COMMUN. MATH. SCI.
Vol. 15, No. 4, pp. 1073–1106
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# STABILITY OF TRAVELING WAVE SOLUTIONS OF NONLINEAR CONSERVATION LAWS FOR IMAGE PROCESSING*

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**Abstract.** This paper studies the stability of smooth traveling wave solutions to a nonlinear PDE problem in reducing image noise. Specifically, we prove that the solution to the Cauchy problem approaches to the traveling wave solution if the initial data is a small perturbation of the traveling wave. We use a weighted energy method to show that if the initial perturbation decays algebraically or exponentially as $|x| \to \infty$, then the Cauchy problem solution approaches to the traveling wave at corresponding rates as $t \to \infty$.

**Keywords.** nonlinear stability; nonlinear conservation laws; rate of decay; weighted energy estimates; image processing.

**AMS subject classifications.** 35L50, 35L65, 35C07, 35B35, 35B40, 35K15, 92C35.

## 1. Introduction

In image processing and computer vision, PDE-based techniques have led to a large variety of research areas. Especially, nonlinear PDEs have been applied to denoising images, edge detection and image inpainting; see, eg., Perona and Malik [27], Rudin, Osher and Fatemi [28], Tumblin and Turk [29], and You and Kaveh [33].

Before the development of nonlinear PDE methods, a linear-filtering technique was introduced by Marr and Hildreth [20] to address the problem of noise reduction in images. The idea of the linear-filtering is that the image intensity function is convolved with a Gaussian as the optimal smoothing filter. It was further refined by a Gaussian scale-space filtering method. Here, a scale-space is a necessary concept when one deals with images. For instance, scale-spaces have been used in image segmentation and edge detection (see [9,30]). As a typical type of scale-spaces, the Gaussian scale-space method was initiated by Witkin [31], and subsequently developed by Koenderink [8] and Canny [2]. Those main ideas are based on solving the heat equation with initial data given by a noisy image intensity function.

However, the above linear diffusion filtering for reducing noise in the image has a major drawback; the resulting images become badly blurred, and it is difficult to determine the edges of the objects in the original image. Also the idea that a process depends on local properties of the image should be applied. To deal with the issues, Perona and Malik [27] proposed the idea of anisotropic diffusion, which is a technique to reduce image noise without losing important image contents such as edges or lines of the image. This method is one of the most well-known and is a relatively simple case of nonlinear scale-spaces. Interestingly, due to the novelty in that diffusion and edge detection interact in one single process, this model is related to the neural dynamics of brightness perception (see [3]).

In Perona and Malik [27], $I(\cdot,t):\Omega\to\mathbb{R}$, where $\Omega\subset\mathbb{R}^2$ denotes a picture domain and $I$ a family of gray scale images. Anisotropic diffusion, or Perona-Malik diffusion, is defined as

$$I_t = \operatorname{div}(c(x,y,t)\nabla I),$$

*Received: May 16, 2016; accepted (in revised form): December 9, 2016. Communicated by Mikhail Feldman.

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where $c(x,y,t)$ is the diffusion coefficient given by

$$c(x,y,t) = g(|\nabla I(x,y,t)|),$$

and the edges are preserved and sharpened, provided that the function $g$ is chosen properly. The thresholding function $g$ is small in regions of sharp gradients. A typical thresholding function $g$ is

$$g(s) = \frac{1}{1+\left(\frac{s}{K}\right)^2},$$

where the constant $K$ controls the sensitivity to edges.

In this paper, we study the Perona-Malik (PM) equation which combines a Burgers-type convection term and the anisotropic diffusion, introduced by Kurganov et al. [10]: for any $x \in \mathbb{R}$ and $t > 0$

$$u_t + uu_x = (g(u_x)u_x)_x, \tag{1.1}$$

where $g$ satisfies

$$g(s) = \frac{1}{1+s^2}. \tag{1.2}$$

The initial data is given by

$$u(x,0) = u_0(x) \quad \text{for } x \in \mathbb{R}, \quad \text{where} \quad u_0(x) \to \begin{cases} u_l & \text{as} \quad x \to -\infty, \\ u_r & \text{as} \quad x \to +\infty, \end{cases} \tag{1.3}$$

where $u_l$ and $u_r$ are far left and far right states.

Previously, Kurganov et al. [10] investigated shock or jump-type behavior which is typical of edges in images. Specifically, Kurganov et al. [10] addressed well-posedness of classical solutions to the Cauchy problem. Goodman et al. [4] proved that solutions with a certain large initial data blow up in a finite time.

Kurganov et al. [10] also showed that there exist smooth traveling waves for $0 < |u_l - u_r| < 2$ and a critical threshold of discontinuity occurs at $|u_l - u_r| = 2$. Wu [32] proved the existence of the smooth traveling wave solutions as well. Moreover, in Greer and Bertozzi [5], the existence and smoothness of traveling waves were shown for a more general thresholding function $g(s)$ which satisfies the physical properties required in Perona and Malik [27].

Kurganov et al. [10] demonstrated numerically that both the smooth and discontinuous forms of traveling wave solutions are strong attractors. Also Greer and Bertozzi [5] showed through numerical results that the traveling wave solutions in [10] are stable. A rigorous proof of the stability of the smooth traveling waves in an exponential weighted space was obtained by Wu [32] by spectral analysis. To achieve the spectral gap needed in the spectral stability, this exponential weight is essential in [32].

In the current paper, we study nonlinear stability in a non-weighted space $H^2$ by energy estimate arguments. We show that the solution to the Cauchy problem (1.1)-(1.3) converges to a traveling wave if the initial perturbation is small. Furthermore, we investigate algebraic and exponential rates of convergence by weighted energy estimate methods, where the exponential decay result in Theorem 2.3 in the current paper is qualitatively similar to that of the Main Theorem in [32]. Regarding literature on the energy estimate arguments, we refer the readers to [6,13–15,17,18] for details. The weighted energy estimate methods were initiated by Kawashima and Matsumura [6] and by Matsumura and Nishihara [21] with further developments in [12,18,19,22,23,25,26].

This paper is organized as follows. After listing some notation, we introduce the main theorems in Section 2. In Section 3, we briefly describe properties of the traveling wave solutions. We also reformulate our problem and state theorems for the reformulated problem. The proofs will be given in Section 4. When it comes to algebraic decay rates, after establishing a priori estimates, we complete the proof of the theorems in Section 5. We state and prove theorems of exponential decay rates in Section 6.

Notation 1.1. *We denote $C>0$ as a generic constant so each $C$ can be a different number at different context. We write $f(x)\sim g(x)$ as $x\to a$ if $C^{-1}g<f<Cg$ in a neighborhood of $a$ for some $C>0$. Let $\|f\|$ be the $L^2$ norm of a function $f\in L^2$ which is*

$$\|f\|=\Big(\int |f(x)|^2 dx\Big)^{\frac{1}{2}}.$$

*We denote $H^k$ the usual Sobolev space $W^{k,2}$ with the norm $\|f\|_k$ for any function $f\in H^k$, $k\geq 1$,*

$$\|f\|_k=\Big(\int \sum_{i=0}^{k}\Big|\frac{d^i f(x)}{dx^i}\Big|^2 dx\Big)^{\frac{1}{2}}.$$

*We also denote $L^2_w$ the space of measurable functions on $\mathbb{R}$ which satisfy $\sqrt{w(x)}f\in L^2$, where $w(x)>0$ is a weight function. Here, the space is endowed with the norm*

$$\|f\|_w=\Big(\int w(x)|f(x)|^2 dx\Big)^{\frac{1}{2}}.$$

*Similarly, $H^k_w$ denotes the weighted Sobolev space of $L^2_w$-functions $f$ whose derivatives $\partial^i_x f$, $i=1,\dots,k$, are also $L^2_w$-functions, with the norm*

$$\|f\|_{k,w}=\Big(\sum_{i=0}^{k}\|\partial^i_x f\|^2_w\Big)^{\frac{1}{2}}.$$

*We also define*

$$\langle x\rangle=\sqrt{1+x^2}. \tag{1.4}$$

*Particularly, in Section 5, a weight function is of the form $w(x)\sim\langle x\rangle^{\alpha}$ for $\alpha\geq 0$ and we denote $L^2_w=L^2_\alpha$. In Section 6, a weight function is defined as $w(x)\sim e^{a\langle z\rangle}$ for some $a>0$.*

## 2. The main results

In this section, we are going to state our main theorems about the asymptotic stability of smooth traveling wave solutions of the problem (1.1)-(1.3) under small perturbations.

Traveling wave solutions are of the form $u(x,t)=U(x-ct)=U(z)$, where $c$ is the traveling wave speed and $z=x-ct$ the traveling wave variable. From the entropy condition, $u_l>u_r$. Without loss of generality, we assume that $u_r=0$ in the rest of the paper Then $0<u_l<2$.

Assume that the initial data $u_0 - U$ is a small perturbation of the smooth traveling wave and that is integrable over $\mathbb{R}$. As in [7,11,16,17], there exists a unique $x_0$ satisfying the conservation of mass principle

$$\int_{-\infty}^{\infty} (u_0(x) - U(x))dx = x_0(u_r - u_l).$$

Since the translated $U(x-ct+x_0)$ is also a traveling wave solution of (1.1), (1.2) connecting $u_l$ and $u_r$, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{-\infty}^{\infty} (u(x,t) - U(x-ct+x_0))dx \\
=& -\frac{1}{2}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\Big(u^2(x,t) - U^2(x-ct+x_0)\Big)dx \\
&\quad + \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\Big(g(u(x,t))u(x,t) - g(U(x-ct+x_0))U(x-ct+x_0)\Big)dx \\
=&0.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} (u(x,t) - U(x-ct+x_0))dx = \int_{-\infty}^{\infty} (u_0(x) - U(x+x_0))dx \\
=&\int_{-\infty}^{\infty} (u_0(x) - U(x))dx + \int_{-\infty}^{\infty} (U(x) - U(x+x_0))dx \\
=&\int_{-\infty}^{\infty} (u_0(x) - U(x))dx - x_0(u_r - u_l) = 0. \qquad (2.1)
\end{aligned}$$

Before we mention the main results of this paper, we will decompose the solution $u$ of the problem (1.1)-(1.3) into the traveling wave and its perturbation as

$$u(x,t) = U(x-ct+x_0) + \phi_x(x,t) \qquad (2.2)$$

for any $x \in \mathbb{R}$ and $t \geq 0$, where the antiderivative of $\phi_x$ satisfies

$$\phi(x,t) = \int_{-\infty}^{x} (u(y,t) - U(y-ct+x_0))dy. \qquad (2.3)$$

Then it follows from (2.1) that for any $t \geq 0$

$$\phi(\pm\infty,t) = \int_{-\infty}^{\infty} (u(x,t) - U(x-ct+x_0))dx = 0.$$

For simplicity of presentation, let $x_0 = 0$. Now we have

$$\phi(\pm\infty,t) = \int_{-\infty}^{\infty} (u_0(x,t) - U(x-ct))dx = 0. \qquad (2.4)$$

Also the initial data $\phi_0$ of $\phi(x,t)$ is given by

$$\phi_0(x) = \phi(x,0) = \int_{-\infty}^{x} (u_0(y) - U(y))dy. \qquad (2.5)$$

Based on the above construction in Equations (2.2) and (2.3), the main theorems of this paper are stated as follows.

THEOREM 2.1. *Let $0<u_l\le 1.8$. There exists a constant $\delta_0>0$ such that if $\|u_0-U\|_1+\|\phi_0\|\le\delta_0$, then the problem (1.1)-(1.3) has a unique global solution $u(x,t)$ satisfying*

$$u-U\in C([0,\infty);H^1)\cap L^2([0,\infty);H^1)$$

*and there exist some $\nu>0$ and $C>0$ such that*

$$\|\phi(\cdot,t)\|_2^2+\int_0^t\|\sqrt{|U'|}\phi(\cdot,\tau)\|^2d\tau+\nu\int_0^t\|\phi_z(\cdot,\tau)\|_2^2d\tau\le C\|\phi_0\|_2^2,$$

*for any $t>0$. Furthermore, the solution satisfies*

$$\sup_{x\in\mathbb{R}}|u(x,t)-U(x-ct)|\to 0 \quad as \quad t\to+\infty. \tag{2.6}$$

The following theorems show that if the initial perturbation decays algebraically or exponentially as $|x|\to\infty$, then the perturbation decays at corresponding rates in time.

In the following theorem, a weight function is defined as

$$w(z)=\langle z-z_*\rangle^\alpha$$

for $\alpha\ge 0$, where $z_*$ is defined in (5.1).

THEOREM 2.2 (Algebraic rates). *Let $0<u_l\le 1.8$. Suppose $\phi_0\in L^2_\alpha\cap H^2$ for some $\alpha\ge 0$. Then there is a constant $\delta_\alpha>0$ such that if $\|u_0-U\|_1+\|\phi_0\|_{0,\alpha}\le\delta_\alpha$, then the problem (1.1)-(1.3) has a unique global solution $u(x,t)$ satisfying*

$$u-U\in C([0,\infty);H^1)\cap L^2([0,\infty);H^1),$$

*and there are some $\nu_\alpha>0$ and $C>0$ such that for all $t>0$*

$$(1+t)^\gamma\|\phi(\cdot,t)\|_2^2+\nu_\alpha\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_2^2d\tau\le C(\|\phi_0\|_{0,\alpha}^2+\|\phi_{z,0}\|_1^2),$$

*where $0\le\gamma\le\alpha$ when $\alpha$ is an integer and $0\le\gamma<\alpha$ when $\alpha$ is a non-integer.*

*Moreover, if $\alpha$ is an integer, there is a constant $C>0$ such that for any $t>0$*

$$\sup_{\mathbb{R}}|u(x,t)-U(x-ct)|\le C(1+t)^{-\frac{\alpha}{2}}(\|\phi_0\|_{0,\alpha}+\|\phi_{z,0}\|_1). \tag{2.7}$$

*On the other hand, if $\alpha$ is a non-integer, it holds*

$$\sup_{\mathbb{R}}|u(x,t)-U(x-ct)|\le C_\varepsilon(1+t)^{-\frac{\alpha}{2}+\varepsilon}(\|\phi_0\|_{0,\alpha}+\|\phi_{z,0}\|_1) \tag{2.8}$$

*for all $t>0$, where $C_\varepsilon\to\infty$ if $\varepsilon\to 0$.*

For the next theorem, we employ a weight function

$$w(z)=e^{a\langle z-z_*\rangle},$$

where $z_*$ is given in (5.1) and $a>0$ is determined in Lemma 6.1.

