligned}$$

Therefore, if we choose $\epsilon_0 \leq (\frac{1}{100C\|\Theta_0\|_{L^\infty}^2})^{\frac{1}{1+2\beta}}$, we get

$$I_3^\theta \leq \frac{\epsilon_0^{\alpha-\beta}}{100}\|\nabla F\|_{L^2}^2 + \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2.$$

3.3.6. Estimate for I_5.

Estimate for I_5^f:

$$\begin{aligned} I_5^f &= \epsilon_0^\alpha|\langle[\Lambda^{\frac{5\beta}{2}}, U_F.\nabla]\Theta, \Lambda^{\frac{5\beta}{2}}\Theta\rangle| \leq \epsilon_0^\alpha\|\Lambda^{3\beta}\Theta\|_{L^2}\|\Lambda^{-\frac{\beta}{2}}[\Lambda^{\frac{5\beta}{2}}, U_F.\nabla]\Theta\|_{L^2} \\ &\leq \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2 + C\epsilon_0^{2\alpha}\|\Lambda^{-\frac{\beta}{2}}[\Lambda^{\frac{5\beta}{2}}, U_F.\nabla]\Theta\|_{L^2}^2 \\ &\leq \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2 + C\epsilon_0^{2\alpha}\|F\|_{L^6}^2\|\Lambda^{2\beta}\Theta\|_{L^3}^2 \\ &\leq \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2 + \frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|_{L^6}^6 + C\epsilon_0^{\frac{1+4\alpha}{2}}\|\Lambda^{2\beta}\Theta\|_{L^2}^3 \\ &\leq \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2 + \frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|_{L^6}^6 + C\epsilon_0^{\frac{1+4\alpha}{2}}\|\Lambda^{3\beta}\Theta\|_{L^2}^2\|\Theta\|_{L^\infty} \\ &\leq \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2 + \frac{\epsilon_0^{\alpha-\beta}}{100}\|F\|_{L^6}^6 + \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2 \end{aligned}$$

where we took $\epsilon_0 \leq (\frac{1}{100C\|\Theta_0\|_{L^\infty}})^{\frac{2}{1+4\alpha}}$.

Estimate for I_5^θ: a using of inequality ((2.5)) and Gagliardo-Nirenberg gives

$$\begin{aligned} I_5^f &= \epsilon_0^\alpha|\langle[\Lambda^{\frac{5\beta}{2}}, U_\Theta.\nabla]\Theta, \Lambda^{\frac{\beta}{2}}\Theta\rangle| \leq \epsilon_0^\alpha\|\Lambda^{3\beta}\Theta\|_{L^2}\|\Lambda^{-\frac{\beta}{2}}[\Lambda^{\frac{5\beta}{2}}, U_\Theta.\nabla]\Theta\|_{L^2} \\ &\leq \epsilon_0^\alpha\|\Lambda^{3\beta}\Theta\|_{L^2}^2\|\Theta\|_{L^\infty} \leq \frac{1}{100}\|\Lambda^{3\beta}\Theta\|_{L^2}^2, \end{aligned}$$

where we took $\epsilon_0 \leq (\frac{1}{100C\|\Theta_0\|_{L^\infty}})^{\frac{1}{\alpha}}$. This completes the proof. $\square$

Appendix A. Commutator estimates. Before we proceed with the proofs of Lemma (2.2) and Lemma (2.3), we would like to present some classical estimates for maximal functions, which will be used frequently in this section. First, there is the point-wise control of Littlewood-Paley operators by the maximal function, namely

$$(\Delta_k f)(x)+(\Delta_{<k-10}f)(x)\leq C\mathcal{M}[f](x).$$

Another useful result is the Fefferman-Stein estimate for the maximal function (see Theorem 4.6.6, p. 331, [9]), which states that $\mathcal{M}$ is a bounded operator from $L^p(l^r)$ into itself. More explicitly, for every $r,p\in(1,\infty)$, there is $C_{p,r}$, so that

$$\|(\sum_k(\mathcal{M}g_k)^r)^{1/r}\|_{L^p}\leq C_{p,r}\|(\sum_k|g_k|^r)^{1/r}\|_{L^p}.$$

Another basic tool is the following standard para product decomposition

$$\Delta_k(fg)=\Delta_k(f_{<k-10}g_{\sim k})+\Delta_k(f_{\sim k}g_{<k+10})+\Delta_k(\sum_{l=k+10}^{\infty}f_l g_{\sim l}),$$

available for say every pair of Schwartz functions f,g. We refer to the corresponding terms as low-high, high-low and high-high interaction terms.

