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ON THE CAUCHY PROBLEM WITH LARGE DATA FOR A SPACE-DEPENDENT BOLTZMANN-NORDHEIM BOSON EQUATION*

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Abstract. This paper studies a Boltzmann Nordheim equation in a slab with two-dimensional velocity space and pseudo-Maxwellian forces. Strong solutions are obtained for the Cauchy problem with large initial data in an $L^1\cap L^\infty$ setting. The main results are existence, uniqueness and stability of solutions conserving mass, momentum and energy that explode in L^∞ if they are only local in time. The solutions are obtained as limits of solutions to corresponding anyon equations.

Keywords. bosonic Boltzmann-Nordheim equation; low temperature kinetic theory; quantum Boltzmann equation.

AMS subject classifications. 82C10; 82C22; 82C40.

1. Introduction

In a previous paper [1], we have studied the Cauchy problem for a space-dependent anyon Boltzmann equation,

$$\partial_t f(t,x,v)+v_1\partial_x f(t,x,v)=Q_\alpha(f)(t,x,v),\quad f(0,x,v)=f_0(x,v),(t,x)\in\mathbb{R}_+\times[0,1],\\ v=(v_1,v_2)\in\mathbb{R}^2. \quad (1.1)$$

The collision operator Q_α in [1] depends on a parameter $\alpha\in]0,1[$ and is given by

$$Q_\alpha(f)(v)=\int_{\mathbb{R}^2\times S^1}B(|v-v_*|,n)[f'f'_*F_\alpha(f)F_\alpha(f_*)-ff_*F_\alpha(f')F_\alpha(f'_*)]dv_*dn,$$

with the kernel B of Maxwellian type, f', f'_*, f, f_* the values of f at v', v'_*, v, and v_* respectively, where

$$v'=v-(v-v_*,n)n,\quad v'_*=v_*+(v-v_*,n)n,$$

and the filling factor F_α

$$F_\alpha(f)=(1-\alpha f)^\alpha(1+(1-\alpha)f)^{1-\alpha}.$$

Anyons are other types of particles that occur in one and two-dimensions besides fermions and bosons. The exchange of two identical anyons may cause a phase shift different from π (fermions) and 2π (bosons). In [1], also the limiting case $\alpha=1$ is discussed, a Boltzmann-Nordheim (BN) equation [11] for fermions. In the present paper we shall consider the other limiting case, $\alpha=0$, which is a BN equation for bosons.

For the bosonic BN equation general existence results were first obtained by X. Lu in [7] in the space-homogeneous isotropic boson large data case. It was followed by a number of interesting studies in the same isotropic setting, by X. Lu [8–10], and by M. Escobedo and J.L. Velázquez [5,6]. Results with the isotropy assumption removed, were recently obtained by M. Briant and A. Einav [3]. Finally a space-dependent case

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close to equilibrium has been studied by G. Royat in [12]. The papers [7–10] by Lu, study the isotropic, space-homogeneous BN equation both for Cauchy data leading to mass and energy conservation, and for data leading to mass loss when time tends to infinity. Escobedo and Velásquez in [5,6], again in the isotropic space-homogeneous case, study initial data leading to concentration phenomena and blow-up in finite time of the L^∞-norm of the solutions. The paper [3] by Briant and Einav removes the isotropy restriction and obtain in polynomially weighted spaces of $L^1 \cap L^\infty$ type, existence and uniqueness on a time interval $[0,T_0)$. In [3] either $T_0=\infty$, or for finite T_0 the L^∞-norm of the solution tends to infinity, when time tends to T_0. Finally the paper [12] considers the space-dependent problem, for a particular setting close to equilibrium, and proves well-posedness and convergence to equilibrium.

In the papers cited above, the velocity space is $\mathbb{R}^3$. The present paper on the other hand studies a space-dependent, large data problem for the BN equation with velocities in $\mathbb{R}^2$. The analysis is based on the anyon results in [1], which are restricted to a slab set-up, since the proofs in [1] use an estimate for the Bony functional only valid in one space dimension. Due to the filling factor $F_\alpha(f)$, those proofs also depend on the two-dimensional velocity frame in an essential way. By a limiting procedure relying on the anyon case when $\alpha \to 0$, well-posedness and conservation laws are obtained in the present paper for the BN problem.

With

$$\cos\theta = n \cdot \frac{v-v_*}{|v-v_*|},$$

the kernel $B(|v-v_*|,n)$ will from now on be written as $B(|v-v_*|,\theta)$ and assumed measurable with

$$0 \le B \le B_0, \tag{1.2}$$

for some $B_0>0$. It is also assumed for some $\gamma,\gamma',c_B>0$, that

$$B(|v-v_*|,\theta)=0 \text{ for } |\cos\theta|<\gamma', \quad 1-|\cos\theta|<\gamma', \quad \text{and } |v-v_*|<\gamma, \tag{1.3}$$

and that

$$\int B(|v-v_*|,\theta)d\theta \ge c_B>0 \quad \text{for } |v-v_*|\ge\gamma. \tag{1.4}$$

These strong cut-off conditions on B are made for mathematical reasons and assumed throughout the paper. For a more general discussion of cut-offs in the collision kernel B, see [8]. Notice that contrary to the classical Boltzmann operator where rigorous derivations of B from various potentials have been made, little is known about collision kernels in quantum kinetic theory (cf [13]).

With v_1 denoting the component of v in the x-direction, the initial value problem for the Boltzmann Nordheim equation in a periodic in space setting is

$$\partial_t f(t,x,v) + v_1 \partial_x f(t,x,v) = Q(f)(t,x,v), \tag{1.5}$$

where

$$Q(f)(v) = \int_{IR^2\times[0,\pi]} B(|v-v_*|,\theta)[f'f'_*F(f)F(f_*) - ff_*F(f')F(f'_*)]dv_*d\theta, \tag{1.6}$$

and

$$F(f)=1+f. \tag{1.7}$$

Denote by

$$f^\sharp(t,x,v)=f(t,x+tv_1,v) \quad (t,x,v)\in\mathbb{R}_+\times[0,1]\times\mathbb{R}^2. \tag{1.8}$$

Strong solutions to the Boltzmann-Nordheim equation are considered in the following sense.

Definition 1.1. *f is a strong solution to (1.5) on the time interval I if*

$$f\in\mathcal{C}^1(I;L^1([0,1]\times\mathbb{R}^2)),$$

and

$$\frac{d}{dt}f^\sharp=\big(Q(f)\big)^\sharp, \quad \text{on } I\times[0,1]\times\mathbb{R}^2. \tag{1.9}$$

The main result of this paper is the following.

Theorem 1.1. *Assume (1.2)-(1.3)-(1.4). Let $f_0\in L^\infty([0,1]\times\mathbb{R}^2)$ and satisfy*

$$(1+|v|^2)f_0(x,v)\in L^1([0,1]\times\mathbb{R}^2),\ \int\sup_{x\in[0,1]}f_0(x,v)dv=c_0<\infty,$$
$$\inf_{x\in[0,1]}f_0(x,v)>0,\ a.a.v\in\mathbb{R}^2. \tag{1.10}$$

There exist a time $T_\infty>0$ and a strong solution f to (1.5) on $[0,T_\infty)$ with initial value f_0.
For $0<T<T_\infty$, it holds

$$f^\sharp\in\mathcal{C}^1([0,T_\infty);L^1([0,1]\times\mathbb{R}^2))\cap L^\infty([0,T]\times[0,1]\times\mathbb{R}^2). \tag{1.11}$$

If $T_\infty<+\infty$ then

$$\overline{\lim}_{t\to T_\infty}\parallel f(t,\cdot,\cdot)\parallel_{L^\infty([0,1]\times\mathbb{R}^2)}=+\infty. \tag{1.12}$$

The solution is unique, and conserves mass, momentum, and energy. For equibounded families in $L^\infty([0,1]\times\mathbb{R}^2)$ of initial values, the solution depends continuously in L^1 on the initial value f_0.