THEOREM 2.3 (Exponential rates). *Let $0<u_l\le 1.8$. Suppose $\phi_0\in H_w^2$. There exists a constant $\delta_w>0$ such that if $\|u-U\|_{1,w}+\|\phi_0\|_{0,w}\le\delta_w$, then the problem (1.1)-(1.3) has a unique global solution $u(x,t)$ satisfying*

$$u-U\in C([0,\infty);H_w^1)\cap L^2([0,\infty);H_w^1)$$

*and there are positive constants $\nu_w$, $\theta$ and $C$ such that for any $t>0$*

$$\|\phi(\cdot,t)\|_{2,w}^2+\theta\int_0^t\|\phi(\cdot,\tau)\|_{2,w}^2+\nu_w\int_0^t\|\phi_z(\cdot,\tau)\|_{2,w}^2d\tau\le C\|\phi_0\|_{2,w}^2.$$

*Consequently, there is some $C>0$ such that for any $t>0$*

$$\sup_{z\in\mathbb{R}}|u(x,t)-U(x-ct)|\le Ce^{-\frac{\theta}{2}t}\|\phi_0\|_{2,w}. \tag{2.9}$$

## 3. Reformulation of the problem

Theorem 2.1 is proved by showing the global existence of $\phi$, which is the antiderivative of a perturbation defined in Equation (2.3). Thus, in this section, after reformulating the problem (1.1)-(1.3) in terms of $\phi$, we introduce Theorem 3.1 regarding the global existence of $\phi$. Theorem 2.1 is a consequence of Theorem 3.1. The proof of Theorem 3.1 is followed from the local existence and a priori estimates.

To begin with, we shall list some properties of traveling wave solutions of the problem (1.1)-(1.3). These properties are going to be used to establish the desired a priori estimates for the energy/weighted energy estimates in Sections 4, 5 and 6.

Let $U$ be the traveling wave solutions. Substituting the traveling wave $U$ into Equation (1.1), it satisfies the ordinary differential equation

$$-cU'+UU'=\Big(\frac{U'}{1+U'^2}\Big)', \tag{3.1}$$

and the Rankine-Hugoniot condition gives us

$$c=\frac{u_l}{2}. \tag{3.2}$$

Moreover, integrating Equation (3.1) with respect to $z$ over $(z,+\infty)$, we have

$$-cU+\frac{1}{2}U^2=-\frac{u_l}{2}U+\frac{1}{2}U^2=\frac{U'}{1+U'^2}. \tag{3.3}$$

Based on Equations (3.2) and (3.3), the following proposition is derived.

PROPOSITION 3.1. *Let $0<|u_l-u_r|<2$. The traveling wave solution $U(z)$ satisfies as follows:*

*(1) In Wu [32], it is proved that $U$ is monotonically decreasing over $(-\infty,\infty)$ and $u_r=0<U<u_l$.*

*(2) For $z\in\mathbb{R}$,*

$$-1<-\frac{u_l^2}{4}<-\frac{u_l^2}{4+\sqrt{16-u_l^4}}\le U'<0. \tag{3.4}$$

*From inequality (3.4), there exists a constant $m_0>0$ such that*

$$\min_{z\in\mathbb{R}}\Big\{\frac{1-U'^2}{(1+U'^2)^2}\Big\}\ge m_0>0. \tag{3.5}$$

(3) *The traveling wave solution $U$ decays exponentially to its end states as $z\to\pm\infty$, and*

$$\begin{cases} C_1e^{-u_lz}\le U\le C_2e^{-\frac{u_l}{2}z} & \text{if} \quad z\to+\infty, \\ C_3e^{\frac{u_l}{2}z}\le u_l-U\le C_4e^{u_lz} & \text{if} \quad z\to-\infty, \end{cases} \tag{3.6}$$

*for some positive constants $C_1,C_2,C_3,C_4$.*

*Proof.*
(2) From Equation (3.3), one can get

$$U'=\frac{1-\sqrt{1-U^2(U-u_l)^2}}{U(U-u_l)}. \tag{3.7}$$

With Equation (3.7), direct calculation from Equation (3.3) gives us

$$U''=\frac{(U-u_l/2)U'(1+U'^2)^2}{1-U'^2}=\frac{(U-c)U'(1+U'^2)^2}{1-U'^2}. \tag{3.8}$$

Observing the minimum of $U'$, one can obtain (3.4) and (3.5).
(3) From Equation (3.7), it is easy to show

$$\frac{1}{2}|U(U-u_l)|<|U'|<|U(U-u_l)|.$$

By the comparison principle, one has

$$\begin{cases} \dfrac{u_l}{1+c_1e^{u_lz}}\le U\le\dfrac{u_l}{1+c_2e^{\frac{u_l}{2}z}} & \text{if } z\to+\infty, \\ \dfrac{u_l}{1+c_3e^{-\frac{u_l}{2}z}}\le u_l-U\le\dfrac{u_l}{1+c_4e^{-u_lz}} & \text{if } z\to-\infty, \end{cases}$$

for some constants $c_1,c_2,c_3,c_4$ . ☐

Now we shall reformulate the problem (1.1)-(1.3). In view of Equation (2.2), the solution $u$ of the problem (1.1)-(1.3) can be written as

$$u(x,t)=U(x-ct)+\phi_x(x,t)=U(z)+\bar{\phi}_z(z,t), \tag{3.9}$$

where $z=x-ct$ is the traveling wave variable. For simplicity of notation, we will omit a bar in $\bar{\phi}_z$. Substituting Equation (3.9) into Equation (1.1), using Equation (3.1), and changing variable $(x,t)$ to (z,t), we have

$$(-c)\phi_{zz}+\phi_{zt}+\Big(U\phi_z\Big)_z+\Big(\frac{1}{2}\phi_z^2\Big)_z=\Big(g(U'+\phi_{zz})(U'+\phi_{zz})-g(U')U'\Big)_z. \tag{3.10}$$

Integrating Equation (3.10) with respect to $z$, we obtain

$$\phi_t+(U-c)\phi_z=-\frac{1}{2}\phi_z^2+g(U'+\phi_{zz})(U'+\phi_{zz})-g(U')U'. \tag{3.11}$$

On the right hand side of Equation (3.11), one can derive

$$g(U'+\phi_{zz})(U'+\phi_{zz})-g(U')U'=\frac{U'+\phi_{zz}}{1+(U'+\phi_{zz})^2}-\frac{U'}{1+U'^2}$$

$$= \frac{1-U'^2}{(1+U'^2)^2}\phi_{zz} + F(U',\phi_{zz}), \tag{3.12}$$

where

$$F(U',\phi_{zz}) = -\frac{3U' - U'^3 + (1-U'^2)\phi_{zz}}{(1+U'^2)^2(1+(U'+\phi_{zz})^2)}\phi_{zz}^2, \tag{3.13}$$

and $F = O(\phi_{zz}^2)$ when $|\phi_{zz}|$ is small.

In conclusion, $\phi(z,t)$ satisfies

$$\phi_t + (U-c)\phi_z - \frac{1-U'^2}{(1+U'^2)^2}\phi_{zz} = -\frac{1}{2}\phi_z^2 + F \tag{3.14}$$

with the initial data given by

$$\phi(z,0) = \phi_0(z) = \int_{-\infty}^{z}(u_0(y) - U(y))dy \tag{3.15}$$

for any $z \in \mathbb{R}$ from Equation (2.5).

Define the solution space of the problem (3.14), (3.15) as

$$X(0,T) = \{\phi(z,t) : \phi \in C([0,T);H^2),\ \ \phi_z \in L^2((0,T);H^2)\} \tag{3.16}$$

with $0 < T < \infty$. Recall the Sobolev embedding theorem: if $f \in H^1(\mathbb{R})$,

$$\sup_{z\in\mathbb{R}} |f(z)|^2 \le 2\|f\|\|f_z\|, \tag{3.17}$$

since

$$f^2(z) = 2\int_{-\infty}^{z} f(z)f_z(z)dz \le 2\int_{\mathbb{R}} |f||f_z|dz \le 2\|f\|\|f_z\|$$

for all $z \in \mathbb{R}$. Then if we let

$$N(t) = \sup_{0\le\tau\le t}\{\|\phi(\cdot,\tau)\|_2\} \tag{3.18}$$

for all $t \ge 0$, the Sobolev inequality (3.17) implies

$$\sup_{z\in\mathbb{R}} |\phi|^2 \le 2\|\phi\|\|\phi_z\| \quad \text{and} \quad \sup_{z\in\mathbb{R}} |\phi_z|^2 \le 2\|\phi_z\|\|\phi_{zz}\|,$$

and hence, for any $t \ge 0$, there is $C > 0$ such that

$$\sup_{z\in\mathbb{R}}\{|\phi|, |\phi_z|\} \le CN(t). \tag{3.19}$$

Then the global existence of the solution of the problem (1.1)-(1.3) in Theorem 2.1 is a consequence of the following theorem.

Theorem 3.1. *Suppose that the assumptions of Theorem 2.1 hold. Then there are constants $\delta_1 > 0$ and $C > 0$ such that if*

$$N(0) < \delta_1, \tag{3.20}$$

*where $N(t)$ is defined in Equation (3.18), then the problem (3.14), (3.15) has a unique global solution $\phi \in X(0,+\infty)$ satisfying that for all $t>0$,*

$$\|\phi(\cdot,t)\|_2^2+\int_0^t \|\sqrt{|U'|}\phi(\cdot,\tau)\|^2 d\tau+\nu\int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau \le CN^2(0) \tag{3.21}$$

*for some $\nu=\nu(m_0)>0$, where $m_0$ is defined in inequality (3.5). Consequently, it follows*

$$\sup_{z\in\mathbb{R}}|\phi_z(z,t)| \to 0 \quad as \quad t\to+\infty. \tag{3.22}$$

The proof of Theorem 3.1 is based on the following local existence and a priori estimates.

PROPOSITION 3.2 (Local existence). *For any $\delta>0$, there exists a constant $T>0$ depending on $\delta$ such that if $\phi_0\in H^2(\mathbb{R})$ and $2N(0)<\delta$, the problem (3.14), (3.15) has a unique solution $\phi\in X(0,T)$ satisfying*

$$N(t)<2N(0) \tag{3.23}$$

*for any $0\le t\le T$.*

PROPOSITION 3.3 (A priori estimates). *Assume that $\phi\in X(0,T)$ is a solution obtained from Proposition 3.2 for a constant $T>0$. Then there exist constants $\delta_2>0$ and $C>0$, which are independent of $T$, such that for any $0\le t\le T$ if*

$$N(t)<\delta_2 \tag{3.24}$$

*then the solution $\phi$ of the problem (3.14), (3.15) satisfies*

$$\|\phi(\cdot,t)\|_2^2+\int_0^t \|\sqrt{|U'|}\phi(\cdot,\tau)\|^2 d\tau+\nu\int_0^t \|\phi_z(\cdot,\tau)\|_2^2 d\tau \le CN^2(0) \tag{3.25}$$

*for some $\nu=\nu(m_0)>0$, where $m_0$ is given in (3.5).*

The proof of Theorem 3.1 is a consequence of the local existence in Proposition 3.2 and the a priori estimates in Proposition 3.3. Since the local existence can be shown in a standard way [24], we will establish only the a priori estimates.

## 4. Basic energy estimates and proofs of stability theorems

In this section, we need to establish the desired a priori energy estimates for the solution of the problem (3.14), (3.15) to prove Theorem 3.1.

LEMMA 4.1 ($L^2$-estimates). *Under the conditions of Theorem 2.1, if $0<u_l\le 1.8$, it satisfies for all $t>0$,*

$$\begin{aligned}&\|\phi(\cdot,t)\|^2+\frac{1}{2}\int_0^t \|\sqrt{|U'|}\phi(\cdot,\tau)\|^2 d\tau+\frac{m_0}{2}\int_0^t \|\phi_z(\cdot,\tau)\|^2 d\tau\\ \le&\|\phi_0\|^2+C\int_0^t\int_{\mathbb{R}} G_1(\phi,\phi_z,\phi_{zz})dz d\tau\end{aligned} \tag{4.1}$$

*for some $C>0$, where $m_0>0$, which is defined in (3.5), and*

$$G_1(\phi,\phi_z,\phi_{zz}) := |\phi|(\phi_z^2+\phi_{zz}^2) \tag{4.2}$$

*is a collection of cubic terms in inequality (4.1).*

*Proof.* Multiplying Equation (3.14) by $2\phi$, we have

$$2\phi\phi_t+2(U-c)\phi\phi_z-2\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_{zz}=(-\phi_z^2+2F)\phi. \tag{4.3}$$

The left hand side of Equation (4.3) can be expressed as

$$\begin{aligned}&2\phi\phi_t+2(U-c)\phi\phi_z-2\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_{zz}\\=&(\phi^2)_t+\Big((U-c)\phi^2-2\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_z\Big)_z+(-U')\phi^2\\&+2\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi\phi_z+2\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2.\end{aligned}$$

Substituting the above result into (4.3), we deduce, after integrating the resulting equation with respect to $z$ over $\mathbb{R}$, that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\phi^2dz+\int_{\mathbb{R}}(-U')\phi^2dz+2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2dz\\=&-2\int_{\mathbb{R}}\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi\phi_zdz+\int_{\mathbb{R}}(-\phi_z^2+2F)\phi dz.\end{aligned} \tag{4.4}$$

Here, since $-1<U'<0$ for all $z\in\mathbb{R}$ by inequality (3.4), the coefficients of $\phi^2$ and $\phi_z^2$ on the left had side of Equation (4.4) are positive for all $z\in\mathbb{R}$. From Equation (3.13), denote

$$F(U',\phi_{zz})=K_1(U',\phi_{zz})\phi_{zz}^2, \tag{4.5}$$

where $K_1$ is a continuous function of its variables. Since $U'$ is bounded for all $z\in\mathbb{R}$ by (3.4), there is a constant $M_1>0$ such that

$$M_1=\max_{z\in\mathbb{R}}|K_1(U',\phi_{zz})|. \tag{4.6}$$

Hence, it follows

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\phi^2dz+\int_{\mathbb{R}}|U'|\phi^2dz+2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2dz\\\le&\int_{\mathbb{R}}\Big|2\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi\phi_z\Big|dz+\int_{\mathbb{R}}|\phi\phi_z^2|dz+2M_1\int_{\mathbb{R}}|\phi\phi_{zz}^2|dz.\end{aligned} \tag{4.7}$$

Next we estimate the first term on the right hand side of inequality (4.7). Using Equation (3.8), one can directly calculate as

$$\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'=\frac{-2U'(3-U'^2)}{(1+U'^2)^3}U''=\frac{-2U'^2(U-c)(3-U'^2)}{1-U'^4}, \tag{4.8}$$

and employ the Cauchy-Schwarz inequality to estimate as

$$\begin{aligned}\Big|2\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi\phi_z\Big|=&\Big|\frac{4U'^2(U-c)(3-U'^2)}{1-U'^4}\phi\phi_z\Big|\\\le&\varepsilon_1|U'|\phi^2+\frac{1}{\varepsilon_1}\frac{4|U'|^3(U-c)^2(3-U'^2)^2}{(1-U'^4)^2}\phi_z^2\end{aligned}$$

$$=: I_1 + I_2, \tag{4.9}$$

where $0 < \varepsilon_1 < 1$ to be determined. Further, we have $(U-c)^2 = \frac{u_l^2}{4} + \frac{2U'}{1+U'^2}$ from Equation (3.3). Hence, for

$$0 < u_l \leq 1.8, \tag{4.10}$$

we can find $0 < \varepsilon_2 < 2$ such that

$$\begin{aligned} I_2 &= \frac{4}{\varepsilon_1}\Big(\frac{u_l^2}{4} + \frac{2U'}{1+U'^2}\Big)\frac{|U'|^3(3-U'^2)^2}{(1-U'^4)^2}\phi_z^2 \\ &\leq \varepsilon_2 \frac{1-U'^2}{(1+U'^2)^2}\phi_z^2. \end{aligned} \tag{4.11}$$

For simplicity for notation, we adopt

$$\varepsilon_1 = \frac{1}{2} \quad \text{and} \quad \varepsilon_2 = \frac{3}{2} \tag{4.12}$$

in the rest of the paper. By inequalities (4.9) and (4.11), we have

$$\Big|2\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi\phi_z\Big| \leq \frac{1}{2}|U'|\phi^2 + \frac{3}{2}\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2 \tag{4.13}$$

for all $z \in \mathbb{R}$. Plugging inequality (4.13) into inequality (4.7), we can deduce

$$\frac{d}{dt}\int_{\mathbb{R}}\phi^2 dz + \frac{1}{2}\int_{\mathbb{R}}|U'|\phi^2 dz + \frac{1}{2}\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2 dz \leq \int_{\mathbb{R}}|\phi\phi_z^2|dz + 2M_1\int_{\mathbb{R}}|\phi\phi_{zz}^2|dz. \tag{4.14}$$

From inequality (3.5), we derive

$$\frac{d}{dt}\int_{\mathbb{R}}\phi^2 dz + \frac{1}{2}\int_{\mathbb{R}}|U'|\phi^2 dz + \frac{m_0}{2}\int_{\mathbb{R}}\phi_z^2 dz \leq \int_{\mathbb{R}}|\phi\phi_z^2|dz + 2M_1\int_{\mathbb{R}}|\phi\phi_{zz}^2|dz. \tag{4.15}$$

Inequality (3.5) is a key element in order to carry out the energy estimates. Consequently, inequality (4.1) is proved by integrating inequality (4.15) with respect to $t$. □

Remark 4.1. Among the smooth traveling waves with $0 < |u_l - u_r| < 2$, we are able to prove the stability of large amplitude waves $0 < |u_l - u_r| \leq 1.8$. This is due to our technique. With a proper choice of $\varepsilon_1$ and $\varepsilon_2$, the upper bound of $u_l$ in Lemma 4.1 can be improved to be beyond 1.8. For simplicity of presentation, we adopt an upper bound of $u_l$ as 1.8.

