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TRANSPORT-COLLAPSE SCHEME FOR SCALAR CONSERVATION LAWS – INITIAL-BOUNDARY VALUE PROBLEM*

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Abstract. We extend Brenier's transport collapse scheme on the initial-boundary value problem for scalar conservation laws. It is based on averaging out the solution to the corresponding kinetic equation, and it leads to a new solution concept for the problem under consideration. We also provide numerical examples.

Keywords. scalar conservation laws; well posedness; mixed problem; transport-collapse scheme; kinetic formulation.

AMS subject classifications. 35L65; 65M25.

1. Introduction

The subject of the paper is the construction of a new numerical method for the initial-boundary problem for scalar conservation laws. The method is a generalization of the transport-collapse scheme introduced in [5]. A consequence of the analysis of the scheme is a new solution concept of the initial-boundary value problem for scalar conservation laws.

In order to introduce it, let $\Omega \subset \mathbf{R}^d$ be a bounded smooth domain and $\mathbf{R}^+ = [0,\infty)$. We consider

$$\partial_t u + \mathrm{div}_{\mathbf{x}} f(u) = 0, \quad (t,\mathbf{x}) \in \mathbf{R}^+ \times \Omega, \tag{1.1}$$

$$u|_{t=0} = u_0(\mathbf{x}), \tag{1.2}$$

$$u|_{\mathbf{R}^+ \times \partial\Omega} = u_B(t,\mathbf{x}). \tag{1.3}$$

where $f \in C^2(\mathbf{R};\mathbf{R}^d)$. If not stated otherwise, we assume that $u_0 \in L^1(\mathbf{R}^d)$, $u_B \in L^1_{loc}(\mathbf{R}^+ \times \partial\Omega)$. We also assume that

$$a \le u_0, u_B \le b \text{ for some constants } a \le b. \tag{1.4}$$

A typical problem described by (1.1), (1.2), (1.3) arises e.g. in traffic flow models. Namely, if we aim to describe a flow on a finite highway (required to model on and off ramps) we need to use boundary conditions [22]. For instance, optimization of travel time and cost between two points can be obtained by controlling incoming and outgoing car densities [2].

Nevertheless, it is clear that the boundary conditions cannot be prescribed if the characteristics corresponding to Equation (1.1) and emerging from the boundary leave Ω. This means that one needs to introduce a new concept defining what conditions the unknown function u should satisfy in order to be a solution to (1.1), (1.2), (1.3). This was first done in [4] where the existence of strong traces at the boundary of solutions is assumed; see [1,20,23]. The weak formulation which does not require existence of strong traces was later proposed by F.Otto [19] and the corresponding numerical method was developed in [24]. The concept is further extended in [21] in a more general setting

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(on manifolds necessarily implying that the flux depends on $\mathbf{x}$). Before we recall its Euclidean version, let us introduce the notations

$$|z|_- = \begin{cases} 0, & z \geq 0 \\ z, & z \leq 0 \end{cases}, \quad |z|_+ = \begin{cases} z, & z \geq 0 \\ 0, & z \leq 0 \end{cases}.$$

Definition 1.1. *A function $u \in L^\infty(\mathbf{R}^+ \times \Omega)$ is said to be the weak entropy solution to* (1.1), (1.2), (1.3) *if there exists a constant $L \in \mathbf{R}$ such that for every $k \in \mathbf{R}$ and every non-negative $\varphi \in C^1_c(\mathbf{R}^d_+;\mathbf{R}^+)$, $\mathbf{R}^d_+ = \mathbf{R}^+ \times \mathbf{R}^d$, it holds*

$$\int_{\mathbf{R}^d_+} \big(|u-k|_+ \partial_t \varphi + \operatorname{sgn}_+(u-k)(f(u)-f(k))\nabla_{\mathbf{x}}\varphi\big)\, d\mathbf{x}dt \\ + \int_{\mathbf{R}^d} |u_0 - k|_+ \varphi(0,\cdot) d\mathbf{x} + L \int_{\mathbf{R}^+ \times \partial\Omega} \varphi |u_B - k|_+ d\gamma(\mathbf{x}) dt \geq 0, \tag{1.5}$$

and

$$\int_{\mathbf{R}^d_+} \big(|u-k|_- \partial_t \varphi + \operatorname{sgn}_-(u-k)(f(u)-f(k))\nabla_{\mathbf{x}}\varphi\big)\, d\mathbf{x}dt \\ + \int_{\mathbf{R}^d} |u_0 - k|_- \varphi(0,\cdot) d\mathbf{x} + L \int_{\mathbf{R}^+ \times \partial\Omega} \varphi |u_B - k|_- d\gamma(\mathbf{x}) dt \geq 0, \tag{1.6}$$

where γ is $(d-1)$-dimensional Hausdorff measure on $\partial\Omega$.

As it comes to the refinement of the latter concept, we actually base it on an interesting observation from [21] roughly stating that if the characteristics emerging from $\{t=0\} \times \Omega$ hit the boundary then the corresponding boundary value should not affect the solution. Thus, we are going to construct the definition of solution so that it involves somehow only those parts of the boundary which essentially influence on solutions.

To be more precise, assume that we are dealing with the flux depending on $\mathbf{x}$ i.e. $f = f(\mathbf{x}, \lambda)$. Denote by

$$S^- = \{\mathbf{x} \in \partial\Omega : \langle f'_\lambda(\mathbf{x},\lambda), \vec{\nu}(\mathbf{x})\rangle \leq 0 \ \text{ a.e. } \ \lambda \in I\},$$

where I contains all essential values of the functions u_B and u_0 (i.e. of appropriate entropy solution u), and $\vec{\nu}$ is the outer unit normal on $\partial\Omega$. The set S^- actually consists of all points such that all possible characteristics from that point enter into the (interior of the) set Ω. Therefore, for every $\mathbf{x} \in S^-$, the trace of the corresponding entropy solution is actually $u_B(\mathbf{x})$.

