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GLOBAL EXISTENCE FOR VISCOELASTIC FLUIDS WITH INFINITE WEISSENBERG NUMBER*

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Abstract. In this article, we show that the recently studied compressible and incompressible models for viscoelastic fluids with infinite Weissenberg number can be well regarded as specific examples of general hyperbolic-parabolic systems studied by Shizuta and Kawashima. It will be seen that two physically motivated compatibility conditions compensate the breaking of the Kawashima condition. Thus, the global existences of classical small solutions near equilibrium can be easily proved by following the general framework.

Keywords. viscoelastic fluids; global existence; Kawashima condition; compatibility conditions.

AMS subject classifications. 35A01; 35Q35; 76A10.

1. Introduction

Consider both the incompressible model

$$\nabla\cdot\mathbf{v}=0, \tag{1.1a}$$

$$\mathbf{v}_t+\mathbf{v}\cdot\nabla\mathbf{v}+\nabla p=\nu\Delta\mathbf{v}+\nabla\cdot(FF^T), \tag{1.1b}$$

$$F_t+\mathbf{v}\cdot\nabla F=\nabla\mathbf{v}F \tag{1.1c}$$

and compressible model

$$\rho_t+\nabla\cdot(\rho\mathbf{v})=0, \tag{1.2a}$$

$$(\rho\mathbf{v})_t+\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})+\nabla p=\nabla\cdot(\rho FF^T)+\mu\Delta\mathbf{v}+\mu'\nabla\nabla\cdot\mathbf{v}, \tag{1.2b}$$

$$(\rho F)_t+\nabla\cdot(F\otimes\rho\mathbf{v})=(\nabla\mathbf{v})\rho F \tag{1.2c}$$

for viscoelastic fluids with infinite Weissenberg number in the whole space $\mathbb{R}^d$ or on the torus $\mathbb{T}^d$ [10,12]. Here $\rho,\mathbf{v},p=p(\rho),F=[F_{ij}]_{d\times d}$ are the respective density, velocity, pressure and deformation tensor of the fluid, μ and μ' are the viscosity and second viscosity, and ν is the kinematic viscosity. The superscript T denotes for transpose of matrices or vectors.

By its definition [2,10], the deformation tensor F automatically satisfies the compatibility conditions

$$\nabla\cdot(\rho F^T)=0, \tag{1.3}$$

$$F_{lk}\partial_{x_l}F_{ij}=F_{lj}\partial_{x_l}F_{ik}. \tag{1.4}$$

Here the summation over repeated indices will always be understood. Mathematically, it was also shown in [6,10,12,18] that the smooth solutions to Equations (1.1) and (1.2) satisfy these two conditions if so do the initial data. It will be seen that these two conditions are crucial for the analysis of the viscoelastic models above.

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In this paper, we show that both the compressible and incompressible viscoelastic fluid models together with the compatibility conditions can be well regarded as specific examples of hyperbolic-parabolic systems studied by Shizuta and Kawashima [19]. Thus, the local and global existences of classical small solutions near equilibrium can be easily proved by following the general framework [9]. As an example, we state the following global existence theorem, where $U=(\mathbf{v},F)$ for the incompressible model, $U=(\rho,\mathbf{v},F)$ for the compressible model and the equilibrium point U_e is defined with $\rho=\rho_e>0, \mathbf{v}=0$ and $F=I_{d^2}$ (I_k: the unit matrix of order k).

THEOREM 1.1. *Let $s>\frac{d}{2}+1$ be a positive integer. Suppose $U_0-U_e\in H^s$, $\|U_0-U_e\|_{H^s}$ is sufficiently small, and U_0 satisfies the compatibility conditions* (1.3) *and* (1.4). *Then there exists a unique global solution $U=U(x,t)$, to the system* (1.1) *or* (1.2) *with U_0 as initial data, satisfying*

$$U-U_e\in C([0,+\infty),H^s)\cap L^2([0,+\infty),H^s), \quad \mathbf{v}\in L^2([0,+\infty),H^{s+1}),$$

$$\|U(T)-U_e\|_{H^s}^2+\int_0^T \left[\|\nabla\mathbf{v}(t)\|_{H^s}^2+\|\nabla U(t)\|_{H^{s-1}}^2\right]dt\le C\|U_0-U_e\|_{H^s}^2.$$

Actually, the conclusions in this theorem were already established in [12] for the 2-dimensional incompressible model, in [1,10] for the 3-dimensional incompressible model, and in [8] for the compressible model. Moreover, similar conclusions in critical spaces can be found in [6, 18] for the compressible model. Many other mathematical results on the systems (1.1) and (1.2) have also been obtained. The interested reader can consult [3–5, 7, 11, 13, 14, 16, 17, 21] for details.

The proofs of Theorem 1.1 in above works rely heavily on the specific structure of the viscoelastic models and involve subtle estimates. Here we will put the models into Kawashima's general framework and then give a different proof of the theorem. Precisely, we will construct strictly convex entropies for systems (1.1) and (1.2) and show that they are symmetrizable hyperbolic under the compatibility condition (1.3) when neglecting the viscosity terms. On the other hand, it will be seen that the viscoelastic models do not satisfy the Kawashima condition [19]. However, we can exploit the compatibility conditions (1.3) and (1.4) to obtain some estimates enough for the global existence. Having these, the theorem can be easily proved with the standard arguments [9, 20].