Next the estimates for the first derivative of $\phi$ are shown.

Lemma 4.2 ($H^1$-estimates). *Under the same assumptions in Lemma 4.1, there is some $C > 0$ such that*

$$\begin{aligned} &\|\phi_z(\cdot,t)\|^2 + 2m_0\int_0^t \|\phi_{zz}(\cdot,\tau)\|^2 d\tau \\ \leq& C\|\phi_0\|_1^2 + C\int_0^t\int_{\mathbb{R}}(G_1(\phi,\phi_z,\phi_{zz}) + G_2(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau \end{aligned} \tag{4.16}$$

*for all $t>0$, where $m_0>0$, which is given in inequality (3.5), and $G_1(\phi,\phi_z,\phi_{zz})$ is defined in Equation (4.2), and we denote*

$$G_2(\phi_z,\phi_{zz},\phi_{zzz})=|\phi_z\phi_{zz}^2|+|\phi_z\phi_{zz}\phi_{zzz}|. \tag{4.17}$$

*Proof.* Differentiating Equation (3.14) with respect to $z$ and multiplying the result by $2\phi_z$, we obtain

$$2\phi_{tz}\phi_z+2((U-c)\phi_z)_z\phi_z-2\Big(\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}\Big)_z\phi_z=(-\phi_z^2)_z\phi_z+2F_z\phi_z. \tag{4.18}$$

Integrating Equation (4.18) with respect to $z$ over $\mathbb{R}$ and using integration by parts give us

$$\frac{d}{dt}\int_{\mathbb{R}}\phi_z^2dz+2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}^2dz=\int_{\mathbb{R}}(-U')\phi_z^2+2\int_{\mathbb{R}}F_z\phi_zdz, \tag{4.19}$$

where we have used that $\int_{\mathbb{R}}-(\phi_z)_z^2\phi_zdz=0$ since both $(-\phi_z^2)_z\phi_z=(-\phi_z^2\phi_z)_z+\phi_z^2\phi_{zz}$ and $(-\phi_z^2)_z\phi_z=-2\phi_z^2\phi_{zz}$ hold.

Next, let us estimate the right hand side of Equation (4.19). By direct calculation, one obtains

$$\begin{aligned}F_z&=K_{1z}\phi_{zz}^2+2K_1\phi_{zz}\phi_{zzz}\\&=:K_2(U',U'',\phi_{zz})\phi_{zz}^2+K_3(U',\phi_{zz})\phi_{zz}\phi_{zzz},\end{aligned} \tag{4.20}$$

where $K_1$ is given in Equation (4.5). Then one can easily check that $K_2$ and $K_3$ are continuous functions of its variables and its variables are bounded. Let

$$M_2=\max_{z\in\mathbb{R}}|K_2(U',U'',\phi_{zz})| \quad\text{and}\quad M_3=\max_{z\in\mathbb{R}}|K_3(U',\phi_{zz})|, \tag{4.21}$$

where $M_2$ and $M_3$ are positive. By inequality (3.5) and bounds(4.21), we deduce

$$\frac{d}{dt}\int_{\mathbb{R}}\phi_z^2dz+2m_0\int_{\mathbb{R}}\phi_{zz}^2dz\le\int_{\mathbb{R}}|U'|\phi_z^2dz+2M_2\int_{\mathbb{R}}|\phi_z\phi_{zz}^2|dz+2M_3\int_{\mathbb{R}}|\phi_z\phi_{zz}\phi_{zzz}|dz$$

from Equation (4.19).

Since $U'$ is bounded for $z\in\mathbb{R}$, if we integrate the above result with respect to $t$, there is some $C>0$ such that

$$\begin{aligned}&\|\phi_z(\cdot,t)\|^2+2m_0\int_0^t\|\phi_{zz}(\cdot,\tau)\|^2d\tau\\&\le C\|\phi_{z,0}\|^2+C\int_0^t\|\phi_z(\cdot,\tau)\|^2d\tau+C\int_0^t\int_{\mathbb{R}}G_2(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\&\le C\|\phi_0\|_1^2+C\int_0^t\int_{\mathbb{R}}(G_1(\phi,\phi_z,\phi_{zz})+G_2(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau\end{aligned} \tag{4.22}$$

for any $t>0$, where $\int_0^t\|\phi_z(\cdot,\tau)\|^2d\tau$ is estimated by the $L^2$-estimates (4.1), and $G_1(\phi,\phi_z,\phi_{zz})$ and $G_2(\phi_z,\phi_{zz},\phi_{zzz})$ are defined in Equations (4.2) and (4.17), respectively. Therefore, we get the desired $H^1$-estimates (4.16). ∎

Finally, we derive the estimates for the second derivative of $\phi$.

Lemma 4.3 ($H^2$-estimates). *Under the same assumptions of Lemma 4.1, there exists some $C>0$ such that for any $t>0$*

$$\begin{aligned}&\|\phi_{zz}(\cdot,t)\|^2+2m_0\int_0^t\|\phi_{zzz}(\cdot,\tau)\|^2d\tau\\ \le&C\|\phi_0\|_2^2+C\int_0^t\int_{\mathbb{R}}(G_1(\phi,\phi_z,\phi_{zz})+G_2(\phi_z,\phi_{zz},\phi_{zzz})+G_3(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau,\end{aligned}\tag{4.23}$$

*where $m_0>0$, which is defined in (3.5), and $G_1(\phi,\phi_z,\phi_{zz})$ and $G_2(\phi_z,\phi_{zz},\phi_{zzz})$ are given in Equations (4.2) and (4.17), respectively, and we denote*

$$G_3(\phi_z,\phi_{zz},\phi_{zzz})=|\phi_z\phi_{zz}\phi_{zzz}|+|\phi_{zz}^2\phi_{zzz}|+|\phi_{zz}\phi_{zzz}^2|.\tag{4.24}$$

*Proof.* Differentiating Equation (3.14) with respect to $z$ twice and multiplying the result by $2\phi_{zz}$, we have

$$2\phi_{tzz}\phi_{zz}+2((U-c)\phi_z)_{zz}\phi_{zz}-2\Big(\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}\Big)_{zz}\phi_{zz}=(-\phi_z^2)_{zz}\phi_{zz}+2F_{zz}\phi_{zz}.\tag{4.25}$$

Integrate Equation (4.25) with respect to $z$ over $\mathbb{R}$ and use integration by parts. Then it holds

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}\phi_{zz}^2dz+2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\phi_{zzz}^2dz\\ =&\int_{\mathbb{R}}U'''\phi_z^2dz+\int_{\mathbb{R}}\Big(-3U'+\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)''\Big)\phi_{zz}^2dz\\ &+2\int_{\mathbb{R}}(\phi_z\phi_{zz}\phi_{zzz}-K_2\phi_{zz}^2\phi_{zzz}-K_3\phi_{zz}\phi_{zzz}^2)dz,\end{aligned}\tag{4.26}$$

where $K_2$ and $K_3$ are given in Equation (4.20).

Indeed, integrating Equation (4.26) with respect to $t$, we deduce, by using (3.5), (4.21), and the boundedness of $U'$, $U'''$ and $\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)''$, that there exists some $C>0$ such that for any $t>0$

$$\begin{aligned}&\|\phi_{zz}(\cdot,t)\|^2+2m_0\int_0^t\|\phi_{zzz}(\cdot,\tau)\|^2d\tau\\ \le&\|\phi_{zz,0}\|^2+C\int_0^t\|\phi_z(\cdot,\tau)\|_1^2d\tau+C\int_0^t\int_{\mathbb{R}}G_3(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\ \le&C\|\phi_0\|_2^2+C\int_0^t\int_{\mathbb{R}}(G_1(\phi,\phi_z,\phi_{zz})+G_2(\phi_z,\phi_{zz},\phi_{zzz})+G_3(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau,\end{aligned}\tag{4.27}$$

where $\int_0^t\|\phi_z(\cdot,\tau)\|_1^2d\tau$ is estimated by the $L^2$-estimates (4.1) and the $H_1$-estimates (4.16), and $G_1(\phi,\phi_z,\phi_{zz})$, $G_2(\phi_z,\phi_{zz},\phi_{zzz})$ and $G_3(\phi_z,\phi_{zz},\phi_{zzz})$ are given in Equations (4.2), (4.17), and (4.24), respectively. Therefore the proof of (4.23) is completed. □

Now combining the results from the $L^2$-estimates (4.1), the $H^1$-estimates (4.16) and the $H^2$-estimates (4.23), we conclude that there is some $C>0$ such that

$$\|\phi(\cdot,t)\|_2^2+\frac{1}{2}\int_0^t\|\sqrt{|U'|}\phi(\cdot,\tau)\|^2d\tau+\frac{m_0}{2}\int_0^t\|\phi_z(\cdot,\tau)\|_2^2d\tau$$

$$\le C\|\phi_0\|_2^2 + C\int_0^t\int_{\mathbb{R}}(G_1(\phi,\phi_z,\phi_{zz})+G_2(\phi_z,\phi_{zz},\phi_{zzz})+G_3(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau \quad (4.28)$$

for all $t>0$, where $m_0>0$ is defined in (3.5), and $G_1(\phi,\phi_z,\phi_{zz}),G_2(\phi_z,\phi_{zz},\phi_{zzz})$ and $G_3(\phi_z,\phi_{zz},\phi_{zzz})$ are collections of cubic terms in Equations (4.2), (4.17), and (4.24), respectively.

By using the Cauchy-Schwarz inequality and and (3.19), the cubic terms are majored by $CN(t)\int_0^t\|\phi_z(\cdot,\tau)\|_2^2 d\tau$ for some $C>0$. For instance, one obtains for $0\le t<T$,

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}|\phi_z\phi_{zz}\phi_{zzz}|dzd\tau\\ \le&\frac{1}{2}\Big(\int_0^t\int_{\mathbb{R}}|\phi_z|\phi_{zz}^2dzd\tau+\int_0^t\int_{\mathbb{R}}|\phi_z|\phi_{zzz}^2dzd\tau\Big)\\ \le&CN(t)\int_0^t\|\phi_{zz}(\cdot,\tau)\|_1^2d\tau\end{aligned} \quad (4.29)$$

for some $C>0$. The other cubic terms can be estimated in a similar manner. Therefore inequality (4.28) is estimated by

$$\begin{aligned}&\|\phi(\cdot,t)\|_2^2+\frac{1}{2}\int_0^t\|\sqrt{|U'|}\phi(\cdot,\tau)\|^2d\tau+\frac{m_0}{2}\int_0^t\|\phi_z(\cdot,\tau)\|_2^2d\tau\\ \le&C\|\phi_0\|_2^2+CN(t)\int_0^t\|\phi_z(\cdot,\tau)\|_2^2d\tau\end{aligned} \quad (4.30)$$

for any $0\le t\le T$ and for some $C>0$.

Now take a small $\delta_2>0$ in Proposition 3.3 so that $4C\delta_2<m_0$. Then, for any $0\le t\le T$, we finally have

$$\|\phi(\cdot,t)\|_2^2+\int_0^t\|\sqrt{|U'|}\phi(\cdot,\tau)\|^2d\tau+\nu\int_0^t\|\phi_z(\cdot,\tau)\|_2^2d\tau\le CN^2(0)$$

for some $\nu=\nu(m_0)>0$ and $C>0$, which is the desired result of (3.25) in Proposition 3.3.

By the local existence in Proposition 3.2, the a priori estimates in Proposition 3.3 and the continuation arguments, the proof of Theorem 3.1 is finally completed. Therefore Theorem 2.1 is proved.

## 5. Rate of convergence: algebraic decay

In this section, we prove Theorem 2.2. To be specific, we are going to prove that the perturbation decays algebraically in time if the initial perturbation converges algebraically in space by a weighted energy method. This way of showing the rate of asymptotic convergence was introduced by Kawashima and Matsumura in [6] and Matsumura and Nishihara in [21]. We also refer to other papers [12,18,19,22,23,25,26].

To define a weight function $w(z)$, we choose a constant $z_*\in\mathbb{R}$ such that

$$U(z_*)=\frac{u_l}{2}=c. \quad (5.1)$$

Indeed, since $U'<0$ and $0<U<u_l$ for any $z\in\mathbb{R}$ by inequality (3.4), $z_*$ is uniquely determined and satisfies

$$(U-c)(z-z_*)\le 0 \quad \text{for any } z\in\mathbb{R}, \quad (5.2)$$

which plays an essential role in the following estimates (see Lemmas 5.1, 5.2, 5.3, 5.4).

Let $\alpha \geq 0$. Define a weight function

$$w(z) = \langle z - z_* \rangle^{\alpha}, \tag{5.3}$$

the weighted solution space of the problem (3.14), (3.15)

$$X_\alpha = \{\phi(z,t) : \phi \in C([0,T); H^2_\alpha),\ \ \phi_z \in L^2((0,T); H^2_\alpha)\} \tag{5.4}$$

with $0 < T < \infty$, and set

$$N_\alpha(t) = \sup_{0 \leq \tau \leq t} \{\|\phi(\cdot,\tau)\|_{0,\alpha} + \|\phi_z(\cdot,\tau)\|_1\} \tag{5.5}$$

for all $t \geq 0$. Theorem 2.2 is a result of the following theorem.