Similarly, for

$$S^+ = \{\mathbf{x} \in \partial\Omega : \langle f'_\lambda(\mathbf{x},\lambda), \vec{\nu}\rangle \geq 0 \ \text{ a.e. } \ \lambda \in I\},$$

all possible characteristics issuing from $\mathbf{x} \in S^+$ leave the set Ω, and $u_B(\mathbf{x})$ does not influence on the weak entropy solution u to (1.1), (1.2), (1.3).

However, both sets S^- and S^+ can be empty since for some $\lambda \in I$, it can be $\langle f'_\lambda(\mathbf{x},\lambda), \vec{\nu}\rangle \leq 0$ and for other $\lambda \in I$ we could have $\langle f'_\lambda(\mathbf{x},\lambda), \vec{\nu}\rangle > 0$. Therefore, in order to refine former arguments, we need to rewrite considered conservation laws so that we can more accurately take into account behaviour of the flux f with respect to λ. A natural choice is the kinetic formulation to (1.1) since it includes the variable λ in a desired way. Let us first recall the Kruzhkov entropy admissibility conditions to the Cauchy problem for scalar conservation laws.

DEFINITION 1.2. *A bounded function u is called an entropy admissible solution to* (1.1) *with the initial conditions* (1.2) *if for every convex function $V\in C^2(\mathbf{R})$ and every non-negative $\varphi\in C^1_c(\mathbf{R}^d_+)$, it holds*

$$\iint\limits_{\mathbf{R}^+\times\mathbf{R}^d}\big[V(u)\partial_t\varphi+\int_a^u f'_\lambda(v)V'(v)dv\cdot\nabla\varphi\big]d\mathbf{x}dt+\int_{\mathbf{R}^d}V(u_0(\mathbf{x}))\varphi(0,\mathbf{x})d\mathbf{x}\leq 0. \tag{1.7}$$

Equivalent and more usual definition of an admissible solution is given by the Kruzhkov entropies $V(\lambda)=|u-\lambda|$, $\lambda\in\mathbf{R}$, and it states that a bounded function u is called an entropy admissible solution to (1.1), (1.2) if for every $\lambda\in\mathbf{R}$ it holds

$$\partial_t|u-\lambda|+\mathrm{div}_{\mathbf{x}}[\mathrm{sgn}(u-\lambda)(f(u)-f(\lambda))]\leq 0 \tag{1.8}$$

in the sense of distributions on $\mathcal{D}'(\mathbf{R}^d_+)$, and it holds

$$esslim_{t\to 0}\int_\Omega|u(t,\mathbf{x})-u_0(\mathbf{x})|d\mathbf{x}=0.$$

Roughly speaking, by finding derivative with respect to λ in (1.8) one reaches to the kinetic formulation provided below (see e.g. [8, 12, 14, 16] for different variants).

THEOREM 1.1 ([16]). *The function $u\in C([0,\infty);L^1(\mathbf{R}^d))\cap L^\infty_{loc}((0,\infty);L^\infty(\mathbf{R}^d))$ is the entropy admissible solution to* (1.1), (1.2) *if and only if there exists a non-negative Radon measure $m(t,\mathbf{x},\lambda)$ such that $m((0,T)\times\mathbf{R}^{d+1})<\infty$ for all $T>0$ and such that the function* $\chi(\lambda,u)=\begin{cases}1, & 0<\lambda\leq u\\ -1, & u\leq\lambda\leq 0\\ 0, & else\end{cases}$, *represents the distributional solution to*

$$\partial_t\chi+\mathrm{div}_{\mathbf{x}}(f'(\lambda)\chi)=\partial_\lambda m(t,\mathbf{x},\lambda),\quad (t,\mathbf{x})\in\mathbf{R}^+\times\mathbf{R}^d, \tag{1.9}$$

$$\chi(\lambda,u(t=0,\mathbf{x}))=\chi(\lambda,u_0(\mathbf{x})). \tag{1.10}$$

In the next section, we shall provide properties of the function χ.

Remark that through the kinetic concept, one reduces the nonlinear Equation (1.1) on the linear (so called kinetic) equation. However, derivative of a measure figures in the equation (see the right-hand side of (1.9)) and it has one more variable which is usually called kinetic or velocity variable. Due to the former reason, the kinetic equation is not convenient for numerical implementation. Nevertheless, if we neglect the derivative of the measure, and then average out the solution to the obtained linear equation with respect to the kinetic variable, we obtain entropy solution to the considered problem. Such a procedure is proposed in [5] for Cauchy problems corresponding to Equation (1.1).

The power of the method to be presented is in its ability to transform nonlinear problem into linear. Linear scalar conservation laws are easy to solve numerically since there are a lot of robust numerical schemes available. The cost of that "transformation" in practical computing is adding one more dimension (see (1.9)).

Moreover, we shall use the transport-collapse techniques to construct the bounded function u satisfying the following definition.

Definition 1.3. *We say that the function* $u\in L^\infty(\mathbf{R}^+\times\Omega;[a,b])$ *is a weak entropy admissible solution to* (1.1), (1.2), (1.3) *if for every* $k\in\mathbf{R}$ *and every non-negative* $\varphi\in C_c^1(\mathbf{R}^+\times\Omega)$ *it holds*

$$\begin{aligned}&\int_{\Omega\times\mathbf{R}^+}\big(|u-k|_+\partial_t\varphi+\operatorname{sgn}_+(u-k)(f(u)-f(k))\cdot\nabla_{\mathbf{x}}\varphi\big)\,d\mathbf{x}dt\\&-\int_a^b\int_{\substack{\mathbf{R}^+\times\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\varphi(t,\mathbf{x})\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle\operatorname{sgn}_+(\lambda-k)\operatorname{sgn}_+(u_B(t,\mathbf{x})-\lambda)d\gamma(\mathbf{x})dtd\lambda\\&+\int_{\mathbf{R}^d}|u_0-k|_+\varphi(0,\cdot)d\mathbf{x}\geq 0,\end{aligned}\tag{1.11}$$