Additionally, we show that the compatibility condition (1.4) and the well-known condition

$$\rho\det F=1, \tag{1.5}$$

in continuum mechanics [2] imply the condition (1.3). Furthermore, we show that the condition (1.5) together with (1.3) can also compensate the breaking of the Kawashima condition for the 2-dimensional case but not for higher dimensions. This interprets why the compatibility condition (1.4) was not involved in [12] for the 2-dimensional model.

The paper is organized as follows. In Section 2 we show that the models (1.1) and (1.2) allow convex entropies and are symmetrizable hyperbolic under the compatibility condition (1.3) when neglecting the viscosity terms. In Section 3 we show that the viscoelastic models do not satisfy the Kawashima condition but the corresponding estimates based on the Kawahsima condition can still be obtained under the compatibility conditions (1.3) and (1.4). A proof of Theorem 1.1 is given in the last section.

2. Entropy and symmetrizability

In this section, we show that the models (1.1) and (1.2) allow convex entropies and are symmetrizable hyperbolic under the compatibility condition (1.3) when neglecting the viscosity terms. We start by defining

$$S(\mathbf{v},F)=\frac{1}{2}|\mathbf{v}|^2+\frac{1}{2}|F|^2 \tag{2.1}$$

for the incompressible model (1.1) and

$$S(\rho,\rho\mathbf{v},\rho F)=\rho\int_{\rho_0}^{\rho}\frac{p(\zeta)}{\zeta^2}d\zeta+\frac{1}{2}\rho|\mathbf{v}|^2+\frac{1}{2}\rho|F|^2 \tag{2.2}$$

for the compressible model (1.2) with $\rho_0>0$ a possible density value. Obviously, the first function in (2.1) is strictly convex with respect to its arguments. Moreover, it is not difficult to see that the second one (2.2) is also strictly convex if $p=p(\rho)$ is increasing for $\rho>0$.

To show that the second function is an entropy for the compressible model (1.2), we firstly notice that it satisfies

$$p=\rho S_\rho+\rho\mathbf{v}\cdot S_{\rho\mathbf{v}}+\rho F:S_{\rho F}-S.$$

Here and below we use the convention $A:B=\sum_{ij}a_{ij}b_{ij}$ for contraction of two tensors A and B. Then we set

$$\sigma=S_{\rho F}\rho F^T+\mu\nabla\mathbf{v}+\mu'\nabla\cdot\mathbf{v}I_{d^2}$$

and compute

$$\begin{aligned}
S_t&=S_\rho\rho_t+S_{\rho\mathbf{v}}\cdot(\rho\mathbf{v})_t+S_{\rho F}:(\rho F)_t\\
&=-S_\rho\nabla\cdot(\rho\mathbf{v})-S_{\rho\mathbf{v}}\cdot[\nabla\cdot(\rho\mathbf{v}\otimes\mathbf{v})+\nabla p]+S_{\rho\mathbf{v}}\cdot\nabla\cdot\sigma\\
&\quad-S_{\rho F}:\nabla\cdot(\rho F\otimes\mathbf{v})+S_{\rho F}:(\nabla\mathbf{v})\rho F\\
&=-\nabla\cdot(S\mathbf{v}+p\mathbf{v}+\sigma^T\cdot\mathbf{v})+(p+S-\rho S_\rho-\rho\mathbf{v}\cdot S_{\rho\mathbf{v}}-\rho F:S_{\rho F})\nabla\cdot\mathbf{v}\\
&\quad-(\sigma-S_{\rho F}\rho F^T):\nabla\mathbf{v}\\
&=-\nabla\cdot(S\mathbf{v}+p\mathbf{v}+\sigma^T\cdot\mathbf{v})-\mu|\nabla\mathbf{v}|^2-\mu'|\nabla\cdot\mathbf{v}|^2.
\end{aligned} \tag{2.3}$$

Namely, $S=S(W)$ with $W=(\rho,\rho\mathbf{v},\rho F)$ for the compressible model (1.2) satisfies the balance equation

$$S_t(W)=-\nabla\cdot J(W)+\sigma^S \tag{2.4}$$

with entropy flux $J(W)\equiv S\mathbf{v}+p\mathbf{v}+\sigma^T\cdot\mathbf{v}$ and entropy production rate $\sigma^S\equiv-\mu|\nabla\mathbf{v}|^2-\mu'|\nabla\cdot\mathbf{v}|^2$. The non-positiveness of the entropy production rate corresponds to the second law of thermodynamics.

Similarly, for the incompressible model we have

$$S_t=-\nabla\cdot(S\mathbf{v}+p\mathbf{v}+\sigma^T\cdot\mathbf{v})-\nu|\nabla\mathbf{v}|^2. \tag{2.5}$$

Furthermore, under the compatibility condition (1.3), the incompressible model (1.1) in its component form can be written as

$$\begin{pmatrix} v_i \\ F_{kl}\end{pmatrix}_t+\begin{pmatrix} v_j\delta_{ii'} & -F_{jl'}\delta_{ik'}\\ -F_{jl}\delta_{ki'} & v_j\delta_{kk'}\delta_{ll'}\end{pmatrix}\begin{pmatrix} v_{i'}\\ F_{k'l'}\end{pmatrix}_{x_j}+\begin{pmatrix} p\\ 0\end{pmatrix}_{x_i}=\begin{pmatrix}\nu\Delta v_i\\ 0\end{pmatrix}$$

and the compressible model (1.2) can be written as

$$\begin{pmatrix} \rho \\ v_i \\ F_{kl} \end{pmatrix}_t + \begin{pmatrix} v_j & \rho\delta_{i'j} & 0 \\ \frac{p_\rho}{\rho}\delta_{ij} & v_j\delta_{ii'} & -F_{jl'}\delta_{ik'} \\ 0 & -F_{jl}\delta_{ki'} & v_j\delta_{kk'}\delta_{ll'} \end{pmatrix} \begin{pmatrix} \rho \\ v_{i'} \\ F_{k'l'} \end{pmatrix}_{x_j} = \begin{pmatrix} 0 \\ \frac{\mu}{\rho}\Delta v_i + \frac{\mu'}{\rho}\partial_{x_i}\partial_{x_j}v_j \\ 0 \end{pmatrix}.$$