THEOREM 5.1. *Suppose that the assumptions of Theorem 2.2 hold. Let $\gamma$ be any number such that $0 \leq \gamma \leq \alpha$ if $\alpha$ is an integer and that $0 \leq \gamma < \alpha$ if $\alpha$ is a non-integer. Then there are positive constants $\delta_3$ and $C$ such that if*

$$N_\alpha(0) < \delta_3,$$

*then the problem (3.14), (3.15) has a unique global solution $\phi \in X_\alpha(0,+\infty)$ satisfying that for all $t > 0$,*

$$(1+t)^\gamma \|\phi(\cdot,t)\|_2^2 + \nu_\alpha \int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_2^2 d\tau \leq C N_\alpha^2(0) \tag{5.6}$$

*for some $\nu_\alpha = \nu_\alpha(m_0) > 0$, where $m_0$ is given in (3.5).*

PROPOSITION 5.1 (Local existence). *For any $\delta > 0$, there exists a constant $T > 0$ depending on $\delta$ such that if $\phi_0 \in H^2_\alpha(\mathbb{R})$ and $2N_\alpha(0) < \delta$, the problem (3.14), (3.15) has a unique solution $\phi \in X_\alpha(0,T)$ such that for any $0 \leq t \leq T$*

$$N_\alpha(t) < 2N_\alpha(0).$$

PROPOSITION 5.2 (A priori estimates). *Assume that $\phi \in X_\alpha(0,T)$ is a solution obtained from Proposition 5.1 for a constant $T > 0$. Then there are positive constants $\delta_4$ and $C$ independent of $T$ such that for any $0 \leq t \leq T$ if*

$$N_\alpha(t) < \delta_4,$$

*then the solution $\phi$ of the problem* (3.14), (3.15) *satisfies*

$$(1+t)^\gamma \|\phi(\cdot,t)\|_2^2 + \nu_\alpha \int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_2^2 d\tau \leq C N_\alpha^2(0) \tag{5.7}$$

*for some $\nu_\alpha = \nu_\alpha(m_0) > 0$, where $m_0$ is defined in (3.5).*

Now we are going to prove the above a priori estimates (5.7) in the following subsections. In Subsection 5.1, we establish weighted energy estimates. With a result from Subsection 5.1, we continue to establish the desired a priori estimates (5.7) in Subsection 5.2.

**5.1. Weighted a priori estimates.** To proceed to weighted a priori estimates of the solution $\phi$ of the problem (3.14), (3.15), let us fist establish the weighted $L^2$-estimates.

Multiplying Equation (3.14) by a time dependent weight $2(1+t)^\gamma\langle z-z_*\rangle^\beta\phi$ for some $0\le\gamma,\beta\le\alpha$, where $z_*$ is defined in (5.1), we have

$$\begin{aligned}&2(1+t)^\gamma\langle z-z_*\rangle^\beta\phi\phi_t+2(1+t)^\gamma\langle z-z_*\rangle^\beta(U-c)\phi\phi_z-2(1+t)^\gamma\langle z-z_*\rangle^\beta\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_{zz}\\&=-(1+t)^\gamma\langle z-z_*\rangle^\beta\phi\phi_z^2+2(1+t)^\gamma\langle z-z_*\rangle^\beta F\phi.\end{aligned}\tag{5.8}$$

The left hand side of Equation (5.8) can be expressed as

$$\begin{aligned}&2(1+t)^\gamma\langle z-z_*\rangle^\beta\phi\phi_t+2(1+t)^\gamma\langle z-z_*\rangle^\beta(U-c)\phi\phi_z-2(1+t)^\gamma\langle z-z_*\rangle^\beta\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_{zz}\\&=((1+t)^\gamma\langle z-z_*\rangle^\beta\phi^2)_t\\&\quad+\Big((1+t)^\gamma\langle z-z_*\rangle^\beta(U-c)\phi^2-2(1+t)^\gamma\langle z-z_*\rangle^\beta\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_z\Big)_z\\&\quad-\gamma(1+t)^{\gamma-1}\langle z-z_*\rangle^\beta\phi^2-\beta(1+t)^\gamma\langle z-z_*\rangle^{\beta-2}(z-z_*)(U-c)\phi^2\\&\quad-(1+t)^\gamma\langle z-z_*\rangle^\beta U'\phi^2+2\beta(1+t)^\gamma\langle z-z_*\rangle^{\beta-2}(z-z_*)\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_z\\&\quad+2(1+t)^\gamma\langle z-z_*\rangle^\beta\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi\phi_z+2(1+t)^\gamma\langle z-z_*\rangle^\beta\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2.\end{aligned}$$

Substituting the above result into Equation (5.8) and integrating the resulting equation with respect to $z$ over $\mathbb{R}$ give us

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta\phi^2dz-\beta\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^{\beta-2}(z-z_*)(U-c)\phi^2dz\\&\qquad+\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta(-U')\phi^2dz+2\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2dz\\&=\gamma\int_{\mathbb{R}}(1+t)^{\gamma-1}\langle z-z_*\rangle^\beta\phi^2dz-2\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi\phi_zdz\\&\qquad-2\beta\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^{\beta-2}(z-z_*)\frac{1-U'^2}{(1+U'^2)^2}\phi\phi_zdz\\&\qquad+\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta(-\phi\phi_z^2+2F\phi)dz.\end{aligned}\tag{5.9}$$

Now we shall further estimate the right hand side of Equation (5.9). For $0<u_l\le 1.8$, the second term on the right hand side of Equation (5.9) is estimated by using

$$\begin{aligned}&2\int_{\mathbb{R}}\langle z-z_*\rangle^\beta\Big|\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\Big||\phi\phi_z|dz\\&\le\frac{1}{2}\int_{\mathbb{R}}\langle z-z_*\rangle^\beta(-U')\phi^2dz+\frac{3}{2}\int_{\mathbb{R}}\langle z-z_*\rangle^\beta\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2dz\end{aligned}$$

from inequality (4.13). Plugging the above result into Equation (5.9), we arrive at

$$\frac{d}{dt}\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta\phi^2dz+\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^{\beta-1}A_\beta(z)\phi^2dz$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}\frac{1-U'^2}{(1+U'^2)^2}\phi_z^2dz\\
\leq&\gamma\int_{\mathbb{R}}(1+t)^{\gamma-1}\langle z-z_*\rangle^{\beta}\phi^2dz+2\beta\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta-1}\frac{1-U'^2}{(1+U'^2)^2}|\phi\phi_z|dz\\
&+\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}(|\phi\phi_z^2|+2|F\phi|)dz,
\end{aligned}
\tag{5.10}
$$

where

$$
A_\beta(z):=-\beta\frac{(z-z_*)(U-c)}{\langle z-z_*\rangle}+\frac{1}{2}\langle z-z_*\rangle(-U'). \tag{5.11}
$$

LEMMA 5.1. *For $\alpha\geq 0$, let $\beta\in[0,\alpha]$. Then there is $c_0>0$, which is independent of $\beta$, such that for any $z\in\mathbb{R}$*

$$
A_\beta(z)\geq c_0\beta \tag{5.12}
$$

*where $A_\beta(z)$ is defined in (5.11).*

*Proof.* By inequalities (3.4) and (5.2), we have $-\beta\frac{(z-z_*)(U-c)}{\langle z-z_*\rangle}\geq 0$ and $\frac{1}{2}\langle z-z_*\rangle(-U')>0$ for any $z\in\mathbb{R}$. Particularly, it holds that $-\frac{(z-z_*)(U-c)}{\langle z-z_*\rangle}=0$ if and only if $z=z_*$. Hence, for any $\eta>0$ there exists some $c(\eta)>0$ such that

$$
-\frac{(z-z_*)(U-c)}{\langle z-z_*\rangle}\geq c(\eta)>0 \quad \text{for} \quad |z-z_*|\geq\eta. \tag{5.13}
$$

On the other hand, since $-U'$ has a maximum at $z=z_*$ by inequality (3.4), we can pick positive constants $\eta_0$ and $m_1$ satisfying

$$
\frac{1}{2}\langle z-z_*\rangle(-U')\geq m_1\geq\frac{\beta}{\alpha}m_1>0 \quad \text{for} \quad |z-z_*|\leq\eta_0. \tag{5.14}
$$

Letting

$$
c_0=\min\left\{c(\eta_0),\frac{m_1}{\alpha}\right\}>0, \tag{5.15}
$$

we conclude that

$$
A_\beta(z)=-\beta\frac{(z-z_*)(U-c)}{\langle z-z_*\rangle}+\frac{1}{2}\langle z-z_*\rangle(-U')\geq c_0\beta
$$

for all $z\in\mathbb{R}$. ∎

Now we integrate inequality (5.10) with respect to $t$. Using (4.6), (3.5), (5.12) and the boundedness of $U'$ and estimating $F$ by (4.6) yield

$$
\begin{aligned}
&(1+t)^{\gamma}\|\phi(\cdot,t)\|_{0,\beta}^2+c_0\beta\int_0^t(1+\tau)^{\gamma}\|\phi(\cdot,\tau)\|_{0,\beta-1}^2d\tau+\frac{m_0}{2}\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_{0,\beta}^2d\tau\\
\leq&\|\phi_0\|_{0,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{0,\beta}^2d\tau+C\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta-1}|\phi\phi_z|dzd\tau\\
&+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta}G_1(\phi,\phi_z,\phi_{zz})dzd\tau
\end{aligned}
\tag{5.16}
$$

for all $t>0$ and for some $C>0$, where $G_1(\phi,\phi_z,\phi_{zz})$ is defined in Equation (4.2).

Let us estimate the third term on the right hand side of inequality (5.16). By the Cauchy-Schwarz inequality, we have

$$C\beta\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta-1}|\phi\phi_z|dz\le\frac{c_0\beta}{2}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta-1}\phi^2dz+\frac{C^2\beta}{2c_0}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta-1}\phi_z^2dz. \tag{5.17}$$

In particular, the second term on the right hand side of inequality (5.17) can be further estimated as

$$\begin{aligned}&\frac{C^2\beta}{2c_0}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta-1}\phi_z^2dz\\&=\frac{C^2\beta}{2c_0}\int_{|z-z_*|>r}\frac{1}{\langle z-z_*\rangle}\langle z-z_*\rangle^{\beta}\phi_z^2dz+\frac{C^2\beta}{2c_0}\int_{|z-z_*|\le r}\langle z-z_*\rangle^{\beta-1}\phi_z^2dz\\&\le\frac{m_0}{4}\|\phi_z(\cdot,t)\|_{0,\beta}^2+\beta C_r\|\phi_z(\cdot,t)\|^2\end{aligned} \tag{5.18}$$

for a suitable $r>0$ and some $C_r>0$. That is, the third term on the right side of (5.16) is estimated by

$$\begin{aligned}&\frac{c_0\beta}{2}\int_0^t(1+\tau)^\gamma\|\phi(\cdot,\tau)\|_{0,\beta-1}^2d\tau\\&\qquad+\frac{m_0}{4}\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{0,\beta}^2d\tau+\beta C_r\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|^2d\tau\end{aligned} \tag{5.19}$$

for all $t>0$. Insert (5.19) into (5.16) and estimate the first and second terms in (5.19) by the left hand side of (5.16). Then we finally obtain the following lemma.

LEMMA 5.2 (Weighted $L^2$-estimates). *Let $\alpha\ge0$ and $\beta$ and $\gamma$ be any numbers in $[0,\alpha]$. Under the same conditions in Theorem 5.1, there is some $C>0$, which is independent of $\beta$ and $\gamma$,*

$$\begin{aligned}&(1+t)^\gamma\|\phi(\cdot,t)\|_{0,\beta}^2+\frac{c_0\beta}{2}\int_0^t(1+\tau)^\gamma\|\phi(\cdot,\tau)\|_{0,\beta-1}^2d\tau+\frac{m_0}{4}\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{0,\beta}^2d\tau\\&\le\|\phi_0\|_{0,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{0,\beta}^2d\tau+C\beta\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|^2d\tau\\&\quad+C\int_0^t\int_{\mathbb{R}}(1+\tau)^\gamma\langle z-z_*\rangle^\beta G_1(\phi,\phi_z,\phi_{zz})dzd\tau\end{aligned} \tag{5.20}$$

*for any $t>0$, where positive constants $c_0$ and $m_0$ are defined in (5.15) and (3.5), respectively, and $G_1(\phi,\phi_z,\phi_{zz})$ is given in (4.2).*

Next we derive the weighted $H^1$- and $H^2$-estimates.

LEMMA 5.3 (Weighted $H^1$-estimates). *Under the same assumptions in Lemma 5.2, it satisfies that for all $t>0$*

$$\begin{aligned}&(1+t)^\gamma\|\phi_z(\cdot,t)\|_{0,\beta}^2+\frac{c_0\beta}{2}\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{0,\beta-1}^2d\tau+m_0\int_0^t(1+\tau)^\gamma\|\phi_{zz}(\cdot,\tau)\|_{0,\beta}^2d\tau\\&\le C\Big(\|\phi_0\|_{1,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{1,\beta}^2d\tau+\beta\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_1^2d\tau\end{aligned}$$

$$+\int_0^t\int_{\mathbb{R}}(1+\tau)^\gamma\langle z-z_*\rangle^\beta(G_1(\phi,\phi_z,\phi_{zz})+G_4(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau\Big), \tag{5.21}$$

*where a positive constant $C$ is independent of $\beta$ and $\gamma$, $c_0>0$ and $m_0>0$, which are given in (5.15) and (3.5), respectively, $G_1(\phi,\phi_z,\phi_{zz})$ is defined in (4.2), and we denote*

$$G_4(\phi_z,\phi_{zz},\phi_{zzz})=|\phi_z^2\phi_{zz}|+|\phi_z\phi_{zz}^2|+|\phi_z\phi_{zz}\phi_{zzz}|. \tag{5.22}$$

*Proof.* Differentiating Equation (3.14) with respect to $z$ and multiplying the resulting equation by $2(1+t)^\gamma\langle z-z_*\rangle^\beta\phi_z$, we have

$$\begin{aligned}&2(1+t)^\gamma\langle z-z_*\rangle^\beta\phi_{tz}\phi_z\\&\quad+2(1+t)^\gamma\langle z-z_*\rangle^\beta((U-c)\phi_z)_z\phi_z-2(1+t)^\gamma\langle z-z_*\rangle^\beta\Big(\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}\Big)_z\phi_z\\&=-2(1+t)^\gamma\langle z-z_*\rangle^\beta\phi_z^2\phi_{zz}+2(1+t)^\gamma\langle z-z_*\rangle^\beta F_z\phi_z.\end{aligned} \tag{5.23}$$

Integrate Equation (5.23) with respect to $z$ over $\mathbb{R}$ and use integration by parts. It then follows

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta\phi_z^2dz\\&\quad+\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^{\beta-1}A_\beta(z)\phi_z^2dz+2\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}^2dz\\&=\gamma\int_{\mathbb{R}}(1+t)^{\gamma-1}\langle z-z_*\rangle^\beta\phi_z^2dz+\frac{3}{2}\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta(-U')\phi_z^2dz\\&\quad-2\beta\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^{\beta-2}(z-z_*)\frac{1-U'^2}{(1+U'^2)^2}\phi_z\phi_{zz}dz\\&\quad+2\int_{\mathbb{R}}(1+t)^\gamma\langle z-z_*\rangle^\beta(-\phi_z^2\phi_{zz}+F_z\phi_z)dz,\end{aligned} \tag{5.24}$$

where $A_\beta$ is given in Equation (5.11).