and

$$\int_{\Omega\times\mathbf{R}^+}\big(|u-k|_-\partial_t\varphi+\operatorname{sgn}_-(u-k)(f(u)-f(k))\nabla_{\mathbf{x}}\varphi\big)\,d\mathbf{x}dt\tag{1.12}$$

$$-\int_a^b\int_{\substack{\mathbf{R}^+\times\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\varphi(t,\mathbf{x})\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle\operatorname{sgn}_-(\lambda-k)\operatorname{sgn}_-(u_B(t,\mathbf{x})-\lambda)d\gamma(\mathbf{x})dtd\lambda\tag{1.13}$$

$$+\int_{\mathbf{R}^d}|u_0-k|_-\varphi(0,\cdot)d\mathbf{x}\geq 0.$$

It is not difficult to see that if u satisfies conditions of Definition 1.3 then u also satisfies Definition 1.1. This will be proved in the last section.

Let us briefly comment Definition 1.3. The first and last terms on the left-hand sides of (1.11) and (1.12) are standard in the entropy admissibility concept (compare with Definition 1.1 and (1.8)) and they are related to the behaviour of the solution u in the interior of Ω and on $t=0$. The middle terms on the left-hand sides of (1.11) and (1.12) simply say that when the characteristics enter Ω (i.e. when the angle between the outer normal $\vec{\nu}$ and $f'(\lambda)$ is greater than $\pi/2$, i.e. when $\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0$) then we shall take the boundary data into account.

Finally, let us remark that work in the field of numerical methods for conservation laws mostly deals with Cauchy problems (scalar conservation laws or systems; see e.g. classical books [11, 15] and references therein). As for (1.1), (1.2), (1.3), there are not so many results since the interest for this kind of problem has arisen relatively recently. We mention [3, 24] and references therein. For results in the case of systems, one can consult [18] where one can also find thorough overview of the subject.

The paper is organized as follows. In Section 2, we shall recall necessary properties of of the transport-collapse scheme for initial value problems corresponding to (1.1) from [5]. In Section 3, we shall introduce a transport-collapse type operator for (1.1), (1.2), (1.3), and prove its convergence toward the entropy solution in the sense of Definition 1.3.

2. Transport collapse scheme for the Cauchy problem

In this section, we recall basic facts concerning the transport-collapse scheme for the Cauchy problem. The results are taken from [5].

The idea of the transport collapse scheme for the initial value problem (1.1), (1.2) is to solve problem (1.9), (1.10) when we omit the right-hand side in (1.9):

$$\partial_t h+\operatorname{div}_{\mathbf{x},\lambda}[f'(\lambda)h]=0,\quad h|_{t=0}=\chi(\lambda,u_0(\mathbf{x})).\tag{2.1}$$

The solution of this equation is obtained via the method of characteristics and, since the equation is linear, it is given by

$$h(t,\mathbf{x},\lambda)=\chi(\lambda,u_0(\mathbf{x}-f'(\lambda)t)). \tag{2.2}$$

In order to explain how to use it for solving (1.1), (1.2), we need some properties of the kinetic function χ.

Proposition 2.1 ([5], page 1018). *It holds*

a) $\forall u,v\in L^1(\mathbf{R}^d)$ *such that* $u\geq v \Longrightarrow \chi(\lambda,u)\geq\chi(\lambda,v)$*;*

b) $\forall u\in L^1(\mathbf{R}^d)$, $\forall g\in L^\infty(\mathbf{R})$, *it holds* $\iint\chi(\lambda,u)g(\lambda)d\mathbf{x}d\lambda=\int\left(\int_a^u g(\lambda)d\lambda\right)d\mathbf{x}$*;*

Let us now define the transport-collapse operator T.

Definition 2.1. *The transport collapse operator* $T(t)$ *is defined for every* $u\in L^1(\mathbf{R}^d)$ *by*

$$T(t)u(\mathbf{x})=\int\chi(\lambda,u(\mathbf{x}-f'(\lambda)t))d\lambda. \tag{2.3}$$

It satisfies the following properties given in [5, Proposition 1].

Proposition 2.2. *It holds for every* $u,v\in L^1(\mathbf{R}^d)$

a) $u\leq v$ *a.e. implies* $T(t)u\leq T(t)v$ *a.e;*

b) $\int T(t)u(\mathbf{x})d\mathbf{x}=\int u(\mathbf{x})d\mathbf{x}$*;*

c) the operator $T(t)$ *is non-expansive*

$$\|T(t)u-T(t)v\|_{L^1(\mathbf{R}^d)}\leq\|u-v\|_{L^1(\mathbf{R}^d)},$$

and, in particular, $\|T(t)u\|_{L^1(\mathbf{R}^d)}\leq\|u\|_{L^1(\mathbf{R}^d)}$*;*

We also need the following result whose proof is based on the following inequality for convex functions (cf. from [9, p.50]) which we shall also need. Let V be a convex function. Then it holds [5, A.1.5]:

$$V(T(t)u(\mathbf{x}))\leq\int V'(\lambda)\chi(\lambda,u(\mathbf{x}-f'(\lambda)t))d\lambda+V(0). \tag{2.4}$$

It holds:

Proposition 2.3 ([5], Proposition 2). *For any smooth positive test function* φ*, any* $u\in L^1(\mathbf{R})$ *such that* $a\leq u\leq b$*, and convex Lipschitz function* $V:\mathbf{R}\to\mathbf{R}$*, we have*

$$\int(V(T(t)u)-V(u))(\mathbf{x})\varphi(\mathbf{x})d\mathbf{x}\leq\int_0^t\int B_V(t',\mathbf{x},u(\mathbf{x}))\nabla\varphi d\mathbf{x}dt'+Ct^2\|u\|_{L^\infty(\mathbf{R})}$$

where $B_V(t,\mathbf{x},u)=\int_0^u f'(\lambda)V'(\lambda)d\lambda$.