They can be rewritten into the unified form

$$U_t + \sum_{j=1}^{d} A_j(U)U_{x_j} = Q[U] \tag{2.6}$$

with

$$Q[U] = \begin{pmatrix} \nu\Delta\mathbf{v} - \nabla p \\ 0_{d^2} \end{pmatrix},$$

$$Q[U] = \frac{1}{\rho}\begin{pmatrix} 0 \\ \mu\Delta\mathbf{v} + \mu'\nabla\nabla\cdot\mathbf{v} \\ 0_{d^2} \end{pmatrix}$$

for models (1.1) and (1.2), respectively. Referring to the component forms, $A_j(U)$ for the incompressible model is obviously symmetric. On the other hand, we take

$$A_0(U) = \begin{pmatrix} \frac{p_\rho}{\rho^2} & 0 & 0 \\ 0 & I_d & 0 \\ 0 & 0 & I_{d^2} \end{pmatrix}, \tag{2.7}$$

which is symmetric and is positive definite if $p_\rho > 0$. Clearly, this $A_0 = A_0(U)$ symmetrizes the compressible model, that is, $A_0(U)A_j(U)$ is symmetric.

Remark 2.1. Under the compatibility condition (1.3), the F-equation in model (1.2) can be rewritten in the conservative form

$$(\rho F)_t + \nabla\cdot(F\otimes\rho\mathbf{v}) = \nabla\cdot(\rho\mathbf{v}\otimes F^T). \tag{2.8}$$

Thus, the viscoelastic models are of the conservation form with strictly convex entropies.

3. Kawashima condition

In this section, we show that the hyperbolic-parabolic system (2.6) does not satisfy the Kawashima condition at equilibrium U_e where $\rho = \rho_e, \mathbf{v} = 0$ and $F = I_{d^2}$. However, the corresponding estimates based on the Kawashima condition can be obtained under the compatibility conditions (1.3) and (1.4).

To check the Kawashima condition [19], we consider the linearized form of the hyperbolic-parabolic system (2.6):

$$U_t + \sum_{j=1}^{d} A_j(U_e)U_{x_j} - \sum_{j=1}^{d}\sum_{k=1}^{d} D_{jk}(U_e)U_{x_jx_k} = 0. \tag{3.1}$$

Here

$$D_{jk}(U) = \begin{pmatrix} \nu\delta_{jk}\delta_{ii'} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad (1.1),$$

$$D_{jk}(U)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu}{\rho}\delta_{jk}\delta_{ii'}+\frac{\mu'}{\rho}\delta_{ij}\delta_{ki'} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for} \quad (1.2).$$

Let $\xi=(\xi_1,\xi_2,\cdots,\xi_d)\in S^{d-1}$ (the unit sphere). Direct calculations show that the eigenvalues of the symbol matrix

$$\left[i\sum_{j=1}^{d}\xi_j A_j(U_e)+\sum_{j=1}^{d}\sum_{k=1}^{d}\xi_j\xi_k D_{jk}(U_e) \right] \tag{3.2}$$

are either zero or with negative real parts (this fact requires $d>1$ for the incompressible model). Therefore, the hyperbolic-parabolic system (2.6) does not satisfy the Kawashima condition at the equilibrium state U_e, for it says that the symbol matrix has only eigenvalues with negative real parts. In addition, it is not difficult to see that the eigenvectors $\hat{U}=(\hat{\rho},\hat{\mathbf{v}},\hat{F})$ associated with the zero eigenvalue satisfy

$$\hat{\mathbf{v}}=0, \quad \xi_i\frac{p_\rho(\rho_e)}{\rho_e}\hat{\rho}-\xi_{l'}\hat{F}_{il'}=0. \tag{3.3}$$

On the other hand, we define linearized differential operators

$$\begin{aligned} \mathcal{C}_1(U)=&-\nabla\rho-\rho_e\nabla\cdot F^T, \\ [\mathcal{C}_2(U)]_{kmj}=&\partial_{x_m}F_{kj}-\partial_{x_j}F_{km} \end{aligned} \tag{3.4}$$

corresponding to the compatibility conditions (1.3) and (1.4), respectively. Suppose the zero eigenvector $\hat{U}$ is in the kernels of the symbol matrices of the operators $\mathcal{C}_1(U)$ and $\mathcal{C}_2(U)$. Then we have

$$\xi_i\hat{\rho}+\rho_e\xi_j\hat{F}_{ji}=0, \tag{3.5}$$

$$\xi_m\hat{F}_{kj}-\xi_j\hat{F}_{km}=0. \tag{3.6}$$

Multiplying Equations (3.3) with ξ_i, taking the summation over i and using Equation (3.5), we deduce that

$$|\xi|^2\frac{p_\rho(\rho_e)}{\rho_e}\hat{\rho}=\xi_i\xi_{l'}\hat{F}_{il'}=-\xi_{l'}\frac{\xi_{l'}}{\rho_e}\hat{\rho}=-\frac{|\xi|^2}{\rho_e}\hat{\rho}$$

