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HIGH-ORDER QUASI-COMPACT DIFFERENCE SCHEMES FOR FRACTIONAL DIFFUSION EQUATIONS*

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Abstract. The continuous time random walk (CTRW) underlies many fundamental processes in non-equilibrium statistical physics. When the jump length of CTRW obeys a power-law distribution, its corresponding Fokker–Planck equation has a space fractional derivative, which characterizes Lévy flights. Sometimes the infinite variance of Lévy flight discourages it as a physical approach; exponentially tempering the power-law jump length of CTRW makes it more 'physical' and the tempered space fractional diffusion equation appears. This paper provides the basic strategy of deriving the high-order quasi-compact discretizations for the space fractional derivative and the tempered space fractional derivative. The fourth-order quasi-compact discretization for the space fractional derivative is applied to solve a space fractional diffusion equation, and the unconditional stability and convergence of the scheme are theoretically proved and numerically verified. Furthermore, the tempered space fractional diffusion equation is effectively solved by its counterpart, the fourth-order quasi-compact scheme, and the convergence orders are verified numerically.

Key words. space fractional derivative; tempered space fractional derivative; WSGD discretization; quasi-compact difference scheme; numerical stability and convergence.

AMS subject classifications. 65M06; 65M12; 26A33.

1. Introduction

In recent years, more and more scientific and engineering problems are involved in fractional calculus. They range from relaxation oscillation phenomena [14] to viscoelasticity [2] and from control theory [24] to transport problem [18]. The fractional diffusion equation has been put forward as a more suitable model for describing ion channel gating dynamics [10] and subdiffusive anomalous transport in an external field [3], which are the results of the continuous time random walk (CTRW) in the scaling limit. The CTRW is a mathematical formalization of a path that consists of a succession of random steps including the elements of random waiting time and jump length, and it underlies many fundamental stochastic processes in statistical physics. When the first moment of the distribution of waiting time and the second moment of jump length are finite, the probability density function (PDF) of the particle's location and time satisfy the classical diffusion equation. However, if the jump length obeys the power-law distribution, the PDF of the particle's location and time is the solution of the space fractional diffusion equation, and the corresponding dynamics is called Lévy flight. Sometimes the jumps of the particles are limited by the finite size of the physical system and the infinite variance of Lévy flight discourages it as a physical approach. So the power-law distribution of the jump length is expected to be truncated [15] or exponentially tempered [5]. For the CTRW with the distribution of the tempered jump

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length $|x|^{-(1+\alpha)}e^{-\lambda|x|}$ [6], the corresponding PDF of the particles satisfies the tempered space fractional diffusion equation [5].

It seems that there are fewer works on the numerical solutions of tempered space fractional diffusion equation [12]. However, for the space fractional diffusion or advection-diffusion equation, much progress has been made for its numerical methods, e.g., [13, 17, 21, 22, 19, 23, 27, 9]. Transforming the Riemann–Liouville fractional derivative to the Caputo fractional derivative, the space fractional Fokker–Planck equation is solved by the method of lines in [13]. Using the superconvergence of Grünwald discretization at a particular point, a second-order finite difference scheme is proposed in [19]. Based on the difference discretization and spline approximation to the Riemann–Liouville fractional derivative, a second-order scheme is presented for the three-dimensional space fractional partial differential equations in [9]. Currently, the most popular discretization scheme for the space Riemann–Liouville fractional derivative seems to be the weighted and shifted Grünwald (WSGD) operator. The first-order WSGD operator is first presented and discussed in detail in [17], and the second-order convergence is obtained by using the extrapolation method [21, 22]. The second-order WSGD operator is given in [23], and the third-order compact WSGD (CWSGD) is presented in [27]. The positivity and boundedness-preserving WSGD schemes for the space-time fractional reaction-diffusion equation appear in [25], and the related schemes for the time fractional equation can be found in [26]. Following the idea of the weighting and shifting Grünwald operator, this paper provides the basic strategy of deriving the quasi-compact scheme with any desired convergence orders for the space fractional diffusion equation, and it can also be extended to solve the tempered space fractional diffusion equation. The fourth-order quasi-compact scheme is discussed in detail when solving the space fractional diffusion equation, including stability and convergence analysis and numerical verification of convergence orders. The fourth-order quasi-compact scheme for the tempered space fractional diffusion equation is also proposed and effectively used to solve the equation, and the convergence orders are numerically verified.

The outline of this paper is as follows. In Section 2, the high-order quasi-compact discretizations are presented to approximate the space Riemann–Liouville fractional derivative. In Section 3, following the obtained quasi-compact discretizations, the high-order quasi-compact scheme for the one-dimensional space fractional diffusion equation is designed, and its stability and convergence analysis are performed. Section 4 focuses on the quasi-compact scheme and the corresponding stability and convergence analysis in two-dimensional case. The high-order quasi-compact discretizations is extended to the tempered space fractional derivative in Section 5, and the corresponding scheme is derived to solve the tempered space fractional diffusion equation. In Section 6, numerical experiments are performed to test the efficiency and verify the convergence orders of the schemes. We conclude the paper with some discussion in the last section.

2. Quasi-compact discretizations for the Riemann–Liouville space fractional derivatives

We first introduce some definitions and lemmas, including the Riemann–Liouville fractional derivatives and the shifted Grünwald–Letnikov formulations.

DEFINITION 2.1 ([20]). *If the function $u(x)$ is defined in the interval (a,b) and sufficiently regular, then the α-th-order left and right Riemann–Liouville fractional derivatives are, respectively, defined as*

$$ {}_aD_x^{\alpha}u(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-s)^{n-\alpha-1}u(s)ds, \quad n-1<\alpha<n \tag{2.1}$$

and

$$ {}_xD_b^\alpha u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (s-x)^{n-\alpha-1} u(s)ds, \quad n-1<\alpha<n, \tag{2.2} $$

where a can be $-\infty$ and b can be $+\infty$.

And the standard left and right Grünwald–Letnikov formulations, which can be potentially used to approximate the left and right Riemann–Liouville fractional derivatives, are, respectively, given as

$$ {}_aD_x^\alpha u(x) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[\frac{x-a}{h}]} g_k^{(\alpha)} u(x-kh) \tag{2.3} $$

and

$$ {}_xD_b^\alpha u(x) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{[\frac{b-x}{h}]} g_k^{(\alpha)} u(x+kh), \tag{2.4} $$

where the Grünwald weights $g_k^{(\alpha)} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}$ are the coefficients of the power series expansion of $(1-z)^\alpha$. To get the stable scheme, a shifted Grünwald–Letnikov operator is proposed to approximate the left Riemann–Liouville fractional derivative with first-order accuracy [21].

LEMMA 2.2 ([21]). *Let $1<\alpha<2$, $u \in C^{n+3}(R)$, and $D^k u(x) \in L^1(R)$, $k=0,1,\cdots,n+3$. For any integer p, define the left shifted Grünwald–Letnikov operator by*

$$ \Delta_p^\alpha u(x) := \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} u(x-(k-p)h). \tag{2.5} $$

Then we have

$$ \Delta_p^\alpha u(x) = {}_{-\infty}D_x^\alpha u(x) + \sum_{l=1}^{n-1} a_{p,l}^\alpha \, {}_{-\infty}D_x^{\alpha+l} u(x) h^l + O(h^n), \tag{2.6} $$

uniformly in $x \in R$, where the weights $a_{p,l}^\alpha$ are the coefficients of the power series expansion of the function $(\frac{1-e^{-z}}{z})^\alpha e^{pz}$ and the first four terms of the coefficients are $a_{p,0}^\alpha = 1$, $a_{p,1}^\alpha = p-\alpha/2$, $a_{p,2}^\alpha = (\alpha+3\alpha^2-12\alpha p+12p^2)/24$, and $a_{p,3}^\alpha = (8p^3+2p\alpha-12p^2\alpha-\alpha^2+6p\alpha^2-\alpha^3)/48$.

To approximate the right Riemann–Liouville fractional derivative, ${}_xD_\infty^\alpha u(x)$, the right shifted Grünwald–Letnikov operator is given by $\Lambda_p^\alpha f(x) := \frac{1}{h^\alpha} \sum_{k=0}^{\infty} g_k^{(\alpha)} f(x+(k-p)h)$. In the finite interval $[a,b]$, the left and right shifted Grünwald–Letnikov fractional derivatives are, respectively,

$$ \tilde{\Delta}_p^\alpha u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{[\frac{x-a}{h}]+p} g_k^{(\alpha)} u(x-(k-p)h) \tag{2.7} $$

and

$$ \tilde{\Lambda}_p^\alpha u(x) = \frac{1}{h^\alpha} \sum_{k=0}^{[\frac{b-x}{h}]+p} g_k^{(\alpha)} u(x+(k-p)h). \tag{2.8} $$

In the remaining analysis of the paper, for a function defined in the bounded interval, we suppose that it has been zero extended to R whenever the value of $u(x)$ outside of the bounded interval is used.

2.1. Fourth-order quasi-compact approximation to the Riemann–Liouville fractional derivative. According to the definitions of the shifted Grünwald–Letnikov fractional derivatives, we know that p can be any integer. In order to ensure that the nodes in Equations (2.7) or (2.8) are within the bounded interval, we need to choose the integer $p \in \{1,0,-1\}$ when approximating the non-periodic fractional differential equation in the bounded interval. Inspired by the shifted Grünwald–Letnikov operator and the Taylor expansion, we derive the following fourth-order combined quasi-compact approximations.

THEOREM 2.3. *Let $u(x) \in C^7(R)$ and suppose all the derivatives of $u(x)$ up to order 7 belong to $L^1(R)$. Then the following quasi-compact approximation has fourth-order accuracy; i.e.,*

$$P_x\, {}_{-\infty}D_x^{\alpha} u(x) = \mu_1 \Delta_1^{\alpha} u(x) + \mu_0 \Delta_0^{\alpha} u(x) + \mu_{-1} \Delta_{-1}^{\alpha} u(x) + O(h^4), \tag{2.9}$$

where $P_x = 1 + h^2 b_2^{\alpha} \delta_x^2$, called the CWSGD operator, δ_x^2 is the centered difference operator; and the coefficients b_2^{α}, μ_1, μ_0, and μ_{-1} are functions of α and

$$\begin{cases} \mu_1 = (1+\alpha)(2+\alpha)/12, \\ \mu_0 = -(-2+\alpha)(2+\alpha)/6, \\ \mu_{-1} = (-2+\alpha)(-1+\alpha)/12, \\ b_2^{\alpha} = (4+\alpha-\alpha^2)/24. \end{cases} \tag{2.10}$$

Proof. Taking $n=4$ in Equation (2.6) of Lemma 2.2, we require that the following conditions are satisfied: $u \in C^7(R)$ and assume all the derivatives of $u(x)$ up to order 7 belong to $L^1(R)$. For a different parameter $p \in \{1,0,-1\}$, there exists

$$\Delta_p^{\alpha} u(x) = {}_{-\infty}D_x^{\alpha} u(x) + \sum_{l=1}^{3} a_{p,l}^{\alpha}\, {}_{-\infty}D_x^{\alpha+l} u(x) h^l + O(h^4), \quad p = 1,0,-1.$$

Next we choose three suitable variables, μ_1, μ_0, and μ_{-1}, to eliminate two lower-order terms in the above equations corresponding to h^l $(l=1,3)$; i.e.,

$$\mu_1 \Delta_1^{\alpha} u(x) + \mu_0 \Delta_0^{\alpha} u(x) + \mu_{-1} \Delta_{-1}^{\alpha} u(x) = {}_{-\infty}D_x^{\alpha} u(x) + b_2^{\alpha}\, {}_{-\infty}D_x^{\alpha+2} u(x) h^2 + O(h^4),$$

where $b_2^{\alpha} = \mu_1 a_{1,2}^{\alpha} + \mu_0 a_{0,2}^{\alpha} + \mu_{-1} a_{-1,2}^{\alpha}$, which can be obtained by solving the following algebraic equation:

$$\begin{cases} \mu_1 + \mu_0 + \mu_{-1} = 1, \\ \mu_1 a_{1,1}^{\alpha} + \mu_0 a_{0,1}^{\alpha} + \mu_{-1} a_{-1,1}^{\alpha} = 0, \\ \mu_1 a_{1,3}^{\alpha} + \mu_0 a_{0,3}^{\alpha} + \mu_{-1} a_{-1,3}^{\alpha} = 0. \end{cases}$$

Let $\delta_x^2 u(x) = (u(x-h) - 2u(x) + u(x+h))/h^2$. Then we have

$$
\begin{aligned}
\mu_1\Delta_1^\alpha u(x)+\mu_0\Delta_0^\alpha u(x)+\mu_{-1}\Delta_{-1}^\alpha u(x) &= {}_{-\infty}D_x^\alpha u(x)+b_2^\alpha\, {}_{-\infty}D_x^{\alpha+2}u(x)h^2+O(h^4)\\
&=\left(1+h^2 b_2^\alpha\frac{\partial^2}{\partial x^2}\right) {}_{-\infty}D_x^\alpha u(x)+O(h^4)\\
&=\left(1+h^2 b_2^\alpha \delta_x^2\right) {}_{-\infty}D_x^\alpha u(x)+O(h^4)\\
&=P_x\, {}_{-\infty}D_x^\alpha u(x)+O(h^4),
\end{aligned}
\tag{2.11}
$$

where $P_x=1+h^2b_2^\alpha\delta_x^2$. The proof is complete. □

REMARK 2.4. The assumption of requiring $u\in C^7(R)$ in Theorem 2.3 can be weakened by using recently introduced techniques; see Equation (3.14) of [8].