The following theorem was central in [5].

Theorem 2.1 ([5], Theorem 1). *For any initial value* $u_0\in L^1(\mathbf{R}^d)$*, the unique entropy solution to* (1.1), (1.2) *is given by*

$$u(t,\mathbf{x})=S(t)u_0(\mathbf{x})=L^1-\lim_{n\to\infty}T(\frac{t}{n})^n u_0(\mathbf{x}).$$

3. Boundary value problem

In this section, we shall consider boundary value problem for homogeneous scalar conservation law (1.1) on the domain Ω, which is a bounded open smooth subset of $\mathbf{R}^d$.

In order to simplify the presentation, we shall assume that $a=0$ in (1.4), i.e. that the solution to the considered problem is non-negative. In particular, this implies that the kinetic function χ corresponding to such a solution satisfies

$$\chi(\lambda,u)=\operatorname{sgn}_+(u-\lambda)\geq 0. \tag{3.1}$$

First, notice that the kinetic formulation from Theorem 2 still holds in the interior of $\mathbf{R}^+\times\Omega$. However, we cannot use the method of characteristics from the previous section directly since the characteristics entering the boundary determine the value at the boundary. Nevertheless, since we are re-iterating the procedure after a short period of time, we can modify the transport collapse scheme so that we take into account the boundary data.

Accordingly, recall that the kinetic reformulation for (1.1) has the form:

$$\partial_t\chi(\lambda,u)+f'(\lambda)\operatorname{div}_{\mathbf{x}}\chi(\lambda,u)=\partial_\lambda m_+(t,\mathbf{x},\lambda) \tag{3.2}$$

where m_+ is a non-negative measure. Assume that Ω is an open set such that for some $\sigma\in(0,1)$, no two outer normals from $\partial\Omega$ intersect in the set

$$\Omega_\sigma=\{\mathbf{x}\in\mathbf{R}^d:\ \ dist(\mathbf{x},\Omega)<\sigma\}\ \text{ i.e. in the set}$$
$$\Omega^\sigma=\Omega_\sigma\backslash\Omega$$

(i.e. we assume that Ω is of C^1-class). In order to augment (3.2) (with neglected right-hand side) with appropriate initial data, denote by $\vec{\nu}(\mathbf{x})$, $\mathbf{x}\in\Omega_\sigma\setminus\Omega$ the unit outer normal on $\partial\Omega$ passing through the point $\mathbf{x}$. We then extend the boundary data $u_B(t,\mathbf{x})$ for every fixed $t\geq 0$ to be constants along the outer normals $\vec{\nu}(\mathbf{x})$ in the set Ω_σ.

More precisely, we set for $\mathbf{x}\in\Omega^\sigma=\Omega_\sigma\backslash\Omega$ (slightly abusing the notation)

$$u_B(t,\mathbf{x})=u_B(t,\mathbf{x}_0),\ \ \text{for } \mathbf{x}_0\in\partial\Omega \text{ such that } \mathbf{x}=\mathbf{x}_0+r\vec{\nu}(\mathbf{x}_0), \tag{3.3}$$

for some $r>0$. Finally, introduce the function

$$w_{u(t,\cdot)}(\mathbf{x})=\begin{cases}0, & \mathbf{x}\notin\Omega_\sigma\\ u(t,\mathbf{x}), & \mathbf{x}\in\Omega\\ u_B(t,\mathbf{x}), & \mathbf{x}\in\Omega_\sigma\setminus\Omega=\Omega^\sigma,\end{cases} \tag{3.4}$$

which is actually the extension of u along the outer normals $\vec{\nu}$. If the function u does not depend on t, then we put $u(\mathbf{x})$ instead of $u(t,\mathbf{x})$, and $u_B(0,\mathbf{x})$ instead of $u_B(t,\mathbf{x})$ on the right-hand side of (3.4). Remark that we can rewrite the function $w_{u(t,\cdot)}(\mathbf{x})$ in the form

$$w_{u(t,\cdot)}(\mathbf{x})=u(t,\mathbf{x})\kappa_\Omega(\mathbf{x})+u_B(t,\mathbf{x})\kappa_{\Omega^\sigma}(\mathbf{x}),$$

where κ_A is the characteristic function of the set A.

Now, we are ready to introduce a modification of the transport collapse scheme from the previous section. Fix $t>0$ and $n\in\mathbf{N}$. We neglect the right-hand side of (3.2) and, on the first step, we take $\chi(\lambda,w_{u_0}(\mathbf{x}))$ as the initial data.

$$\partial_t h+f'(\lambda)\operatorname{div}_{\mathbf{x}}h=0, \tag{3.5}$$

$$h|_{t=0} = \chi(\lambda, w_{u_0}(\mathbf{x})). \tag{3.6}$$

The solution to (3.5) is given by $h(t,\mathbf{x},\lambda) = \chi(\lambda, w_{u_0}(\mathbf{x} - f'(\lambda)t))$ (see (2.2)). We construct the approximate solution u_n to (1.1), (1.2), (1.3) by the following procedure. For any $\mathbf{x} \in \Omega$, we define (recursively):

- $$u_n(t',\mathbf{x}) = T(t'/n)(w_{u_0}(\mathbf{x})) := \int_0^b \chi(\lambda, \omega_{u_0}(\mathbf{x} - f'(\lambda)t'))d\lambda, \quad t' \in (0, t/n]. \tag{3.7}$$
- For $k = 1, \dots, n-1$, we take
$$u_n(kt/n + t', \mathbf{x}) = \int_0^b \chi(\lambda, \omega_{u_n(kt/n,\cdot)}(\mathbf{x} - f'(\lambda)t'))d\lambda, \quad t' \in (0, t/n]. \tag{3.8}$$

Remark that here, we have actually applied the transport collapse operator from (2.3). Roughly speaking, the approximate solution in $[0,t] \times \Omega$ is given by the transport-collapse operator, while in $[0,t] \times \Omega^C$ the sequence (u_n) is equal to the boundary data extended along the outer normals on $\partial\Omega$.