and thereby $\hat{\rho}=0$. Plugging this into Equations (3.3) gives $\xi_{l'}\hat{F}_{il'}=0$. Multiplying this $\hat{F}_{im}$, taking the summation over i and using Equation (3.6) leads to

$$0=\xi_{l'}\hat{F}_{il'}\hat{F}_{im}=\xi_m\hat{F}_{il'}\hat{F}_{il'}.$$

From this equality it follows that $\hat{F}=0$ due to $\xi\neq 0$. Thus we have shown that the zero eigenvectors are excluded with the linearized operators $\mathcal{C}_1(U)$ and $\mathcal{C}_2(U)$. With this fact, we expect that the breaking of the Kawashima condition can be compensated with the compatibility conditions (1.3) and (1.4).

In [19], it was shown that the Kawashima condition is equivalent to the existence of a skew-symmetric matrix $K=K(U_e,\xi)$ such that the matrix

$$K\sum_j\xi_jA_j(U_e)-\sum_j\xi_jA_j^T(U_e)K+\sum_{j,k}\xi_j\xi_kD_{jk}(U_e)$$

is positive definite for any $\xi \in S^{d-1}$. Our above checking indicates the impossibility to find such a skew-symmetric matrix. However, we can use the compatibility conditions (1.3) and (1.4) to formulate the following lemma.

LEMMA 3.1. *For the compressible model set*

$$K_j = diag\left(\frac{p'(\rho_e)}{\rho_e^2}, -I_d, I_{d^2}\right) A_j(U_e),$$

which is a skew-symmetric matrix. Then there are positive constants η *and* C_S *such that for any smooth function* $U = U(x)$

$$\begin{aligned}&\sum_{j,m=1}^{d} [(\eta K_m A_j(U_e) U_{x_j}, U_{x_m}) + (D_{mj}(U_e) U_{x_j}, U_{x_m})]\\ &\geq C_S \|\nabla U\|_{L^2}^2 + \eta \frac{2p'(\rho_e)}{\rho_e^2} (\mathcal{C}_1(U - U_e), \nabla \rho) + \eta (\mathcal{C}_2(U - U_e), \nabla F).\end{aligned} \tag{3.7}$$

Similarly, for the incompressible model we have

$$\begin{aligned}&\sum_{j,m=1}^{d} [(\eta K_m A_j(U_e) U_{x_j}, U_{x_m}) + (D_{mj}(U_e) U_{x_j}, U_{x_m})]\\ &\geq C_S \|\nabla U\|_{L^2}^2 + \eta (\mathcal{C}_2(U - U_e), \nabla F)\end{aligned} \tag{3.8}$$

with

$$K_j = diag(-I_d, I_{d^2}) A_j(U_e)$$

a skew-symmetric matrix.

Proof. For the compressible case, we have

$$\sum_j A_j(U_e) U_{x_j} = \begin{pmatrix} \rho_e \nabla \cdot \mathbf{v} \\ \frac{p'(\rho_e)}{\rho_e} \nabla \rho - \nabla \cdot F \\ -\nabla \mathbf{v} \end{pmatrix}$$

and thereby

$$-\sum_j K_j U_{x_j} = -\text{diag}\left(\frac{p'(\rho_e)}{\rho_e^2}, -I_d, I_{d^2}\right) \sum_j A_j(U_e) U_{x_j} = \begin{pmatrix} -\frac{p'(\rho_e)}{\rho_e} \nabla \cdot \mathbf{v} \\ \frac{p'(\rho_e)}{\rho_e} \nabla \rho - \nabla \cdot F \\ \nabla \mathbf{v} \end{pmatrix}.$$

With these, we compute (we drop the subscript e in all the coefficients below)

$$\begin{aligned}&\sum_{m,j=1}^{d} (K_m A_j(U_e) U_{x_j}, U_{x_m}) = -(\sum_j A_j(U_e) U_{x_j}, \sum_m K_m U_{x_m})\\ &= -p' \|\nabla \cdot \mathbf{v}\|_{L^2}^2 - \|\nabla \mathbf{v}\|_{L^2}^2 + \|\frac{p'}{\rho} \nabla \rho\|_{L^2}^2 - 2\frac{p'}{\rho} (\nabla \rho, \nabla \cdot F) + \|\nabla \cdot F\|_{L^2}^2\\ &= -p' \|\nabla \cdot \mathbf{v}\|_{L^2}^2 - \|\nabla \mathbf{v}\|_{L^2}^2 + \|\frac{p'}{\rho} \nabla \rho\|_{L^2}^2 - \frac{2p'}{\rho} (\partial_{x_j} \rho, \partial_{x_m} F_{jm}) + (\partial_{x_j} F_{kj}, \partial_{x_m} F_{km})\end{aligned}$$