In what follows, we present the proof of bounds ((2.4)) and ((2.8)). The difference between the two estimates is only in the dependence on the derivatives $\pm s_1$ taken on the commutators. Below, we take Λ^{-s_1} (matching the setup in ((2.4))), but we assume $s_1\in(-1,1)$ as to cover both ((2.4)) and ((2.8)). A crucial condition that needs to be met though is that $s_2-s_1\leq 1$. As far as (2.5) is concerned, note that it is an endpoint result of ((2.4)) (as sup norm is allowed on the right-hand side) and as such, only minor modifications are needed, details are provided in Appendix (A.2).

A.1. Proof of ((2.4)) and ((2.8)). We first present the proof for the hardest case $a=1$. We then discuss the necessary adjustments for the general case $a\in[s_2-s_1,1)$. Start with

$$\Lambda^{-s_1}[\Lambda^{s_2},V\cdot\nabla]\varphi=\sum_k\Delta_k[\Lambda^{-s_1}[\Lambda^{s_2},V\cdot\nabla]\varphi]].$$

Each one of these terms generates a separate entry for the estimate ((2.4)).

A.1.1. Low-high terms. For the low-high term, which is usually the hardest one in commutator estimates theory, we need to estimate $\|I_{low,high}\|_{L^p}$, where

$$I_{low,high}(x)=\sum_k\Delta_k[\Lambda^{-s_1}[\Lambda^{s_2},V_{<k-10}\cdot\nabla]\varphi_{\sim k}]].$$

In fact, we will show the estimate only under the restriction $2<q\leq\infty$ and **no restrictions on** s_2,s_1. More precisely, $q=\infty$ and any s_1,s_2 are allowed for the low-high interaction terms. Below, we tacitly assume $q<\infty$, the proof for $q=\infty$ requires minor modifications, which are left to the reader. By Littlewood-Paley theory, it suffices to control $\|S\|_{L^p}$, where the Littlewood-Paley square function S is given by

$$\begin{aligned}S^2(x)&=\sum_k|\Delta_k[\Lambda^{-s_1}[\Lambda^{s_2},V_{<k-10}\cdot\nabla]\varphi_{\sim k}]](x)|^2\\&=\sum_k 2^{2k(s_2-s_1)}|\Delta_k^1[[\Delta_k^2,V_{<k-10}\cdot\nabla]\varphi_{\sim k}]](x)|^2,\end{aligned}$$

where $\Delta_k^j, j=1,2$ are modified Littlewood-Paley operators similar to Δ_k. We will show that for $p_1,q_1\in(1,\infty):\frac{1}{p_1}+\frac{1}{q_1}=1$, we have the pointwise bound

$$|[\Delta_k^2, g\cdot\nabla]f](x)|\leq C\mathcal{M}[|\nabla g|^{q_1}](x)^{1/q_1}\mathcal{M}[|f|^{p_1}](x)^{1/p_1}. \tag{A.1}$$

where $\nabla\cdot g=0$ and $\mathcal{M}$ is the Hardy-Littlewood maximal function.