We shall show that the sequence (u_n) strongly converges in $L^1([0,t] \times \Omega)$ toward a function u which represents the solution to (1.1), (1.2), (1.3) in the sense of Definition 1.3 (see Corollary 3.3). In order to prove the later fact, we shall use the kinetic formulation similar to [21]. We introduce the following definition.

Definition 3.1. *We say that a non-negative function $p_+ = p_+(t,\mathbf{x},k) \in L^\infty(\mathbf{R}^+ \times \Omega \times \mathbf{R})$ decreasing with respect to $k \in \mathbf{R}$, is the kinetic super-solution to* (1.1), (1.2), (1.3) *if it satisfies the following equation for any $\varphi \in C_c^1(\mathbf{R}^+ \times \Omega)$ and $\rho \in C_c^1((0,b))$ (recall that we assumed $a = 0$ in* (1.4)*):*

$$\begin{aligned}
&\int_{\mathbf{R}^+ \times \Omega \times \mathbf{R}} \rho(k) p_+(t,\mathbf{x},k)\left(\partial_t \varphi + f'(k)\nabla_{\mathbf{x}}\varphi\right) d\mathbf{x}dtdk \\
&\quad + \int_{\mathbf{R}} \int_{\Omega} \rho(k) \mathrm{sgn}_+(u_0(\mathbf{x}) - k)\varphi(0,\mathbf{x}) d\mathbf{x}dk \\
&\quad - \int_{\mathbf{R}} \int_0^b \int_{\substack{\mathbf{R}^+ \times \partial\Omega \\ \langle f'(\lambda), \vec{\nu}(\mathbf{x})\rangle < 0}} \rho'(k)\varphi(t,\mathbf{x})\langle f'(\lambda), \vec{\nu}(\mathbf{x})\rangle \mathrm{sgn}_+(\lambda - k)\mathrm{sgn}_+(u_B(t,\mathbf{x}) - \lambda) d\gamma(\mathbf{x}) dt d\lambda dk \\
&= -\int_{\mathbf{R}} \int_{\mathbf{R}^+ \times \Omega} \varphi(t,\mathbf{x})\rho'(k) dm^+(t,\mathbf{x},k),
\end{aligned} \tag{3.9}$$

for a non-negative measure m^+.

We say that a non-positive function $p_- = p_-(t,\mathbf{x},k) \in L^\infty(\mathbf{R}^+ \times \Omega \times \mathbf{R})$ decreasing with respect to $k \in \mathbf{R}$, is the kinetic sub-solution to (1.1), (1.2), (1.3) *if it satisfies the following equation for any $\varphi \in C_c^1(\mathbf{R}^+ \times \Omega)$ and $\rho \in C_c^1(\mathbf{R})$:*

$$\begin{aligned}
&\int_{\mathbf{R}^+ \times \Omega \times \mathbf{R}} \rho(k) p_-(t,\mathbf{x},k)\left(\partial_t \varphi + f'(k)\nabla_{\mathbf{x}}\varphi\right) d\mathbf{x}dtdk \\
&\quad + \int_{\mathbf{R}} \int_{\Omega} \rho(k) \mathrm{sgn}_+(u_0(\mathbf{x}) - k)\varphi(0,\mathbf{x}) d\mathbf{x}dk \\
&\quad - \int_{\mathbf{R}} \int_0^b \int_{\substack{\mathbf{R}^+ \times \partial\Omega \\ \langle f'(\lambda), \vec{\nu}(\mathbf{x})\rangle < 0}} \rho'(k)\varphi(t,\mathbf{x})\langle f'(\lambda), \vec{\nu}(\mathbf{x})\rangle \mathrm{sgn}_-(\lambda - k)\mathrm{sgn}_-(u_B(t,\mathbf{x}) - \lambda) d\gamma(\mathbf{x}) dt d\lambda dk
\end{aligned}$$

$$=-\int_{\mathbf{R}}\int_{\mathbf{R}^+\times\Omega}\varphi(t,\mathbf{x})\rho'(k)dm^-(t,\mathbf{x},k), \tag{3.10}$$

for a non-negative measure m^-.

The function $p\in L^\infty(\mathbf{R}^+\times\Omega\times\mathbf{R})$ is a kinetic super-solution if it is a kinetic supersolution and $(p-1)$ is a kinetic sub-solution.

Let us notice that we tacitly keep in mind that for the kinetic solution we need to have $p=\operatorname{sgn}_+(u(t,\mathbf{x})-\lambda)$ and thus $p-1=\operatorname{sgn}_-(u(t,\mathbf{x})-\lambda)$ for the entropy solution u. The following theorem holds.

THEOREM 3.1. *There exists the kinetic solution to* (1.1), (1.2), (1.3) *in the sense of Definition 3.1.*