$$
\begin{aligned}
&= -p'\|\nabla\cdot\mathbf{v}\|_{L^2}^2 - \|\nabla\mathbf{v}\|_{L^2}^2 + \|\frac{p'}{\rho}\nabla\rho\|_{L^2}^2 - \frac{2p'}{\rho}(\partial_{x_m}\rho, \partial_{x_j}F_{jm}) + (\partial_{x_m}F_{kj}, \partial_{x_j}F_{km})\\
&= -p'\|\nabla\cdot\mathbf{v}\|_{L^2}^2 - \|\nabla\mathbf{v}\|_{L^2}^2 + \|\frac{p'}{\rho}\nabla\rho\|_{L^2}^2 - \frac{2p'}{\rho}(\nabla\rho, \nabla\cdot F^T)\\
&\quad + \|\nabla F\|_{L^2}^2 + (\partial_{x_m}F_{kj} - \partial_{x_j}F_{km}, \partial_{x_j}F_{km})\\
&= -p'\|\nabla\cdot\mathbf{v}\|_{L^2}^2 - \|\nabla\mathbf{v}\|_{L^2}^2 + \frac{p'^2 + 2p'}{\rho^2}\|\nabla\rho\|_{L^2}^2 + \|\nabla F\|_{L^2}^2\\
&\quad + \frac{2p'}{\rho^2}(\nabla\rho, \mathcal{C}_1(U - U_e)) + (\mathcal{C}_2(U - U_e), \nabla F).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\sum_{j,m=1}^{d}[(\eta K_m A_j(U_e)U_{x_j}, U_{x_m}) + (D_{mj}(U_e)U_{x_j}, U_{x_m})]\\
&= \eta\frac{p'^2 + 2p'}{\rho^2}\|\nabla\rho\|_{L^2}^2 + (\frac{\mu}{\rho} - \eta)\|\nabla\mathbf{v}\|_{L^2}^2 + (\frac{\mu'}{\rho} - \eta p')\|\nabla\cdot\mathbf{v}\|_{L^2}^2 + \eta\|\nabla F\|_{L^2}^2\\
&\quad + \eta\frac{2p'}{\rho^2}(\nabla\rho, \mathcal{C}_1(U - U_e)) + \eta(\mathcal{C}_2(U - U_e), \nabla F)
\end{aligned}
$$

and thereby inequality (3.7) with $\eta = \min\{\frac{\mu}{2\rho}, \frac{\mu'}{2\rho p'}\}$.

Similarly, for the incompressible case we have

$$
\begin{aligned}
&\sum_{j,m=1}^{d}[(\eta K_m A_j(U_e)U_{x_j}, U_{x_m}) + (D_{mj}(U_e)U_{x_j}, U_{x_m})]\\
&= (\nu - \eta)\|\nabla\mathbf{v}\|_{L^2}^2 + \eta\|\nabla F\|_{L^2}^2 + \eta(\mathcal{C}_2(U - U_e), \nabla F).
\end{aligned}
$$

This completes the proof. ∎

We conclude this section with the following remark.

Remark 3.1. Here we first show that the condition (1.5), together with (1.4), implies (1.3). Indeed, this fact can simply be shown with Jacobi's formula

$$\partial_{x_i}\det A = \det A \operatorname{Tr}(A^{-1}\partial_{x_i}A)$$

as follows:

$$
\begin{aligned}
\nabla\cdot(\rho F^T) &= \partial_{x_i}(\rho F_{ij}) = \partial_{x_i}(\frac{1}{\det F}F_{ij})\\
&= \frac{1}{\det F}\partial_{x_i}F_{ij} - \frac{1}{(\det F)^2}F_{ij}\det F \operatorname{Tr}(F^{-1}\partial_{x_i}F)\\
&= \frac{1}{\det F}(\partial_{x_i}F_{ij} - F_{ij}(F^{-1})_{mn}\partial_{x_i}F_{nm})\\
&= \frac{1}{\det F}(\partial_{x_i}F_{ij} - (F^{-1})_{mn}F_{ij}\partial_{x_i}F_{nm})\\
&= \frac{1}{\det F}(\partial_{x_i}F_{ij} - (F^{-1})_{mn}F_{im}\partial_{x_i}F_{nj})\\
&= \frac{1}{\det F}(\partial_{x_i}F_{ij} - \delta_{in}\partial_{x_i}F_{nj})
\end{aligned}
$$

$$=\frac{1}{\det F}(\partial_{x_i}F_{ij}-\partial_{x_i}F_{ij})=0,$$

where the sixth equality is due to the condition (1.4). Namely, we arrive at the compatibility condition (1.3). Hence, the breaking of the Kawashima condition can also be compensated with the conditions (1.4) and (1.5).

Furthermore, we can show in the two-dimensional case that the breaking of the Kawashima condition can also be compensated with the compatibility conditions (1.3) and (1.5). (Notice that the condition (1.4) naturally holds in the one-dimensional case). Indeed, differentiating the condition (1.5) gives

$$0=\nabla(\rho\det F)=\nabla\rho\det F+\rho\det F\operatorname{Tr}(F^{-1}\nabla F).$$

Linearizing the right-hand side yields the differential operator

$$\mathcal{C}_3(U):=-\nabla\rho-\rho_e\nabla\operatorname{Tr}(F).$$

With this, the only two independent components of $\mathcal{C}_2(U)$ (for $d=2$) can be expressed in terms of $\mathcal{C}_1$ and $\mathcal{C}_3$:

$$\begin{aligned}\mathcal{C}_2(U)_{112}&=\partial_{x_1}F_{12}-\partial_{x_2}F_{11}=\mathcal{C}_3(U)_2-\mathcal{C}_1(U)_2,\\ \mathcal{C}_2(U)_{221}&=\partial_{x_2}F_{21}-\partial_{x_1}F_{22}=\mathcal{C}_3(U)_1-\mathcal{C}_1(U)_1.\end{aligned}$$

These indicate that operators $\mathcal{C}_1(U)$ and $\mathcal{C}_3(U)$ can also exclude the zero eigenvectors of the symbolic matrix (3.2).