Assuming (A.1), let us show the estimate for the low-high piece of ((2.4)). We have for all $p_1,q_1\in(1,\infty):\frac{1}{p_1}+\frac{1}{q_1}=1$

$$\begin{aligned}S^2(x)&\leq\sum_k 2^{2k(s_2-s_1)}|\Delta_k^1[[\Delta_k^2, V_{<k-10}\cdot\nabla]\varphi_{\sim k}]](x)|^2\\&\leq\sum_k 2^{2k(s_2-s_1)}\mathcal{M}[[\Delta_k^2, V_{<k-10}\cdot\nabla]\varphi_{\sim k}]^2\\&\leq C\sum_k 2^{2k(s_2-s_1)}|\mathcal{M}[\mathcal{M}[|\nabla V_{<k-10}|^{q_1}]^{1/q_1}\mathcal{M}[|\varphi_{\sim k}|^{p_1}]^{1/p_1}]^2.\end{aligned}$$

Clearly, $\mathcal{M}[|\nabla V_{<k-10}|^{q_1}]\leq C\mathcal{M}[[\mathcal{M}(\nabla V)]^{q_1}]$. Thus, by the Fefferman-Stein estimates and by the Hölder's inequality

$$\begin{aligned}\|S\|_{L^p}&\leq C\|(\sum_k 2^{2k(s_2-s_1)}|\mathcal{M}[\mathcal{M}[|\nabla V_{<k-10}|^{q_1}]^{1/q_1}\mathcal{M}[|\varphi_{\sim k}|^{p_1}]^{1/p_1}|^2)^{1/2}\|_{L^p}\\&\leq C\|\mathcal{M}[\mathcal{M}(\nabla V)]^{q_1}])^{1/q_1}(\sum_k 2^{2k(s_2-s_1)}\mathcal{M}[|\varphi_{\sim k}|^{p_1}]^{2/p_1})^{1/2}\|_{L^p}\\&\leq\|\mathcal{M}[[\mathcal{M}(\nabla V)]^{q_1}]^{1/q_1}\|_{L^q}\|(\sum_k 2^{2k(s_2-s_1)}\mathcal{M}[|\varphi_{\sim k}|^{p_1}]^{2/p_1}])^{1/2}\|_{L^r}.\end{aligned}$$

Here, we need to select $q_1<q$, so that we can estimate (by the boundedness of $\mathcal{M}$ on L^{q/q_1})

$$\begin{aligned}\|\mathcal{M}[[\mathcal{M}(\nabla V)]^{q_1}]^{1/q_1}\|_{L^q}&=\|\mathcal{M}[[\mathcal{M}(\nabla V)]^{q_1}]\|_{L^{q/q_1}}^{1/q_1}\leq C\|\mathcal{M}(\nabla V)^{q_1}\|_{L^{q/q_1}}^{1/q_1}\\&\leq C\||\nabla V|^{q_1}\|_{L^{q/q_1}}^{1/q_1}=C\|\nabla V\|_{L^q}.\end{aligned}$$

For the other term, let $p_1:p_1<2,p_1<r$. Upon introducing $g_k:=[2^{k(s_2-s_1)}|\varphi_{\sim k}|]^{p_1}$, we have by Fefferman-Stein and Littlewood-Paley theory that

$$\begin{aligned}&\|(\sum_k 2^{2k(s_2-s_1)}\mathcal{M}[|\varphi_{\sim k}|^{p_1}]^{2/p_1}])^{1/2}\|_{L^r}=\|(\sum_k|\mathcal{M}g_k|^{2/p_1})^{1/2}\|_{L^r}\\&\leq\|(\sum_k|\mathcal{M}g_k|^{2/p_1})^{p_1/2}\|_{L^{r/p_1}}^{1/p_1}\leq C\|(\sum_k|g_k|^{2/p_1})^{p_1/2}\|_{L^{r/p_1}}^{1/p_1}\\&=C\|(\sum_k 2^{2k(s_2-s_1)}|\varphi_{\sim k}|^2)^{p_1/2}\|_{L^{r/p_1}}^{1/p_1}=C\|(\sum_k 2^{2k(s_2-s_1)}|\varphi_{\sim k}|^2)^{1/2}\|_{L^r}\leq C\|\Lambda^{s_2-s_1}\varphi\|_{L^r}.\end{aligned}$$