Proof. Let us consider behaviour of $V(T(t)v)-V(v)$ for a convex function V whose special form will be chosen later, and for the function $v\geq 0$ playing the role of $u_n(t_s,\cdot)$ (recall that we have assumed that $a=0$ implying that our sequence of approximate solutions is non-negative) such that $v(t,\mathbf{x})=u_B(t,\mathbf{x})$, $\mathbf{x}\in\Omega_\sigma\backslash\Omega$. Accordingly, denote by

$$\Omega-f'(\lambda)t=\{\mathbf{x}\in\mathbf{R}^d\,|\,\exists\mathbf{x}_0\in\partial\Omega\ \ \mathbf{x}=\mathbf{x}_0-tf'(\lambda)\},$$

and by $\vec{\nu}_{\Omega-f'(\lambda)t}(\mathbf{x})$ the unit outer normal to $\partial(\Omega-f'(\lambda)t)$, assume that we fixed $t<\sigma$ for σ given in (3.3). Consider for $\varphi\in C^2_c(\Omega_\sigma)$:

$$\begin{aligned}
&\int_\Omega (V(T(t)v)(\mathbf{x})-V(v)(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}\\
&\overset{(2.4)}{\leq}\int_\Omega\int_0^b V'(\lambda)(\chi(\lambda,v(\mathbf{x}-f'(\lambda)t))-\chi(\lambda,v(\mathbf{x})))\varphi(\mathbf{x})d\lambda d\mathbf{x}=\Big(\mathbf{y}=\mathbf{x}-f'(\lambda)t\Big)\\
&=\int_0^b\int_{\Omega-f'(\lambda)t} V'(\lambda)\chi(\lambda,v(\mathbf{y}))\varphi(\mathbf{y}+f'(\lambda)t)d\mathbf{y}d\lambda-\int_0^b\int_\Omega V'(\lambda)\chi(\lambda,v(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}d\lambda\\
&=\int_0^b\int_{(\Omega-f'(\lambda)t)\backslash\Omega} V'(\lambda)\chi(\lambda,u_B(0,\mathbf{x}))\varphi(\mathbf{x}+tf'(\lambda))d\mathbf{x}d\lambda\\
&\quad-\int_0^b\int_{\Omega\backslash(\Omega-f'(\lambda)t)} V'(\lambda)\chi(\lambda,v(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}d\lambda\\
&\quad+\int_0^b\int_{(\Omega-f'(\lambda)t)\cap\Omega} V'(\lambda)\chi(\lambda,v(\mathbf{x}))(\varphi(\mathbf{x}+tf'(\lambda))-\varphi(\mathbf{x}))d\mathbf{x}d\lambda.
\end{aligned}\tag{3.11}$$

Let us now prove that for almost every $\lambda\in(a,b)$

$$\begin{aligned}
&\lim_{t\to 0}\frac{1}{t}\int_{(\Omega-f'(\lambda)t)\backslash\Omega}\chi(\lambda,u_B(0,\mathbf{x}))\varphi(\mathbf{x}+tf'(\lambda))d\mathbf{x}\\
&=-\int_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\chi(\lambda,u_B(0,\mathbf{x}))\varphi(\mathbf{x})\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle d\gamma(\mathbf{x}).
\end{aligned}\tag{3.12}$$

The conclusion follows from the coarea formula [10] (which expresses the integral of a function over an open set in Euclidean space in terms of the integral of the level sets of another function). To this end, for $\mathbf{x}\in(\Omega-f'(\lambda)t)\backslash\Omega$ let us denote by $Y(\mathbf{x})\in\partial\Omega$ the point such that for some $r\geq 0$

$$\mathbf{x}=Y(\mathbf{x})-rf'(\lambda) \text{ and denote } z(\mathbf{x})=r. \tag{3.13}$$

In other words,

$$\mathbf{x}=Y(\mathbf{x})-z(\mathbf{x})f'(\lambda).$$

Finding derivative with respect to x_i, $i=1,\dots,d$, here and then multiplying the obtained expression by the unit outer normal $\vec{\nu}(Y(\mathbf{x}))=(\nu_1(Y(\mathbf{x})),\dots,\nu_d(Y(\mathbf{x})))$ on $\partial\Omega$ at $Y(\mathbf{x})$, we get

$$\nu_i(Y(\mathbf{x}))=\langle\frac{\partial Y(\mathbf{x})}{\partial x_i},\vec{\nu}(Y(\mathbf{x}))\rangle+\langle\vec{\nu}(Y(\mathbf{x})),f'(\lambda)\rangle\frac{\partial z(\mathbf{x})}{\partial x_i}.$$

Since $\frac{\partial Y(\mathbf{x})}{\partial x_i}$ belongs to the tangential hyper-plane to $\partial\Omega$ at $Y(\mathbf{x})$, it must be $\langle\frac{\partial Y(\mathbf{x})}{\partial x_i},\vec{\nu}(Y(\mathbf{x}))\rangle=0$, and therefore

$$|\nabla z(\mathbf{x})|=\frac{1}{|\langle\vec{\nu}(Y(\mathbf{x})),f'(\lambda)\rangle|}.$$

Now, since

$$(\Omega-f'(\lambda)t)\backslash\Omega=\{\mathbf{x}|\,z(\mathbf{x})=r\in(0,t)\}$$

the coarea formula provides

$$\begin{aligned}&\frac{1}{t}\int\limits_{(\Omega-f'(\lambda)t)\backslash\Omega}\chi(\lambda,u_B(0,\mathbf{x}))\varphi(\mathbf{x}+tf'(\lambda))|\langle f'(\lambda),\vec{\nu}(Y(\mathbf{x}))\rangle||\nabla z(\mathbf{x})|d\mathbf{x}\\ =&\frac{1}{t}\int_0^t\int\limits_{z^{-1}(t)}\chi(\lambda,u_B(0,\mathbf{x}))\varphi(\mathbf{x}+tf'(\lambda))|\langle f'(\lambda),\vec{\nu}(Y(\mathbf{x}))\rangle|d\gamma(\mathbf{x})dr\end{aligned} \tag{3.14}$$

and this immediately gives (3.12) after letting $t\to 0$ (since $|\langle f'(\lambda),\vec{\nu}(Y(\mathbf{x}))\rangle||\nabla z(\mathbf{x})|=1$ and $z^{-1}(0)=\{\mathbf{x}\in\partial\Omega|\,\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0\}$). Indeed, according to the definition of u_B, for $\mathbf{x}\in z^{-1}(t)$ we have

$$u_B(0,\mathbf{x})=u_B(0,\tilde{\mathbf{x}}-tf'(\lambda))=u_B(0,Y(\mathbf{x}_t)),$$