Remark 1.1. A finite T_∞ may not correspond to a condensation. In the isotropic space-homogeneous case considered in [5,6], additional assumptions on the concentration of the initial value are considered in order to obtain condensation.

The paper is organized as follows. In the following section, solutions f_α to the Cauchy problem for the anyon Boltzmann equation in the above setting are recalled, and their Bony functionals are uniformly controlled with respect to α. In Section 3 the mass density of f_α is studied with respect to uniform control in α. Theorem 1.1 is proven in Section 4 except for the conservations of mass, momentum and energy that are proven in Section 5.

2. Preliminaries on anyons and the Bony functional

The Cauchy problem for a space-dependent anyon Boltzmann equation in a slab was studied in [1]. That paper will be the starting point for the proof of Theorem 1.1, so we recall the main results from [1].

THEOREM 2.1. *Assume (1.2)-(1.3)-(1.4). Let the initial value f_0 be a measurable function on $[0,1]\times\mathbb{R}^2$ with values in $]0,\frac{1}{\alpha}]$, and satisfying (1.10). For every $\alpha\in]0,1[$, there exists a strong solution f_α of equation (1.1) with*

$$f_\alpha^\sharp\in\mathcal{C}^1([0,\infty[;L^1([0,1]\times\mathbb{R}^2)), \qquad 0<f_\alpha(t,\cdot,\cdot)<\frac{1}{\alpha} \quad \textit{for } t>0,$$

and

$$\int \sup_{(s,x)\in[0,t]\times[0,1]} f_\alpha^\sharp(s,x,v)dv\le c_\alpha(t), \tag{2.1}$$

for some function $c_\alpha(t)>0$ only depending on mass and energy. There is $t_m>0$ such that for any $T>t_m$, there is $\eta_T>0$ so that

$$f_\alpha(t,\cdot,\cdot)\le\frac{1}{\alpha}-\eta_T, \quad t\in[t_m,T].$$

The solution is unique and depends continuously in $\mathcal{C}([0,T];L^1([0,1]\times\mathbb{R}^2))$ on the initial L^1-datum. It conserves mass, momentum and energy.

The conditions $f_0\in L^\infty([0,1]\times\mathbb{R}^2)$ and (1.10) are assumed throughout the paper.

To obtain Theorem 1.1 for the boson BN equation from the anyon results, we start from a fixed initial value f_0 bounded by 2^L with $L\in\mathbb{N}$. We shall prove that there is a time $T>0$ independent of $0<\alpha<2^{-L-1}$, so that the solutions are bounded by 2^{L+1} on $[0,T]$. For that, some lemmas from the anyon paper are sharpened to obtain control in terms of only mass, energy and L. We then prove that the limit f of the solutions f_α when $\alpha\to0$ solves the corresponding bosonic BN problem. Iterating the result from T on, it follows that f exists up to the first time T_∞ when $\overline{\lim}_{t\to T_\infty}\parallel f_\alpha(t,\cdot,\cdot)\parallel_{L^\infty([0,1]\times\mathbb{R}^2)}=\infty$.

We observe that

LEMMA 2.1. *Given $f_0\le 2^L$ and satisfying condition (1.10), there is for each $\alpha\in]0,2^{-L-1}[$ a time $T_\alpha>0$ so that the solution f_α to equation (1.1) is bounded by 2^{L+1} on $[0,T_\alpha]$.*

Proof. Split the Boltzmann anyon operator Q_α into $Q_\alpha=Q_\alpha^+-Q_\alpha^-$, where the gain (resp. loss) term Q_α^+ (resp. Q_α^-) is defined by

$$\begin{aligned}&Q_\alpha^+(f)(v)=\int Bf'f'_*F_\alpha(f)F_\alpha(f_*)dv_*d\theta\\ &(\textit{resp. } Q_\alpha^-(f)(v)=\int Bff_*F_\alpha(f')F_\alpha(f'_*)dv_*d\theta).\end{aligned} \tag{2.2}$$

The solution f_α to equation (1.1) satisfies

$$f_\alpha^\sharp(t,x,v)=f_0(x,v)+\int_0^t Q_\alpha(f_\alpha)(s,x+sv_1,v)ds\le f_0(x,v)+\int_0^t Q_\alpha^+(f_\alpha)(s,x+sv_1,v)ds.$$

Hence

$$\sup_{s\le t} f_\alpha^\sharp(s,x,v)\le f_0(x,v)+\int_0^t Q_\alpha^+(f_\alpha)(s,x+sv_1,v)ds$$

$$=f_0(x,v)+\int_0^t\int Bf_\alpha(s,x+sv_1,v')f_\alpha(s,x+sv_1,v'_*)$$
$$F_\alpha(f_\alpha)(s,x+sv_1,v)F_\alpha(f_\alpha)(s,x+sv_1,v_*)dv_*d\theta ds$$
$$\leq 2^L+\frac{B_0}{\alpha}\Big(\frac{1}{\alpha}-1\Big)^{2(1-2\alpha)}\int_0^t\int f_\alpha(s,x+sv_1,v')dv_*d\theta ds, \tag{2.3}$$

since the maximum of F_α on $[0,\frac{1}{\alpha}]$ is $(\frac{1}{\alpha}-1)^{1-2\alpha}$ for $\alpha\in]0,\frac{1}{2}[$. With the angular cut-off (2.2), $v_*\to v'$ is a change of variables. Using it and inequality (2.1) for $t\leq 1$ leads to

$$\sup_{s\leq t,x} f^\sharp_\alpha(s,x,v)\leq 2^L+c\frac{B_0c_\alpha(1)}{\alpha}\Big(\frac{1}{\alpha}-1\Big)^{2(1-2\alpha)}t$$
$$\leq 2^{L+1} \qquad\qquad \text{for } t\leq\min\{1,\frac{2^L\alpha^{3-4\alpha}(1-\alpha)^{2(2\alpha-1)}}{cB_0c_\alpha(1)}\}.$$

The lemma follows. ∎

The estimate of the Bony functional

$$\bar{B}_\alpha(t):=\int_0^1\int|v-v_*|^2Bf_\alpha f_{\alpha*}F_\alpha(f'_\alpha)F_\alpha(f'_{\alpha*})dvdv_*d\theta dx,\quad t\geq 0,$$

from the proof of Theorem 2.1 for $f_\alpha\leq 2^{L+1}$, can be sharpened.

Lemma 2.2. *For $\alpha\leq 2^{-L-1}$ and $T>0$ such that $f_\alpha(t)\leq 2^{L+1}$ for $0\leq t\leq T$, it holds*

$$\int_0^T\bar{B}_\alpha(t)dt\leq c'_0(1+T),$$

with c'_0 independent of T and α, and only depending on $\int f_0(x,v)dxdv$, $\int|v|^2f_0(x,v)dxdv$ and L.

Proof. Denote f_α by f for simplicity. The proof is an extension of the classical one (cf [2, 4]), together with the control of the filling factor F_α when $v\in\mathbb{R}^2$, as follows.