Since $\delta_x^2 u(x)=\frac{\partial^2 u(x)}{\partial x^2}+O(h^2)$, for any function u we have

$$
P_x u=\left(1+h^2 b_2^\alpha\frac{\partial^2}{\partial x^2}\right)u+O(h^4).
$$

In a similar way, we can derive the fourth-order quasi-compact approximation for the right Riemann–Liouville fractional derivative:

$$
P_x\, {}_xD_{+\infty}^\alpha u(x)=\mu_1\Lambda_1^\alpha u(x)+\mu_0\Lambda_0^\alpha u(x)+\mu_{-1}\Lambda_{-1}^\alpha u(x)+O(h^4). \tag{2.12}
$$

For $u(x)$ defined in a bounded interval, supposing its zero extension to R satisfies the assumptions of Theorem 2.3, the following approximations hold:

$$
P_x\, {}_aD_x^\alpha u(x)=\mu_1\tilde{\Delta}_1^\alpha u(x)+\mu_0\tilde{\Delta}_0^\alpha u(x)+\mu_{-1}\tilde{\Delta}_{-1}^\alpha u(x)+O(h^4) \tag{2.13}
$$

and

$$
P_x\, {}_xD_b^\alpha u(x)=\mu_1\tilde{\Lambda}_1^\alpha u(x)+\mu_0\tilde{\Lambda}_0^\alpha u(x)+\mu_{-1}\tilde{\Lambda}_{-1}^\alpha u(x)+O(h^4). \tag{2.14}
$$

Now using the CWSGD operator, we solve a two-point boundary value problem to numerically verify the above statements.

EXAMPLE 2.5. *Consider the steady state fractional diffusion problem*

$$
{}_0D_x^\alpha u(x)=\frac{720x^{6-\alpha}}{\Gamma(7-\alpha)},\quad x\in(0,1),
$$

with $1<\alpha<2$ *and the boundary conditions* $u(0)=0$, $u(1)=1$. *Its exact solution is* $u(x)=x^6$.

Using the quasi-compact scheme (2.13) to solve Example 2.5 leads to the desired convergence orders; see Table 2.1.

2.2. Fifth-order quasi-compact approximation to the Riemann–Liouville fractional derivative. In this subsection, we present a fifth-order quasi-compact approximation given as follows.

THEOREM 2.6. *Let* $u(x)\in C^8(R)$ *and assume all the derivatives of* $u(x)$ *up to order 8 belong to* $L^1(R)$. *Then the following quasi-compact approximation has fifth-order accuracy; i.e.,*

$$
P_x^5\, {}_{-\infty}D_x^\alpha u(x)=\mu_1\Delta_1^\alpha f(x)+\mu_0\Delta_0^\alpha f(x)+\mu_{-1}\Delta_{-1}^\alpha f(x)+O(h^5), \tag{2.15}
$$

α	h_x	$\|u-U\|_2$	rate	$\|u-U\|_\infty$	rate
1.1	1/8	$6.0879e-04$		$1.0551e-03$	
	1/16	$2.7715e-05$	4.4572	$5.1569e-05$	4.3548
	1/32	$1.5024e-06$	4.2054	$2.8244e-06$	4.1905
	1/64	$9.0430e-08$	4.0543	$1.6385e-07$	4.1075
	1/128	$5.5808e-09$	4.0183	$9.5651e-09$	4.0984
1.5	1/8	$2.9459e-04$		$3.9380e-04$	
	1/16	$1.8470e-05$	3.9955	$2.4150e-05$	4.0274
	1/32	$1.1590e-06$	3.9942	$1.5252e-06$	3.9850
	1/64	$7.2639e-08$	3.9960	$9.5671e-08$	3.9948
	1/128	$4.5471e-09$	3.9977	$5.9911e-09$	3.9972
1.9	1/8	$1.1926e-04$		$1.6198e-04$	
	1/16	$7.4913e-06$	3.9927	$1.0174e-05$	3.9928
	1/32	$4.6919e-07$	3.9970	$6.3722e-07$	3.9970
	1/64	$2.9352e-08$	3.9986	$3.9899e-08$	3.9974
	1/128	$1.8352e-09$	3.9994	$2.4947e-09$	3.9994

TABLE 2.1. *Numerical errors and convergence rates in the L_∞-norm and the L_2-norm by using Equation (2.13) to solve Example 2.5, where U denotes the numerical solution and h_x is the space step size.*

where $P_x^5\,{}_{-\infty}D_x^\alpha u(x) = \gamma_1\,{}_{-\infty}D_x^\alpha u(x-h) + {}_{-\infty}D_x^\alpha u(x) + \gamma_2\,{}_{-\infty}D_x^\alpha u(x+h)$, *called 5-CWSGD operator, and*

$$\begin{cases} \gamma_1 = \dfrac{350+331\alpha-15\alpha^2-75\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3+30\alpha^4}, \\ \gamma_2 = \dfrac{566-329\alpha-135\alpha^2+105\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3+30\alpha^4}, \\ \mu_1 = \dfrac{566+329\alpha-135\alpha^2-105\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3+30\alpha^4}, \\ \mu_0 = \dfrac{862+\alpha-285\alpha^2+15\alpha^3+15\alpha^4}{862-a-285\alpha^2-15\alpha^3+15\alpha^4}, \\ \mu_{-1} = \dfrac{350-331\alpha-15\alpha^2+75\alpha^3-15\alpha^4}{1724-2\alpha-570\alpha^2-30\alpha^3+30\alpha^4}. \end{cases} \tag{2.16}$$

The method for deriving Equation (2.15) is similar to the derivation of the fourth-order quasi-compact approximation. On one hand, from Equation (2.6), we know for a different parameter $p \in \{1,0,-1\}$ there exists

$$\Delta_p^\alpha u(x) = {}_{-\infty}D_x^\alpha u(x) + \sum_{k=1}^{4} a_{p,k}^\alpha\,{}_{-\infty}D_x^{\alpha+k} u(x) h^k + O(h^5), \quad p=1,0,-1. \tag{2.17}$$

On the other hand, in view of the Taylor expansion, we have

$$\begin{aligned} {}_{-\infty}D_x^\alpha u(x-h) &= {}_{-\infty}D_x^\alpha u(x) + (-1)^k \sum_{k=1}^{4} \frac{1}{k!}\,{}_{-\infty}D_x^{\alpha+k} u(x) h^k + O(h^5), \\ {}_{-\infty}D_x^\alpha u(x+h) &= {}_{-\infty}D_x^\alpha u(x) + \sum_{k=1}^{4} \frac{1}{k!}\,{}_{-\infty}D_x^{\alpha+k} u(x) h^k + O(h^5). \end{aligned} \tag{2.18}$$

So in order to get the fifth-order quasi-compact approximation, combining Equations (2.17) and (2.18), we need to eliminate the lower-order terms corresponding to h^k ($k=1,2,3,4$) which can be done by solving the algebraic equation

$$\begin{cases} \mu_1+\mu_0+\mu_{-1}-\gamma_1-\gamma_2=1, \\ \mu_1 a_{1,1}^{\alpha}+\mu_0 a_{0,1}^{\alpha}+\mu_{-1}a_{-1,1}^{\alpha}+\gamma_1-\gamma_2=0, \\ \mu_1 a_{1,2}^{\alpha}+\mu_0 a_{0,2}^{\alpha}+\mu_{-1}a_{-1,2}^{\alpha}-\gamma_1/2-\gamma_2/2=0, \\ \mu_1 a_{1,3}^{\alpha}+\mu_0 a_{0,3}^{\alpha}+\mu_{-1}a_{-1,3}^{\alpha}+\gamma_1/3!-\gamma_2/3!=0, \\ \mu_1 a_{1,4}^{\alpha}+\mu_0 a_{0,4}^{\alpha}+\mu_{-1}a_{-1,4}^{\alpha}-\gamma_1/4!-\gamma_2/4!=0. \end{cases} \tag{2.19}$$

Equation (2.16) is the solution of Equation (2.19). Then we get Theorem 2.6. Next we utilize the 5-CWSGD operator to solve Example 2.7, and the numerical results are presented in Table 2.2, from which the accuracy of the 5-CWSGD operator is verified.

EXAMPLE 2.7. We again consider the steady state fractional diffusion problem simulated in Example 2.5; i.e.,

$${}_0D_x^{\alpha}u(x)=\frac{720x^{6-\alpha}}{\Gamma(7-\alpha)}, \quad x\in(0,1),$$

with $1<\alpha<2$ and the boundary conditions $u(0)=0$, $u(1)=1$, and the exact solution $u(x)=x^6$.

α	h_x	$\|u-U\|_2$	rate	$\|u-U\|_\infty$	rate
1.1	1/8	$2.3456e-05$		$5.2058e-05$	
	1/16	$6.8783e-07$	5.0918	$1.6758e-06$	4.9572
	1/32	$2.0903e-08$	5.0403	$5.3410e-08$	4.9716
	1/64	$6.4355e-10$	5.0215	$1.6852e-09$	4.9861
	1/128	$1.9956e-11$	5.0112	$5.2916e-11$	4.9931
1.5	1/8	$9.0595e-06$		$1.9904e-05$	
	1/16	$2.8200e-07$	5.0057	$6.7018e-07$	4.8924
	1/32	$8.9299e-09$	4.9809	$2.2033e-08$	4.9268
	1/64	$2.8313e-10$	4.9791	$7.1095e-10$	4.9538
	1/128	$8.9603e-12$	4.9818	$2.2661e-11$	4.9714

TABLE 2.2. *Numerical errors and convergence rates in the L_∞-norm and the L_2-norm of the scheme (2.15) to solve Example 2.7, where U denotes the numerical solution and h_x is space step size.*

REMARK 2.8. Since the fifth-order quasi-compact scheme is not stable in solving the time-dependent space fractional differential equation, we discuss in detail the fourth-order quasi-compact schemes in sections 3 and 4.

3. Quasi-compact scheme for the one-dimensional space fractional diffusion equation

Based on the fourth-order quasi-compact discretization to the Riemann-Liouville space fractional derivative, we develop the Crank–Nicolson (C-N) quasi-compact scheme of the two-sided space fractional diffusion equations. Here, we consider the initial boundary value problem of the space fractional diffusion equation

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = K_1\,{}_aD_x^\alpha u(x,t) + K_2\,{}_xD_b^\alpha u(x,t) + f(x,t), & (x,t)\in(a,b)\times(0,T], \\ u(x,0) = u_0(x), & x\in[a,b], \\ u(a,t)=\phi_a(t),\ u(b,t)=\phi_b(t), & t\in[0,T], \end{cases} \tag{3.1}$$

where $1<\alpha\le 2$. The diffusion coefficients K_1 and K_2 are nonnegative constants and they satisfy $K_1^2+K_2^2\neq 0$. If $K_1\neq 0$, then $\phi_a(t)\equiv 0$ and $K_2\neq 0$, then $\phi_b(t)\equiv 0$. In the following analysis of the numerical method, we suppose that the problem (3.1) has a unique and sufficiently smooth solution.

3.1. CN-CWSGD scheme. The time interval $[0,T]$ is partitioned into a uniform mesh with step size $\tau=T/N$ and the space interval $[a,b]$ is partioned into another uniform mesh with step size $h=(b-a)/M$, where N and M are two positive integers. Then the set of grid points can be denoted by $x_j=a+jh$ $(0\le j\le M)$ and $t_n=n\tau$ $(0\le n\le N)$. Let $u_j^n=u(x_j,t_n)$, $t_{n+1/2}=(t_n+t_{n+1})/2$, and $f_j^{n+1/2}=f(x_j,t_{n+1/2})$ for $0\le n\le N-1$. The maximum norm and the discrete L_2-norm are defined as

$$\|u\|_\infty = \max_{1\le j\le M-1}|u_j|, \quad \|u\|^2 = h\sum_{j=1}^{M-1}u_j^2. \tag{3.2}$$

We use the C-N technique for the time discretization of the problem (3.1) and get

$$\begin{aligned} \frac{u_j^{n+1}-u_j^n}{\tau} =& \frac{1}{2}\left(K_1({}_aD_x^\alpha u)_j^n + K_1({}_aD_x^\alpha u)_j^{n+1} + K_2({}_xD_b^\alpha u)_j^n + K_2({}_xD_b^\alpha u)_j^{n+1}\right) \\ &+ f_j^{n+1/2} + O(\tau^2). \end{aligned} \tag{3.3}$$

In space, the fourth-order quasi-compact discretizations are used to approximate the Riemann–Liouville fractional derivatives. This implies that

$$\begin{aligned} P_x\frac{u_j^{n+1}-u_j^n}{\tau} =& \frac{K_1\tau}{2}\,{}_LD_h^\alpha u_j^n + \frac{K_2\tau}{2}\,{}_RD_h^\alpha u_j^n + \frac{K_1\tau}{2}\,{}_LD_h^\alpha u_j^{n+1} + \frac{K_2\tau}{2}\,{}_RD_h^\alpha u_j^{n+1} \\ &+ P_x f_j^{n+1/2} + R_j^{n+1/2}, \end{aligned} \tag{3.4}$$

where

$${}_LD_h^\alpha u_j^n =: \mu_1\tilde{\Delta}_1^\alpha u_j^n + \mu_0\tilde{\Delta}_0^\alpha u_j^n + \mu_{-1}\tilde{\Delta}_{-1}^\alpha u_j^n = \frac{1}{h^\alpha}\sum_{k=0}^{j+1} w_k^{(\alpha)} u_{j-k+1}^n,$$

$${}_RD_h^\alpha u_j^n =: \mu_1\tilde{\Lambda}_1^\alpha u_j^n + \mu_0\tilde{\Lambda}_0^\alpha u_j^n + \mu_{-1}\tilde{\Lambda}_{-1}^\alpha u_j^n = \frac{1}{h^\alpha}\sum_{k=0}^{M-j+1} w_k^{(\alpha)} u_{j+k-1}^n,$$

the coefficients $w_0^{(\alpha)}=\mu_1 g_0^{(\alpha)}$, $w_1^{(\alpha)}=\mu_0 g_0^{(\alpha)}+\mu_1 g_1^{(\alpha)}$, and $w_k^{(\alpha)}=\mu_1 g_k^{(\alpha)}+\mu_0 g_{k-1}^{(\alpha)}+\mu_{-1}g_{k-2}^{(\alpha)}$, $k=2,\cdots,M$ and $R_j^{n+1/2}\le C(\tau^2+h^4)$.