However, for $d\geq 3$, the compatibility conditions (1.3) and (1.5) are not sufficient to exclude the zero eigenvectors. For example, for $d=3$ and $\xi=(1,0,0)$, we can verify that $(0,0,G)$ with

$$G=\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

is an eigenvector associated with the zero eigenvalue of the symbol matrix (3.2) and, simultaneously, in the kernels of the symbolic matrices for both $\mathcal{C}_1(U)$ and $\mathcal{C}_3(U)$. Consequently, we cannot expect an analog of Lemma 3.1 under the compatibility conditions (1.3) and (1.5) merely.

4. Proof of Theorem 1.1

Since the viscoelastic models (1.1) and (1.2) can be written into a symmetrizable form, the local existence of smooth solutions can be established as in [9]. See also [8,10,12]. Here we only derive the *a priori* estimates for Theorem 1.1 in the compressible case. The incompressible case is similar (and simpler).

As in [20], our *a priori* estimates will be derived with three steps: entropy estimate, energy estimate for higher-order derivatives and Kawashima's estimate.

Step 1. For the entropy estimate, we set

$$G(W)=S(W)-S(W_e)-S_W(W_e)(W-W_e)$$

with $W_e=(\rho_e,0_d,\rho_e I_{d^2})$. Thanks to the strict convexity of the entropy function, there are positive generic constants c and C such that

$$c|W-W_e|^2\leq G(W)\leq C|W-W_e|^2,\quad c|U-U_e|^2\leq|W-W_e|^2\leq C|U-U_e|^2$$

for U close to U_e. Then it follows from (2.3) and the original equations with (2.8) that

$$G(W)_t+\nabla\cdot(J(W)+\phi(\rho_e)\rho\mathbf{v}+\rho(\mathrm{Tr}(F)-F)\mathbf{v})=-\mu|\nabla\mathbf{v}|^2-\mu'|\nabla\cdot\mathbf{v}|^2$$

with $J(W)$ being that in (2.4) and $\phi(\rho)=\frac{p(\rho)}{\rho}+\int_{\rho_0}^{\rho}\frac{p(\zeta)}{\zeta^2}d\zeta-\frac{d}{2}$. Integrating this equation for $x\in\mathbb{R}^d$ and $t\in[0,T]$ gives

$$\|U(T)-U_e\|_{L^2}^2+c\int_0^T[\mu\|\nabla\mathbf{v}(t)\|_{L^2}^2+\mu'\|\nabla\cdot\mathbf{v}(t)\|_{L^2}^2]dt\le C\|U_0-U_e\|_{L^2}^2. \tag{4.1}$$

Step 2. Differentiating the two sides of Equation (2.6) with ∂_x^α for multi-index α satisfying $|\alpha|\le s$ leads to

$$\partial_x^\alpha U_t+\sum_{j=1}^d A_j(U)\partial_x^\alpha U_{x_j}=\partial_x^\alpha Q[U]+\sum_{j=1}^d[A_j(U),\partial_x^\alpha]U_{x_j} \tag{4.2}$$

with $[X,Y]=XY-YX$ the commutator. Recall that $A_0(U)$ is symmetric positive definite and $A_0(U)A_j(U)$ is symmetric. We take L^2-inner products with $A_0(U)\partial_x^\alpha U$ in both sides of the last equation to obtain

$$\begin{aligned}(A_0(U)\partial_x^\alpha U,\partial_x^\alpha U)_t&=2(A_0(U)\partial_x^\alpha Q[U],\partial_x^\alpha U)+2\sum_{j=1}^d(A_0(U)[A_j(U),\partial_x^\alpha]U_{x_j},\partial_x^\alpha U)\\&\quad+\left(\big(\partial_t A_0(U)+\sum_{j=1}^d\partial_{x_j}(A_0(U)A_j(U))\big)\partial_x^\alpha U,\partial_x^\alpha U\right)\\&\equiv I_1+I_2+I_3.\end{aligned} \tag{4.3}$$

The right-hand sides can be estimated by using the Sobolev calculus inequalities [15] as follows:

$$\begin{aligned}I_1&=2(\partial_x^\alpha(\frac{\mu}{\rho}\Delta\mathbf{v}+\frac{\mu'}{\rho}\nabla\nabla\cdot\mathbf{v}),\partial_x^\alpha\mathbf{v})\\&=2(\partial_x^\alpha(\frac{\mu}{\rho_e}\Delta\mathbf{v}+\frac{\mu'}{\rho_e}\nabla\nabla\cdot\mathbf{v}),\partial_x^\alpha\mathbf{v})+2(\partial_x^\alpha(\mu(\frac{1}{\rho}-\frac{1}{\rho_e})\Delta\mathbf{v}+\mu'(\frac{1}{\rho}-\frac{1}{\rho_e})\nabla\nabla\cdot\mathbf{v}),\partial_x^\alpha\mathbf{v})\\&=2(\partial_x^\alpha(\frac{\mu}{\rho_e}\Delta\mathbf{v}+\frac{\mu'}{\rho_e}\nabla\nabla\cdot\mathbf{v}),\partial_x^\alpha\mathbf{v})-2(\partial_x^{\alpha-1}(\mu(\frac{1}{\rho}-\frac{1}{\rho_e})\Delta\mathbf{v}+\mu'(\frac{1}{\rho}-\frac{1}{\rho_e})\nabla\nabla\cdot\mathbf{v}),\partial_x^{\alpha+1}\mathbf{v})\\&\le-\frac{2\mu}{\rho_e}\|\partial_x^\alpha\nabla\mathbf{v}\|_{L^2}-\frac{2\mu'}{\rho_e}\|\partial_x^\alpha\nabla\cdot\mathbf{v}\|_{L^2}+C(\mu+\mu')\|\rho-\rho_e\|_{H^s}\|\nabla\mathbf{v}\|_{H^s}^2,\end{aligned}$$