The integral over time of the momentum $\int v_1f(t,0,v)dv$ (resp. the momentum flux $\int v_1^2f(t,0,v)dv$) is first controlled. Let $\beta\in C^1([0,1])$ be such that $\beta(0)=-1$ and $\beta(1)=1$. Multiply equation (1.1) by $\beta(x)$ (resp. $v_1\beta(x)$) and integrate over $[0,t]\times[0,1]\times\mathbb{R}^2$. It gives

$$\int_0^t\int v_1f(\tau,0,v)dvd\tau$$
$$=\frac{1}{2}\Big(\int\beta(x)f_0(x,v)dxdv-\int\beta(x)f(t,x,v)dxdv+\int_0^t\int\beta'(x)v_1f(\tau,x,v)dxdvd\tau\Big),$$

$\Bigg($ resp.

$$\int_0^t\int v_1^2f(\tau,0,v)dvd\tau$$
$$=\frac{1}{2}\Big(\int\beta(x)v_1f_0(x,v)dxdv-\int\beta(x)v_1f(t,x,v)dxdv+\int_0^t\int\beta'(x)v_1^2f(\tau,x,v)dxdvd\tau\Big)\Bigg).$$

Consequently, using the conservation of mass and energy of f,

$$\Big|\int_0^t\int v_1f(\tau,0,v)dvd\tau\Big|+\int_0^t\int v_1^2f(\tau,0,v)dvd\tau\leq c(1+t). \tag{2.4}$$

Here c is of magnitude of mass plus energy uniformly in α. Let

$$\mathcal{I}(t)=\int_{x<y}(v_1-v_{*1})f(t,x,v)f(t,y,v_*)dxdydvdv_*.$$

It results from

$$\mathcal{I}'(t)=-\int(v_1-v_{*1})^2f(t,x,v)f(t,x,v_*)dxdvdv_*\\+2\int v_{*1}(v_{*1}-v_1)f(t,0,v_*)f(t,x,v)dxdvdv_*,$$

and the conservations of the mass, momentum and energy of f that

$$\begin{aligned}&\int_0^t\int_0^1\int(v_1-v_{*1})^2f(s,x,v)f(s,x,v_*)dvdv_*dxds\\
&\leq 2\int f_0(x,v)dxdv\int|v_1|f_0(x,v)dv+2\int f(t,x,v)dxdv\int|v_1|f(t,x,v)dxdv\\
&\quad+2\int_0^t\int v_{*1}(v_{*1}-v_1)f(\tau,0,v_*)f(\tau,x,v)dxdvdv_*d\tau\\
&\leq 2\int f_0(x,v)dxdv\int(1+|v|^2)f_0(x,v)dv+2\int f(t,x,v)dxdv\int(1+|v|^2)f(t,x,v)dxdv\\
&\quad+2\int_0^t(\int v_{*1}^2f(\tau,0,v_*)dv_*)d\tau\int f_0(x,v)dxdv\\
&\quad-2\int_0^t(\int v_{*1}f(\tau,0,v_*)dv_*)d\tau\int v_1f_0(x,v)dxdv\\
&\leq c\Big(1+\int_0^t\int v_1^2f(\tau,0,v)dvd\tau+|\int_0^t\int v_1f(\tau,0,v)dvd\tau|\Big).\end{aligned}$$

So, by inequality (2.4),

$$\int_0^t\int_0^1\int(v_1-v_{*1})^2f(s,x,v)f(s,x,v_*)dvdv_*dxds\leq c(1+t). \tag{2.5}$$

Denote by $u_1=\frac{\int v_1fdv}{\int fdv}$. Recalling (1.2) it holds

$$\begin{aligned}&\int_0^t\int_0^1\int(v_1-u_1)^2Bff_*F_\alpha(f')F_\alpha(f'_*)(s,x,v,v_*,\theta)dvdv_*d\theta dxds\\
&\leq c\int_0^t\int_0^1\int(v_1-u_1)^2ff_*(s,x,v,v_*)dvdv_*dxds\\
&=\frac{c}{2}\int_0^t\int_0^1\int(v_1-v_{*1})^2ff_*(s,x,v,v_*)dvdv_*dxds\\
&\leq c(1+t).\end{aligned} \tag{2.6}$$

Here c also contains $\sup F_\alpha(f')F_\alpha(f'_*)$ which is of magnitude bounded by 2^{2L}. So c is of magnitude 2^{2L}(mass+energy) and uniformly in α. Multiply equation (1.1) for f by v_1^2, integrate and use that $\int v_1^2Q_\alpha(f)dv=\int(v_1-u_1)^2Q_\alpha(f)dv$ and inequality (2.6). It results

$$\begin{aligned}&\int_0^t\int(v_1-u_1)^2Bf'f'_*F_\alpha(f)F_\alpha(f_*)dvdv_*d\theta dxds\\=&\int v_1^2f(t,x,v)dxdv-\int v_1^2f_0(x,v)dxdv\\&+\int_0^t\int(v_1-u_1)^2Bff_*F_\alpha(f')F_\alpha(f'_*)dxdvdv_*d\theta ds\\<&c_0(1+t),\end{aligned}$$

where c_0 is a constant of magnitude 2^{2L}(mass+energy). After a change of variables the left hand side can be written

$$\begin{aligned}&\int_0^t\int(v'_1-u_1)^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\=&\int_0^t\int(c_1-n_1[(v-v_*)\cdot n])^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds,\end{aligned}$$

where $c_1=v_1-u_1$. Therefore,

$$\begin{aligned}&\int_0^t\int n_1^2[(v-v_*)\cdot n])^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\\le&c_0(1+t)+2\int_0^t\int c_1n_1[(v-v_*)\cdot n]Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds.\end{aligned}$$

The term containing $n_1^2[(v-v_*)\cdot n]^2$ is estimated from below. When n is replaced by an orthogonal (direct) unit vector $n_\perp$, v' and v'_* are shifted and the product $ff_*F_\alpha(f')F_\alpha(f'_*)$ is unchanged. In $\mathbb{R}^2$ the ratio between the sum of the integrand factors $n_1^2[(v-v_*)\cdot n]^2+n_{\perp 1}^2[(v-v_*)\cdot n_\perp]^2$ and $|v-v_*|^2$, is, outside of the angular cut-off (1.3), uniformly bounded from below by γ'^2. Indeed, if θ (resp. θ_1) denotes the angle between $\frac{v-v_*}{|v-v_*|}$ and n (resp. the angle between e_1 and n, where e_1 is a unit vector in the x-direction),

$$\begin{aligned}n_1^2[\frac{v-v_*}{|v-v_*|}\cdot n]^2+n_{\perp 1}^2[\frac{v-v_*}{|v-v_*|}\cdot n_\perp]^2=&\cos^2\theta_1\cos^2\theta+\sin^2\theta_1\sin^2\theta\\\ge&\gamma'^2\cos^2\theta_1+\gamma'(2-\gamma')\sin^2\theta_1\\\ge&\gamma'^2,\quad \gamma'<|\cos\theta|<1-\gamma',\quad \theta_1\in[0,2\pi].\end{aligned}$$

This is where the condition $v\in\mathbb{R}^2$ is used.