Then the above equation can be rewritten as

$$
\begin{aligned}
&P_x u_j^{n+1} - \frac{K_1\tau}{2}\,{}_LD_h^\alpha u_j^{n+1} - \frac{K_2\tau}{2}\,{}_RD_h^\alpha u_j^{n+1} \\
=&P_x u_j^{n} + \frac{K_1\tau}{2}\,{}_LD_h^\alpha u_j^{n} + \frac{K_2\tau}{2}\,{}_RD_h^\alpha u_j^{n} + \tau P_x f_j^{n+1/2} + \tau R_j^{n+1/2}.
\end{aligned} \tag{3.5}
$$

Denoting by U_j^n the numerical approximation of u_j^n, we obtain the C-N quasi-compact scheme for the problem (3.1)

$$
\begin{aligned}
&P_x U_j^{n+1} - \frac{K_1\tau}{2}\,{}_LD_h^\alpha U_j^{n+1} - \frac{K_2\tau}{2}\,{}_RD_h^\alpha U_j^{n+1} \\
=&P_x U_j^{n} + \frac{K_1\tau}{2}\,{}_LD_h^\alpha U_j^{n} + \frac{K_2\tau}{2}\,{}_RD_h^\alpha U_j^{n} + \tau P_x f_j^{n+1/2}.
\end{aligned} \tag{3.6}
$$

For convenience, the approximation scheme (3.6) can be written in matrix form

$$
(P_\alpha - B_\alpha)U^{n+1} = (P_\alpha + B_\alpha)U^n + \tau P_\alpha F^n + H^n, \tag{3.7}
$$

where $U^n = (U_1^n, U_2^n, \cdots, U_{M-1}^n)^T$, $\quad F^n = (f_1^{n+1/2}, f_2^{n+1/2}, \cdots, f_{M-1}^{n+1/2})^T$, $B_\alpha = \frac{\tau}{2h^\alpha}(K_1 A_\alpha + K_2 A_\alpha^T)$ with

$$
A_\alpha = \begin{pmatrix}
w_1^{(\alpha)} & w_0^{(\alpha)} & & & \\
w_2^{(\alpha)} & w_1^{(\alpha)} & w_0^{(\alpha)} & & \\
\vdots & w_2^{(\alpha)} & w_1^{(\alpha)} & & \\
w_{M-2}^{(\alpha)} & \cdots & \ddots & \ddots & w_0^{(\alpha)} \\
w_{M-1}^{(\alpha)} & w_{M-2}^{(\alpha)} & \cdots & w_2^{(\alpha)} & w_1^{(\alpha)}
\end{pmatrix}, \tag{3.8}
$$

$$
P_\alpha = \begin{pmatrix}
1-2b_2^\alpha & b_2^\alpha & & & \\
b_2^\alpha & (1-2b_2^\alpha) & b_2^\alpha & & \\
 & & \cdots & & \\
 & & b_2^\alpha & 1-2b_2^\alpha & b_2^\alpha \\
 & & & b_2^\alpha & 1-2b_2^\alpha
\end{pmatrix},
$$

and

$$
\begin{aligned}
H^n = &\begin{pmatrix} b_2^\alpha \\ 0 \\ \vdots \\ 0 \end{pmatrix}(U_0^n - U_0^{n+1}) + \frac{\tau}{2h^\alpha}\begin{pmatrix} K_1 w_2^{(\alpha)} + K_2 w_0^{(\alpha)} \\ K_1 w_3^{(\alpha)} \\ \vdots \\ K_1 w_{M-1}^{(\alpha)} \\ K_1 w_M^{(\alpha)} \end{pmatrix}(U_0^n + U_0^{n+1}) \\
&+ \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_2^\alpha \end{pmatrix}(U_M^n - U_M^{n+1}) + \frac{\tau}{2h^\alpha}\begin{pmatrix} K_2 w_M^{(\alpha)} \\ K_2 w_{M-1}^{(\alpha)} \\ \vdots \\ K_2 w_3^{(\alpha)} \\ K_1 w_0^{(\alpha)} + K_2 w_2^{(\alpha)} \end{pmatrix}(U_M^n + U_M^{n+1}).
\end{aligned} \tag{3.9}
$$

3.2. Stability and convergence analysis. In this subsection, we prove that the CN quasi-compact scheme has fourth-order accuracy in space and is unconditionally stable. Now we give some important lemmas to be used in the analyses.

LEMMA 3.1 ([7]). *Let H be a Toeplitz matrix with generating function $f \in C_{2\pi}$. Let $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ denote the smallest and largest eigenvalues of H, respectively. Then we have*

$$f_{\min} \leq \lambda_{\min}(H) \leq \lambda_{\max}(H) \leq f_{\max},$$

where $f_{\min}$ and $f_{\max}$ denote the minimum and maximum values of $f(x)$, respectively. In particular, if $f_{\max} \leq 0$ and $f_{\min} \neq f_{\max}$, then H is negative definite.

LEMMA 3.2 ([4]). *Let A be a positive semi-definite matrix. Then there exists a unique n-square positive semi-definite matrix B such that $B^2 = A$. Such a matrix B is called the square root of A, denoted by $A^{\frac{1}{2}}$.*

THEOREM 3.3. *The matrix $A_\alpha + A_\alpha^T$ is negative definite, and $B_\alpha + B_\alpha^T$ is also negative definite, where A_α is given by Equation (3.8) and $B_\alpha = \frac{\tau}{2h^\alpha}(K_1 A_\alpha + K_2 A_\alpha^T)$.*

In fact, the generating function [7] of $A + A^T$ satisfies

$$\begin{aligned}
&f(\alpha,x)\\
=&f_{A_\alpha}(x) + f_{A_\alpha^T}(x) = \left(\sum_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)x} + \sum_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)x}\right)\\
=&\mu_1\left(\sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-i(k-1)x} + \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{i(k-1)x}\right) + \mu_0\left(\sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-ikx} + \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{ikx}\right)\\
&+\mu_{-1}\left(\sum_{k=0}^{\infty} g_k^{(\alpha)} e^{-i(k+1)x} + \sum_{k=0}^{\infty} g_k^{(\alpha)} e^{i(k+1)x}\right)\\
=&\mu_1((1-e^{-ix})^\alpha e^{ix} + (1-e^{ix})^\alpha e^{-ix}) + \mu_0((1-e^{-ix})^\alpha + (1-e^{ix})^\alpha)\\
&+\mu_{-1}((1-e^{-ix})^\alpha e^{-ix} + (1-e^{ix})^\alpha e^{ix})\\
=&\left(2\sin\left(\frac{x}{2}\right)\right)^\alpha (\mu_1(e^{i(\frac{\alpha\pi}{2}-\frac{\alpha x}{2}+x)} + e^{-i(\frac{\alpha\pi}{2}-\frac{\alpha x}{2}+x)}) + \mu_0(e^{i(\frac{\alpha\pi}{2}-\frac{\alpha x}{2})} + e^{-i(\frac{\alpha\pi}{2}-\frac{\alpha x}{2})})\\
&+\mu_{-1}(e^{i(\frac{\alpha\pi}{2}-\frac{\alpha x}{2}-x)} + e^{-i(\frac{\alpha\pi}{2}-\frac{\alpha x}{2}-x)}))\\
=&2\left(2\sin\left(\frac{x}{2}\right)\right)^\alpha \left(\mu_1\cos\left(\frac{\alpha}{2}(\pi-x)+x\right) + \mu_0\cos\left(\frac{\alpha}{2}(\pi-x)\right) + \mu_{-1}\cos\left(\frac{\alpha}{2}(\pi-x)-x\right)\right)\\
=&2\left(2\sin\left(\frac{x}{2}\right)\right)^\alpha \left((\mu_1 + \mu_0\cos(x) + \mu_{-1}\cos(2x))\cos\left(\frac{\alpha}{2}(\pi-x)+x\right)\right.\\
&\left.+(\mu_0\sin(x) + \mu_{-1}\sin(2x))\sin\left(\frac{\alpha}{2}(\pi-x)+x\right)\right),
\end{aligned} \tag{3.10}$$

where $f_{A_\alpha}(x)$ and $f_{A_\alpha^T}(x)$ denote the generating functions of the matrices A_α and A_α^T, respectively. Next we check that $f(\alpha;x) \leq 0$ for $1 < \alpha < 2$. Denote

$$f(\alpha;x) = f_1(\alpha;x)\cdot(f_2(\alpha;x) + f_3(\alpha;x)),$$

where $f_1(\alpha;x) = 2\left(2\sin(\frac{x}{2})\right)^\alpha$, $f_2(\alpha;x) = (\mu_1 + \mu_0\cos(x) + \mu_{-1}\cos(2x))\cos(\frac{\alpha}{2}(\pi-x)+x)$, and $f_3(\alpha;x) = (\mu_0\sin(x) + \mu_{-1}\sin(2x))\sin(\frac{\alpha}{2}(\pi-x)+x)$. Since $f(\alpha;x)$ is a real-valued and even function, it's reasonable to consider its principal value on $[0,\pi]$. For $f_1(\alpha;x)$, there exists

$$f_1(\alpha;x) \geq 0.$$

When $\alpha \in [1,2]$, $\frac{\alpha}{2}(\pi - x) + x \in [\pi/2, \pi]$. It's easy to verify that $-\cos(\frac{\alpha}{2}(\pi - x) + x)$ and $(\mu_1 + \mu_0 \cos(x) + \mu_{-1}\cos(2x))$ increase with respect to α and are positive and $\sin(\frac{\alpha}{2}(\pi - x) + x)$ and $(\mu_0 \sin(x) + \mu_{-1}\sin(2x))$ decrease with respect to α and are positive. And they imply that $-f_2(\alpha;x)$ increases with respect to α and $f_3(\alpha;x)$ decreases with respect to α. So

$$-f_2(\alpha;x) \geq -f_2(1;x), \quad f_3(\alpha;x) \leq f_3(1;x).$$

Then we get

$$f_2(\alpha;x) + f_3(\alpha;x) \leq f_2(1;x) + f_3(1;x) = 0,$$

which implies $f(\alpha;x) = f_1(\alpha;x) \cdot (f_2(\alpha;x) + f_3(\alpha;x)) \leq 0$ for $1 < \alpha < 2$ on $[0,\pi]$. Therefore,

$$f(\alpha;x) \leq 0 \tag{3.11}$$

for $1 < \alpha < 2$ on $[-\pi,\pi]$. Then from Lemma 3.1, we know the matrix $A_\alpha + A_\alpha^T$ is negative definite. Rewriting $B_\alpha + B_\alpha^T$ as $(\frac{\tau}{2h^\alpha}(K_1(A_\alpha + A_\alpha^T) + K_2(A_\alpha^T + A_\alpha)))$, it can be clearly seen that $B_\alpha + B_\alpha^T$ is negative definite.