$$\begin{aligned}I_2&\le 2|A_0(U)|_{L^\infty}\|\sum_{j=1}^d[A_j(U),\partial_x^\alpha]U_{x_j}\|_{L^2}\|\partial_x^\alpha U\|_{L^2}\\&\le C\|\partial_x^\alpha U\|_{L^2}\sum_{j=1}^d(|\nabla A_j(U)|_{L^\infty}\|D_x^{s-1}U_{x_j}\|_{L^2}+|U_{x_j}|_{L^\infty}\|D_x^s A_j(U)\|_{L^2})\\&\le C\|\partial_x^\alpha U\|_{L^2}\|\nabla U\|_{H^{s-1}}^2\le C\|\nabla U\|_{H^{s-1}}^3,\end{aligned}$$

and

$$I_3\le C(|\rho_t|_{L^\infty}+|\nabla U|_{L^\infty})(\partial_x^\alpha U,\partial_x^\alpha U)$$

$$\le C|\nabla U|_{L^\infty}\|\partial_x^\alpha U\|_{L^2}^2$$
$$\le C\|\nabla U\|_{H^{s-1}}\|\nabla U\|_{H^{s-1}}^2.$$

Here for I_3 we have used the fact that $A_0(U)$ only depends on ρ and the ρ-equation in (1.2). Substituting the estimates on I_1, I_2 and I_3 into Equation (4.3), we have

$$\begin{aligned}&(A_0(U)\partial_x^\alpha U,\partial_x^\alpha U)_t+\frac{2\mu}{\rho_e}\|\partial_x^\alpha\nabla\mathbf{v}\|_{L^2}+\frac{2\mu'}{\rho_e}\|\partial_x^\alpha\nabla\cdot\mathbf{v}\|_{L^2}\\ &\le C\|\nabla U\|_{H^{s-1}}^3+C(\mu+\mu')\|\rho-\rho_e\|_{H^s}\|\nabla\mathbf{v}\|_{H^s}^2.\end{aligned}$$

Integrating the last inequality over $t\in[0,T]$, adding the results over α for $1\le|\alpha|\le s$ and combining inequality (4.1), we arrive at

$$\begin{aligned}&\|U(T)-U_e\|_{H^s}^2+c\int_0^T\|\nabla\mathbf{v}(t)\|_{H^s}^2dt\\ &\le C\|U_0-U_e\|_{H^s}^2+C(\mu+\mu')\sup_{t\in[0,T]}(\|\rho(t)-\rho_e\|_{H^s})\int_0^T\|\nabla\mathbf{v}(t)\|_{H^s}^2dt\\ &\quad+C\sup_{t\in[0,T]}(\|U(t)-U_e\|_{H^s})\int_0^T\|\nabla U(t)\|_{H^{s-1}}^2dt.\end{aligned}\tag{4.4}$$

Step 3. To control the last term in inequality (4.4), we utilize Lemma 3.1. Rewrite Equation (4.2) as

$$\partial_x^\alpha U_t+\sum_{j=1}^d A_j(U_e)\partial_x^\alpha U_{x_j}=\sum_{j,k=1}^d\partial_x^\alpha(D_{jk}(U)U_{x_ix_j})+\sum_{j=1}^d\partial_x^\alpha((A_j(U_e)-A_j(U))U_{x_j})$$

for α with $0\le|\alpha|\le s-1$. Recall the skew symmetric matrix K_m introduced in Lemma 3.1. We take L^2-inner products with $-K_m\partial_x^\alpha U_{x_m}$ and sum up the results over $m\in[1,d]$ to obtain

$$\begin{aligned}&\sum_{m=1}^d(K_m\partial_x^\alpha U_t,\partial_x^\alpha U_{x_m})+\sum_{m,j=1}^d(K_mA_j(U_e)\partial_x^\alpha U_{x_j},\partial_x^\alpha U_{x_m})\\ &=\sum_{m,j=1}^d(K_m\partial_x^\alpha((A_j(U_e)-A_j(U))U_{x_j}),\partial_x^\alpha U_{x_m})\\ &\quad+\sum_{m,j,k=1}^d(K_m\partial_x^\alpha(D_{jk}(U)U_{x_ix_j}),\partial_x^\alpha U_{x_m})\\ &\le C\sum_j\|\partial_x^\alpha((A_j(U_e)-A_j(U))U_{x_j})\|_{L^2}\|\partial_x^\alpha\nabla U\|_{L^2}\\ &\quad+\epsilon\|\partial_x^\alpha\nabla U\|_{L^2}^2+\frac{C}{\epsilon}\sum_{j,k}\|\partial_x^\alpha([D_{jk}(U_e)+D_{jk}(U)-D_{jk}(U_e)]U_{x_ix_j})\|_{L^2}^2\\ &\le C\|U-U_e\|_{H^s}\|\nabla U\|_{H^{s-1}}^2+\epsilon\|\partial_x^\alpha\nabla U\|_{L^2}^2+\frac{C}{\epsilon}\|\partial_x^{\alpha+1}\nabla\mathbf{v}\|_{L^2}^2\\ &\quad+\frac{C}{\epsilon}\|U-U_e\|_{H^{s-1}}^2\|\nabla\mathbf{v}\|_{H^s}^2.\end{aligned}\tag{4.5}$$