Next, let us integrate Equation (5.24) with respect to $t$. Then, using (3.5), (4.21), (5.12), and the boundedness of $U'$ gives us

$$\begin{aligned}&(1+t)^\gamma\|\phi_{\cdot,t}\|_{0,\beta}^2+c_0\beta\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{0,\beta-1}^2d\tau+2m_0\int_0^t(1+\tau)^\gamma\|\phi_{zz}(\cdot,\tau)\|_{0,\beta}^2d\tau\\&\le\|\phi_{z,0}\|_{0,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi_z(\cdot,\tau)\|_{0,\beta}^2d\tau+C\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{0,\beta}^2d\tau\\&\quad+C\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^\gamma\langle z-z_*\rangle^{\beta-1}|\phi_z\phi_{zz}|dzd\tau\\&\quad+C\int_0^t\int_{\mathbb{R}}(1+\tau)^\gamma\langle z-z_*\rangle^\beta G_4(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\end{aligned} \tag{5.25}$$

for all $t>0$ and for some $C>0$, where $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ is given in Equation (5.22). By a similar argument used in (5.17) and (5.18), the fourth term on the right hand side of (5.25) is estimated as

$$C\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^\gamma\langle z-z_*\rangle^{\beta-1}|\phi_z\phi_{zz}|dzd\tau$$

$$\leq \frac{c_0\beta}{2}\int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_{0,\beta-1}^2 d\tau + m_0\int_0^t (1+\tau)^\gamma \|\phi_{zz}(\cdot,\tau)\|_{0,\beta}^2 d\tau$$
$$+\beta C_{\tilde r}\int_0^t (1+\tau)^\gamma \|\phi_{zz}(\cdot,\tau)\|^2 d\tau \qquad (5.26)$$

for some $C_{\tilde r}>0$. Substituting inequality (5.26) into inequality (5.25), the first and second terms on the right hand side of inequality (5.26) are absorbed in the second and third terms on the left hand side of inequality (5.25). Then there is some $C>0$, independent of $\beta$ and $\gamma$, such that for any $t>0$

$$(1+t)^\gamma \|\phi_z(\cdot,t)\|_{0,\beta}^2 + \frac{c_0\beta}{2}\int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_{0,\beta-1}^2 d\tau + m_0\int_0^t (1+\tau)^\gamma \|\phi_{zz}(\cdot,\tau)\|_{0,\beta}^2 d\tau$$
$$\leq \|\phi_{z,0}\|_{0,\beta}^2 + \gamma\int_0^t (1+\tau)^{\gamma-1}\|\phi_z(\cdot,\tau)\|_{0,\beta}^2 d\tau + C\int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_{0,\beta}^2 d\tau$$
$$+C\beta\int_0^t (1+\tau)^\gamma \|\phi_{zz}(\cdot,\tau)\|^2 d\tau + C\int_0^t\int_{\mathbb{R}} (1+\tau)^\gamma \langle z-z_*\rangle^\beta G_4(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau$$
$$\leq C\Big(\|\phi_0\|_{1,\beta}^2 + \gamma\int_0^t (1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{1,\beta}^2 d\tau + \beta\int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_1^2 d\tau$$
$$+\int_0^t\int_{\mathbb{R}} (1+\tau)^\gamma \langle z-z_*\rangle^\beta (G_1(\phi,\phi_z,\phi_{zz}) + G_4(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau\Big), \qquad (5.27)$$

where $\int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_{0,\beta}^2 d\tau$ is estimated by the weighted $L^2$-estimates (5.20), and $G_1(\phi,\phi_z,\phi_{zz})$ is defined in (4.2). Therefore, the proof of inequality (5.21) is completed. □

LEMMA 5.4 (Weighted $H^2$-estimates). *Under the same assumptions in Lemma 5.2, we have that for any $t>0$*

$$(1+t)^\gamma \|\phi_{zz}(\cdot,t)\|_{0,\beta}^2 + \frac{c_0\beta}{4}\int_0^t (1+\tau)^\gamma \|\phi_{zz}(\cdot,\tau)\|_{0,\beta-1}^2 d\tau + \frac{m_0}{2}\int_0^t (1+\tau)^\gamma \|\phi_{zzz}(\cdot,\tau)\|_{0,\beta}^2 d\tau$$
$$\leq C\Big(\|\phi_0\|_{2,\beta}^2 + \gamma\int_0^t (1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{2,\beta}^2 d\tau + \beta\int_0^t (1+\tau)^\gamma \|\phi_z(\cdot,\tau)\|_2^2 d\tau$$
$$+\beta\int_0^t\int_{\mathbb{R}} (1+\tau)^\gamma \langle z-z_*\rangle^{\beta-1} G_5(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau$$
$$+\int_0^t\int_{\mathbb{R}} (1+\tau)^\gamma \langle z-z_*\rangle^\beta (G_1(\phi,\phi_z,\phi_{zz}) + G_4(\phi_z,\phi_{zz},\phi_{zzz}) + G_6(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau\Big) \qquad (5.28)$$

*for some $C>0$ which is independent of $\beta$ and $\gamma$, where positive constants $c_0$ and $m_0$ are given in (5.15) and (3.5), respectively. In Equations (4.2) and (5.22), $G_1(\phi,\phi_z,\phi_{zz})$ and $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ are defined respectively. We denote*

$$G_5(\phi_z,\phi_{zz},\phi_{zzz}) = |\phi_z\phi_{zz}^2| + |\phi_{zz}^3| + |\phi_{zz}^2\phi_{zzz}| \qquad (5.29)$$

*and*

$$G_6(\phi_z,\phi_{zz},\phi_{zzz}) = |\phi_z\phi_{zz}\phi_{zzz}| + |\phi_{zz}^2\phi_{zzz}| + |\phi_{zz}\phi_{zzz}^2|. \qquad (5.30)$$

*Proof.* Differentiating Equation (3.14) with respect to $z$ twice and multiplying the resulting equation by $2(1+t)^\gamma\langle z-z_*\rangle^\beta \phi_{zz}$, we obtain

$$2(1+t)^\gamma \langle z-z_*\rangle \phi_{tzz}\phi_{zz} + 2(1+t)^\gamma \langle z-z_*\rangle^\beta ((U-c)\phi_z)_{zz}\phi_{zz}$$

$$
\begin{aligned}
&-2(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}\Big(\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}\Big)_{zz}\phi_{zz}\\
=&(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}(-\phi_z^2)_{zz}\phi_{zz}+2(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}F_{zz}\phi_{zz}. 
\end{aligned}\tag{5.31}
$$

Integrating Equation (5.31) with respect to $z$ over $\mathbb{R}$ and using integration by parts, one can show

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}\phi_{zz}^2dz\\
&\quad+\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta-1}A_\beta(z)\phi_{zz}^2dz+2\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}\frac{1-U'^2}{(1+U'^2)^2}\phi_{zzz}^2dz\\
=&\gamma\int_{\mathbb{R}}(1+t)^{\gamma-1}\langle z-z_*\rangle^{\beta}\phi_{zz}^2dz-2\beta\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta-2}(z-z_*)\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi_{zz}^2dz\\
&\quad+\frac{3}{2}\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}(-U')\phi_{zz}^2dz\\
&\quad-2\beta\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta-2}(z-z_*)\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}\phi_{zzz}dz\\
&\quad+2\beta\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta-2}(z-z_*)(\phi_z\phi_{zz}^2-F_z\phi_{zz})dz\\
&\quad+2\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}(\phi_z\phi_{zz}\phi_{zzz}-F_z\phi_{zzz})dz+J_1+J_2+J_3,
\end{aligned}\tag{5.32}
$$

where $A_\beta$ is given in Equation (5.11), and we denote

$$
\begin{aligned}
J_1&=2\beta\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta-2}(z-z_*)U'\phi_z\phi_{zz}dz,\\
J_2&=2\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}U'\phi_z\phi_{zzz}dz,\\
J_3&=-2\int_{\mathbb{R}}(1+t)^{\gamma}\langle z-z_*\rangle^{\beta}\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\phi_{zz}\phi_{zzz}dz.
\end{aligned}
$$

Now we shall estimate the right hand side of Equation (5.32). Firstly, by the Cauchy-Schwarz inequality, $J_1, J_2$ and $J_3$ are estimated as follows:

$$
\begin{aligned}
|J_1|&\le\frac{2\beta}{c_0}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta-1}U'^2\phi_z^2dz+\frac{c_0\beta}{2}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta-1}\phi_{zz}^2dz,\\
|J_2|&\le\frac{2}{m_0}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta}U'^2\phi_z^2dz+\frac{m_0}{2}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta}\phi_{zzz}^2dz,\\
|J_3|&\le\frac{2}{m_0}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta}\Big|\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\Big|^2\phi_{zz}^2dz+\frac{m_0}{2}\int_{\mathbb{R}}\langle z-z_*\rangle^{\beta}\phi_{zzz}^2dz,
\end{aligned}\tag{5.33}
$$

where positive numbers $c_0$ and $m_0$ are defined in (5.15) and (3.5), respectively. After further estimating Equation (5.32) with (3.5), (4.21), (5.12) and the boundedness of $U'$ and $\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'$, insert (5.33) into (5.32). Integrating the result with respect to $t$, one deduces that there exists some $C>0$, independent of $\beta$ and $\gamma$, such that for any $t>0$

$$
\begin{aligned}
&(1+t)^{\gamma}\|\phi_{zz}(\cdot,t)\|_{0,\beta}^2\\
&\quad+\frac{c_0\beta}{2}\int_0^t(1+\tau)^{\gamma}\|\phi_{zz}(\cdot,\tau)\|_{0,\beta-1}^2d\tau+m_0\int_0^t(1+\tau)^{\gamma}\|\phi_{zzz}(\cdot,\tau)\|_{0,\beta}^2d\tau
\end{aligned}
$$

$$\begin{aligned}\le&\|\phi_{zz,0}\|_{0,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi_{zz}(\cdot,\tau)\|_{0,\beta}^2d\tau+C\beta\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_{0,\beta-1}^2d\tau\\&+C\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_{1,\beta}^2d\tau+C\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta-1}|\phi_{zz}\phi_{zzz}|dzd\tau\\&+C\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta-1}G_5(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\&+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta}G_6(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau,\end{aligned}\tag{5.34}$$

where $G_5(\phi_z,\phi_{zz},\phi_{zzz})$ and $G_6(\phi_z,\phi_{zz},\phi_{zzz})$ are given in Equations (5.29) and (5.30), respectively.

We can further estimate the right hand side of inequality (5.34). Using a similar argument from inequalities (5.17) and (5.18), the fifth term of inequality (5.34) is estimated as

$$\begin{aligned}&C\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta-1}|\phi_{zz}\phi_{zzz}|dz\\\le&\frac{c_0\beta}{4}\int_0^t(1+\tau)^{\gamma}\|\phi_{zz}(\cdot,\tau)\|_{0,\beta-1}^2d\tau+\frac{m_0}{2}\int_0^t(1+\tau)^{\gamma}\|\phi_{zzz}(\cdot,\tau)\|_{0,\beta}^2d\tau\\&\quad+\beta C_{\tilde{\tilde{r}}}\int_0^t(1+\tau)^{\gamma}\|\phi_{zzz}(\cdot,\tau)\|^2d\tau\end{aligned}$$

for some $C_{\tilde{\tilde{r}}}>0$ and for any $t>0$. Plugging the above result into inequality (5.34) and estimating it by the second and third terms on the left hand side of inequality (5.34), it leads to

$$\begin{aligned}&(1+t)^{\gamma}\|\phi_{zz}(\cdot,t)\|_{0,\beta}^2\\&\quad+\frac{c_0\beta}{4}\int_0^t(1+\tau)^{\gamma}\|\phi_{zz}(\cdot,\tau)\|_{0,\beta-1}^2d\tau+\frac{m_0}{2}\int_0^t(1+\tau)^{\gamma}\|\phi_{zzz}(\cdot,\tau)\|_{0,\beta}^2d\tau\\\le&\|\phi_{zz,0}\|_{0,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi_{zz}(\cdot,\tau)\|_{0,\beta}^2d\tau+C\beta\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_{0,\beta-1}^2d\tau\\&\quad+C\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_{1,\beta}^2d\tau+C\beta\int_0^t(1+\tau)^{\gamma}\|\phi_{zzz}(\cdot,\tau)\|^2d\tau\\&\quad+C\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta-1}G_5(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\&\quad+C\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta}G_6(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\\le&C\Big(\|\phi_0\|_{2,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{2,\beta}^2d\tau+\beta\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_2^2d\tau\\&\quad+\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta-1}G_5(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\&\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)^{\gamma}\langle z-z_*\rangle^{\beta}(G_1(\phi,\phi_z,\phi_{zz})+G_4(\phi_z,\phi_{zz},\phi_{zzz})+G_6(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau\Big)\end{aligned}$$

for some $C>0$ and for any $t>0$, where $\beta\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_{0,\beta-1}^2d\tau+\int_0^t(1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|_{1,\beta}^2d\tau$ is estimated by the weighted $L^2$-estimates (5.20) and $H^1$-estimates

(5.21), and $G_1(\phi,\phi_z,\phi_{zz})$ and $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ are given in (4.2) and (5.22), respectively. Therefore we finally get the desired result (5.28). ∎

Let us combine all results from the weighted $L^2$-estimates (5.20), $H^1$-estimates (5.21), and $H^2$-estimates (5.28). Then we conclude that there exists some $C>0$ such that for any $t>0$

$$\begin{aligned}&(1+t)^\gamma\|\phi(\cdot,t)\|_{2,\beta}^2+\frac{c_0\beta}{4}\int_0^t(1+\tau)^\gamma\|\phi(\cdot,\tau)\|_{2,\beta-1}^2d\tau+\frac{m_0}{4}\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{2,\beta}^2d\tau\\ &\le C\Big(\|\phi_0\|_{2,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{2,\beta}^2d\tau+\beta\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_2^2d\tau\\ &\quad+\beta\int_0^t\int_{\mathbb{R}}(1+\tau)^\gamma\langle z-z_*\rangle^{\beta-1}G_5(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\ &\quad+\int_0^t\int_{\mathbb{R}}(1+\tau)^\gamma\langle z-z_*\rangle^{\beta}(G_1(\phi,\phi_z,\phi_{zz})+G_4(\phi_z,\phi_{zz},\phi_{zzz})+G_6(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau\Big),\end{aligned} \tag{5.35}$$

where positive constants $c_0$ and $m_0$ are introduced in (5.15) and (3.5), respectively. Also $G_1(\phi,\phi_z,\phi_{zz})$, $G_4(\phi_z,\phi_{zz},\phi_{zzz})$, $G_5(\phi_z,\phi_{zz},\phi_{zzz})$ and $G_6(\phi_z,\phi_{zz},\phi_{zzz})$ are defined in (4.2), (5.22), (5.29) and (5.30), respectively, and are collections of cubic terms.