THEOREM 3.4. *The difference scheme (3.6) with $\alpha \in (1,2)$ is unconditionally stable.*

Proof. Define the round-off error as $\epsilon_j^n = U_j^n - \tilde{U}_j^n$, where $\tilde{U}_j^n$ is the exact solution of the discretized Equation (3.6) and U_j^n the numerical solution of the discretized Equation (3.6) obtained in finite precision arithmetic. Since $\tilde{U}_j^n$ satisfies the discretized equation exactly, round-off error ϵ_j^n must also satisfy the discretized equation [1]. Thus we obtain the following error equation

$$P_x \epsilon_j^{n+1} - \frac{K_1\tau}{2} {}_L D_h^\alpha \epsilon_j^{n+1} - \frac{K_2\tau}{2} {}_R D_h^\alpha \epsilon_j^{n+1} = P_x \epsilon_j^n + \frac{K_1\tau}{2} {}_L D_h^\alpha \epsilon_j^n + \frac{K_2\tau}{2} {}_R D_h^\alpha \epsilon_j^n. \tag{3.12}$$

Since the boundary conditions of the error equation (3.12) are $\epsilon_0^n = \epsilon_M^n = \epsilon_0^{n+1} = \epsilon_M^{n+1} = 0$, we zero extend the solution of the problem (3.12) to the whole real line R. So it's reasonable to replace the symbols $j+1$ and $M-j+1$ in error equation (3.12) with ∞. Now we have

$$\begin{aligned} &b_2^\alpha \epsilon_{j-1}^{n+1} + (1-2b_2^\alpha)\epsilon_j^{n+1} + b_2^\alpha \epsilon_{j+1}^{n+1} - \frac{K_1\tau}{2h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)} \epsilon_{j-k+1}^{n+1} - \frac{K_2\tau}{2h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)} \epsilon_{j+k-1}^{n+1} \\ =& b_2^\alpha \epsilon_{j-1}^{n} + (1-2b_2^\alpha)\epsilon_j^{n} + b_2^\alpha \epsilon_{j+1}^{n} + \frac{K_1\tau}{2h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)} \epsilon_{j-k+1}^{n} + \frac{K_2\tau}{2h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)} \epsilon_{j+k-1}^{n}. \end{aligned} \tag{3.13}$$

Let $\epsilon_j^n = v^n e^{ij\sigma}$ be the solution of Equation (3.13), where $i = \sqrt{-1}$, v^n is the amplitude at time level n, and $\sigma(=2\pi h/k)$ is the phase angle with wavelength k. We just need to prove that the amplification factor $v(\sigma,\alpha)$ satisfies the relation $|v(\sigma,\alpha)| \leq 1$ for all σ in $[-\pi,\pi]$. In fact, by substituting the expressions of $\epsilon_j^n (= v^n e^{ij\sigma})$ and $\epsilon_j^{n+1} (= v^{n+1} e^{ij\sigma})$ into Equation (3.13), we obtain the amplification factor of the CN quasi-compact scheme

$$v(\sigma,\alpha) = \frac{1 - 4b_2^\alpha \sin^2\frac{\sigma}{2} + \frac{K_1\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)\sigma} + \frac{K_2\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)\sigma}}{1 - 4b_2^\alpha \sin^2\frac{\sigma}{2} - \frac{K_1\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)} e^{-i(k-1)\sigma} - \frac{K_2\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)} e^{i(k-1)\sigma}}$$

$$=\frac{Q_1(\sigma,\alpha)+Q_2(\sigma,\alpha)}{Q_1(\sigma,\alpha)-Q_2(\sigma,\alpha)},$$

where $Q_1(\sigma,\alpha)=1-4b_2^\alpha\sin^2\frac{\sigma}{2}$ and
$Q_2(\sigma,\alpha)=\frac{K_1\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty}w_k^{(\alpha)}e^{-i(k-1)\sigma}+\frac{K_2\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty}w_k^{(\alpha)}e^{i(k-1)\sigma}$.

A straightforward calculation yields

$$\begin{aligned}
&Q_2(\sigma,\alpha)\\
=&\frac{K_1\tau}{2h^\alpha}\sum_{k=0}^{\infty}w_k^{(\alpha)}e^{-i(k-1)\sigma}+\frac{K_2\tau}{2h^\alpha}\sum_{k=0}^{\infty}w_k^{(\alpha)}e^{i(k-1)\sigma}\\
=&\frac{\mu_1\tau}{2h^\alpha}(K_1\sum_{k=0}^{\infty}g_k^{(\alpha)}e^{-i(k-1)\sigma}+K_2\sum_{k=0}^{\infty}g_k^{(\alpha)}e^{i(k-1)\sigma})+\frac{\mu_0\tau}{2h^\alpha}(K_1\sum_{k=0}^{\infty}g_k^{(\alpha)}e^{-i(k)\sigma}\\
&+K_2\sum_{k=0}^{\infty}g_k^{(\alpha)}e^{i(k)\sigma})+\frac{\mu_{-1}\tau}{2h^\alpha}(K_1\sum_{k=0}^{\infty}g_k^{(\alpha)}e^{-i(k+1)\sigma}+K_2\sum_{k=0}^{\infty}g_k^{(\alpha)}e^{i(k+1)\sigma})\\
=&\frac{\mu_1\tau}{2h^\alpha}(K_1(1-e^{-i\sigma})^\alpha e^{i\sigma}+K_2(1-e^{i\sigma})^\alpha e^{-i\sigma})+\frac{\mu_0\tau}{2h^\alpha}(K_1(1-e^{-i\sigma})^\alpha+K_2(1-e^{i\sigma})^\alpha)\\
&+\frac{\mu_{-1}\tau}{2h^\alpha}(K_1(1-e^{-i\sigma})^\alpha e^{-i\sigma}+K_2(1-e^{i\sigma})^\alpha e^{i\sigma})\\
=&\frac{\tau}{2h^\alpha}(2\sin(\frac{\sigma}{2}))^\alpha(\mu_1(K_1e^{i(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2}+\sigma)}+K_2e^{-i(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2}+\sigma)})+\mu_0(K_1e^{i(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2})}\\
&+K_2e^{-i(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2})})+\mu_{-1}(K_1e^{i(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2}-\sigma)}+K_2e^{-i(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2}-\sigma)})).
\end{aligned}\tag{3.14}$$

Since $Q_1(\sigma,\alpha)$ is real-valued,

$$|v(\sigma,\alpha)|=\frac{|Q_1+Q_2|}{|Q_1-Q_2|}=\sqrt{\frac{(Q_1+Re(Q_2))^2+(Im(Q_2))^2}{(Q_1-Re(Q_2))^2+(Im(Q_2))^2}},$$

where $Re(Q_2)$ and $Im(Q_2)$ are the real part and the imaginary part of Q_2, respectively. In order to prove that $|v(\sigma,\alpha)|\le 1$, we need to check

$$Q_1\cdot Re(Q_2)\le 0.$$

Note that $b_2^\alpha=(4+\alpha-\alpha^2)/24\le 1/6$ for any $\alpha\in[1,2]$. So $Q_1=1-4b_2^\alpha\sin^2(\frac{\sigma}{2})>0$. From Equation (3.14), we know

$$\begin{aligned}
Re(Q_2)\ =\ &\frac{(K_1+K_2)\tau}{2h^\alpha}(2\sin(\frac{\sigma}{2}))^\alpha(\mu_1\cos(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2}+\sigma)+\mu_0\cos(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2})\\
&+\mu_{-1}\cos(\frac{\alpha\pi}{2}-\frac{\alpha\sigma}{2}-\sigma))\\
=\ &\frac{(K_1+K_2)\tau}{4h^\alpha}f(\alpha;\sigma),
\end{aligned}$$

where $f(\alpha;\sigma)$ is defined by Equation (3.10). Together with $K_1+K_2>0$ and Equation (3.11), we obtain $Re(Q_2)\le 0$. Thus $Q_1\cdot Re(Q_2)\le 0$. Then $|v(\sigma,\alpha)|\le 1$. So the CN quasi-compact difference scheme is unconditionally stable. ∎

THEOREM 3.5. *Let $u(x_j,t_n)$ be the exact solution of the problem (3.1), and U_j^n the solution of the given finite difference scheme (3.6). Then we have*

$$\|u(x_j,t_n)-U_j^n\|\le C(\tau^2+h^4),$$

for all $1 \le n \le N$*, where* C *is a constant independent of* n*,* τ*, and* h*.*

Proof. Denote $\varepsilon_j^n = u(x_j,t_n) - U_j^n$ and $\varepsilon^n = (\varepsilon_1^n, \varepsilon_2^n, \cdots, \varepsilon_{M-1}^n)^T$. According to Equations (3.5)-(3.7), we obtain

$$(P_\alpha - B_\alpha)\varepsilon^{n+1} = (P_\alpha + B_\alpha)\varepsilon^n + \tau R^{n+1/2}, \tag{3.15}$$

where $R^{n+1/2} = (R_1^{n+1/2}, R_2^{n+1/2}, \cdots, R_{M-1}^{n+1/2})^T$. The eigenvalues of P_α are given by

$$\lambda(P_\alpha)_j = 1 - 4b_2^\alpha \sin^2(j\pi/M) > 0, \; j = 1, \cdots, M-1.$$

Since $b_2^\alpha \in (1/12, 1/6)$, we have $\lambda(P_\alpha)_j \in (1/3, 1)$. So the matrix P_α is invertible and positive definite, which means that P_α^{-1} exists and is also positive definite. According to Lemma 3.2, we know that $(P_\alpha^{-1})^{\frac{1}{2}}$ uniquely exists and is positive semi-definite. Multiplying $(P_\alpha^{-1})^{\frac{1}{2}}$ and taking the discrete L_2-norm on both sides of Equation (3.15) imply

$$\|((P_\alpha)^{\frac{1}{2}} - (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)\varepsilon^{n+1}\| \le \|((P_\alpha)^{\frac{1}{2}} + (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)\varepsilon^n\| + \tau \|(P_\alpha^{-1})^{\frac{1}{2}} R^{n+1/2}\|.$$

In view of Theorem 3.3, we know that $B_\alpha + B_\alpha^T$ is a negative definite matrix. Furthermore,

$$\begin{aligned} &((P_\alpha)^{\frac{1}{2}} - (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)^T((P_\alpha)^{\frac{1}{2}} - (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha) \\ =& P_\alpha - B_\alpha - B_\alpha^T + B_\alpha^T P_\alpha^{-1} B_\alpha \ge P_\alpha + B_\alpha^T P_\alpha^{-1} B_\alpha \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &((P_\alpha)^{\frac{1}{2}} + (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha)^T((P_\alpha)^{\frac{1}{2}} + (P_\alpha^{-1})^{\frac{1}{2}} B_\alpha) \\ =& P_\alpha + B_\alpha + B_\alpha^T + B_\alpha^T P_\alpha^{-1} B_\alpha \le P_\alpha + B_\alpha^T P_\alpha^{-1} B_\alpha, \end{aligned} \tag{3.17}$$

where the matrix $A \ge B$ means that $A - B$ is positive semi-definite. Denote

$$E^n = \sqrt{h(\varepsilon^n)^T (P_\alpha + B_\alpha^T P_\alpha^{-1} B_\alpha)\varepsilon^n}. \tag{3.18}$$

Since $B_\alpha^T P_\alpha^{-1} B_\alpha$ is positive definite, we know

$$E^n \ge \sqrt{h(\varepsilon^n)^T P_\alpha \varepsilon^n} \ge \sqrt{\lambda_{\min}(P_\alpha)} \|\varepsilon^n\|, \tag{3.19}$$

where $\lambda_{\min}(P_\alpha)$ is the minimum eigenvalue of matrix P_α. Together with Equations (3.16) and (3.17), we have

$$\begin{aligned} E^{n+1} - E^n \le& \tau \|(P_\alpha^{-1})^{\frac{1}{2}} R^{n+1/2}\| = \tau \sqrt{h(R^{n+1/2})^T (P_\alpha^{-1}) R^{n+1/2}} \\ \le& \tau \sqrt{\lambda_{\max}(P_\alpha^{-1})} \|R^{n+1/2}\| = \frac{\tau}{\sqrt{\lambda_{\min}(P_\alpha)}} \|R^{n+1/2}\|. \end{aligned} \tag{3.20}$$

Summing up Equation (3.20) from 0 to $n-1$ leads to

$$E^n \le \tau \sum_{k=0}^{n-1} \|(P_\alpha^{-1})^{\frac{1}{2}} R^{k+1/2}\| \le \frac{\tau}{\sqrt{\lambda_{\min}(P_\alpha)}} \sum_{k=0}^{n-1} \|R^{k+1/2}\|. \tag{3.21}$$

Combining Equations (3.19) and (3.21) and noticing that $|R_j^{k+1/2}| \le c(\tau^2 + h^4)$ for $1 \le j \le M-1$, we obtain

$$\|\varepsilon^n\| \le \frac{cT}{\lambda_{\min}(P_\alpha)} (\tau^2 + h^4) \le C(\tau^2 + h^4).$$

∎

4. Quasi-compact scheme for the two-dimensional space fractional diffusion equation

To discuss the quasi-compact scheme in the two-dimensional case, we consider the following space fractional diffusion equation:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = K_1^x\, {}_aD_x^{\alpha} u(x,t) + K_2^x\, {}_xD_b^{\alpha} u(x,t) \\ \qquad\qquad + K_1^y\, {}_cD_y^{\beta} u(x,t) + K_2^y\, {}_yD_d^{\beta} u(x,t) + f(x,t), \; (x,y,t)\in\Omega\times(0,T], \\ u(x,y,0) = u_0(x,y), \qquad (x,y)\in\Omega, \\ u(x,y,t) = \phi(x,y,t), \qquad (x,y,t)\in\partial\Omega\times(0,T], \end{cases} \tag{4.1}$$

where $\Omega=(a,b)\times(c,d)$ and the fractional orders $1<\alpha,\beta\le 2$. The diffusion coefficients K_j^x and K_j^y $(j=1,2)$ are non-negative and satisfy $(K_1^j)^2+(K_2^j)^2\ne 0$ $(j=x,y)$. The boundary function ϕ satisfies the following conditions: if $K_1^x\ne 0$, then $\phi(a,y,t)=0$, if $K_1^y\ne 0$, then $\phi(x,c,t)=0$, if $K_2^x\ne 0$, then $\phi(b,y,t)=0$, and if $K_2^y\ne 0$, then $\phi(x,d,t)=0$. We assume that Equation (4.1) has a unique and sufficiently smooth solution.