Analyzing the inequalities $p_1<2,p_1<r$ and $q_1<q$ shows that as long as $q>2$, we can always select $p_1,q_1:\frac{1}{p_1}+\frac{1}{q_1}=1$ with the required properties. This is easily seen by selecting $q_1=q-\epsilon, p_1=\frac{q_1}{q_1-1}=\frac{q-\epsilon}{q-1-\epsilon}$ for some small ϵ. Thus, we have shown

$$\|I_{low,high}\|_{L^p}\leq C\|\nabla V\|_{L^q}\|\Lambda^{s_2-s_1}\varphi\|_{L^r}. \tag{A.2}$$

To finish the proof in this case, we need to prove ((A.1)). But this is a simple application of the following representation formula for commutators

$$[\Delta_k^2, g\cdot\nabla]f(x) = 2^k[\Delta_k^3, g\cdot]f(x) = 2^{3k}\int_{\mathbf{R}^2}\chi_3(2^k(x-y))[g(x)-g(y)]f(y)dy$$
$$= 2^{3k}\int_{\mathbf{R}^2}\chi_3(2^k(x-y))(\int_0^1\langle\nabla g(y+z(x-y)), x-y\rangle dz)f(y)dy.$$

Clearly, after estimating this last expression,

$$|[\Delta_k^2, V_{<k-10}\cdot\nabla]\varphi_{\sim k}](x)|$$
$$\leq C2^{2k}\int_0^1\int_{\mathbf{R}^2}|\chi_4(2^k(x-y))||\nabla g(y+z(x-y)||f(y)|dydz$$
$$\leq C\int_0^1(\int_{\mathbf{R}^2}2^{2k}|\chi_4(2^k(x-y))||f(y)|^{p_1}dy)^{1/p_1}$$
$$\times(\int_{\mathbf{R}^2}2^{2k}|\chi_4(2^k(x-y))||\nabla g(y+z(x-y)|^{q_1}dy)^{1/q_1},$$

where $\chi_4(w) = \chi_3(w)w_i, i = 1,2$. Clearly,

$$\int_{\mathbf{R}^2}2^{2k}|\chi_4(2^k(x-y))||f(y)|^{p_1}dy \leq C\mathcal{M}[|f|^{p_1}](x),$$

Also,

$$\int_{\mathbf{R}^2}2^{2k}|\chi_4(2^k(x-y))||\nabla g(y+z(x-y)|^{q_1}dy = \int_{\mathbf{R}^2}2^{2k}|\chi_4(2^k l)||\nabla g(x-(1-z)l)|^{q_1}dl$$
$$= \int_{\mathbf{R}^2}\frac{2^{2k}}{(1-z)^2}|\chi_4(\frac{2^k}{1-z}m)||\nabla g(x-m)|^{q_1}dm \leq C\mathcal{M}[|\nabla g|^{q_1}](x).$$

This establishes ((A.1)).

A.1.2. High-low term. Here, we need the assumption $s_2 - s_1 \leq 1$, but q, r may be arbitrary (i.e. one does not have $2 < q$), as long as $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

In this case, the commutator structure does not play much role, so we just deal with the two terms separately. In fact, the term $\Lambda^{-s_1}\Delta_k[V_{\sim k}\cdot\nabla\Lambda^{s_2}\varphi_{<k+10}]$ is simpler, so we omit its analysis. For the other term, we have by Littlewood-Paley theory (and its vector-valued version) and Hölder's

$$\|\sum_k\Lambda^{s_2-s_1}\Delta_k[V_{\sim k}\cdot\nabla\varphi_{<k+10}]\|_{L^p} \sim \|(\sum_k 2^{2k(s_2-s_1)}|\Delta_k[V_{\sim k}\cdot\nabla\varphi_{<k+10}]|^2)^{1/2}\|_{L^p}$$
$$\leq C\|(\sum_k 2^{2k(s_2-s_1)}|V_{\sim k}\cdot|\nabla\varphi_{<k+10}|^2)^{1/2}\|_{L^p}$$
$$\leq C\|(\sum_k 2^{2k}|V_{\sim k}|^2)^{1/2}\|_{L^q}\|\sup_k 2^{k(s_2-s_1-1)}|\nabla\varphi_{<k+10}|\|_{L^r}.$$