That leads to the lower bound

$$\begin{aligned}&\int_0^t\int n_1^2[(v-v_*)\cdot n]^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\\ge&\frac{\gamma'^2}{2}\int_0^t\int|v-v_*|^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\gamma'^2\int_0^t\int |v-v_*|^2 Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&\leq 2c_0(1+t)+4\int_0^t\int (v_1-u_1)n_1[(v-v_*)\cdot n]Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&\leq 2c_0(1+t)+4\int_0^t\int \Big(v_1(v_2-v_{*2})n_1n_2\Big)Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds,
\end{aligned}$$

since

$$\begin{aligned}
&\int u_1(v_1-v_{*1})n_1^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dx\\
&=\int u_1(v_2-v_{*2})n_1n_2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dx=0,
\end{aligned}$$

by an exchange of the variables v and v_*. Moreover, exchanging first the variables v and v_*,

$$\begin{aligned}
&2\int_0^t\int v_1(v_2-v_{*2})n_1n_2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&=\int_0^t\int (v_1-v_{*1})(v_2-v_{*2})n_1n_2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&\leq\frac{8}{\gamma'^2}\int_0^t\int (v_1-v_{*1})^2n_1^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&\quad+\frac{\gamma'^2}{8}\int_0^t\int (v_2-v_{*2})^2n_2^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\\
&\leq\frac{8\pi c_0}{\gamma'^2}(1+t)+\frac{\gamma'^2}{8}\int_0^t\int (v_2-v_{*2})^2n_2^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds.
\end{aligned}$$

It follows that

$$\int_0^t\int |v-v_*|^2Bff_*F_\alpha(f')F_\alpha(f'_*)dvdv_*d\theta dxds\leq c'_0(1+t),$$

with c'_0 uniformly with respect to α, of the same magnitude as c_0, only depending on $\int f_0(x,v)dxdv$, $\int |v|^2f_0(x,v)dxdv$ and L. This completes the proof of the lemma. ◻

3. Control of phase space density

This section is devoted to obtaining a time $T>0$, such that

$$\sup_{t\in[0,T],\,x\in[0,1]} f^\sharp_\alpha(t,x,v)\leq 2^{L+1},$$

uniformly with respect to $\alpha\in]0,2^{-L-1}[$. We start from the case of a fixed $\alpha\leq 2^{-L-1}$. Up to Lemma 3.3 the time interval when the solution does not exceed 2^{L+1}, may be α-dependent. Lemma 3.4 implies that this time interval can be chosen independent of α.

Lemma 3.1. *Given $T>0$ such that $f_\alpha(t)\leq 2^{L+1}$ for $0\leq t\leq T$, the solution f_α of equation (1.1) satisfies*

$$\int \sup_{t\in[0,T]} f^\sharp_\alpha(t,x,v)dxdv<c'_1+c'_2T,\quad \alpha\in]0,2^{-L-1}[,$$

where c'_1 *and* c'_2 *are independent of* T *and* α*, and only depend on* $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ *and* L.

Proof. Denote f_α by f for simplicity. By (2.3),

$$\sup_{t\in[0,T]} f^\sharp(t,x,v) \le f_0(x,v) + \int_0^T Q_\alpha^+(f)(t,x+tv_1,v)dt.$$

Integrating the previous inequality with respect to (x,v) and using Lemma 2.2, it gives

$$\int \sup_{0\le t\le T} f^\sharp(t,x,v)dxdv \le \int f_0(x,v)dxdv + \int_0^T \int B$$
$$f(t,x+tv_1,v')f(t,x+tv_1,v'_*)F_\alpha(f)(t,x+tv_1,v)F_\alpha(f)(t,x+tv_1,v_*)dvdv_*d\theta dxdt$$
$$\le \int f_0(x,v)dxdv + \frac{1}{\gamma^2}\int_0^T \int B|v-v_*|^2$$
$$f(t,x,v')f(t,x,v'_*)F_\alpha(f)(t,x,v)F_\alpha(f)(t,x,v_*)dvdv_*d\theta dxdt$$
$$\le \int f_0(x,v)dxdv + \frac{c'_0(1+T)}{\gamma^2} := \frac{C_1+C_2T}{\gamma^2}.$$

□

LEMMA 3.2. *Given* $T>0$ *such that* $f(t)\le 2^{L+1}$ *for* $0\le t\le T$*, and* $\delta_1>0$*, there exist* $\delta_2>0$ *and* $t_0>0$ *independent of* T *and* α *and only depending on* $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ *and* L*, such that*

$$\sup_{x_0\in[0,1]} \int_{|x-x_0|<\delta_2} \sup_{t\le s\le t+t_0} f_\alpha^\sharp(s,x,v)dxdv < \delta_1, \quad \alpha\in]0,2^{-L-1}[, \quad t\in[0,T].$$

Proof. Denote f_α by f for simplicity. For $s\in[t,t+t_0]$ it holds,

$$f^\sharp(s,x,v) = f^\sharp(t+t_0,x,v) - \int_s^{t+t_0} Q_\alpha(f)(\tau,x+\tau v_1,v)d\tau$$
$$\le f^\sharp(t+t_0,x,v) + \int_s^{t+t_0} Q_\alpha^-(f)(\tau,x+\tau v_1,v)d\tau.$$

And so

$$\sup_{t\le s\le t+t_0} f^\sharp(s,x,v) \le f^\sharp(t+t_0,x,v) + \int_t^{t+t_0} Q_\alpha^-(f)(s,x+sv_1,v)ds.$$

Integrating with respect to (x,v), using Lemma 2.2 and the bound 2^{L+1} from above for f, gives

$$\int_{|x-x_0|<\delta_2} \sup_{t\le s\le t+t_0} f^\sharp(s,x,v)dxdv$$
$$\le \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv + \int_t^{t+t_0}\int Bf^\sharp(s,x,v)f(s,x+sv_1,v_*)$$
$$F_\alpha(f)(s,x+sv_1,v')F_\alpha(f)(s,x+sv_1,v'_*)dvdv_*d\theta dxds$$
$$\le \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv + \frac{1}{\lambda^2}\int_t^{t+t_0}\int_{|v-v_*|\ge\lambda} B|v-v_*|^2 f^\sharp(s,x,v)f(s,x+sv_1,v_*)$$

$$F_\alpha(f)(s,x+sv_1,v')F_\alpha(f)(s,x+sv_1,v'_*)dvdv_*d\theta dxds$$
$$+c2^{2L}\int_t^{t+t_0}\int_{|v-v_*|<\lambda} Bf^\sharp(s,x,v)f(s,x+sv_1,v_*)dvdv_*d\theta dxds$$
$$\leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0,x,v)dxdv+\frac{c_0'(1+t_0)}{\lambda^2}+c2^{3L}t_0\lambda^2\int f_0(x,v)dxdv$$
$$\leq \frac{1}{\Lambda^2}\int v^2 f_0dxdv+c\delta_2 2^L\Lambda^2+\frac{c_0'(1+t_0)}{\lambda^2}+c2^{3L}t_0\lambda^2\int f_0(x,v)dxdv.$$

Depending on δ_1, suitably choosing Λ and then δ_2, λ and then t_0, the lemma follows. □

The previous lemmas imply for fixed $\alpha\leq 2^{-L-1}$ a bound for the v-integral of $f_\alpha^\#$ only depending on $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ and L.

Lemma 3.3. *With T'_α defined as the maximum time for which $f_\alpha(t)\leq 2^{L+1}$, $t\in[0,T'_\alpha]$, take $T_\alpha=\min\{1,T'_\alpha\}$.*
The solution f_α of equation (1.1) satisfies

$$\int \sup_{(t,x)\in[0,T_\alpha]\times[0,1]} f_\alpha^\sharp(t,x,v)dv\leq c_1, \tag{3.1}$$

where c_1 is independent of $\alpha\leq 2^{-L-1}$ and only depends on $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ and L.