**5.2. Rate of convergence.** In this subsection, we aim to prove Theorem 5.1. Similar to proof of Theorem 3.1, we only need to establish the a priori estimates (5.7). With (5.35), we continue to establish the desired a priori estimates (5.7) through the following lemmas.

LEMMA 5.5. *For any $\gamma\le\alpha$ when $\alpha$ is an integer, or $\gamma<\alpha$ when $\alpha$ is a non-integer, there are positive constants $\delta_4$ and $C$ independent of $T$ such that if $N(t)<\delta_4$ in Proposition 5.2, it holds for $0\le t\le T$,*

$$\begin{aligned}&(1+t)^\gamma\|\phi(\cdot,t)\|_{2,\beta}^2+\beta\int_0^t(1+\tau)^\gamma\|\phi(\cdot,\tau)\|_{2,\beta-1}^2d\tau+\nu_\alpha\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{2,\beta}^2d\tau\\ &\le C\Big(\|\phi_0\|_{2,\beta}^2+\gamma\int_0^t(1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_{2,\beta}^2d\tau+\beta\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_2^2d\tau\Big),\end{aligned} \tag{5.36}$$

*for some $\nu_\alpha=\nu_\alpha(m_0)>0$, where $m_0$ is given in (3.5).*

*Proof.* To prove this lemma, we need to further estimate (5.35) to obtain inequality (5.36). Since the fourth and fifth terms on the right hand side of estimate (5.35) contain cubic terms, it remains to estimate the cubic terms to show inequality (5.36). Indeed, by the Cauchy-Schwarz inequality, (3.18) and (3.19), the fourth and fifth terms containing the cubic terms are bounded by

$$CN(t)\Big(\beta\int_0^t(1+\tau)^\gamma\|\phi_{zz}(\cdot,\tau)\|_{0,\beta-1}^2d\tau+\int_0^t(1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_{2,\beta}^2d\tau\Big) \tag{5.37}$$

for some $C>0$. Take a sufficiently small $\delta_4>0$ in Proposition 5.2 so that $8CN(t)<8C\delta_4<\min\{c_0,m_0\}$, where positive constants $c_0$ and $m_0$ are defined in (5.15) and (3.5), respectively. Then, we complete the proof of inequality (5.36). ∎

Based on Lemma 5.5, we are going to prove the desired a priori estimates in Proposition 5.2. Dividing into two cases $0\le\gamma\le[\alpha]$ and $[\alpha]<\gamma<\alpha$, we will complete the proof of Proposition 5.2. In the following lemma in the case $0 \le \gamma \le [\alpha]$, we use the principle of induction from Pan and Liu [26].

Lemma 5.6. *Let $\gamma \in [0,\alpha] \cap \mathbb{Z}$. Then there exists a constant $C > 0$ such that for any $0 \le t \le T$*

$$\begin{aligned}&(1+t)^{\gamma}\|\phi(\cdot,t)\|^2_{2,\alpha-\gamma}\\ &\qquad +(\alpha-\gamma)\int_0^t (1+\tau)^{\gamma}\|\phi(\cdot,\tau)\|^2_{2,\alpha-\gamma-1}d\tau+\nu_\alpha\int_0^t (1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|^2_{2,\alpha-\gamma}d\tau\\ &\le CN^2_\alpha(0),\end{aligned} \tag{5.38}$$

*where $\nu_\alpha = \nu_\alpha(m_0) > 0$ is from Lemma 5.5, and $m_0$ is given in (3.5). Consequently, we have for any $0 \le \gamma \le [\alpha]$ and for any $0 \le t \le T$,*

$$(1+t)^{\gamma}\|\phi(\cdot,t)\|^2_2+\nu_\alpha\int_0^t (1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|^2_2 d\tau \le CN^2_\alpha(0). \tag{5.39}$$

*Proof.* First, we will show the induction on $\gamma = 0,1,2,...,[\alpha]$ in

$$\begin{aligned}&(1+t)^{\gamma}\|\phi(\cdot,t)\|^2_{2,\alpha-\gamma}\\ &\qquad +(\alpha-\gamma)\int_0^t (1+\tau)^{\gamma}\|\phi(\cdot,\tau)\|^2_{2,\alpha-\gamma-1}d\tau+\nu_\alpha\int_0^t (1+\tau)^{\gamma}\|\phi_z(\cdot,\tau)\|^2_{2,\alpha-\gamma}d\tau\\ &\le C\|\phi_0\|^2_{2,\alpha}\end{aligned} \tag{5.40}$$

for some $C > 0$ and for any $0 \le t \le T$, where $\nu_\alpha = \nu_\alpha(m_0) > 0$ from Lemma 5.5.

*Step 1.* From inequality (5.36), letting $\gamma = 0$ and $\beta = 0$, we have

$$\|\phi(\cdot,t)\|^2_2+\nu_\alpha\int_0^t \|\phi_z(\cdot,\tau)\|^2_2 d\tau \le C\|\phi_0\|^2_2. \tag{5.41}$$

Now put $\gamma = 0$ and $\beta = \alpha$ in inequality (5.36). By inequality (5.41), one obtains

$$\begin{aligned}&\|\phi(\cdot,t)\|^2_{2,\alpha}+\alpha\int_0^t \|\phi(\cdot,\tau)\|^2_{2,\alpha-1}d\tau+\nu_\alpha\int_0^t \|\phi_z(\cdot,\tau)\|^2_{2,\alpha}d\tau\\ &\le C\Big(\|\phi_0\|^2_{2,\alpha}+\alpha\int_0^t \|\phi_z(\cdot,\tau)\|^2_2 d\tau\Big)\\ &\le C\|\phi_0\|^2_{2,\alpha},\end{aligned} \tag{5.42}$$

so the case $\gamma = 0$ is true.

*Step 2.* Assume that inequality (5.40) holds with $\gamma = k-1$, where $k \le \alpha$. That is,

$$\begin{aligned}&(1+t)^{k-1}\|\phi(\cdot,t)\|^2_{2,\alpha-k+1}\\ &\qquad +(\alpha-k+1)\int_0^t (1+\tau)^{k-1}\|\phi(\cdot,\tau)\|^2_{2,\alpha-k}d\tau+\nu_\alpha\int_0^t (1+\tau)^{k-1}\|\phi_z(\cdot,\tau)\|^2_{2,\alpha-k+1}d\tau\\ &\le C\|\phi_0\|^2_{2,\alpha}.\end{aligned} \tag{5.43}$$

Now letting $\beta = 0$ and $\gamma = k$ in inequality (5.36), and using inequality (5.43), it holds

$$(1+t)^{k}\|\phi(\cdot,\tau)\|^2_2+\nu_\alpha\int_0^t (1+\tau)^{k}\|\phi_z(\cdot,\tau)\|^2_2 d\tau$$

$$\leq C\Big(\|\phi_0\|_2^2 + k\int_0^t (1+\tau)^{k-1}\|\phi(\cdot,\tau)\|_{2,\alpha-k}^2 d\tau\Big)$$
$$\leq C\|\phi_0\|_{2,\alpha}^2. \tag{5.44}$$

Moreover, putting $\beta = \alpha - k$ in inequality (5.36), it ends up with

$$\begin{aligned}&(1+t)^k\|\phi(\cdot,t)\|_{2,\alpha-k}^2\\ &\quad + (\alpha-k)\int_0^t (1+\tau)^k\|\phi(\cdot,\tau)\|_{2,\alpha-k-1}^2 d\tau + \nu_\alpha\int_0^t (1+\tau)^k\|\phi_z(\cdot,\tau)\|_{2,\alpha-k}^2 d\tau\\ &\leq C\Big(\|\phi_0\|_{2,\alpha-k}^2 + k\int_0^t (1+\tau)^{k-1}\|\phi(\cdot,\tau)\|_{2,\alpha-k}^2 d\tau + (\alpha-k)\int_0^t (1+\tau)^k\|\phi_z(\cdot,\tau)\|_2^2 d\tau\Big)\\ &\leq C\|\phi_0\|_{2,\alpha}^2\end{aligned} \tag{5.45}$$

for any $0\leq t\leq T$ and for some $C>0$, where the last inequality holds by inequalities (5.43) and (5.44). Thus estimate (5.40) is true when $\gamma = k$. By the principle of induction, (5.40) holds for any $\gamma = 0,1,...,[\alpha]$.

Furthermore, if $\phi_0 \in L_\alpha^2$, then $\phi_{z,0}\in L_\alpha^2$. Thus we have $\|\phi_0\|_{2,\alpha}^2 \leq CN_\alpha^2(0)$ for some $C>0$. Therefore we complete the proof of (5.38), which implies (5.39). ∎

It remains to show inequality (5.39) in the case $[\alpha] < \gamma < \alpha$ when $\alpha$ is a non-integer. We will use a technique, introduced by Matsumura and Nishihara in [21].

LEMMA 5.7. *If $\alpha$ is a non-integer, there exists some $C_\varepsilon > 0$ such that for any $\gamma < \alpha$ and for any $0 \leq t\leq T$,*

$$(1+t)^\gamma\|\phi(\cdot,t)\|_2^2 + \nu_\alpha\int_0^t (1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_2^2 d\tau \leq C_\varepsilon N_\alpha^2(0), \tag{5.46}$$

*where $\nu_\alpha = \nu_\alpha(m_0) > 0$ is from Lemma 5.5 and $m_0$ is given in (3.5), and $C_\varepsilon\to\infty$ as $\varepsilon\to 0$.*

*Proof.* Considering $\beta = 0$ in (5.36), we have

$$\begin{aligned}&(1+t)^\gamma\|\phi(\cdot,t)\|_2^2 + \nu_\alpha\int_0^t (1+\tau)^\gamma\|\phi_z(\cdot,\tau)\|_2^2 d\tau\\ &\leq C\Big(\|\phi_0\|_2^2 + \gamma\int_0^t (1+\tau)^{\gamma-1}\|\phi(\cdot,\tau)\|_2^2 d\tau\Big).\end{aligned} \tag{5.47}$$

We only need to estimate the last term in inequality (5.47). To estimate this term, we will use (5.38) with $\gamma = [\alpha]$,

$$\begin{aligned}&(1+t)^{[\alpha]}\|\phi(\cdot,t)\|_{2,\alpha-[\alpha]}^2\\ &\quad + (\alpha-[\alpha])\int_0^t (1+\tau)^{[\alpha]}\|\phi(\cdot,\tau)\|_{2,\alpha-[\alpha]-1}^2 d\tau + \nu_\alpha\int_0^t (1+\tau)^{[\alpha]}\|\phi_z(\cdot,\tau)\|_{2,\alpha-[\alpha]}^2 d\tau\\ &\leq CN_\alpha^2(0),\end{aligned} \tag{5.48}$$

and Hölder's inequality,

$$\int |fg| \leq \|f\|_{L^p}\|g\|_{L^q}, \quad \text{where} \quad p = \frac{1}{\alpha-[\alpha]},\ q = \frac{1}{[\alpha]+1-\alpha}.$$

Then for $i=0,1,2$, we have, for any $[\alpha]\le\gamma<\alpha$ and for any $0\le t\le T$,

$$\begin{aligned}
&\int_0^t(1+\tau)^{\gamma-1}\|\partial_z^i\phi(\cdot,\tau)\|^2d\tau\\
&=\int_0^t(1+\tau)^{\gamma-1}\Big(\int_{\mathbb{R}}\Big(\langle z-z_*\rangle^{(\alpha-[\alpha])([\alpha]+1-\alpha)}(\partial_z^i\phi)^{2([\alpha]+1-\alpha)}\Big)\\
&\qquad\qquad\Big(\langle z-z_*\rangle^{-(\alpha-[\alpha])([\alpha]+1-\alpha)}(\partial_z^i\phi)^{2(\alpha-[\alpha])}\Big)dz\Big)d\tau\\
&\le\int_0^t(1+\tau)^{\gamma-1}\Big(\Big(\int_{\mathbb{R}}\langle z-z_*\rangle^{(\alpha-[\alpha])}(\partial_z^i\phi)^2dz\Big)^{[\alpha]+1-\alpha}\\
&\qquad\qquad\Big(\int_{\mathbb{R}}\langle z-z_*\rangle^{-([\alpha]+1-\alpha)}(\partial_z^i\phi)^2dz\Big)^{\alpha-[\alpha]}\Big)d\tau\\
&\le\int_0^t(1+\tau)^{\gamma-1-[\alpha]}\Big((1+\tau)^{[\alpha]}\Big(\|\phi(\cdot,\tau)\|^2_{2,\alpha-[\alpha]}\Big)^{[\alpha]+1-\alpha}\\
&\qquad\qquad\Big(\int_{\mathbb{R}}(1+\tau)^{[\alpha]}\langle z-z_*\rangle^{-([\alpha]+1-\alpha)}(\partial_z^i\phi)^2dz\Big)^{\alpha-[\alpha]}\Big)d\tau\\
&\le CN_\alpha^{2([\alpha]+1-\alpha)}(0)\int_0^t(1+\tau)^{\gamma-1-[\alpha]}\Big((1+\tau)^{[\alpha]}\|\phi(\cdot,\tau)\|^2_{2,\alpha-[\alpha]-1}\Big)^{\alpha-[\alpha]}d\tau\\
&\le CN_\alpha^{2([\alpha]+1-\alpha)}(0)\Big(\int_0^t(1+\tau)^{-\frac{[\alpha]+1-\gamma}{[\alpha]+1-\alpha}}d\tau\Big)^{[\alpha]+1-\alpha}\\
&\qquad\qquad\Big(\int_0^t(1+\tau)^{[\alpha]}\|\phi(\cdot,\tau)\|^2_{2,\alpha-[\alpha]-1}d\tau\Big)^{\alpha-[\alpha]}\\
&\le CN_\alpha^2(0)\Big(\frac{[\alpha]+1-\alpha}{\gamma-\alpha}(1+t)^{\frac{\gamma-\alpha}{[\alpha]+1-\alpha}}-\frac{[\alpha]+1-\alpha}{\gamma-\alpha}\Big)^{[\alpha]+1-\alpha}\\
&\le C_\varepsilon N_\alpha^2(0),
\end{aligned}$$

where $\varepsilon=\frac{\alpha-\gamma}{2}$, and $C_\varepsilon\to\infty$ as $\varepsilon\to 0$.

Plugging the above result into inequality (5.47), the proof of estimate (5.46) is completed. ∎

From Lemmas 5.6 and 5.7, the a priori estimates (5.7) have been derived. Using the local existence in Proposition 5.1, the a priori estimates in Proposition 5.2 and the continuation arguments, the proof of Theorem 5.1 is finally completed. Therefore, Theorem 2.2 is proved.

## 6. Rate of convergence: exponential decay

In this section, we prove Theorem 6.1 when the initial perturbation decays exponentially in space, it will be shown that the perturbation does in time as well.

The exponential time decay in the problem (1.1)-(1.3) was proved by Wu [32], using spectral analysis in some weighted spaces. In the current paper, we prove the rate of convergence by using weighted energy estimates. For the previous works using the weighted energy estimates with different exponential weights, we refer to Li [12], Mei [22], and Mei and Yang [23].

In the following lemma, we determine the decay rate of a weight function $w(z)$ which will be used in this section.