4.1. CN-CWSGD scheme. Let us denote $x_j=a+jh_x$, $y_s=c+sh_y$, and $t_n=n\tau$ for $0\le j\le M_x$, $0\le s\le M_y$, and $0\le n\le N$, where the space step size $h_x=(b-a)/M_x$, $h_y=(d--c)/M_y$ and time step size $\tau=T/N$. Here we take $u_{j,s}^n=u(x_j,y_s,t_n)$ and $f_{j,s}^{n+1/2}=f(x_j,y_s,t_{n+1/2})$. The maximum norm and the discrete L_2-norm are defined as

$$\|u\|_\infty = \max_{\substack{1\le j\le M_x-1,\\ 1\le s\le M_y-1}} |u_{j,s}|, \quad \|u\|^2 = h_x h_y \sum_{j=1}^{M_x-1}\sum_{s=1}^{M_y-1} u_{j,s}^2. \tag{4.2}$$

We still use the C-N technique for the time discretization of Equation (4.1) and get

$$\begin{aligned} \frac{u_{j,s}^{n+1}-u_{j,s}^n}{\tau} =& \frac{1}{2}\Big(K_1^x({}_aD_x^{\alpha}u)_{j,s}^n + K_1^x({}_aD_x^{\alpha}u)_{j,s}^{n+1} + K_2^x({}_xD_b^{\alpha}u)_{j,s}^n + K_2^x({}_xD_b^{\alpha}u)_{j,s}^{n+1} \\ &+K_1^y({}_cD_y^{\beta}u)_{j,s}^n + K_1^y({}_cD_y^{\beta}u)_{j,s}^{n+1} + K_2^y({}_yD_d^{\beta}u)_{j,s}^n + K_2^y({}_yD_d^{\beta}u)_{j,s}^{n+1}\Big) \\ &+ f_{j,s}^{n+1/2} + O(\tau^2). \end{aligned} \tag{4.3}$$

In space, the fourth-order quasi-compact discretizations are used to approximate the Riemann–Liouville fractional derivatives. This implies that

$$\begin{aligned} &(P_xP_y - \frac{K_1^x\tau}{2}P_y\,{}_LD_{h_x}^{\alpha} - \frac{K_2^x\tau}{2}P_y\,{}_RD_{h_x}^{\alpha} - \frac{K_1^y\tau}{2}P_x\,{}_LD_{h_y}^{\alpha} - \frac{K_2^y\tau}{2}P_x\,{}_RD_{h_y}^{\alpha})u_{j,s}^{n+1} \\ =&(P_xP_y + \frac{K_1^x\tau}{2}P_y\,{}_LD_{h_x}^{\alpha} + \frac{K_2^x\tau}{2}P_y\,{}_RD_{h_x}^{\alpha} + \frac{K_1^y\tau}{2}P_x\,{}_LD_{h_y}^{\alpha} + \frac{K_2^y\tau}{2}P_x\,{}_RD_{h_y}^{\alpha})u_{j,s}^{n} \\ &+ \tau P_xP_yf_{j,s}^{n+1/2} + \tau R_{j,s}^{n+1/2}, \end{aligned} \tag{4.4}$$

where

$$R_{j,s}^{n+1/2} \le C(\tau^2+h_x^4+h_y^4).$$

For convenience, we introduce the following discrete operator which works for two variables x,y,

$$\delta_x^{\alpha}u_{j,s} = K_1^x\,{}_LD_{h_x}^{\alpha}u_{j,s} + K_2^x\,{}_RD_{h_x}^{\alpha}u_{j,s}.$$

Then Equation (4.4) can be rewritten as

$$\begin{aligned}&(P_xP_y-\frac{\tau}{2}P_y\delta_x^\alpha-\frac{\tau}{2}P_x\delta_y^\beta)u_{j,s}^{n+1}\\&=(P_xP_y+\frac{\tau}{2}P_y\delta_x^\alpha+\frac{\tau}{2}P_x\delta_y^\beta)u_{j,s}^{n}+\tau P_xP_yf_{j,s}^{n+1/2}+\tau R_{j,s}^{n+1/2}.\end{aligned}\tag{4.5}$$

Adding the splitting term

$$\frac{\tau^2}{4}\delta_x^\alpha\delta_y^\beta(u_{j,s}^{n+1}-u_{j,s}^{n}),\tag{4.6}$$

which is equal to $\frac{\tau^3}{4}\Big((K_1^x\,{}_aD_x^\alpha+K_2^x\,{}_xD_b^\alpha)(K_1^y\,{}_cD_y^\beta+K_2^y\,{}_yD_d^\beta)u_t\Big)_{j,s}^{n+1/2}+\tau^3O(\tau^2+h_x^4+h_y^4)$, to Equation (4.5), we obtain

$$(P_x-\frac{\tau}{2}\delta_x^\alpha)(P_y-\frac{\tau}{2}\delta_y^\beta)u_{j,s}^{n+1}=(P_x+\frac{\tau}{2}\delta_x^\alpha)(P_y+\frac{\tau}{2}\delta_y^\beta)u_{j,s}^{n}+\tau P_xP_yf_{j,s}^{n+1/2}+\tau R_{j,s}^{n+1/2}.\tag{4.7}$$

Thus the quasi-compact finite difference scheme for Equation (4.1) is given by

$$(P_x-\frac{\tau}{2}\delta_x^\alpha)(P_y-\frac{\tau}{2}\delta_y^\beta)U_{j,s}^{n+1}=(P_x+\frac{\tau}{2}\delta_x^\alpha)(P_y+\frac{\tau}{2}\delta_y^\beta)U_{j,s}^{n}+\tau P_xP_yf_{j,s}^{n+1/2}.\tag{4.8}$$

As an efficient way to implement, we give the following equivalent schemes:

- quasi-compact Douglas–ADI scheme:

$$\begin{aligned}&(P_x-\frac{\tau}{2}\delta_x^\alpha)U_{j,s}^{*}=(P_xP_y+\frac{\tau}{2}P_y\delta_x^\alpha+\tau P_x\delta_y^\beta)U_{j,s}^{n}+\tau P_xP_yf_{j,s}^{n+1/2},\\&(P_y-\frac{\tau}{2}\delta_y^\beta)U_{j,s}^{n+1}=U_{j,s}^{*}-\frac{\tau}{2}\delta_y^\beta U_{j,s}^{n},\end{aligned}\tag{4.9}$$

- quasi-compact D'yakonov–ADI scheme:

$$\begin{aligned}&(P_x-\frac{\tau}{2}\delta_x^\alpha)U_{j,s}^{*}=(P_x+\frac{\tau}{2}\delta_x^\alpha)(P_y+\frac{\tau}{2}\delta_y^\beta)U_{j,s}^{n}+\tau P_xP_yf_{j,s}^{n+1/2},\\&(P_y-\frac{\tau}{2}\delta_y^\beta)U_{j,s}^{n+1}=U_{j,s}^{*}.\end{aligned}\tag{4.10}$$

4.2. Stability and convergence analysis. The following stability analysis and accuracy analysis indicate that the two-dimensional CN quasi-compact scheme has fourth-order accuracy in space and is unconditionally stable.

LEMMA 4.1 ([4]). *Let A, B be two positive semi-definite matrices, symbolized $A\geq 0$, $B\geq 0$. Then $A\otimes B\geq 0$.*

LEMMA 4.2 ([11]). *Let $A\in R^{n\times n}$ have eigenvalues $\{\tilde\rho_j\}_{j=1}^n$ and $B\in R^{m\times m}$ have eigenvalues $\{\rho_j\}_{j=1}^m$. Then the mn eigenvalues of $A\otimes B$ are*

$$\tilde\rho_1\rho_1,\cdots,\tilde\rho_1\rho_m,\tilde\rho_2\rho_1,\cdots,\tilde\rho_2\rho_m,\cdots,\tilde\rho_n\rho_1,\cdots,\tilde\rho_n\rho_m.$$

LEMMA 4.3 ([11]). *Let $A\in R^{m\times n}$, $B\in R^{r\times s}$, $C\in R^{n\times p}$, and $D\in R^{s\times t}$. Then*

$$(A\otimes B)(C\otimes D)=AC\otimes BD,$$

where $\otimes$ denotes the Kronecker product. Moreover, if $A,B\in R^{n\times n}$ and I is a unit matrix of order n, then the matrices $I\otimes A$ and $B\otimes I$ commute.

LEMMA 4.4 ([11]). *Let A be a $m\times n$ matrix and B a $p\times q$ matrix. We have that the transposition is distributive over the Kronecker product:*

$$(A\otimes B)^T = A^T\otimes B^T.$$

THEOREM 4.5. *For any $1<\alpha,\beta<2$, the finite different scheme (4.8) is unconditionally stable.*

Proof. Define the round-off error as $\epsilon_{j,s}^n = U_{j,s}^n - \tilde{U}_{j,s}^n$. The error equation is given by

$$(P_x - \frac{\tau}{2}\delta_x^\alpha)(P_y - \frac{\tau}{2}\delta_y^\beta)\epsilon_{j,s}^{n+1} = (P_x + \frac{\tau}{2}\delta_x^\alpha)(P_y + \frac{\tau}{2}\delta_y^\beta)\epsilon_{j,s}^n. \tag{4.11}$$

Since the boundary conditions of the above error equation are homogeneous, we zero extend the solution of the problem (4.11) to the whole real plane $R\times R$. It's reasonable to replace the symbols $j+1$ and $M-j+1$ in the error equation (4.11) with ∞. Now we have

$$(P_x - \frac{\tau}{2}\delta_x^{\alpha'})(P_y - \frac{\tau}{2}\delta_y^{\beta'})\epsilon_{j,s}^{n+1} = (P_x + \frac{\tau}{2}\delta_x^{\alpha'})(P_y + \frac{\tau}{2}\delta_y^{\beta'})\epsilon_{j,s}^n, \tag{4.12}$$

where

$$\delta_x^{\alpha'}\epsilon_{j,s} = \frac{K_1^x}{h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)}\epsilon_{j-k+1,s} + \frac{K_2^x}{h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)}\epsilon_{j+k-1,s},$$

which works for two variables x,y. Let $\epsilon_{j,s}^n = v^n e^{i(j\sigma_1+s\sigma_2)}$, where $i=\sqrt{-1}$, v^n is the amplitude at time level n, and $\sigma_1 = 2\pi h_x/k_x$, $\sigma_2 = 2\pi h_y/k_y$ are the phase angles with wavelength k_x and k_y, respectively. Next we just need to prove that the amplification factor $G(\sigma_1,\sigma_2) = v^{n+1}/v^n$ satisfies the relation $|G(\sigma_1,\sigma_2)|\le 1$ for all σ_1 and σ_2 in $[-\pi,\pi]$. In fact, substituting the expressions of $\epsilon_{j,s}^n$ and $\epsilon_{j,s}^{n+1}$ into Equation (4.12), we get the amplification factor

$$\begin{aligned}
G(\sigma_1,\sigma_2) &= \frac{(1-4b_2^\alpha\sin^2\frac{\sigma_1}{2} + \frac{K_1^x\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)}e^{-i(k-1)\sigma_1} + \frac{K_2^x\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)}e^{i(k-1)\sigma_1})}{(1-4b_2^\alpha\sin^2\frac{\sigma_1}{2} - \frac{K_1^x\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)}e^{-i(k-1)\sigma_1} - \frac{K_2^x\tau}{2h^\alpha}\sum\limits_{k=0}^{\infty} w_k^{(\alpha)}e^{i(k-1)\sigma_1})} \\
&\quad\cdot\frac{(1-4b_2^\beta\sin^2\frac{\sigma_2}{2} + \frac{K_1^y\tau}{2h^\beta}\sum\limits_{k=0}^{\infty} w_k^{(\beta)}e^{-i(k-1)\sigma_2} + \frac{K_2^y\tau}{2h^\beta}\sum\limits_{k=0}^{\infty} w_k^{(\beta)}e^{i(k-1)\sigma_2})}{(1-4b_2^\beta\sin^2\frac{\sigma_2}{2} - \frac{K_1^y\tau}{2h^\beta}\sum\limits_{k=0}^{\infty} w_k^{(\beta)}e^{-i(k-1)\sigma_2} - \frac{K_2^y\tau}{2h^\beta}\sum\limits_{k=0}^{\infty} w_k^{(\beta)}e^{i(k-1)\sigma_2})} \\
&= \frac{Q_1(\sigma_1,\alpha)+Q_2(\sigma_1,\alpha)}{Q_1(\sigma_1,\alpha)-Q_2(\sigma_1,\alpha)}\cdot\frac{Q_1(\sigma_2,\beta)+Q_2(\sigma_2,\beta)}{Q_1(\sigma_2,\beta)-Q_2(\sigma_2,\beta)} \\
&= v(\sigma_1,\alpha)\cdot v(\sigma_2,\beta),
\end{aligned}$$

where

$$Q_1(\sigma_1,\alpha) = 1-4b_2^\alpha\sin^2\frac{\sigma_1}{2}$$

and

$$Q_2(\sigma_1,\alpha) = \frac{K_1^x\tau}{2h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)}e^{-i(k-1)\sigma_1} + \frac{K_2^x\tau}{2h^\alpha}\sum_{k=0}^{\infty} w_k^{(\alpha)}e^{i(k-1)\sigma_1},$$

which work for two pairs of variables (σ_1,α) and (σ_2,β). According to the analysis of Theorem 3.4, we know that $|v(\sigma_1,\alpha)|\le 1$ and $|v(\sigma_2,\beta)|\le 1$ hold for any $\alpha,\beta\in(1,2)$. Then

$$|G(\sigma_1,\sigma_2)|=|v(\sigma_1,\alpha)|\cdot|v(\sigma_2,\beta)|\le 1.$$

So the CN quasi-compact scheme is unconditionally stable. □

THEOREM 4.6. *Let $u(x_j,y_s,t_n)$ be the exact solution of Equation (4.1), and let $U^n_{j,s}$ be the solution of the given finite difference scheme (4.8). Then we have*

$$\|u(x_j,y_s,t_n)-U^n_{j,s}\|\le C(\tau^2+h_x^4+h_y^4),$$

for all $1\le n\le N$, where C is a constant independent of τ, h_x, and h_y.