Proof. Denote by $E(x)$ the integer part of $x\in\mathbb{R}$, $E(x)\leq x<E(x)+1$.
By (2.3),

$$\begin{aligned}\sup_{s\leq t} f^\sharp(s,x,v)&\leq f_0(x,v)+\int_0^t Q_\alpha^+(f)(s,x+sv_1,v)ds\\ &=f_0(x,v)+\int_0^t\int Bf(s,x+sv_1,v')f(s,x+sv_1,v'_*)\\ &\quad F_\alpha(f)(s,x+sv_1,v)F_\alpha(f)(s,x+sv_1,v_*)dv_*d\theta ds\\ &\leq f_0(x,v)+c2^{2L}A,\end{aligned} \tag{3.2}$$

where

$$A=\int_0^t\int B\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v_1'),v')\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_{*1}),v'_*)dv_*d\theta ds.$$

For θ outside of the angular cutoff (2.2), let n be the unit vector in the direction $v-v'$, and $n_\perp$ the orthogonal unit vector in the direction $v-v'_*$. With e_1 a unit vector in the x-direction,

$$\max(|n\cdot e_1|,|n_\perp\cdot e_1|)\geq\frac{1}{\sqrt{2}}.$$

For $\delta_2>0$ that will be fixed later, split A into $A_1+A_2+A_3+A_4$, where

$$A_1=\int_0^t\int_{|n\cdot e_1|\geq\frac{1}{\sqrt{2}},\, t|v_1-v_1'|>\delta_2} B\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v_1'),v')$$
$$\sup_{\tau\in[0,t]} f^\#(\tau,x+s(v_1-v'_{*1}),v'_*)dv_*d\theta ds,$$

$$A_2 = \int_0^t \int_{|n\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v_1'|<\delta_2} B \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v_1'), v') \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v_*')dv_*d\theta ds,$$

$$A_3 = \int_0^t \int_{|n_\perp\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v_1'|>\delta_2} B \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v_1'), v') \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v_*')dv_*d\theta ds,$$

$$A_4 = \int_0^t \int_{|n_\perp\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v_1'|<\delta_2} B \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-v_1'), v') \sup_{\tau\in[0,t]} f^{\#}(\tau, x+s(v_1-{v'}_{*1}), v_*')dv_*d\theta ds.$$

In A_1 and A_2, bound the factor $\sup_{\tau\in[0,t]} f^{\sharp}(\tau, x+s(v_1-{v'}_{*1}), v_*')$ by its supremum over $x\in[0,1]$, and make the change of variables

$$s \to y = x+s(v_1-v_1'),$$

with Jacobian

$$\frac{Ds}{Dy} = \frac{1}{|v_1-v_1'|} = \frac{1}{|v-v_*|\,|(n, \frac{v-v_*}{|v-v_*|})|\,|n_1|} \leq \frac{\sqrt{2}}{\gamma\gamma'}.$$

It holds that

$$A_1 \leq \int_{t|v_1-v_1'|>\delta_2} \frac{B}{|v_1-v_1'|}\Big(\int_{y\in(x,x+t(v_1-v_1'))} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v')dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*')dv_*d\theta,$$

and

$$A_2 \leq \frac{\sqrt{2}}{\gamma\gamma'} \int_{|n\cdot e_1|\geq \frac{1}{\sqrt{2}},\, t|v_1-v_1'|<\delta_2} B\Big(\int_{|y-x|<\delta_2} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v')dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*')dv_*d\theta.$$

Then, performing the change of variables $(v,v_*,n)\to(v',v_*',-n)$,

$$\int \sup_{x\in[0,1]} A_1 dv \leq \int_{t|v_1-v_1'|>\delta_2} \frac{B}{|v_1-v_1'|} \sup_{x\in[0,1]}\Big(\int_{y\in(x,x+t(v_1'-v_1))} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy\Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*d\theta,$$

so that

$$\int \sup_{x\in[0,1]} A_1 dv$$

$$\leq \int_{t|v_1-v'_1|>\delta_2} \frac{B}{|v_1-v'_1|} \sup_{x\in[0,1]} \Big(\int_{y\in(x,x+E(t(v'_1-v_1)+1))} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy \Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*d\theta$$

$$= \int_{t|v_1-v'_1|>\delta_2} \frac{B}{|v_1-v'_1|} |E(t(v'_1-v_1)+1)| \Big(\int_0^1 \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy \Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*d\theta$$

$$\leq t(1+\frac{1}{\delta_2}) \int B \Big(\int_0^1 \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dy \Big) \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dvdv_*d\theta$$

$$\leq B_0\pi t(1+\frac{1}{\delta_2}) \int \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v)dydv \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dv_*.$$

Apply Lemma 3.1, so that

$$\int \sup_{x\in[0,1]} A_1 dv \leq B_0\pi t(1+\frac{1}{\delta_2})(c'_1+c'_2) \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v_*)dv_*. \tag{3.3}$$

Moreover, performing the change of variables $(v,v_*,n)\to(v'_*,v',-n)$,

$$\int \sup_{x\in[0,1]} A_2 dv$$
$$\leq \frac{B_0\pi\sqrt{2}}{\gamma\gamma'} \sup_{x\in[0,1]} \Big(\int_{|y-x|<\delta_2} \sup_{\tau\in[0,t]} f^{\#}(\tau,y,v_*)dydv_* \Big) \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v)dv.$$

Given $\delta_1 = \frac{\gamma\gamma'}{4B_0\pi\sqrt{2}}$, apply Lemma 3.2 with the corresponding δ_2 and t_0, so that for $t\leq\min\{T,t_0\}$,

$$\int \sup_{x\in[0,1]} A_2 dv \leq \frac{1}{4} \int \sup_{(\tau,X)\in[0,t]\times[0,1]} f^{\#}(\tau,X,v)dv. \tag{3.4}$$

The terms A_3 and A_4 are treated similarly, with the change of variables $s\to y=x+s(v_1-v'_{*1})$. Using (3.3)-(3.4) and the corresponding bounds obtained for A_3 and A_4 leads to

$$\int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv$$
$$\leq 2\int \sup_{x\in[0,1]} f_0(x,v)dv + 4B_0\pi t(1+\frac{1}{\delta_2})(c'_1+c'_2) \int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv,$$
$$t\leq\min\{T,t_0\}.$$

Hence

$$\int \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv \leq 4\int \sup_{x\in[0,1]} f_0(x,v)dv,$$
$$t\leq\min\{t_0, \frac{\delta_2}{8B_0\pi(\delta_2+1)(c'_1+c'_2)}\}.$$

Since t_0, c_1' and c_2' are independent of $\alpha \le 2^{-L-1}$ and only depend on $\int f_0(x,v)dxdv$, $\int |v|^2 f_0(x,v)dxdv$ and L, it follows that the argument can be repeated up to $t=T_\alpha$ with the number of steps uniformly bounded with respect to $\alpha \le 2^{-L-1}$. This completes the proof of the lemma. ∎

We now prove that the positive time T_α used above, such that $f_\alpha(t) \le 2^{L+1}$ for $t \in [0,T_\alpha]$, can be taken independent of α.