LEMMA 6.1. *Let*

$$B_a(z)=-a\frac{(z-z_*)(U-c)}{\langle z-z_*\rangle}-\frac{1}{2}U'-4a^2\frac{1-U'^2}{(1+U'^2)^2}\frac{(z-z_*)^2}{\langle z-z_*\rangle^2},\tag{6.1}$$

*where $z_*$ is defined in Equation (5.1). Then we can find positive constants $a=a(m_0,U')$ and $\theta=\theta(a,m_0,U')$, where $m_0$ is defined in (3.5), so that*

$$B_a(z)\geq\theta>0 \tag{6.2}$$

*for any $z\in\mathbb{R}$.*

*Proof.* Similar to the derivation of inequalities (5.13) and (5.14), we can pick positive constants $\eta_1$ and $m_2$ such that

$$\begin{cases} -a\dfrac{(z-z_*)(U-c)}{\langle z-z_*\rangle}\geq ac(\eta_1)>0 \quad \text{for} \quad |z-z_*|\geq\eta_1, \\ -\frac{1}{2}U'\geq m_2>0 \quad \text{for} \quad |z-z_*|\leq\eta_1, \end{cases} \tag{6.3}$$

where $c(\eta_1)>0$ is determined by (5.14). From inequalities (6.3), we can find suitable $a=a(m_0,U')>0$ and $\theta=\theta(a,m_0,U')>0$, where $m_0$ is defined in (3.5), such that

$$0<\theta\leq\begin{cases} ac(\eta_1)-4a^2\dfrac{1-U'^2}{(1+U'^2)^2}\dfrac{(z-z_*)^2}{\langle z-z_*\rangle^2} \quad \text{for} \quad |z-z_*|\geq\eta_1, \\ m_2-4a^2\dfrac{1-U'^2}{(1+U'^2)^2}\dfrac{(z-z_*)^2}{\langle z-z_*\rangle^2} \quad \text{for} \quad |z-z_*|\leq\eta_1. \end{cases} \tag{6.4}$$

By Equation (6.1) and inequality (6.4), we obtain inequality (6.2). □

Now define a weight function

$$w(z)=e^{a\langle z-z_*\rangle}, \tag{6.5}$$

where $z_*$ is given in Equation (5.1) and $a>0$ is determined in Lemma 6.1, and the weighted solution space of the problem (3.14), (3.15)

$$X_w(0,T)=\{\phi(z,t):\phi\in C([0,T);H^2_w),\ \phi_z\in L^2((0,T);H^2_w)\},$$

where $0<T<\infty$. Let

$$N_w(t)=\sup_{0\leq\tau\leq t}\{\|\phi(\cdot,\tau)\|_{2,w}\} \tag{6.6}$$

for all $t\geq 0$. Theorem 2.3 is a consequence of the following theorem.

THEOREM 6.1. *Suppose that the assumptions of Theorem 2.3 hold. Then there exist positive constants $\delta_5$ and $C$ such that if*

$$N_w(0)<\delta_5,$$

*then the problem (3.14), (3.15) has a unique global solution $\phi\in X_w(0,+\infty)$ satisfying for all $t>0$,*

$$\|\phi(\cdot,t)\|^2_{2,w}+\theta\int_0^t\|\phi(\cdot,\tau)\|^2_{2,w}d\tau+\nu_w\int_0^t\|\phi_z(\cdot,\tau)\|^2_{2,w}d\tau\leq CN^2_w(0) \tag{6.7}$$

*for some $\nu_w=\nu_w(m_0)>0$, where $m_0$ is given in (3.5) and $\theta>0$ is determined in (6.4). Consequently, there is some $C>0$ such that for any $t>0$*

$$\sup_{z\in\mathbb{R}}|\phi_z(z,t)|\leq C\|\phi_0\|_{2,w}e^{-\frac{\theta}{2}t}. \tag{6.8}$$

The local existence and the a priori estimates are stated as follows.

Proposition 6.1 (Local existence). *For any $\delta>0$, there exists a constant $T>0$ depending on $\delta$ such that if $\phi_0\in H_w^2(\mathbb{R})$ and $2N_w(0)<\delta$, the problem (3.14), (3.15) has a unique solution $\phi\in X_w(0,T)$ satisfying*

$$N_w(t)<2N_w(0)$$

*for any $0\le t\le T$.*

Proposition 6.2 (A priori estimates). *Assume that $\phi\in X_w(0,T)$ is a solution obtained from Proposition 6.1 for a constant $T>0$. Then there are positive constants $\delta_6$ and $C$, which are independent of $T$, such that if*

$$N_w(t)<\delta_6$$

*for any $0\le t\le T$, then the solution $\phi$ of (3.14), (3.15) satisfies*

$$\|\phi(\cdot,t)\|_{2,w}^2+\theta\int_0^t\|\phi(\cdot,\tau)\|_{2,w}^2d\tau+\nu_w\int_0^t\|\phi_z(\cdot,\tau)\|_{2,w}^2d\tau\le CN_w^2(0) \tag{6.9}$$

*for some $\nu_w=\nu_w(m_0)>0$, where $m_0$ is given in (3.5) and $\theta>0$ is determined in (6.4).*

Theorem 6.1 can be proved by the local existence in Proposition 6.1 and the a priori estimates in Proposition 6.2.

Lemma 6.2 (Weighted $L^2$-estimates). *Under the conditions of Theorem 6.1, there is some $C>0$ such that for any $t>0$*

$$\begin{aligned}&\|\phi(\cdot,t)\|_w^2+\theta\int_0^t\|\phi(\cdot,\tau)\|_w^2d\tau+\frac{m_0}{4}\int_0^t\|\phi_z(\cdot,\tau)\|_w^2d\tau\\ \le&\|\phi_0\|_w^2+C\int_0^t\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}G_1(\phi,\phi_z,\phi_{zz})dzd\tau,\end{aligned} \tag{6.10}$$

*where positive constants $\theta$ and $m_0$ are given in (6.4) and (3.5), respectively, and $G_1(\phi,\phi_z,\phi_{zz})$ is defined in (4.2).*

*Proof.* Multiplying Equation (3.14) by $2e^{a\langle z-z_*\rangle}\phi$, we have

$$\begin{aligned}&2e^{a\langle z-z_*\rangle}\phi\phi_t+2(U-c)e^{a\langle z-z_*\rangle}\phi\phi_z-2\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi\phi_{zz}\\ =&-e^{a\langle z-z_*\rangle}\phi\phi_z^2+2e^{a\langle z-z_*\rangle}F\phi.\end{aligned} \tag{6.11}$$

The left hand side of Equation (6.11) can be expressed as

$$\begin{aligned}&2e^{a\langle z-z_*\rangle}\phi\phi_t+2(U-c)e^{a\langle z-z_*\rangle}\phi\phi_z-2\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi\phi_{zz}\\ =&(e^{a\langle z-z_*\rangle}\phi^2)_t+\Big((U-c)e^{a\langle z-z_*\rangle}\phi^2-2\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi\phi_z\Big)_z\\ &-U'e^{a\langle z-z_*\rangle}\phi^2-a(U-c)\frac{z-z_*}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi^2+2\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'e^{a\langle z-z_*\rangle}\phi\phi_z\\ &+2\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_z^2+2a\frac{1-U'^2}{(1+U'^2)^2}\frac{z-z_*}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi\phi_z.\end{aligned}$$

Substituting the above result into Equation (6.11) and integrating the resulting equation with respect to $z$ over $\mathbb{R}$ give us

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}\phi^2 dz+\int_{\mathbb{R}}|U'|e^{a\langle z-z_*\rangle}\phi^2 dz\\&\quad -a\int_{\mathbb{R}}(U-c)\frac{z-z_*}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi^2 dz+2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_z^2 dz\\&=-2\int_{\mathbb{R}}\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)_z e^{a\langle z-z_*\rangle}\phi\phi_z dz-2a\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\frac{z-z_*}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi\phi_z dz\\&\quad +\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}(-\phi\phi_z^2+2F\phi)dz. \qquad (6.12)\end{aligned}$$

For $0<u_l\le 1.8$, similar to the derivation of inequality (4.13), the first term on the right hand side of Equation (6.12) is estimated as

$$\begin{aligned}&\Big|2\int_{\mathbb{R}}\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)' e^{a\langle z-z_*\rangle}\phi\phi_z dz\Big|\\&\le\frac{1}{2}\int_{\mathbb{R}}|U'|e^{a\langle z-z_*\rangle}\phi^2 dz+\frac{3}{2}\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_z^2 dz. \qquad (6.13)\end{aligned}$$

Also, by the Cauchy-Schwarz inequality, the second term on the right hand side of Equation (6.12) is estimated as

$$\begin{aligned}&\Big|2a\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\frac{(z-z_*)}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi\phi_z dz\Big|\\&\le 4a^2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\frac{(z-z_*)^2}{\langle z-z_*\rangle^2}e^{a\langle z-z_*\rangle}\phi^2 dz+\frac{1}{4}\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_z^2 dz. \qquad (6.14)\end{aligned}$$

Substituting estimates (6.13) and (6.14) into Equation (6.12), it ends up with

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}\phi^2 dz+\int_{\mathbb{R}}B_a(z)e^{a\langle z-z_*\rangle}\phi^2 dz+\frac{1}{4}\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_z^2 dz\\&\le\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}(|\phi\phi_z^2|+2|F\phi|)dz, \qquad (6.15)\end{aligned}$$

where $B_a$ is defined in Equation (6.1).

Now, after integrating inequality (6.15) with respect to $t$ and using (4.6), (3.5) and (6.2), we arrive at

$$\begin{aligned}&\|\phi(\cdot,t)\|_w^2+\theta\int_0^t\|\phi(\cdot,\tau)\|_w^2 d\tau+\frac{m_0}{4}\int_0^t\|\phi_z(\cdot,\tau)\|_w^2 d\tau\\&\le\|\phi_0\|_w^2+C\int_0^t\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}G_1(\phi,\phi_z,\phi_{zz})dz d\tau\end{aligned}$$

for some $C>0$ and for all $t>0$, where $G_1(\phi,\phi_z,\phi_{zz})$ is given (4.2), which is the desired result (6.10). ∎

Now we show the weighted $H^1$- and $H^2$-estimates.

Lemma 6.3 (Weighted $H^1$-estimates). *Under the same assumptions in Lemma 6.2, there is some $C>0$ satisfying*

$$\|\phi_z(\cdot,t)\|_w^2+\theta\int_0^t\|\phi_z(\cdot,\tau)\|_w^2 d\tau+\frac{7m_0}{4}\int_0^t\|\phi_{zz}(\cdot,\tau)\|_w^2 d\tau$$

$$\le C\|\phi_0\|_{1,w}^2 + C\int_0^t \int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(G_1(\phi,\phi_z,\phi_{zz}) + G_4(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau \tag{6.16}$$

*for any $t>0$, where positive constants $\theta$ and $m_0$ are defined in (6.4) and (3.5), respectively. In (4.2) and (5.22), $G_1(\phi,\phi_z,\phi_{zz})$ and $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ are defined respectively.*

*Proof.* Differentiating Equation (3.14) with respect to $z$ and multiplying the resulting equation by $2e^{a\langle z-z_*\rangle}\phi_z$, we have

$$2e^{a\langle z-z_*\rangle}\phi_z\phi_{zt} + 2U'e^{a\langle z-z_*\rangle}\phi_z^2 + 2(U-c)e^{a\langle z-z_*\rangle}\phi_z\phi_{zz} - 2\Big(\frac{1-U'^2}{(1+U'^2)^2}\phi_{zz}\Big)_z e^{a\langle z-z_*\rangle}\phi_z$$
$$= -2e^{a\langle z-z_*\rangle}\phi_z^2\phi_{zz} + 2F_z e^{a\langle z-z_*\rangle}\phi_z. \tag{6.17}$$

If we integrate Equation (6.17) with respect to $z$ over $\mathbb{R}$ and use integration by parts, then we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}\phi_z^2 dz + \frac{1}{2}\int_{\mathbb{R}}(-U')|e^{a\langle z-z_*\rangle}\phi_z^2 dz\\
&\quad - a\int_{\mathbb{R}}(U-c)\frac{z-z_*}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi_z^2 dz + 2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_{zz}^2 dz\\
=&\frac{3}{2}\int_{\mathbb{R}}(-U')e^{a\langle z-z_*\rangle}\phi_z^2 dz - 2a\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}\frac{z-z_*}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi_z\phi_{zz}dz\\
&\quad + 2\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(-\phi^2\phi_{zz} + F_z\phi_z)dz.
\end{aligned} \tag{6.18}$$

Similar to the derivation of inequality (6.14), estimate the second term on the right hand side of Equation (6.18) and plug the estimated result into Equation (6.18). It ends up with

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}\phi_z^2 dz + \int_{\mathbb{R}} B_a(z)e^{a\langle z-z_*\rangle}\phi_z^2 dz + \frac{7}{4}\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_{zz}^2 dz\\
\le&\frac{3}{2}\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}|U'|\phi_z^2 dz + 2\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(|\phi^2\phi_{zz}| + |F_z\phi_z|)dz,
\end{aligned} \tag{6.19}$$

where $B_a$ is introduced in (6.1).