Clearly, $\|(\sum_k 2^{2k}|V_{\sim k}|^2)^{1/2}\|_{L^q} \sim \|\nabla V\|_{L^q}$. For $s_2 - s_1 = 1$, we have $\|\sup_k 2^{k(s_2-s_1-1)}|\nabla\varphi_{<k+10}|\|_{L^r} \leq C\|\mathcal{M}[\nabla\varphi]\|_{L^r} \leq C\|\Lambda\varphi\|_{L^r}$.

For $s_2 - s_1 < 1$, we can estimate point-wise

$$2^{k(s_2-s_1-1)}|\nabla\varphi_{<k+10}| \leq 2^{k(s_2-s_1-1)}\sum_{l<k+10}|\nabla\varphi_l|$$

$$\le C \sum_{l<k+10} 2^{(l-k)(1-(s_2-s_1))} 2^{l(s_2-s_1)} 2^{-l} |\nabla \varphi_l| \le C \mathcal{M}[\Lambda^{s_2-s_1}\varphi].$$

From these estimate, we conclude

$$\|\sup_k 2^{k(s_2-s_1-1)} |\nabla \varphi_{<k+10}| \|_{L^r} \le C \|\mathcal{M}[\Lambda^{s_2-s_1}\varphi]\|_{L^r} \le C \|\Lambda^{s_2-s_1}\varphi\|_{L^r}.$$

A.1.3. High-high interactions. For this term, we need $s_1<1$ and $q>2$. Again that the commutator structure is not important and one term is simpler. So, we concentrate on

$$\Lambda^{-s_1} \sum_k \Delta_k [\sum_{l=k+10}^{\infty} V_l \cdot \nabla \Lambda^{s_2} \varphi_{\sim l}] = \Lambda^{-s_1} \sum_k \nabla \Delta_k [\sum_{l=k+10}^{\infty} V_l \cdot \Lambda^{s_2} \varphi_{\sim l}].$$

The contribution of these terms is bounded by

$$\|(\sum_k 2^{2k(1-s_1)} |\Delta_k [\sum_{l=k+10}^{\infty} V_l \cdot \Lambda^{s_2} \varphi_{\sim l}]|^2)^{1/2}\|_{L^p}.$$

By Littlewood-Paley theory the last expression is bounded by

$$I = \|(\sum_k 2^{2k(1-s_1)} | \sum_{l=k+10}^{\infty} V_l \cdot \Lambda^{s_2} \varphi_{\sim l}|^2)^{1/2}\|_{L^p}.$$

But for $s_1<1$, we have

$$\begin{aligned}
&\sum_k 2^{2k(1-s_1)} | \sum_{l=k+10}^{\infty} V_l \cdot \Lambda^{s_2} \varphi_{\sim l}|^2 \\
&= \sum_{l_1} V_{l_1} \cdot \Lambda^{s_2} \varphi_{\sim l_1} \sum_{l_2} V_{l_2} \cdot \Lambda^{s_2} \varphi_{\sim l_2} \sum_{k<\min(l_1,l_2)-10} 2^{2k(1-s_1)} \\
&\le C \sum_{l_1} V_{l_1} \cdot \Lambda^{s_2} \varphi_{\sim l_1} \sum_{l_2} V_{l_2} \cdot \Lambda^{s_2} \varphi_{\sim l_2} 2^{2\min(l_1,l_2)(1-s_1)} \\
&\le C \sum_l |V_l|^2 2^{2l} \sum_l |\Lambda^{s_2-s_1} \varphi_{\sim l}|^2.
\end{aligned}$$

By Hölder's

$$I \le \|(\sum_l |V_l|^2 2^{2l})^{1/2}\|_{L^q} \|(\sum_l |\Lambda^{s_2-s_1}\varphi_{\sim l}|^2)^{1/2}\|_{L^r} \le C \|\nabla V\|_{L^q} \|\Lambda^{s_2-s_1}\varphi\|_{L^r}.$$