for appropriate $\tilde{\mathbf{x}}\in\partial\Omega$ and $Y(\mathbf{x}_t)\in\partial\Omega$ such that $\mathbf{x}=r\vec{\nu}(Y(\mathbf{x}_t))$ for some $r>0$. Clearly, $Y(\mathbf{x}_t)\to\tilde{\mathbf{x}}$ as $t\to 0$ and from here, since $u_B(0,\cdot)\in L^1(\partial\Omega)$ and almost every point of an integrable function is the Lebesgue point, we have for almost every $\lambda\in(a,b)$ and $\mathcal{H}^{d-1}$-almost every $\tilde{\mathbf{x}}\in\partial\Omega$

$$\chi(\lambda,u_B(0,\mathbf{x}))=\chi(\lambda,u_B(0,\tilde{\mathbf{x}}-tf'(\lambda)))=\chi(\lambda,u_B(0,Y(\mathbf{x}_t)))\underset{t\to 0}{\longrightarrow}\chi(\lambda,u_B(0,\tilde{\mathbf{x}})).$$

The test function φ is continuous and therefore $\varphi(\mathbf{x}+tf'(\lambda))\to\varphi(\tilde{\mathbf{x}})$ as $t\to 0$.

Notice that the condition $\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0$ appears since only in that case the parameter r in (3.13) will be greater than zero (the angle between the outer normal $\vec{\nu}$ at $\partial\Omega$ and $f'(\lambda)$ is greater than $\pi/2$ and thus $-tf'(\lambda)$ goes in the same direction as $\vec{\nu}$).

To continue, notice that since $v \geq 0$ it will be $\chi(\lambda,v) = \mathrm{sgn}_+(v-\lambda) \geq 0$ (see (3.1)). Next, for a fixed $k \in \mathbf{R}$, choose $V(\lambda) = V_+(\lambda) = |\lambda - k|_+$ in (3.11).

After expanding the function φ in the Taylor expansion around $\mathbf{x}$ and taking into account (3.12) and Proposition 2.1 b), we get from (3.11) (keep in mind that $V_+'(\lambda) = \mathrm{sgn}_+(\lambda - k) \geq 0$ and $\chi(\lambda,u) = \mathrm{sgn}_+(u-\lambda)$ if $u \geq 0$)

$$\begin{aligned}
&\int_\Omega (|T(t)(v)(\mathbf{x}) - k|_+ - |v(\mathbf{x}) - k|_+)\varphi(\mathbf{x}) d\mathbf{x} \\
&\leq -t \int_0^b \int_{\substack{\partial\Omega \\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle < 0}} \mathrm{sgn}_+(\lambda - k)\mathrm{sgn}_+(u_B(t,\mathbf{x}) - \lambda)\langle f'(\lambda), \vec{\nu}(\mathbf{x})\rangle \varphi(\mathbf{x}) d\gamma(\mathbf{x}) d\lambda \\
&\quad + t \int_\Omega \int_0^b f'(\lambda)\mathrm{sgn}_+(\lambda - k)\mathrm{sgn}_+(v(\mathbf{x}) - \lambda) \cdot \nabla\varphi(\mathbf{x}) d\lambda d\mathbf{x} + o(t) \\
&= -t \int_0^b \int_{\substack{\partial\Omega \\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle < 0}} \mathrm{sgn}_+(\lambda - k)\mathrm{sgn}_+(u_B(t,\mathbf{x}) - \lambda)\langle f'(\lambda), \vec{\nu}(\mathbf{x})\rangle \varphi(\mathbf{x}) d\gamma(\mathbf{x}) d\lambda \\
&\quad + t \int_\Omega \mathrm{sgn}_+(v(\mathbf{x}) - k)(f(v(\mathbf{x})) - f(k)) \cdot \nabla\varphi(\mathbf{x}) d\mathbf{x} + o(t),
\end{aligned} \tag{3.15}$$

where $o(t)$ is the Landau symbol meaning that $\lim_{t\to 0} \frac{o(t)}{t} = 0$. Since for every n, the function $u_n(t,\cdot)$ has the same properties as the function v from the above, we see that $u_n(t,\cdot)$ satisfies (3.15). More precisely, denote by

$$u_B^n(\tau,\mathbf{x}) = \sum_{j=0}^{n-1} u_B^j(\mathbf{x}) \kappa_{[jt/n,(j+1)t/n)}(\tau)$$

a piecewise constant with respect to $\tau \in \mathbf{R}^+$ approximation of the function u_B satisfying

$$\lim_{n\to\infty} \|u_B^n - u_B\|_{L^1([0,t)\times\partial\Omega)} = 0. \tag{3.16}$$

Recall that we can choose for instance $u_B^j(\mathbf{x}) = \frac{t}{n}\int_{jt/n}^{(j+1)t/n} u_B(t',\mathbf{x}) dt'$. Then, taking in (3.15) u_B^n instead of u_B and having in mind that

$$u_n(\frac{(j+1)t}{n},\mathbf{x}) = T(\frac{t}{n})(u_n(\frac{jt}{n},\cdot)(\mathbf{x}),$$

we have for any $t > 0$ and any $j,n \in \mathbf{N}$, $j < n$, $\varphi \in C_c^2(\Omega_\sigma)$, and $\psi \in C_c^2((0,t))$

$$\begin{aligned}
&\int_\Omega \left(|u_n(\frac{(j+1)t}{n},\mathbf{x}) - k|_+ - |u_n(\frac{jt}{n},\mathbf{x}) - k|_+ \right) \psi(\frac{jt}{n})\varphi(\mathbf{x}) d\mathbf{x} \overset{(3.15)}{\leq} \\
&- \frac{t}{n} \int_0^b \int_{\substack{\partial\Omega \\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle < 0}} \mathrm{sgn}_+(\lambda - k)\chi(\lambda, u_B^n(\frac{jt}{n},\cdot)(\mathbf{x}))\langle f'(\lambda),\vec{\nu}\rangle \psi(\frac{jt}{n})\varphi(\mathbf{x}) d\gamma(\mathbf{x}) d\lambda \\
&+ \frac{t}{n} \int_\Omega \mathrm{sgn}_+(u_n(\frac{jt}{n},\mathbf{x}) - k)(f(u_n(\frac{jt}{n},\mathbf{x})) - f(k))\psi(\frac{jt}{n}) \cdot \nabla\varphi(\mathbf{x}) d\mathbf{x} + o(t/n),
\end{aligned}$$