Proof. Denote $\varepsilon^n_{j,s}=u(x_j,y_s,t_n)-U^n_{j,s}$, and

$$P_{(\alpha)}=I_\beta\otimes P_\alpha,\quad P_{(\beta)}=P_\beta\otimes I_\alpha,$$

$$(P_{(\alpha)})^{\frac12}=I_\beta\otimes(P_\alpha)^{\frac12},\quad (P_{(\beta)})^{\frac12}=(P_\beta)^{\frac12}\otimes I_\alpha,$$

$$B_{(\alpha)}=\frac{K_1^x\tau}{2h_x^\alpha}I_\beta\otimes A_\alpha+\frac{K_2^x\tau}{2h_x^\alpha}I_\beta\otimes A_\alpha^T,\quad B_{(\beta)}=\frac{K_1^y\tau}{2h_y^\beta}A_\beta\otimes I_\alpha+\frac{K_2^y\tau}{2h_y^\beta}A_\beta^T\otimes I_\alpha, \tag{4.13}$$

where A_α and A_β are defined in Equation (3.8) corresponding to α and β. In view of Equations (4.7)–(4.8), we obtain

$$(P_{(\alpha)}-B_{(\alpha)})(P_{(\beta)}-B_{(\beta)})\varepsilon^{n+1}=(P_{(\alpha)}+B_{(\alpha)})(P_{(\beta)}+B_{(\beta)})\varepsilon^n+\tau R^{n+1/2}, \tag{4.14}$$

where

$$\varepsilon=(\varepsilon_{1,1},\varepsilon_{2,1},\cdots,\varepsilon_{M_x-1,1},\varepsilon_{1,2},\varepsilon_{2,2},\cdots,\varepsilon_{M_x-1,2},\varepsilon_{1,M_y-1},\varepsilon_{2,M_y-1},\cdots,\varepsilon_{M_x-1,M_y-1})^T.$$

Multiplying $(P^{-1}_{(\alpha)})^{\frac12}(P^{-1}_{(\beta)})^{\frac12}$ and taking the discrete L_2-norm on both sides of Equation (4.14) imply

$$\begin{aligned}&\|(P^{-1}_{(\alpha)})^{\frac12}(P^{-1}_{(\beta)})^{\frac12}(P_{(\alpha)}-B_{(\alpha)})(P_{(\beta)}-B_{(\beta)})\varepsilon^{n+1}\|\\ \le&\|(P^{-1}_{(\alpha)})^{\frac12}(P^{-1}_{(\beta)})^{\frac12}(P_{(\alpha)}+B_{(\alpha)})(P_{(\beta)}+B_{(\beta)})\varepsilon^{n}\|+\tau\|(P^{-1}_{(\alpha)})^{\frac12}(P^{-1}_{(\beta)})^{\frac12}R^{n+1/2}\|.\end{aligned} \tag{4.15}$$

Using lemmas 4.3 and 4.4, it is easy to check that the matrix $(P^{-1}_{(\beta)})^{\frac12}$ can commute with $(P^{-1}_{(\alpha)})^{\frac12}$ and $P_{(\alpha)}\pm B^T_{(\alpha)}$; i.e.,

$$(P^{-1}_{(\beta)})^{\frac12}(P^{-1}_{(\alpha)})^{\frac12}=(P^{-1}_{(\alpha)})^{\frac12}(P^{-1}_{(\beta)})^{\frac12}=(P^{-1}_{\beta})^{\frac12}\otimes(P^{-1}_{\alpha})^{\frac12},$$

$$(P^{-1}_{(\beta)})^{\frac12}(P_{(\alpha)}\pm B^T_{(\alpha)})=(P_{(\alpha)}\pm B^T_{(\alpha)})(P^{-1}_{(\beta)})^{\frac12}=(P^{-1}_{\beta})^{\frac12}\otimes\left(P_\alpha\pm\frac{K_1^x\tau}{2h_x^\alpha}A_\alpha^T\pm\frac{K_2^x\tau}{2h_x^\alpha}A_\alpha\right).$$

After some similar calculations, we also get that $P_{(\beta)}-B_{(\beta)}$ commutes with $P_{(\alpha)}-B_{(\alpha)}$, $(P^{-1}_{(\alpha)})^{\frac12}$, and $P_{(\alpha)}-B^T_{(\alpha)}$, and $P_{(\beta)}+B_{(\beta)}$ commutes with $P_{(\alpha)}+B_{(\alpha)}$, $(P^{-1}_{(\alpha)})^{\frac12}$, and $P_{(\alpha)}+B^T_{(\alpha)}$. In view of Theorem 3.3, we know that $B_\alpha+B_\alpha^T$ and $B_\beta+B_\beta^T$ are negative definite matrixes. Together with Lemma 4.2, it yields that $B_{(\alpha)}+B_{(\alpha)}^T$ and $B_{(\beta)}+B_{(\beta)}^T$ are also negative definite matrixes. Using Lemma 4.1, there exist

$$\begin{aligned}&((P_{(\alpha)}^{-1})^{\frac{1}{2}}(P_{(\beta)}^{-1})^{\frac{1}{2}}(P_{(\alpha)}-B_{(\alpha)})(P_{(\beta)}-B_{(\beta)}))^T(P_{(\alpha)}^{-1})^{\frac{1}{2}}(P_{(\beta)}^{-1})^{\frac{1}{2}}(P_{(\alpha)}-B_{(\alpha)})(P_{(\beta)}-B_{(\beta)})\\ \geq&(P_{(\beta)}+B_{(\beta)}^TP_{(\beta)}^{-1}B_{(\beta)})(P_{(\alpha)}+B_{(\alpha)}^TP_{(\alpha)}^{-1}B_{(\alpha)})+(B_{(\beta)}+B_{(\beta)}^T)(B_{(\alpha)}+B_{(\alpha)}^T)\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}&((P_{(\alpha)}^{-1})^{\frac{1}{2}}(P_{(\beta)}^{-1})^{\frac{1}{2}}(P_{(\alpha)}+B_{(\alpha)})(P_{(\beta)}+B_{(\beta)}))^T(P_{(\alpha)}^{-1})^{\frac{1}{2}}(P_{(\beta)}^{-1})^{\frac{1}{2}}(P_{(\alpha)}+B_{(\alpha)})(P_{(\beta)}+B_{(\beta)})\\ \leq&(P_{(\beta)}+B_{(\beta)}^TP_{(\beta)}^{-1}B_{(\beta)})(P_{(\alpha)}+B_{(\alpha)}^TP_{(\alpha)}^{-1}B_{(\alpha)})+(B_{(\beta)}+B_{(\beta)}^T)(B_{(\alpha)}+B_{(\alpha)}^T),\end{aligned} \tag{4.17}$$

where the matrix $A\geq B$ means that $A-B$ is positive semi-definite. Denoting

$$E^n=\sqrt{h(\varepsilon^n)^T((P_{(\beta)}+B_{(\beta)}^TP_{(\beta)}^{-1}B_{(\beta)})(P_{(\alpha)}+B_{(\alpha)}^TP_{(\alpha)}^{-1}B_{(\alpha)})+(B_{(\beta)}+B_{(\beta)}^T)(B_{(\alpha)}+B_{(\alpha)}^T))\varepsilon^n},$$

we have

$$E^n\geq\sqrt{h(\varepsilon^n)^T(P_{(\alpha)})(P_{(\beta)})\varepsilon^n}\geq\sqrt{\lambda_{\min}(P_\alpha)\lambda_{\min}(P_\beta)}||\varepsilon^n||, \tag{4.18}$$

where $\lambda_{\min}(P_\alpha)$ and $\lambda_{\min}(P_\beta)$ are the minimum eigenvalues of the matrices P_α and P_β, respectively. Together with Equations (4.16) and (4.17), we have

$$\begin{aligned}E^{n+1}\leq&E^0+\tau\sum_{k=0}^{n}\|(P_{(\alpha)}^{-1})^{\frac{1}{2}}(P_{(\beta)}^{-1})^{\frac{1}{2}}R^{n+1/2}\|\leq\tau\sum_{k=0}^{n}\sqrt{\lambda_{\max}(P_{(\alpha)}^{-1}P_{(\beta)}^{-1})}\|R^{n+1/2}\|\\ =&\frac{\tau}{\sqrt{\lambda_{\min}(P_\alpha)\lambda_{\min}(P_\beta)}}\sum_{k=0}^{n}\|R^{n+1/2}\|.\end{aligned}$$

Using Equation (4.18) and noticing that $|R_{j,s}^{k+1/2}|\leq c(\tau^2+h_x^4+h_y^4)$ for $1\leq j\leq M_x-1$ and $1\leq s\leq M_y-1$, we obtain

$$\|\varepsilon^n\|\leq\frac{cT}{\lambda_{\min}(P_\alpha)\lambda_{\min}(P_\beta)}(\tau^2+h_x^4+h_y^4)\leq C(\tau^2+h_x^4+h_y^4).$$

□

5. Extending quasi-compact discretizations and schemes to the tempered space fractional derivative and equation

This section focuses on developing the high-order quasi-compact schemes of the tempered fractional differential equation with Dirichlet boundary condition. We begin with the definitions of α-th-order left and right Riemann–Liouville tempered fractional derivatives.

DEFINITION 5.1 ([12]). *If the function $u(x)$ is defined in a finite interval $[a,b]$ and sufficiently regular, then for any $\lambda\geq 0$ the α-th-order left and right Riemann–Liouville tempered fractional derivatives are, respectively, defined as*

$${}_aD_x^{\alpha,\lambda}u(x)=e^{-\lambda x}\,{}_aD_x^{\alpha}(e^{\lambda x}u(x))=\frac{e^{-\lambda x}}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x(x-s)^{n-\alpha-1}e^{\lambda s}u(s)ds \tag{5.1}$$

and

$$ {}_xD_b^{\alpha,\lambda}u(x)=e^{\lambda x}\,{}_xD_b^{\alpha}(e^{-\lambda x}u(x))=\frac{(-1)^n e^{\lambda x}}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (s-x)^{n-\alpha-1}e^{-\lambda s}u(s)ds, \tag{5.2}$$

where $n-1<\alpha<n$. Moreover, if $\lambda=0$, then the derivatives ${}_aD_x^{\alpha,\lambda}u(x)$ and ${}_xD_b^{\alpha,\lambda}u(x)$ reduce to the derivatives ${}_aD_x^{\alpha}u(x)$ and ${}_xD_b^{\alpha}u(x)$ defined in Definition 2.1.

To get the stable scheme, we introduce a shifted Grünwald–Letnikov operator to approximate the left tempered Riemann–Liouville fractional derivative with first-order accuracy.

LEMMA 5.2 ([12]). *Let $1<\alpha<2$, $u\in C^{n+3}(R)$ such that $D^k u(x)\in L^1(R)$, $k=0,1,\cdots,n+3$. For any integer p and $\lambda\geq 0$, define the left shifted tempered Grünwald–Letnikov operator by*

$$\Delta_p^{\alpha,\lambda}u(x):=\frac{1}{h^\alpha}\sum_{k=0}^{\infty} g_k^{(\alpha)}e^{-(k-p)\lambda h}u(x-(k-p)h). \tag{5.3}$$

Then we have

$$\Delta_p^{\alpha,\lambda}u(x)={}_{-\infty}D_x^{\alpha,\lambda}u(x)+\sum_{l=1}^{n-1}a_{p,l}^{\alpha}\,{}_{-\infty}D_x^{\alpha+l,\lambda}u(x)h^l+O(h^n) \tag{5.4}$$

uniformly in $x\in R$, where the weights $a_{p,l}^{\alpha}$ are the same as in Lemma 2.2.

To approximate the right Riemann–Liouville tempered fractional derivative ${}_xD_{+\infty}^{\alpha,\lambda}u(x)$, the right shifted tempered Grünwald–Letnikov operator is defined as $\Lambda_p^{\alpha,\lambda}f(x):=\frac{1}{h^\alpha}\sum\limits_{k=0}^{\infty}g_k^{(\alpha)}e^{-(k-p)\lambda h}u(x+(k-p)h)$. If the function $u(x)$ is defined on the bounded interval $[a,b]$, then the shifted tempered Grünwald–Letnikov formulae approximating the tempered fractional derivative at the point x are written as

$$\begin{aligned}\tilde{\Delta}_p^{\alpha,\lambda}u(x)&=\frac{1}{h^\alpha}\sum_{k=0}^{[\frac{x-a}{h}]+p} g_k^{(\alpha)}e^{-(k-p)h\lambda}u(x-(k-p)h),\\ \tilde{\Lambda}_p^{\alpha,\lambda}u(x)&=\frac{1}{h^\alpha}\sum_{k=0}^{[\frac{b-x}{h}]+p} g_k^{(\alpha)}e^{-(k-p)h\lambda}u(x+(k-p)h).\end{aligned} \tag{5.5}$$

Next we establish some suitable high-order finite difference discretizations to approximate the tempered fractional derivative.

5.1. Quasi-compact discretizations to the tempered Riemann–Liouville space fractional derivative. Now from the Taylor's expansions of the shifted tempered Grünwald–Letnikov operator, similar to get the CWSGD operator given in Section 2, we derive the fourth and fifth-order quasi-compact difference operators for Riemann–Liouville tempered fractional derivative.