In order to extend the results to the case $a \in [s_2-s_1, 1)$, it suffice to go over the different terms. For the low-high interaction term, we have, by our previous estimates

$$\begin{aligned}
S^2(x) &\le C \sum_k 2^{2k(s_2-s_1)} |\mathcal{M}[\mathcal{M}[|\nabla V_{<k-10}|^{q_1}]^{1/q_1} \mathcal{M}[|\varphi_{\sim k}|^{p_1}]^{1/p_1}]^2 \\
&= C \sum_k |\mathcal{M}[\mathcal{M}[2^{k(s_2-s_1)} 2^{-k} |\nabla V_{<k-10}|^{q_1}]^{1/q_1} \mathcal{M}[2^k |\varphi_{\sim k}|^{p_1}]^{1/p_1}]^2 \\
&\le C \sum_k |\mathcal{M}[\mathcal{M}[\mathcal{M}|\Lambda^{s_2-s_1} V|^{q_1}]^{1/q_1} \mathcal{M}[2^k |\varphi_{\sim k}|^{p_1}]^{1/p_1}]|^2.
\end{aligned}$$

Applying the Fefferman-Stein estimates yields (assuming $p_1<2, p_1<r, q_1<q$)

$$\|I_{low,high}\|_{L^p} \sim \|S\|_{L^p} \leq C\|\mathcal{M}[\mathcal{M}|\Lambda^{s_2-s_1}V|^{q_1}]^{1/q_1}\|_{L^q}\|(\sum_k \mathcal{M}[2^k|\varphi_{\sim k}|^{p_1}]^{1/p_1})^{1/2}\|_{L^r}$$
$$\leq C\|\Lambda^{s_2-s_1}V\|_{L^q}\|\Lambda\varphi\|_{L^r}.$$

An interpolation between the last estimate and ((A.2)) yields the required estimate

$$\|I_{low,high}\|_{L^p} \leq \|\Lambda^a V\|_{L^q}\|\Lambda^{s_2-s_1+1-a}\varphi\|_{L^r}.$$

Next, for the high-low terms, we clearly have the following bound

$$2^{k(s_2-s_1)}|\Delta_k[V_{\sim k}\cdot\nabla\varphi_{<k+10}]|(x) \leq C\mathcal{M}[\mathcal{M}[\Lambda^{s_2-s_1}V_{\sim_k}]\mathcal{M}[\Lambda^1\varphi]],$$

Applying the same arguments as above yields the bound
$\|I_{high,low}\|_{L^p} \leq C\|\Lambda^{s_2-s_1}V\|_{L^q}\|\Lambda^1\varphi\|_{L^r}$, which by interpolation results in

$$\|I_{high,low}\|_{L^p} \leq C\|\Lambda^a V\|_{L^q}\|\Lambda^{s_2-s_1+1-a}\varphi\|_{L^r}$$

for all $a\in[s_2-s_1,1]$.

Finally in the high-high case, one may move all the derivatives between V, φ (since they are both localized at the same frequency l), so in particular

$$\|I_{high,high}\|_{L^p} \leq C\|\Lambda^a V\|_{L^q}\|\Lambda^{s_2-s_1+1-a}\varphi\|_{L^r}.$$

A.2. Proof of ((2.5))**.** We start again with the low-high term. In this case, the estimate for $\|I_{low,high}\|_{L^2}$ is actually already contained in the estimates for $I_{low,high}$, since we have already remarked that in there, one can take $q=\infty$.