LEMMA 3.4. *Given $f_0 \le 2^L$ and satisfying (1.10), there is $T \in]0,1]$ so that for all $\alpha \in]0,2^{-L-1}[$, the solution f_α to equation (1.1) is bounded by 2^{L+1} on $[0,T]$.*

Proof. Given $\alpha \le 2^{-L-1}$, it follows from Lemma 2.1 that the maximum time T_α' for which $f_\alpha \le 2^{L+1}$ on $[0,T_\alpha']$ is positive. By (2.3),

$$\begin{aligned}\sup_{s\le t} f_\alpha^\sharp(s,x,v) &\le f_0(x,v) + \int_0^t Q_\alpha^+(f_\alpha)(s,x+sv_1,v)ds\\ &= f_0(x,v) + \int_0^t \int B f_\alpha(s,x+sv_1,v') f_\alpha(s,x+sv_1,v_*')\\ &\qquad F_\alpha(f_\alpha)(s,x+sv_1,v) F_\alpha(f_\alpha)(s,x+sv_1,v_*) dv_* d\theta ds.\end{aligned}$$

With the angular cut-off (2.2), $v_* \to v'$ and $v_* \to v_*'$ are changes of variables, and so using Lemma 3.3, the functions f_α for $\alpha \in]0,2^{-L-1}[$satisfy

$$\begin{aligned}\sup_{(s,x)\in[0,t]\times[0,1]} f_\alpha^\sharp(s,x,v) &\le f_0(x,v) + cB_0 2^{3L} t \int \sup_{(s,x)\in[0,t]\times[0,1]} f_\alpha(s,x,v')dv'\\ &\le 2^L + cB_0 2^{3L} t c_1\\ &\le 3(2^{L-1}), \qquad t \in [0, \min\{T_\alpha', \frac{1}{cc_1 B_0 2^{2L+1}}\}].\end{aligned}$$

For all $\alpha \le 2^{-L-1}$, it holds that $T_\alpha' \ge \frac{1}{cc_1 B_0 2^{2L+1}}$, else T_α' would not be the maximum time such that $f_\alpha(t) \le 2^{L+1}$ on $[0,T_\alpha']$. Denote by $T = \min\{1, \frac{1}{cc_1 B_0 2^{2L+1}}\}$. The lemma follows since T does not depend on α. ∎

4. Proof of Theorem 1.1

After the above preparations we can now prove Theorem 1.1. The conservations of mass, momentum and energy will be proven in Section 5.

Proof. **(Proof of Theorem 1.1.)** Let us first prove that (f_α) is a Cauchy sequence in $C([0,T];L^1([0,1]\times\mathbb{R}^2))$ with T of Lemma 3.4. For any $(\alpha_1,\alpha_2) \in]0,1[^2$, the function $g = f_{\alpha_1} - f_{\alpha_2}$ satisfies the equation

$$\begin{aligned}\partial_t g + v_1 \partial_x g &= \int B(f_{\alpha_1}' f_{\alpha_1 *}' - f_{\alpha_2}' f_{\alpha_2 *}') F_{\alpha_1}(f_{\alpha_1}) F_{\alpha_1}(f_{\alpha_1 *}) dv_* d\theta\\ &- \int B(f_{\alpha_1} f_{\alpha_1 *} - f_{\alpha_2} f_{\alpha_2 *}) F_{\alpha_1}(f_{\alpha_1}') F_{\alpha_1}(f_{\alpha_1 *}') dv_* d\theta\\ &+ \int B f_{\alpha_2}' f_{\alpha_2 *}' \Big(F_{\alpha_1}(f_{\alpha_1 *})\big(F_{\alpha_1}(f_{\alpha_1}) - F_{\alpha_1}(f_{\alpha_2})\big) + F_{\alpha_2}(f_{\alpha_2})\big(F_{\alpha_1}(f_{\alpha_1 *}) - F_{\alpha_1}(f_{\alpha_2 *})\big)\Big) dv_* d\theta\\ &+ \int B f_{\alpha_2}' f_{\alpha_2 *}' \Big(F_{\alpha_1}(f_{\alpha_1 *})\big(F_{\alpha_1}(f_{\alpha_2}) - F_{\alpha_2}(f_{\alpha_2})\big) + F_{\alpha_2}(f_{\alpha_2})\big(F_{\alpha_1}(f_{\alpha_2 *}) - F_{\alpha_2}(f_{\alpha_2 *})\big)\Big) dv_* d\theta\\ &- \int B f_{\alpha_2} f_{\alpha_2 *} \Big(F_{\alpha_1}(f_{\alpha_1 *}')\big(F_{\alpha_1}(f_{\alpha_1}') - F_{\alpha_1}(f_{\alpha_2}')\big) + F_{\alpha_2}(f_{\alpha_2}')\big(F_{\alpha_1}(f_{\alpha_1 *}') - F_{\alpha_1}(f_{\alpha_2 *}')\big)\Big) dv_* d\theta\\ &- \int B f_{\alpha_2} f_{\alpha_2 *} \Big(F_{\alpha_1}(f_{\alpha_1 *}')\big(F_{\alpha_1}(f_{\alpha_2}') - F_{\alpha_2}(f_{\alpha_2}')\big) + F_{\alpha_2}(f_{\alpha_2}')\big(F_{\alpha_1}(f_{\alpha_2 *}') - F_{\alpha_2}(f_{\alpha_2 *}')\big)\Big) dv_* d\theta.\end{aligned} \tag{4.1}$$

Using Lemma 3.3 and taking $\alpha_1,\alpha_2 < 2^{-L-1}$,

$$\int B\Big(|f_{\alpha_1}f_{\alpha_1*}-f_{\alpha_2}f_{\alpha_2*}|F_{\alpha_1}(f'_{\alpha_1})F_{\alpha_1}(f'_{\alpha_1*})\Big)^\sharp dxdvdv_*d\theta$$

$$\leq c2^{2L}\Big(\int\sup_{x\in[0,1]}f^\sharp_{\alpha_1}(t,x,v)dv+\int\sup_{x\in[0,1]}f^\sharp_{\alpha_2}(t,x,v)dv\Big)\int|(f_{\alpha_1}-f_{\alpha_2})^\sharp(t,x,v)|dxdv$$

$$\leq cc_12^{2L}\int|g^\sharp(t,x,v)|dxdv.$$

We similarly obtain

$$\int B\Big(f'_{\alpha_2}f'_{\alpha_2*}F_{\alpha_1}(f_{\alpha_1*})|(F_{\alpha_1}(f_{\alpha_2})-F_{\alpha_2}(f_{\alpha_2})|)\Big)^\sharp dxdvdv_*d\theta\leq cc_12^{2L}|\alpha_1-\alpha_2|,$$

and

$$\int B\Big(f_{\alpha_2}f_{\alpha_2*}F_{\alpha_1}(f'_{\alpha_1*})|F_{\alpha_1}(f'_{\alpha_1})-F_{\alpha_1}(f'_{\alpha_2})|\Big)^\sharp dxdvdv_*d\theta\leq cc_12^L\int|g^\sharp(t,x,v)|dxdv.$$

The remaining terms are estimated in the same way. It follows

$$\frac{d}{dt}\int|g^\sharp(t,x,v)|dxdv\leq cc_12^{2L}\Big(\int|g^\sharp(t,x,v)|dxdv+|\alpha_1-\alpha_2|\Big).$$

Hence

$$\lim_{(\alpha_1,\alpha_2)\to(0,0)}\sup_{t\in[0,T]}\int|g^\sharp(t,x,v)|dxdv=0.$$

And so (f_α) is a Cauchy sequence in $C([0,T];L^1([0,1]\times\mathbb{R}^2))$. Denote by f its limit. With analogous arguments to the previous ones in the proof of this lemma, it holds that

$$\lim_{\alpha\to0}\int|Q(f)-Q(f_\alpha)|(t,x,v)dtdxdv=0.$$

Hence f is a strong solution to (1.5) on $[0,T]$ with initial value f_0. If there were two solutions, their difference denoted by G would with similar arguments satisfy

$$\frac{d}{dt}\int|G^\sharp(t,x,v)|dxdv\leq cc_12^{2L}\int|G^\sharp(t.x.v)|dxdv,$$

hence be identically equal to its initial value zero.