5.1.1. Fourth-order quasi-compact approximation to the tempered Riemann–Liouville fractional derivative.

THEOREM 5.3. *Let $u(x)\in C^7(R)$ and all the derivatives of $u(x)$ up to order 7 belong to $L_1(R)$. Then the following quasi-compact approximation has fourth-order accuracy, i.e.,*

$$P_x^{\lambda}\,{}_{-\infty}D_x^{\alpha,\lambda}u(x)=\mu_1\Delta_1^{\alpha,\lambda}u(x)+\mu_0\Delta_0^{\alpha,\lambda}u(x)+\mu_{-1}\Delta_{-1}^{\alpha,\lambda}u(x)+O(h^4), \tag{5.6}$$

where $P_x^\lambda u(x) = u(x) + h^2 b_2^\alpha e^{-\lambda x}\delta_x^2(e^{\lambda x}u(x))$ *and the coefficients* b_2^α, μ_1, μ_0 *and* μ_{-1} *are given by Equation (2.10).*

Note that, by Lemma 5.2, the following equation holds:

$$\begin{aligned}
&\mu_1\Delta_1^{\alpha,\lambda}u(x)+\mu_0\Delta_0^{\alpha,\lambda}u(x)+\mu_{-1}\Delta_{-1}^{\alpha,\lambda}u(x)\\
=&{}_{-\infty}D_x^{\alpha,\lambda}u(x)+b_2^\alpha\,{}_{-\infty}D_x^{\alpha+2,\lambda}u(x)h^2+O(h^4)\\
=&(1+h^2b_2^\alpha\,{}_{-\infty}D_x^{2,\lambda})\,{}_{-\infty}D_x^{\alpha,\lambda}u(x)+O(h^4)\\
=&(1+h^2b_2^\alpha e^{-\lambda x}\frac{\partial^2}{\partial x^2}e^{\lambda x})\,{}_{-\infty}D_x^{\alpha,\lambda}u(x)+O(h^4)\\
=&P_x^\lambda\,{}_{-\infty}D_x^{\alpha,\lambda}u(x)+O(h^4).
\end{aligned}\tag{5.7}$$

Then we get Equation (5.6). Since $\delta_x^2 u = \frac{\partial^2}{\partial x^2}u + O(h^2)$, we know that for any function u,

$$P_x^\lambda u = (1+h^2b_2^\alpha\,{}_{-\infty}D_x^{2,\lambda})u + O(h^4).$$

In a similar way, we obtain a quasi-compact approximation of the right Riemann–Liouville tempered fractional derivative:

$$P_x^\lambda\,{}_xD_{+\infty}^{\alpha,\lambda}u(x) = \mu_1\Lambda_1^{\alpha,\lambda}u(x_j)+\mu_0\Lambda_0^{\alpha,\lambda}u(x_j)+\mu_{-1}\Lambda_{-1}^{\alpha,\lambda}u(x_j)+O(h^4). \tag{5.8}$$

For $u(x)$ defined on a bounded interval, supposing its zero extension to R satisfies the assumptions of Theorem 5.3, the following approximations hold:

$$P_x\,{}_aD_x^{\alpha,\lambda}u(x) = \mu_1\tilde{\Delta}_1^{\alpha,\lambda}u(x)+\mu_0\tilde{\Delta}_0^{\alpha,\lambda}u(x)+\mu_{-1}\tilde{\Delta}_{-1}^{\alpha,\lambda}u(x)+O(h^4) \tag{5.9}$$

and

$$P_x\,{}_xD_b^{\alpha,\lambda}u(x) = \mu_1\tilde{\Lambda}_1^{\alpha,\lambda}u(x)+\mu_0\tilde{\Lambda}_0^{\alpha,\lambda}u(x)+\mu_{-1}\tilde{\Lambda}_{-1}^{\alpha,\lambda}u(x)+O(h^4). \tag{5.10}$$

Next we give an example to verify the efficiency and convergence order of the above statement.

EXAMPLE 5.4. Consider the steady state tempered fractional diffusion problem

$${}_0D_x^{\alpha,\lambda}u(x) = \frac{720e^{-\lambda x}x^{6-\alpha}}{\Gamma(7-\alpha)}, \quad x\in(0,1),$$

with the boundary conditions $u(0)=0$ and $u(1)=e^{-\lambda}$, and $\alpha\in(1,2)$. The exact solution is given by $u(x)=e^{-\lambda x}x^6$.

Let us denote by u and U the exact solution and approximate value, respectively. In Table 5.1, we show that the proposed approximation in this subsection has fourth-order accuracy in the L_∞-norm and the L_2-norm.

5.1.2. Fifth-order quasi-compact approximation to the tempered Riemann–Liouville fractional derivative.

THEOREM 5.5. *Let* $u(x)\in C^8(R)$. *Then the quasi-compact approximations corresponding to the left Riemann–Liouville tempered fractional derivative have fifth-order accuracy,*

$$P_x^{\lambda,5}\,{}_{-\infty}D_x^{\alpha,\lambda}u(x) = \mu_1\Delta_1^{\alpha,\lambda}u(x)+\mu_0\Delta_0^{\alpha,\lambda}u(x)+\mu_{-1}\Delta_{-1}^{\alpha,\lambda}u(x)+O(h^5), \tag{5.11}$$

α	h_x	$\|u-U\|_2$	rate	$\|u-U\|_\infty$	rate
1.1	1/8	$3.8735e-04$		$8.7474e-04$	
	1/16	$1.8576e-05$	4.3821	$4.6954e-05$	4.2195
	1/32	$1.0159e-06$	4.1926	$2.6950e-06$	4.1229
	1/64	$6.0438e-08$	4.0712	$1.6005e-07$	4.0737
	1/128	$3.6901e-09$	4.0337	$9.4537e-09$	4.0815
1.9	1/8	$6.2019e-05$		$8.8032e-05$	
	1/16	$3.8991e-06$	3.9915	$5.6382e-06$	3.9647
	1/32	$2.4425e-07$	3.9967	$3.5328e-07$	3.9964
	1/64	$1.5281e-08$	3.9985	$2.2104e-08$	3.9984
	1/128	$9.5548e-10$	3.9994	$1.3822e-09$	3.9993

TABLE 5.1. *Numerical errors and convergence rates in the L_∞-norm and the L_2-norm of the scheme (5.6) to solve Example 5.4, where U denotes the numerical solution, h_x is space step size, and $\lambda=1.5$.*

where the operator $P_x^{\lambda,5}u(x)=me^{-\lambda h}u(x-h)+u(x)+ne^{\lambda h}u(x+h)$ and the coefficientsm, n, μ_1, μ_0 and μ_{-1} satisfy Equation (2.16).

Similar to the discussions in Subsection 2.2, we show three equalities

$$\Delta_p^{\alpha,\lambda}u(x)={}_{-\infty}D_x^{\alpha,\lambda}u(x)+\sum_{l=1}^{4}a_{p,l}^{\alpha}\,{}_{-\infty}D_x^{\alpha+l,\lambda}u(x)h^l+O(h^5),\quad p=1,0,-1. \tag{5.12}$$

In view of the Taylor expansion, we know

$$\begin{aligned}{}_{-\infty}D_x^{\alpha}e^{\lambda(x-h)}u(x-h)&={}_{-\infty}D_x^{\alpha}e^{\lambda x}u(x)+(-1)^l\sum_{l=1}^{4}\frac{1}{l!}\,{}_{-\infty}D_x^{\alpha+l}e^{\lambda x}u(x)h^l+O(h^5),\\ {}_{-\infty}D_x^{\alpha}e^{\lambda(x+h)}u(x+h)&={}_{-\infty}D_x^{\alpha}e^{\lambda x}u(x)+\sum_{l=1}^{4}\frac{1}{l!}\,{}_{-\infty}D_x^{\alpha+l}e^{\lambda x}u(x)h^l+O(h^5).\end{aligned} \tag{5.13}$$

Since $e^{\lambda x}\,{}_{-\infty}D_x^{\alpha,\lambda}u(x)={}_{-\infty}D_x^{\alpha}e^{\lambda x}u(x)$, multiplying by $e^{-\lambda x}$ in the equations of (5.13) we obtain

$$\begin{aligned}e^{-\lambda h}\,{}_{-\infty}D_x^{\alpha,\lambda}u(x-h)&={}_{-\infty}D_x^{\alpha,\lambda}u(x)+(-1)^l\sum_{l=1}^{4}\frac{1}{l!}\,{}_{-\infty}D_x^{\alpha+l,\lambda}u(x)h^l+O(h^5),\\ e^{\lambda h}\,{}_{-\infty}D_x^{\alpha,\lambda}u(x+h)&={}_{-\infty}D_x^{\alpha,\lambda}u(x)+\sum_{l=1}^{4}\frac{1}{l!}\,{}_{-\infty}D_x^{\alpha+l,\lambda}u(x)h^l+O(h^5).\end{aligned} \tag{5.14}$$

So in order to get the fifth-order approximation, combining Equations (5.12) and (5.14), we just need to eliminate the lower-order terms corresponding to $h^l\,(l=1,2,3,4)$. Then we get Equation (5.11).

To show the efficiency of the proposed approximation in this subsection, we numerically solve Example 5.6 and present the numerical results in Table 5.2, where u and U denote the exact solution and approximate value, respectively. Obviously, the approximations have fifth-order accuracy which verify the theoretical analysis.

EXAMPLE 5.6. Here we also consider the steady state tempered fractional diffusion problem

$$ {}_0D_x^{\alpha,\lambda}u(x) = \frac{720e^{-\lambda x}x^{6-\alpha}}{\Gamma(7-\alpha)}, \quad x \in (0,1) $$

with the boundary conditions $u(0)=0$ and $u(1)=e^{-\lambda}$, and $\alpha \in (1,2)$. The exact solution is $u(x)=e^{-\lambda x}x^6$.

α	h_x	$\|u-U\|_2$	rate	$\|u-U\|_\infty$	rate
1.1	1/8	8.2144e−06		1.4011e−05	
	1/16	2.4016e−07	5.0961	4.1068e−07	5.0924
	1/32	7.3703e−09	5.0261	1.2489e−08	5.0392
	1/64	2.2851e−10	5.0114	3.8494e−10	5.0199
	1/128	7.1140e−12	5.0054	1.1945e−11	5.0101
1.5	1/8	3.1463e−06		5.3572e−06	
	1/16	9.7972e−08	5.0051	1.6423e−07	5.0276
	1/32	3.1300e−09	4.9681	5.1523e−09	4.9944
	1/64	9.9944e−11	4.9689	1.6239e−10	4.9876
	1/128	3.1783e−12	4.9748	5.1120e−12	4.9895

TABLE 5.2. *Numerical errors and convergence rates in the L_∞-norm and the L_2-norm of the scheme (5.11) to solve Example 5.6, where U denotes the numerical solution, h_x is space step size, and $\lambda=1.5$.*

5.2. Quasi-compact scheme for tempered space fractional diffusion equation. In this subsection, we present the numerical scheme of the variant of the space fractional diffusion equation whose space fractional derivatives are replaced by the tempered fractional derivatives

$$ \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = K_1\,{}_aD_x^{\alpha,\lambda}u(x,t) + K_2\,{}_xD_b^{\alpha,\lambda}u(x,t) + f(x,t), & (x,t) \in (a,b)\times(0,T], \\ u(x,0) = u_0(x), & x \in [a,b], \\ u(a,t) = \phi_a(t),\ u(b,t) = \phi_b(t), & t \in [0,T], \end{cases} \tag{5.15} $$

where $\lambda \geq 0$. Utilizing the C-N technique for the time discretization of (5.15) and fourth-order quasi-compact discretization in space direction, we get

$$ \begin{aligned} P_x^\lambda \frac{u_j^{n+1}-u_j^n}{\tau} = {}& \frac{K_1\tau}{2}\,{}_LD_h^{\alpha,\lambda}u_j^n + \frac{K_2\tau}{2}\,{}_RD_h^{\alpha,\lambda}u_j^n + \frac{K_1\tau}{2}\,{}_LD_h^{\alpha,\lambda}u_j^{n+1} + \frac{K_2\tau}{2}\,{}_RD_h^{\alpha,\lambda}u_j^{n+1} \\ & + P_x^\lambda f(x_j,t_{n+1/2}) + R_j^{n+1/2}, \end{aligned} \tag{5.16} $$

where

$$ {}_LD_h^{\alpha,\lambda}u_j^n =: \mu_1\tilde{\Delta}_1^{\alpha,\lambda}u_j^n + \mu_0\tilde{\Delta}_0^{\alpha,\lambda}u_j^n + \mu_{-1}\tilde{\Delta}_{-1}^{\alpha,\lambda}u_j^n = \frac{1}{h^\alpha}\sum_{k=0}^{j+1} w_k^{(\alpha,\lambda)}u_{j-k+1}^n, $$

$$ {}_RD_h^{\alpha,\lambda}u_j^n =: \mu_1\tilde{\Lambda}_1^{\alpha,\lambda}u_j^n + \mu_0\tilde{\Lambda}_0^{\alpha,\lambda}u_j^n + \mu_{-1}\tilde{\Lambda}_{-1}^{\alpha,\lambda}u_j^n = \frac{1}{h^\alpha}\sum_{k=0}^{M-j+1} w_k^{(\alpha,\lambda)}u_{j+k-1}^n, $$

the coefficients are $w_0^{(\alpha,\lambda)} = \mu_1 g_0^{(\alpha)} e^{\lambda h}$, $w_1^{(\alpha,\lambda)} = \mu_1 g_1^{(\alpha)} + \mu_0 g_0^{(\alpha)}$, and $w_k^{(\alpha,\lambda)} = (\mu_1 g_k^{(\alpha)} + \mu_0 g_{k-1}^{(\alpha)} + \mu_{-1} g_{k-2}^{(\alpha)}) e^{-(k-1)\lambda h}$, $k = 2, \cdots, M$, and $R_j^{n+1/2} \leq C(\tau^2 + h^4)$. Denoting by U_j^n the numerical approximation of u_j^n, we obtain the C-N quasi-compact scheme for (5.15)

$$\begin{aligned} &P_x^\lambda U_j^{n+1} - \frac{K_1\tau}{2} {}_L D_h^{\alpha,\lambda} U_j^{n+1} - \frac{K_2\tau}{2} {}_R D_h^{\alpha,\lambda} U_j^{n+1} \\ =&P_x^\lambda U_{j,s}^n + \frac{K_1\tau}{2} {}_L D_h^{\alpha,\lambda} U_j^n + \frac{K_2\tau}{2} {}_R D_h^{\alpha,\lambda} U_j^n + \tau P_x^\lambda f_j^{n+1/2}. \end{aligned} \tag{5.17}$$