For the first term on the left-hand side of inequality (4.5), we use the integration by parts to obtain

$$\begin{aligned}&(K_m\partial_x^\alpha U_t,\partial_x^\alpha U_{x_m})\\&=\frac{1}{2}\int(\partial_x^\alpha U_{x_m}^T K_m\partial_x^\alpha U)_t dx-\frac{1}{2}\int(\partial_x^\alpha U^T K_m\partial_x^\alpha U_t)_{x_m}dx\\&=\frac{1}{2}(K_m\partial_x^\alpha U,\partial_x^\alpha U_{x_m})_t.\end{aligned}\tag{4.6}$$

For the second term, we recall that the solution $U=U(x,t)$ satisfies the compatibility conditions (1.3) and (1.4). Then we have

$$\begin{aligned}\mathcal{C}_1(U-U_e)&=\nabla\cdot[(\rho-\rho_e)(F^T-I_{d^2})],\\ [\mathcal{C}_2(U-U_e)]_{kmj}&=(F_{lj}-\delta_{lj})\partial_{x_l}F_{km}-(F_{lm}-\delta_{lm})\partial_{x_l}F_{kj}\end{aligned}\tag{4.7}$$

with $\mathcal{C}_1$ and $\mathcal{C}_2$ the linearized differential operators defined in Equation (3.4). Thus, it follows from Lemma 3.1 and Equation (4.7) that

$$\begin{aligned}&\sum_{j,m}(K_mA_j(U_e)\partial_x^\alpha U_{x_j},\partial_x^\alpha U_{x_m})\\&=\sum_{j,m}[(K_mA_j(U_e)\partial_x^\alpha U_{x_j},\partial_x^\alpha U_{x_m})+(D_{mj}(U_e)\partial_x^\alpha U_{x_j},\partial_x^\alpha U_{x_m})]\\&\quad-\sum_{m,j}(D_{mj}(U_e)\partial_x^\alpha U_{x_j},\partial_x^\alpha U_{x_m})\\&\geq C_S\|\partial_x^\alpha\nabla U\|_{L^2}^2-\frac{\mu}{\rho_e}\|\partial_x^\alpha\nabla\mathbf{v}\|_{L^2}^2-\frac{\mu'}{\rho_e}\|\partial_x^\alpha\nabla\cdot\mathbf{v}\|_{L^2}^2\\&\quad+\eta\frac{2p'(\rho_e)}{\rho_e^2}(\partial_x^\alpha\mathcal{C}_1(U-U_e),\nabla\partial_x^\alpha\rho)+\eta(\partial_x^\alpha\mathcal{C}_2(U-U_e),\nabla\partial_x^\alpha F),\\&\geq C_S\|\partial_x^\alpha\nabla U\|_{L^2}^2-\frac{\mu}{\rho_e}\|\partial_x^\alpha\nabla\mathbf{v}\|_{L^2}^2-\frac{\mu'}{\rho_e}\|\partial_x^\alpha\nabla\cdot\mathbf{v}\|_{L^2}^2\\&\quad-C\|(\partial_x^\alpha\mathcal{C}_1(U-U_e)\|_{L^2}\|\nabla\partial_x^\alpha\rho\|_{L^2}-C\|\partial_x^\alpha\mathcal{C}_2(U-U_e)\|_{L^2}\|\nabla\partial_x^\alpha F\|_{L^2}\\&\geq C_S\|\partial_x^\alpha\nabla U\|_{L^2}^2-\frac{\mu}{\rho_e}\|\partial_x^\alpha\nabla\mathbf{v}\|_{L^2}^2-\frac{\mu'}{\rho_e}\|\partial_x^\alpha\nabla\cdot\mathbf{v}\|_{L^2}^2\\&\quad-C\|\nabla U\|_{H^{s-1}}^2(\|\nabla\partial_x^\alpha\rho\|_{L^2}+\|\nabla\partial_x^\alpha F\|_{L^2}).\end{aligned}\tag{4.8}$$

Plugging Equation (4.6) and inequality (4.8) into inequality (4.5) with $\epsilon=\frac{C_S}{2}$ and adding over α for $0\leq|\alpha|\leq s-1$ lead to

$$\begin{aligned}C_S\|\nabla U\|_{H^{s-1}}^2\leq&-\sum_m(K^mD_x^lU,D_x^lU_{x_m})_t+C\|U-U_e\|_{H^s}\|\nabla U\|_{H^{s-1}}^2\\&+C\|\nabla\mathbf{v}\|_{H^s}^2+C\|U-U_e\|_{H^s}^2\|\nabla\mathbf{v}\|_{H^s}^2.\end{aligned}$$

Integrate this inequality over $t\in[0,T]$ to get

$$\begin{aligned}\int_0^T\|\nabla U(t)\|_{H^{s-1}}^2&\leq C\|U(T)-U_e\|_{H^s}^2+C\|U_0-U_e\|_{H^s}^2+C\int_0^T\|\nabla\mathbf{v}\|_{H^s}^2dt\\+C\sup_{t\in[0,T]}\|U(t)-U_e\|_{H^s}&\int_0^T\|\nabla U(t)\|_{H^{s-1}}^2dt+C\sup_{t\in[0,T]}\|U(t)-U_e\|_{H^s}^2\int_0^T\|\nabla\mathbf{v}(t)\|_{H^s}^2dt.\end{aligned}$$

Combining this with inequality (4.4) leads to the estimate in Theorem 1.1. The global existence of smooth solutions then can be obtained by the standard argument [20].

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