Denote by $\mathcal{F}$ a given equibounded family of initial values bounded by 2^L. Let f_1 resp. f_2 be the solution to (1.5) with initial value $f_{10}\in\mathcal{F}$ resp. $f_{20}\in\mathcal{F}$. The equation for $\bar{g}=f_1-f_2$ can be written analogously to equation 4.1. Similar arguments lead to

$$\frac{d}{dt}\int|(f_1-f_2)^\sharp(t,x,v)|dxdv\leq cc_12^{2L}\int|(f_1-f_2)^\sharp(t,x,v)|dxdv,$$

so that

$$\|(f_1-f_2)(t,\cdot,\cdot)\|_{L^1([0,1]\times\mathbb{R}^2)}\leq e^{cc_1T2^{2L}}\|f_{10}-f_{20}\|_{L^1([0,1]\times\mathbb{R}^2)},\quad t\in[0,T].$$

This proves the stability statement of Theorem 1.1.

If $\sup_{(x,v)\in[0,1]\times\mathbb{R}^2}f(T,x,v)<2^{L+1}$, then the procedure can be repeated, i.e. the same proof can be carried out from the initial value $f(T)$. It leads to a maximal interval denoted by $[0,\tilde{T}_1]$ on which $f(t,\cdot,\cdot)\leq 2^{L+1}$. By induction there exists an increasing sequence of times $(\tilde{T}_n)$ such that $f(t,\cdot,\cdot)\leq 2^{L+n}$ on $[0,\tilde{T}_n]$. Let $T_\infty=\lim_{n\to+\infty}\tilde{T}_n$. Either $\tilde{T}_\infty=+\infty$ and the solution f is global in time, or T_∞ is finite and $\overline{\lim}_{t\to T_\infty}\|f(t)\|_\infty=\infty$.

□

5. Conservations of mass, momentum and energy

The following two preliminary lemmas are needed for the control of large velocities.

Lemma 5.1.
The solution f of equation (1.5) with initial value f_0, satisfies

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{t\in[0,T]} f^\sharp(t,x,v)dvdx \leq \frac{c_T}{\lambda}, \quad t\in[0,T],$$

where c_T only depends on T, $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

Proof. As in (2.3),

$$\sup_{t\in[0,T]} f^\sharp(t,x,v) \leq f_0(x,v) + \int_0^T Q^+(f)(s,x+sv_1,v)ds.$$

Integration with respect to (x,v) for $|v|>\lambda$, gives

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{t\in[0,T]} f^\sharp(t,x,v)dvdx \leq \int\int_{|v|>\lambda} |v| f_0(x,v)dvdx + \int_0^T \int_{|v|>\lambda} B$$
$$|v| f(s,x+sv_1,v')f(s,x+sv_1,v'_*)F(f)(s,x+sv_1,v)F(f)(s,x+sv_1,v_*)dvdv_*d\theta dxds.$$

Here in the last integral, either $|v'|$ or $|v'_*|$ is the largest and larger than $\frac{\lambda}{\sqrt{2}}$. The two cases are symmetric, and we discuss the case $|v'|\geq |v'_*|$. After a translation in x, the integrand is estimated from above by

$$c|v'| f^\#(s,x,v') \sup_{(t,x)\in[0,T]\times[0,1]} f^\#(t,x,v'_*).$$

The change of variables $(v,v_*,n)\to(v',v'_*,-n)$, the integration over

$$(s,x,v,v_*,\theta)\in[0,T]\times[0,1]\times\{v\in\mathbb{R}^2; |v|>\frac{\lambda}{\sqrt{2}}\}\times\mathbb{R}^2\times[-\frac{\pi}{2},\frac{\pi}{2}],$$

and Lemma 3.3 give the bound

$$\frac{c}{\lambda}\Big(\int_0^T \int |v|^2 f^\#(s,x,v)dxdvds\Big)\Big(\int \sup_{(t,x)\in[0,T]\times[0,1]} f^\#(t,x,v_*)dv_*\Big)$$
$$\leq \frac{cTc_1(T)}{\lambda}\int |v|^2 f_0(x,v)dxdv.$$

The lemma follows. ∎

Lemma 5.2. *The solution f of equation (1.5) with initial value f_0 satisfies*

$$\int_{|v|>\lambda} \sup_{(t,x)\in[0,T]\times[0,1]} f^\sharp(t,x,v)dv \leq \frac{c'_T}{\sqrt{\lambda}}, \quad t\in[0,T],$$

where c'_T only depends on T, $\int f_0(x,v)dxdv$ and $\int |v|^2 f_0(x,v)dxdv$.

Proof. Take $\lambda>2$. As above,

$$\int_{|v|>\lambda} \sup_{(t,x)\in[0,T]\times[0,1]} f^\sharp(t,x,v)dv \leq \int_{|v|>\lambda} \sup_{x\in[0,1]} f_0(x,v)dv + cC, \tag{5.1}$$

where

$$C=\int_{|v|>\lambda}\sup_{x\in[0,1]}\int_0^T\int Bf^{\#}(s,x+s(v_1-v_1'),v')f^{\#}(s,x+s(v_1-v_{*1}'),v_*')dvdv_*d\theta ds.$$

For v',v_*' outside of the angular cutoff (1.3), let n be the unit vector in the direction $v-v'$, and $n_\perp$ the orthogonal unit vector in the direction $v-v_*'$. Let e_1 be a unit vector in the x-direction.

Split C as $C=\sum_{1\le i\le 6}C_i$, where C_1 (resp. C_2, C_3) refers to integration with respect to (v_*,θ) on

$$\{(v_*,\theta);\quad n\cdot e_1\ge\frac{1}{\sqrt{2}},\quad |v'|\ge|v_*'|\},$$

$$\Big(\text{resp. } \{(v_*,\theta);n\cdot e_1\ge\sqrt{1-\frac{1}{\lambda}},\ |v'|\le|v_*'|\},\quad \{(v_*,\theta);n\cdot e_1\in[\frac{1}{\sqrt{2}},\sqrt{1-\frac{1}{\lambda}}],\ |v'|\le|v_*'|\}\Big),$$

and analogously for C_i, $4\le i\le 6$, with n replaced by $n_\perp$. By symmetry, C_i, $4\le i\le 6$ can be treated as C_i, $1\le i\le 3$, so we only discuss the control of C_i, $1\le i\le 3$.