For convenience, the approximation scheme (5.17) may be written in matrix form

$$(P_\alpha^\lambda - B^\lambda) U^{n+1} = (P_\alpha^\lambda + B^\lambda) U^n + \tau P_\alpha^\lambda F^n + H^\lambda, \tag{5.18}$$

where $(P_\alpha^\lambda)_{j,s} = (P_\alpha)_{j,s} e^{(s-j)\lambda h}$, $B^\lambda = \frac{\tau}{2h^\alpha}(K_1 A_\alpha^\lambda + K_2 (A_\alpha^\lambda)^T)$, $(A_\alpha^\lambda)_{j,s} = (A_\alpha)_{j,s} e^{(s-j)\lambda h}$, $U^n = (U_1^n, U_2^n, \cdots, U_{M-1}^n)^T$, and $F^n = (f_1^{n+1/2}, f_2^{n+1/2}, \cdots, f_{M-1}^{n+1/2})^T$.

Remark 5.7. Note that when taking $\lambda = 0$, the tempered fractional diffusion equation (5.15) reduces to the fractional diffusion equation (3.1) and its scheme (5.17) reduces to the scheme (3.6).

6. Numerical experiments

For the numerical schemes of the fractional diffusion equation, we present some numerical results in the one- and two-dimensional cases to verify the theoretical results including the convergence orders and unconditional stability. For the tempered fractional diffusion equation, numerical simulations are also performed which show the effectiveness of the proposed scheme, and the desired fourth-order convergence is also obtained.

Example 6.1. Consider the following tempered space fractional diffusion equation:

$$\frac{\partial u}{\partial t} = {}_0 D_x^{\alpha,\lambda} u(x) - e^{-t-\lambda x}\left(x^6 + \frac{720 x^{6-\alpha}}{\Gamma(7-\alpha)}\right), \quad (x,t) \in (0,1) \times (0,1], \tag{6.1}$$

with the boundary conditions $u(0,t) = 0$ and $u(1,t) = e^{-t-\lambda}$ and the initial value $u(x,0) = e^{-\lambda x} x^6$, $x \in [0,1]$. The exact solution is $u(x) = e^{-t-\lambda x} x^6$.

In Table 6.1, we show that the quasi-compact scheme (5.17) is fourth-order convergent in space.

Example 6.2. Consider the following space fractional diffusion equation:

$$\frac{\partial u}{\partial t} = {}_0 D_x^\alpha u(x) + {}_x D_1^\alpha u(x) + f(x,t), \quad (x,t) \in (0,1) \times (0,1]. \tag{6.2}$$

Then the source term is

$$\begin{aligned} f(x,t) = &- e^{-t}(x^5(1-x)^5 - \Gamma(11)(x^{10-\alpha} + (1-x)^{10-\alpha})/\Gamma(11-\alpha) \\ &+ 5\Gamma(10)(x^{9-\alpha} + (1-x)^{9-\alpha})/\Gamma(10-\alpha) - 10\Gamma(9)(x^{8-\alpha} + (1-x)^{8-\alpha})/\Gamma(9-\alpha) \\ &+ 10\Gamma(8)(x^{7-\alpha} + (1-x)^{7-\alpha})/\Gamma(8-\alpha) - 5\Gamma(7)(x^{6-\alpha} + (1-x)^{6-\alpha})/\Gamma(7-\alpha) \\ &+ \Gamma(6)(x^{5-\alpha} + (1-x)^{5-\alpha})/\Gamma(6-\alpha). \end{aligned}$$

The exact solution is given by $u(x) = e^{-t} x^5 (1-x)^5$. In the domain $t \in [0,1]$, the boundary conditions are $u(0,t) = 0$ and $u(1,t) = 0$. The initial value is $u(x,0) = x^5(1-x)^5$, $x \in [0,1]$.

		$\lambda=0$		$\lambda=1.5$	
α	M_X	$\|u-U\|_2$	rate	$\|u-U\|_2$	rate
1.1	8	$2.4321e-04$		$1.6011e-04$	
	16	$1.3090e-05$	4.2156	$9.1799e-06$	4.1245
	32	$7.4456e-07$	4.1360	$5.3102e-07$	4.1117
	64	$4.4692e-08$	4.0583	$3.1555e-08$	4.0728
	128	$2.7455e-09$	4.0249	$1.9171e-09$	4.0408
1.5	8	$1.2806e-04$		$7.5690e-05$	
	16	$8.0137e-06$	3.9982	$4.7019e-06$	4.0088
	32	$5.0273e-07$	3.9946	$2.9387e-07$	4.0000
	64	$3.1507e-08$	3.9960	$1.8396e-08$	3.9978
	128	$1.9724e-09$	3.9976	$1.1512e-09$	3.9981
1.9	8	$4.4604e-05$		$2.3601e-05$	
	16	$2.8032e-06$	3.9920	$1.4844e-06$	3.9909
	32	$1.7561e-07$	3.9967	$9.2998e-08$	3.9965
	64	$1.0987e-08$	3.9985	$5.8188e-09$	3.9984
	128	$6.8700e-10$	3.9993	$3.6385e-10$	3.9993

TABLE 6.1. *Numerical errors and convergence rates in the L_2-norm to Equation (6.1), approximated by the quasi-compact difference scheme (5.17) at $t=1$ with $\tau=h^2$.*

α	M_x	$\|u-U\|_2$	rate	$\|u-U\|_\infty$	rate
1.1	8	$9.4394e-07$		$1.4488e-06$	
	16	$7.7153e-08$	3.6129	$1.2492e-07$	3.5358
	32	$5.6349e-09$	3.7753	$9.1789e-09$	3.7665
	64	$3.8217e-10$	3.8821	$6.2304e-10$	3.8809
	128	$2.4920e-11$	3.9389	$4.0617e-11$	3.9392
1.5	8	$1.4931e-06$		$2.5326e-06$	
	16	$1.0619e-07$	3.8135	$1.7066e-07$	3.8915
	32	$7.2530e-09$	3.8720	$1.1354e-08$	3.9098
	64	$4.7498e-10$	3.9326	$7.2882e-10$	3.9615
	128	$3.0416e-11$	3.9650	$4.7293e-11$	3.9459
1.9	8	$1.5101e-06$		$2.6288e-06$	
	16	$8.5433e-08$	4.1437	$1.3980e-07$	4.2329
	32	$5.3511e-09$	3.9969	$8.3686e-09$	4.0622
	64	$3.3620e-10$	3.9925	$5.1288e-10$	4.0283
	128	$2.1078e-11$	3.9955	$3.2590e-11$	3.9761

TABLE 6.2. *Numerical errors and convergence rates in the L_∞-norm and the L_2-norm to Equation (6.2), approximated by the quasi-compact difference scheme (3.6) at $t=1$ with $\tau=h^2$.*

Table 6.2 shows that the quasi-compact scheme (3.6) to solve the one-dimensional two sided fractional diffusion equation is also fourth-order convergent.

EXAMPLE 6.3. The following two-dimensional two-sided fractional diffusion problem

$$\frac{\partial u(x,y,t)}{\partial t} = {}_0D_x^\alpha u(x,y,t) + {}_xD_1^\alpha u(x,y,t) + {}_0D_y^\beta u(x,y,t) + {}_yD_1^\beta u(x,y,t) + f(x,y,t), \tag{6.3}$$

is considered in the domain $\Omega=(0,1)^2$ and $t\in(0,1]$. The source term is

$$f(x,t) = -10^6 e^{-t}\left[x^5(1-x)^5 y^5(1-y)^5\right.$$

$$
\begin{aligned}
&-\left(\frac{\Gamma(11)}{\Gamma(11-\alpha)}(x^{10-\alpha}+(1-x)^{10-\alpha})+\frac{5\Gamma(10)}{\Gamma(10-\alpha)}(x^{9-\alpha}+(1-x)^{9-\alpha})\right.\\
&-\frac{10\Gamma(9)}{\Gamma(9-\alpha)}(x^{8-\alpha}+(1-x)^{8-\alpha})+\frac{10\Gamma(8)}{\Gamma(8-\alpha)}(x^{7-\alpha}+(1-x)^{7-\alpha})\\
&\left.-\frac{5\Gamma(7)}{\Gamma(7-\alpha)}(x^{6-\alpha}+(1-x)^{6-\alpha})+\frac{\Gamma(6)}{\Gamma(6-\alpha)}(x^{5-\alpha}+(1-x)^{5-\alpha})\right)y^5(1-y)^5\\
&-\left(\frac{\Gamma(11)}{\Gamma(11-\beta)}(y^{10-\beta}+(1-y)^{10-\beta})+\frac{5\Gamma(10)}{\Gamma(10-\beta)}(y^{9-\beta}+(1-y)^{9-\beta})\right.\\
&-\frac{10\Gamma(9)}{\Gamma(9-\beta)}(y^{8-\beta}+(1-y)^{8-\beta})+\frac{10\Gamma(8)}{\Gamma(8-\beta)}(x^{7-\beta}+(1-x)^{7-\beta})\\
&\left.\left.-\frac{5\Gamma(7)}{\Gamma(7-\beta)}(y^{6-\beta}+(1-y)^{6-\beta})+\frac{\Gamma(6)}{\Gamma(6-\beta)}(y^{5-\beta}+(1-y)^{5-\beta})\right)x^5(1-x)^5\right].
\end{aligned}
$$

The exact solution is given by $u(x)=10^6e^{-t}x^5(1-x)^5y^5(1-y)^5$. The boundary condition is $u(x,y,t)=0$ with $(x,y)\in\partial\Omega$ and $t\in[0,1]$. The initial value is $u(x,y,0)=10^6x^5(1-x)^5y^5(1-y)^5$ with $(x,y)\in[0,1]^2$.

In Table 6.3, we present the numerical errors $\|u-U\|_2$ and the corresponding convergence orders with space step size $h_x=h_y$, where U is the solution of the quasi-compact difference scheme (4.9) or (4.10). It can be noted that the schemes are fourth-order convergent, which is in agreement with the theoretical convergence analysis.

		$(\alpha,\beta)=(1.1,1.5)$		$(\alpha,\beta)=(1.4,1.9)$	
	M_x	$\|u-U\|_2$	rate	$\|u-U\|_2$	rate
D'yakonov	8	$7.2903e-04$		$8.4729e-04$	
	16	$5.3915e-05$	3.7572	$5.7210e-05$	3.8885
	32	$3.7385e-06$	3.8502	$3.8200e-06$	3.9046
	64	$2.4685e-07$	3.9207	$2.4748e-07$	3.9482
	128	$1.5880e-08$	3.9584	$1.5763e-08$	3.9727
Douglas	8	$7.2903e-04$		$8.4729e-04$	
	16	$5.3915e-05$	3.7572	$5.7210e-05$	3.8885
	32	$3.7385e-06$	3.8502	$3.8200e-06$	3.9046
	64	$2.4685e-07$	3.9207	$2.4748e-07$	3.9482
	128	$1.5880e-08$	3.9584	$1.5763e-08$	3.9727

TABLE 6.3. *Numerical errors and convergence rates in the L_2-norm to Equation (6.3), approximated by the quasi-compact difference schemes (4.9) and (4.10), respectively, at $t=1$ with $\tau=h_x^2=h_y^2$.*

7. Conclusions

The continuous time random walk (CTRW) model is the basic stochastic process in statistical physics. The CTRW model characterizes Lévy flight if the first moment of the distribution of the waiting time is finite and the jump length obeys the power law distribution and its second moment is infinite. The corresponding Fokker–Planck equation of the process is the space fractional diffusion equation. Sometimes because of the limit of space size, the power law distribution of the jump length has to be tempered. The Fokker–Planck equation of the new stochastic process is the tempered space fractional diffusion equation. This paper provides the basic strategy for deriving the quasi-compact high-order discretizations of the space fractional derivative and the tempered space fractional derivative. As concrete examples, fourth-order discretizations are discussed in detail and applied to solve the (tempered) space fractional diffusion equation, and extensive numerical simulations confirm the effectiveness of the provided schemes. In fact, strict numerical stability and convergence analysis are also performed for the one- and two-dimensional space fractional diffusion equations.

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