Next, we verify the contribution of the high-low terms interactions. We have by Littlewood-Paley theory that

$$\begin{aligned}
\|I_{high,low}\|_{L^2}^2 &\leq C\sum_k 2^{2k(s_2-s_1-s_3)}\|V_{\sim k}\cdot\nabla\varphi_{<k+10}\|_{L^2}^2\\
&\leq C\|V\|_{L^\infty}^2\sum_k 2^{2k(s_2-s_1-s_3)}\|\nabla\varphi_{<k+10}\|_{L^2}^2\\
&\leq C\|V\|_{L^\infty}^2\sum_k 2^{2k(s_2-s_1-s_3)}\sum_{l<k+10}2^{2l}\|\varphi_l\|_{L^2}^2\\
&\leq C\|V\|_{L^\infty}^2\sum_l 2^{2l(1+s_2-s_1-s_3)}\|\varphi_l\|_{L^2}^2 \leq C\|V\|_{L^\infty}^2\|\Lambda^{s_2-s_1+1-s_3}\varphi\|_{L^2}^2.
\end{aligned}$$

where in the derivation, we have used that $\sum_{k>l-10}2^{2k(s_2-s_1-s_3)} \leq C2^{2l(s_2-s_1-s_3)}$, which requires that $s_2-s_1-s_3<0$.

Finally, we turn our attention to the high-high terms. Again, the commutator structure is unimportant here and we might as well consider the two terms separately. One of them is actually simpler (where Λ^{s_2} is acting on the low frequency outside), so we consider the other term only, namely

$$\Lambda^{-s_1}\sum_k\Delta_k[\sum_{l=k+10}^{\infty}\Lambda^{-s_3}V_l\cdot\nabla\Lambda^{s_2}\varphi_{\sim l}] = \Lambda^{-s_1}\sum_k\nabla\Delta_k[\sum_{l=k+10}^{\infty}\Lambda^{-s_3}V_l\cdot\Lambda^{s_2}\varphi_{\sim l}].$$

Note that here again, we have moved ∇ outside, because $\nabla\cdot V=0$. Taking L^2 norms yields

$$\|I_{high,high}\|_{L^2}^2 \le C\sum_k 2^{2k(1-s_1)}\|\sum_{l=k+10}^{\infty}\Lambda^{-s_3}V_l\cdot\Lambda^{s_2}\varphi_{\sim l}\|_{L^2}^2$$

$$\le C\|V\|_{L^\infty}^2\sum_k 2^{2k(1-s_1)}(\sum_{l=k+10}^{\infty}2^{l(s_2-s_3)}\|\varphi_{\sim l}\|_{L^2})^2$$

$$=C\|V\|_{L^\infty}^2\sum_{l_1}2^{l_1(s_2-s_3)}\|\varphi_{\sim l_1}\|_{L^2}\sum_{l_2}2^{l_2(s_2-s_3)}\|\varphi_{\sim l_2}\|_{L^2}\sum_{k<\min(l_1,l_2)-10}2^{2k(1-s_1)}.$$

Now, since $1-s_1>0$, we have

$$\sum_{k<\min(l_1,l_2)-10}2^{2k(1-s_1)}\le C2^{2\min(l_1,l_2)(1-s_1)}=C2^{l_1(1-s_1)}2^{l_2(1-s_1)}2^{-|l_1-l_2|(1-s_1)}.$$

Plugging this inside our estimate for $\|I_{high,high}\|_{L^2}^2$ and applying Cauchy-Schwartz we obtain

$$\begin{aligned}\|I_{high,high}\|_{L^2}^2 &\le C\|V\|_{L^\infty}^2\sum_{l_1,l_2}2^{(l_1+l_2)(1-s_1+s_2-s_3)}\|\varphi_{\sim l_1}\|_{L^2}\|\varphi_{\sim l_2}\|_{L^2}2^{-|l_1-l_2|(1-s_1)}\\ &\le C\|V\|_{L^\infty}^2\sum_{l_1,l_2}2^{2l_1(1-s_1+s_2-s_3)}\|\varphi_{\sim l_1}\|_{L^2}^2 2^{-|l_1-l_2|(1-s_1)}\le\\ &\le C\|V\|_{L^\infty}^2\|\Lambda^{1-s_1+s_2-s_3}\varphi\|_{L^2}^2.\end{aligned}$$

This concludes the proof of inequality ((2.5)) and thus of Lemma (2.2).

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