By the change of variables $(v,v_*,n)\to(v',v_*',-n)$, and noticing that $|v'|\ge\frac{\lambda}{\sqrt{2}}$ in the domain of integration of C_1, it holds that

$$\begin{aligned}C_1&\le\int_{|v|>\frac{\lambda}{\sqrt{2}}}\sup_{x\in[0,1]}\int_0^T\int_{n\cdot e_1\ge\frac{1}{\sqrt{2}}}Bf^{\#}(s,x+s(v_1'-v_1),v)f^{\#}(s,x+s(v_1'-v_{*1}),v_*)dv_*d\theta dsdv\\&\le\int_{|v|>\frac{\lambda}{\sqrt{2}}}\sup_{x\in[0,1]}\int_0^T\int_{n\cdot e_1\ge\frac{1}{\sqrt{2}}}B\\&\qquad\sup_{\tau\in[0,T]}f^{\#}(\tau,x+s(v_1'-v_1),v)\sup_{(\tau,X)\in[0,T]\times[0,1]}f^{\#}(\tau,X,v_*)dv_*d\theta dsdv.\end{aligned}$$

With the change of variables $s\to y=x+s(v_1'-v_1)$,

$$\begin{aligned}C_1&\le\int_{|v|>\frac{\lambda}{\sqrt{2}}}\sup_{x\in[0,1]}\int_{n\cdot e_1\ge\frac{1}{\sqrt{2}}}\int_{y\in(x,x+T(v_1'-v_1))}\frac{B}{|v_1'-v_1|}\\&\qquad\sup_{\tau\in[0,T]}f^{\#}(\tau,y,v)\sup_{(\tau,X)\in[0,T]\times[0,1]}f^{\#}(\tau,X,v_*)dydv_*d\theta dv\\&\le\int_{|v|>\frac{\lambda}{\sqrt{2}}}\int_{n\cdot e_1\ge\frac{1}{\sqrt{2}}}\frac{|E(T(v_1'-v_1))+1)|}{|v_1'-v_1|}\int_0^1B\\&\qquad\sup_{\tau\in[0,T]}f^{\#}(\tau,y,v)\sup_{(\tau,X)\in[0,T]\times[0,1]}f^{\#}(\tau,X,v_*)dydv_*d\theta dv.\end{aligned}$$

Moreover,

$$|E(T(v_1'-v_1))+1)|\le T|v_1'-v_1|+1\le\big(T+\frac{\sqrt{2}}{\gamma\gamma'}\big)|v_1'-v_1|,$$

where γ and γ' were defined in (2.2). Consequently,

$$\begin{aligned}C_1&\le c(T+1)\int_0^1\int_{|v|>\frac{\lambda}{\sqrt{2}}}\sup_{\tau\in[0,T]}f^{\#}(\tau,y,v)dydv\int\sup_{(\tau,X)\in[0,T]\times[0,1]}f^{\#}(\tau,X,v_*)dv_*\\&\le\frac{c(T+1)}{\lambda}\int_0^1\int_{|v|>\frac{\lambda}{\sqrt{2}}}|v|\sup_{\tau\in[0,T]}f^{\#}(\tau,y,v)dydv\int\sup_{(\tau,X)\in[0,T]\times[0,1]}f^{\#}(\tau,X,v_*)dv_*.\end{aligned}$$

By Lemmas 3.3 and 5.1,

$$C_1 \leq \frac{c}{\lambda^2}(T+1)c_T c_1(T).$$

Moreover,

$$\begin{aligned} C_2 \leq & \int_{|v'|>\lambda, |v_*|>|v|, n\cdot e_1 \geq \sqrt{1-\frac{1}{\lambda}}} \frac{B}{|v_1'-v_1|} \\ & \sup_{x\in[0,1]} \int_{y\in(x,x+T(v_1'-v_1))} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dy dv dv_* d\theta \\ & \leq c(T+1) \int_{n\cdot e_1\geq\sqrt{1-\frac{1}{\lambda}}} d\theta \int \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v) dy dv \int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v_*) dv_* \\ & \leq \frac{c}{\sqrt{\lambda}}(T+1)^2 c_1(T), \end{aligned}$$

by Lemmas 3.1 and 3.3. Finally,

$$\begin{aligned} C_3 \leq & \int_{|v_*|>\frac{\lambda}{\sqrt{2}}, \frac{1}{\sqrt{\lambda}}\leq n_\perp\cdot e_1\leq\frac{1}{\sqrt{2}}} \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v) \frac{B}{|v_1'-v_{*1}|} \\ & \sup_{x\in[0,1]} \Big(\int_{y\in(x,x+T(v_1'-v_{*1}))} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v_*) dy \Big) dv dv_* d\theta \\ & \leq c(T+1)\sqrt{\lambda} \Big(\int \sup_{(\tau,X)\in[0,T]\times[0,1]} f^{\#}(\tau,X,v) dv \Big) \Big(\int_{|v_*|>\frac{\lambda}{\sqrt{2}}} \sup_{\tau\in[0,T]} f^{\#}(\tau,y,v_*) dy dv_* \Big). \end{aligned}$$

By Lemmas 3.3 and 5.1,

$$C_3 \leq \frac{c}{\sqrt{\lambda}}(T+1)c_1(T)c_T.$$

The lemma follows. □

LEMMA 5.3. *The solution f to equation (1.5) with initial value f_0 conserves mass, momentum and energy.*

Proof. The conservation of mass and first momentum of f will follow from the boundedness of the total energy. The energy is non-increasing since the approximations f_α conserve energy and

$$\lim_{\alpha\to 0} \int_0^1 \int_{|v|<V} |(f-f_\alpha)(t,x,v)||v|^2 dx dv = 0, \quad \text{for all } t\in[0,T] \text{ and positive } V.$$

Energy conservation will be satisfied if the energy is non-decreasing. Taking $\psi_\epsilon = \frac{|v^2|}{1+\epsilon|v|^2}$ as approximation for $|v|^2$, it is enough to bound

$$\int Q(f)(t,x,v)\psi_\epsilon(v) dx dv = \int B\psi_\epsilon \Big(f' f_*' F(f) F(f_*) - f f_* F(f') F(f_*') \Big) dx dv dv_* d\theta$$

from below by zero in the limit $\epsilon\to 0$. Similarly to [8],

$$\begin{aligned} \int Q(f)\psi_\epsilon dx dv = & \frac{1}{2} \int B f f_* F(f') F(f_*') \Big(\psi_\epsilon(v') + \psi_\epsilon(v_*') - \psi_\epsilon(v) - \psi_\epsilon(v_*) \Big) dx dv dv_* d\theta \\ & \geq - \int B f f_* F(f') F(f_*') \frac{\epsilon |v|^2 |v_*|^2}{(1+\epsilon|v|^2)(1+\epsilon|v_*|^2)} dx dv dv_* d\theta. \end{aligned}$$

The previous line, with the integral taken over a bounded set in (v,v_*), converges to zero when $\epsilon\to 0$. In integrating over $|v|^2+|v_*|^2\geq 2\lambda^2$, there is symmetry between the subset of the domain with $|v|^2>\lambda^2$ and the one with $|v_*|^2>\lambda^2$. We discuss the first sub-domain, for which the integral in the last line is bounded from below by

$$-c\int |v_*|^2 f(t,x,v_*)dxdv_*\int_{|v|\geq\lambda} B \sup_{(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dvd\theta$$
$$\geq -c\int_{|v|\geq\lambda} \sup_{0\leq(s,x)\in[0,t]\times[0,1]} f^{\#}(s,x,v)dv.$$

It follows from Lemma 5.2 that the right hand side tends to zero when $\lambda\to\infty$. This implies that the energy is non-decreasing, and bounded from below by its initial value. That completes the proof of the lemma. ∎

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