To finish this lemma, using (3.5), (4.21), (6.2) and the boundedness of $U'$, integrate (6.19) with respect to $t$. Then we obtain

$$\begin{aligned}
&\|\phi_z(\cdot,t)\|_w^2 + \theta\int_0^t \|\phi_z(\cdot,\tau)\|_w^2 d\tau + \frac{7m_0}{4}\int_0^t \|\phi_{zz}(\cdot,\tau)\|_w^2 d\tau\\
\le&\|\phi_{z,0}\|_w^2 + C\int_0^t \|\phi_z(\cdot,\tau)\|_w^2 d\tau + C\int_0^t\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}G_4(\phi_z,\phi_{zz},\phi_{zzz})dzd\tau\\
\le&C\|\phi_0\|_{1,w}^2 + C\int_0^t\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(G_1(\phi,\phi_z,\phi_{zz}) + G_4(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau
\end{aligned} \tag{6.20}$$

for some $C>0$ and for any $t>0$, where $\int_0^t \|\phi_z(\cdot,\tau)\|_w^2 d\tau$ is estimated by the weighted $L^2$-estimates (6.10), and $G_1(\phi,\phi_z,\phi_{zz})$ and $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ are introduced in Equations (4.2) and (5.22), respectively. Therefore, we complete the proof of (6.16). □

LEMMA 6.4 (Weighted $H^2$-estimates). *Under the same conditions in Lemma 6.2, we have for any $t>0$*

$$\|\phi_{zz}(\cdot,t)\|_w^2 + \theta\int_0^t \|\phi_{zz}(\cdot,\tau)\|_w^2 d\tau + \frac{3m_0}{4}\int_0^t \|\phi_{zzz}(\cdot,\tau)\|_w^2 d\tau$$

$$\leq C\|\phi_0\|_{2,w}^2 + C\int_0^t\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(G_1(\phi,\phi_z,\phi_{zz})+G_4(\phi_z,\phi_{zz},\phi_{zzz})+G_7(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau \quad (6.21)$$

*for some $C>0$, where positive constants $\theta$ and $m_0$ are defined in (6.4) and (3.5), respectively. Also $G_1(\phi,\phi_z,\phi_{zz})$ and $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ are defined in (4.2) and (5.22), respectively, and we denote*

$$G_7(\phi_z,\phi_{zz},\phi_{zzz}) = |\phi_z\phi_{zz}\phi_{zzz}| + |\phi_{zz}|(\phi_{zz}^2+\phi_{zzz}^2) + |\phi_{zz}^2\phi_{zzz}|. \quad (6.22)$$

*Proof.* Differentiating Equation (3.14) with respect to $z$ twice and multiplying the resulting equation by $2e^{a\langle z-z_*\rangle}\phi_{zz}$, we have

$$\begin{aligned}&2e^{a\langle z-z_*\rangle}\phi_{zz}\phi_{zzt} + 2U''e^{a\langle z-z_*\rangle}\phi_z\phi_{zz} + 4U'e^{a\langle z-z_*\rangle}\phi_{zz}^2\\&\quad + 2(U-c)e^{a\langle z-z_*\rangle}\phi_{zz}\phi_{zzz} - 2\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)''e^{a\langle z-z_*\rangle}\phi_{zz}\\&= -2e^{a\langle z-z_*\rangle}\phi_{zz}^3 - 2e^{a\langle z-z_*\rangle}\phi_z\phi_{zz}\phi_{zzz} + 2e^{a\langle z-z_*\rangle}F_{zz}\phi_{zz}.\end{aligned} \quad (6.23)$$

Integrate Equation (6.23) with respect to $z$ over $\mathbb{R}$ and use integration by parts. Then it follows

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}\phi_{zz}^2dz - a\int_{\mathbb{R}}(U-c)\frac{(z-z_*)}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi_{zz}^2dz\\&\quad + \frac{1}{2}\int_{\mathbb{R}}(-U')e^{a\langle z-z_*\rangle}\phi_{zz}^2dz + 2\int_{\mathbb{R}}\frac{1-U'^2}{(1+U'^2)^2}e^{a\langle z-z_*\rangle}\phi_{zzz}^2dz\\&= -2\int_{\mathbb{R}}U''e^{a\langle z-z_*\rangle}\phi_z\phi_{zz}dz + \int_{\mathbb{R}}\Big(-\frac{7}{4}U' + 2\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)''\Big)e^{a\langle z-z_*\rangle}\phi_{zz}^2dz\\&\quad + 2\int_{\mathbb{R}}\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)_z e^{a\langle z-z_*\rangle}\phi_{zz}\phi_{zzz}dz - 2a\int_{\mathbb{R}}\frac{(1-U'^2)}{(1+U'^2)^2}\frac{z-z_*}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}\phi_{zz}\phi_{zzz}dz\\&\quad - 2\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}(\phi_{zz}^3 + \phi_z\phi_{zz}\phi_{zzz} + F_z\phi_{zzz})dz - 2a\int_{\mathbb{R}}\frac{(z-z_*)}{\langle z-z_*\rangle}e^{a\langle z-z_*\rangle}F_z\phi_{zz}dz.\end{aligned} \quad (6.24)$$

We shall further estimate the right hand side of Equation (6.24). By the Cauchy-Schwarz inequality, the first and third terms are estimated as

$$2\int_{\mathbb{R}}|U''|e^{a\langle z-z_*\rangle}|\phi_z\phi_{zz}|dz \leq \int_{\mathbb{R}}|U''|e^{a\langle z-z_*\rangle}\phi_z^2dz + \int_{\mathbb{R}}|U''|e^{a\langle z-z_*\rangle}\phi_{zz}^2dz$$

and

$$\begin{aligned}&2\int_{\mathbb{R}}\Big|\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\Big|e^{a\langle z-z_*\rangle}|\phi_{zz}\phi_{zzz}|dz\\&\leq \frac{1}{m_0}\int_{\mathbb{R}}\Big(\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\Big)^2e^{a\langle z-z_*\rangle}\phi_{zz}^2dz + m_0\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}\phi_{zzz}^2dz,\end{aligned}$$

where $m_0$ is defined in (3.5). Similar to the derivation of inequality (6.14), we also estimate the fourth term on the right hand side of Equation (6.24). Inserting the above results into Equation (6.24) gives us

$$\frac{d}{dt}\int_{\mathbb{R}}e^{a\langle z-z_*\rangle}\phi_{zz}^2dz + \int_{\mathbb{R}}B_a(z)e^{a\langle z-z_*\rangle}\phi_{zz}^2dz + \int_{\mathbb{R}}\Big(\frac{7}{4}\frac{1-U'^2}{(1+U'^2)^2} - m_0\Big)e^{a\langle z-z_*\rangle}\phi_{zzz}^2dz$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}} |U''| e^{a\langle z-z_*\rangle} \phi_z^2 dz \\
&+ \int_{\mathbb{R}} \Big( \frac{7}{4}|U'| + |U''| + 2\Big| \Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)''\Big| + \frac{1}{m_0}\Big(\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'\Big)^2\Big) e^{a\langle z-z_*\rangle}\phi_{zz}^2 dz \\
&+ 2\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(|\phi_{zz}|^3 + |\phi_z\phi_{zz}\phi_{zzz}| + |F_z\phi_{zzz}|)dz + 2a\int_{\mathbb{R}} \frac{|z-z_*|}{\langle z-z_*\rangle} e^{a\langle z-z_*\rangle}|F_z\phi_{zz}|dz,
\end{aligned}
\tag{6.25}
$$

where $B_a$ is defined in Equation (6.1).

Using (3.5), (4.20), (6.2) and the boundedness of $U'$, $U''$, $\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)'$ and $\Big(\frac{1-U'^2}{(1+U'^2)^2}\Big)''$, integrate inequality (6.25) with respect to $t$. Then there exists some $C>0$ such that for any $t>0$

$$
\begin{aligned}
&\|\phi_{zz}(\cdot,t)\|_w^2 + \theta\int_0^t \|\phi(\cdot,\tau)\|_w^2 d\tau + \frac{3m_0}{4}\int_0^t \|\phi_{zzz}(\cdot,\tau)\|_w^2 d\tau \\
\le& \|\phi_{zz,0}\|_w^2 + C\int_0^t \|\phi_z(\cdot,\tau)\|_{1,w}^2 d\tau + C\int_0^t\int_{\mathbb{R}} e^{a\langle z-z_*\rangle} G_7(\phi_z,\phi_{zz},\phi_{zzz})dz \\
\le& C\|\phi_0\|_{2,w}^2 \\
&+ C\int_0^t\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(G_1(\phi,\phi_z,\phi_{zz}) + G_4(\phi_z,\phi_{zz},\phi_{zzz}) + G_7(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau,
\end{aligned}
\tag{6.26}
$$

where $\int_0^t \|\phi_z(\cdot,\tau)\|_{1,w}^2 d\tau$ is estimated by the weighted $H^1$-estimates (6.16), and $G_1(\phi,\phi_z,\phi_{zz})$, $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ and $G_7(\phi_z,\phi_{zz},\phi_{zzz})$ are defined in (4.2), (5.22) and (6.22), respectively. Thus, the proof of inequality (6.21) is completed. □

To finish proving inequality (6.9), we combine the results from the weighted $L^2$-estimates (6.10), $H^1$-estimates (6.16) and $H^2$-estimates (6.21) to conclude that there is some $C>0$ such that for any $t>0$

$$
\begin{aligned}
&\|\phi(\cdot,t)\|_{2,w}^2 + \theta\int_0^t \|\phi(\cdot,\tau)\|_w^2 d\tau + \frac{m_0}{4}\int_0^t \|\phi_z(\cdot,\tau)\|_{2,w}^2 d\tau \\
\le& C\|\phi_0\|_{2,w}^2 \\
&+ C\int_0^t\int_{\mathbb{R}} e^{a\langle z-z_*\rangle}(G_1(\phi,\phi_z,\phi_{zz}) + G_4(\phi_z,\phi_{zz},\phi_{zzz}) + G_7(\phi_z,\phi_{zz},\phi_{zzz}))dzd\tau,
\end{aligned}
\tag{6.27}
$$

where positive constants $\theta$ and $m_0$ are defined in (6.4) and (3.5) respectively. Collections of cubic terms $G_1(\phi,\phi_z,\phi_{zz})$, $G_4(\phi_z,\phi_{zz},\phi_{zzz})$ and $G_7(\phi_z,\phi_{zz},\phi_{zzz})$ are defined in (4.2), (5.22) and (6.22), respectively.

Notice that the second term on the right hand side of (6.27) contains only cubic terms. By the Cauchy-Schwarz inequality and (3.19), this second term is majored by $CN(t)\int_0^t \|\phi_z(\cdot,\tau)\|_{2,w}^2 d\tau$ for some $C>0$. Take a small $\delta_6>0$ in Proposition 6.2 so that $8C\delta_6<m_0$. That is, $8CN(t)\le 8CN_w(t)<8C\delta_6<m_0$. From (6.27), we conclude that there are $\nu_w=\nu_w(m_0)>0$ and $C>0$ such that for any $0\le t\le T$ in (4.2) and (5.22)

$$
\|\phi(\cdot,t)\|_{2,w}^2 + \theta\int_0^t \|\phi(\cdot,\tau)\|_{2,w}^2 d\tau + \nu_w\int_0^t \|\phi_z(\cdot,\tau)\|_{2,w}^2 d\tau \le C\|\phi_0\|_{2,w}^2,
\tag{6.28}
$$

which is the desired result (6.9).

It only remains to prove estimate (6.8). To this end, we employ Gronwall's inequality to inequality (6.7). That proves Theorem 6.1. Therefore Theorem 2.3 is proved.

## REFERENCES


[1] M. Bertalmio, A. Bertozzi, and G. Sapiro, *Navier–Stokes, fluid dynamics and image and video inpainting*, in Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, Hawaii, I, 355–362, 2001.

[2] J. Canny, *A computational approach to edge detection*, IEEE Trans. Pattern Anal. Machine Intell., PAMI-8:679–698, 1986.

[3] M.A. Cohen and S. Grossberg, *Neural dynamics of brightness perception: Features, boundaries, diffusion and resonance*, Percept Psychophys, 36:428–456, 1984.

[4] J. Goodman, A. Kurganov, and P. Rosenau, *Breakdown in Burgers-type equations with saturating dissipation fluxes*, Nonlinearity, 12:247–268, 1999.

[5] J.B. Greer and A.L. Bertozzi, *Traveling wave solutions of fourth order PDEs for image processing*, SIAM J. Math. Anal., 36:38–68, 2004.

[6] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion*, Comm. Math. Phys, 101:97–127, 1985.

[7] S. Kawashima and A. Matsumura, *Stability of shock profiles in viscoelasticity with non-convex constitutive relations*, Commun. Pure Appl. Math., 47:1547–1569, 1994.

[8] J.J. Koenderink, *The structure of image*, Bio. Cybernet, 50:363–370, 1984.

[9] A. Kuijper, *The deep structure of Gaussian scale spaces images*, PhD Thesis, Utrecht University, 83–90, 2002.

[10] A. Kurganov, D. Levy, and P. Rosenau, *On Burgers-Type equations with nonmonotonic dissipative fluxes*, Commun. Pure Appl. Math., 51:0443–0473, 1998.

[11] J. Li, T. Li, and Z-A. Wang, *Stability of traveling waves of the Keller–Segel system with logarithmic sensitivity*, Math. Model. Meth. Appl. Sci., 24:2819–2849, 2014.

[12] T. Li, *Stability of strong detonation waves and rates of convergence*, EJDE, 9:1–17, 1998.

[13] T. Li, *Stability of traveling waves in quasi-linear hyperbolic systems with relaxation and diffusion*, SIAM J. Math. Anal., 40:1058–1075, 2008.

[14] T. Li and H. Liu, *Stability of a traffic flow model with nonconvex relaxation*, Commun. Math. Sci., 3:101–118, 2005.

[15] T. Li and Z-A. Wang, *Nonlinear stability of traveling waves to a hyperbolic-parabolic system modeling chemotaxis*, SIAM J. Appl. Math., 70:1522–1541, 2009.

[16] T. Li and Z-A. Wang, *Nonlinear stability of large amplitude viscous shock waves of a generalized hyperbolic-parabolic system arising in chemotaxis*, Math. Model. Meth. Appl. Sci., 20:1967–1998, 2010.

[17] T. Li and Z-A. Wang, *Asymptotic nonlinear stability of traveling waves to conservation laws arising from chemotaxis*, J. Diff. Eqs., 250:1310–1333, 2011.

[18] T. Li and H. Yin, *Convergence rate to strong boundary layer solutions for generalized BBM-Burgers equations with non-convex flux*, Commun. Pure Appl. Math., 13:835–858, 2014.

[19] H. Liu, C-W. Woo, and T. Yang, *Decay rate for traveling waves of a relaxation model*, J. Diff. Eqs., 134:347–367, 1998.

[20] D. Marr and E. Hildreth, *Theory of edge detection*, Proc. Roy. Soc. London Ser. B, 207:187–217, 1980.

[21] A. Matsumura and K. Nishihara, *Asymptotic stability of traveling waves for scalar viscous conservation laws with non-convex nonlinearity*, Comm. Math. Phys., 165:83–96, 1994.

[22] M. Mei, *Stability of shock profiles for nonconvex scalar viscous conservation laws*, Math. Model. Meth. Appl. Sci., 5:279–296, 1995.

[23] M. Mei and T. Yang, *Convergence rates to travelling waves for a nonconvex relaxation model*, Proc. Roy. Soc. Edinburgh., 128A:1053–1068, 1998.

[24] T. Nishida, *Nonlinear hyperbolic equations and related topics in fluid dynamics*, Publ. Math. D'osay 78-02, Départment de Mathématique, Université de Paris-Sud, Orsay, France, 78, 1978.

[25] M. Nishikata, *Convergence rate to the traveling wave for viscous conservation laws*, Funkcialoj Ekvacioj, 41:107–132, 1998.

[26] J. Pan and H. Liu, *Convergence rates to traveling waves for viscous conservation laws with dispersion*, J. Diff. Eqs., 187:337–358, 2003.

[27] P. Perona and J. Malik, *Scale-space and edge detection using anisotropic diffusion*, IEEE. Trans. Pattern Anal. Machine Intell., 12:629–639, 1990.

[28] L. Rudin, S. Osher, and E. Fatemi, *Nonlinear total variation based noise removal algorithms*, Phys. D, 60:259–268, 1992.

[29] J. Tumblin and G. Turk, *A boundary hierarchy for detail-preserving contrast reduction*, in Proceedings of the 26th Annual Conference on Computer Graphics and Interactive Techniques, ACM Press/Addison-Wesley, New York, 83–90, 1990.
[30] J.A. Weickert, *Anisotropic Diffusion in Image Processing*, Teubner, Stuttart, 1998.
[31] A. Witkin, *Scale-space filtering*, in Proceedings of the International Joint Conference on Artificial Intelligence, Karlsruhe, Germany, 1019–1021, 1983.
[32] Y. Wu, *The stability of traveling fronts for some quasilinear Burgers-type equations*, Adv. Math., 31:363–371, 2002.
[33] Y.-L. You and M. Kaveh, *Fourth-order partial differential equations for noise removal*, IEEE Trans. Image Process, 9:1723–1730, 2000.