where $o(t/n)$ is the Landau symbol meaning that $\lim_{n\to\infty} \frac{o(t/n)}{t/n} = 0$. We sum the latter relation from $j = 0$ to $j = n-1$ and take into account that $\psi(t) = 0$. After a simple rearranging and expanding the function ψ in the Taylor expansion around $\frac{(j+1)t}{n}$ on the left-hand side of the obtained expression, we get

$$\begin{aligned}
&-\sum_{j=0}^{n-1}\int_\Omega \frac{t}{n}|u_n(\frac{(j+1)t}{n},\mathbf{x}))-k|_+\psi'(\frac{(j+1)t}{n})\varphi(\mathbf{x})d\mathbf{x}\\
&-\int_\Omega |u_0(\mathbf{x})-k|_+\psi(0)\varphi(\mathbf{x})d\mathbf{x}+\mathcal{O}(t/n)\\
&\leq -\sum_{j=0}^{n-1}\frac{t}{n}\int_0^b \int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}} \mathrm{sgn}_+(\lambda-k)\,\mathrm{sgn}_+(u_B^n(\frac{jt}{n},\mathbf{x})-\lambda)\,\langle f'(\lambda),\vec{\nu}\rangle\,\psi(\frac{jt}{n})\varphi(\mathbf{x})d\gamma(\mathbf{x})d\lambda\\
&\quad+\sum_{j=0}^{n-1}\frac{t}{n}\int_\Omega \mathrm{sgn}_+(u_n(\frac{jt}{n},\mathbf{x})-k)(f(u_n(\frac{jt}{n},\mathbf{x}))-f(k))\psi(\frac{jt}{n})\cdot\nabla\varphi(\mathbf{x})d\mathbf{x}+o(1). \qquad (3.17)
\end{aligned}$$

To proceed, notice that according to the Schwartz theorem on non-negative distributions there exists a non-negative measure $m_+^n(\tau,\mathbf{x},k)\in L^\infty(\mathbf{R};\mathcal{M}((0,t)\times\Omega))$ such that

$$\begin{aligned}
&-\sum_{j=0}^{n-1}\int_\Omega \frac{t}{n}|u_n(\frac{(j+1)t}{n},\mathbf{x}))-k|_+\psi'(\frac{(j+1)t}{n})\varphi(\mathbf{x})d\mathbf{x} \qquad (3.18)\\
&-\int_\Omega |u_0(\mathbf{x})-k|_+\psi(0)\varphi(\mathbf{x})d\mathbf{x}+\mathcal{O}(t/n)\\
&= -\sum_{j=0}^{n-1}\frac{t}{n}\int_0^b \int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}} \mathrm{sgn}_+(\lambda-k)\,\mathrm{sgn}_+(u_B^n(\frac{jt}{n},\mathbf{x})-\lambda)\langle f'(\lambda),\vec{\nu}\rangle\,\psi(\frac{jt}{n})\varphi(\mathbf{x})d\gamma(\mathbf{x})d\lambda\\
&\quad+\sum_{j=0}^{n-1}\frac{t}{n}\int_\Omega \mathrm{sgn}_+(u_n(\frac{jt}{n},\mathbf{x})-k)(f(u_n(\frac{jt}{n},\mathbf{x}))-f(k))\psi(\frac{jt}{n})\cdot\nabla\varphi(\mathbf{x})d\mathbf{x}+o(1)\\
&\quad+\langle\psi(\tau)\varphi(\mathbf{x}),m_+^n(\tau,\mathbf{x},k)\rangle. \qquad (3.19)
\end{aligned}$$

By multiplying the latter expression by $\rho'(k)$, $\rho\in C_c^1((0,b))$ and integrating by parts with respect to $k\in(0,b)$ on appropriate places, we get

$$\begin{aligned}
&-\sum_{j=0}^{n-1}\int_\Omega\int_0^b \frac{t}{n}\mathrm{sgn}_+(u_n(\frac{(j+1)t}{n},\mathbf{x}))-k)\rho(k)\psi'(\frac{jt}{n})\varphi(\mathbf{x})dkd\mathbf{x}\\
&-\int_\Omega\int_0^b \rho(k)\mathrm{sgn}_+(u_0(\mathbf{x})-k)\psi(0)\varphi(\mathbf{x})dkd\mathbf{x}+\mathcal{O}(t/n)\\
&=\sum_{j=0}^{n-1}\frac{t}{n}\int_0^b\int_0^b \int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}} \mathrm{sgn}_+(\lambda-k)\mathrm{sgn}_+(u_B^n(\frac{jt}{n},\mathbf{x})-\lambda)\\
&\qquad\qquad\times\langle f'(\lambda),\vec{\nu}\rangle\psi(\frac{jt}{n})\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\lambda dk\\
&\quad+\sum_{j=0}^{n-1}\int_0^b\frac{t}{n}\int_\Omega \mathrm{sgn}_+(u_n(\frac{jt}{n},\mathbf{x})-k)f'(k)\rho(k)\psi(\frac{jt}{n})\cdot\nabla\varphi(\mathbf{x})dkd\mathbf{x}+o(1)\\
&\quad-\langle\rho'(k)\psi(\tau)\varphi(\mathbf{x}),m_+^n(\tau,\mathbf{x},k)\rangle. \qquad (3.20)
\end{aligned}$$

Now, we want to get rid of n in the boundary term. We have

$$\sum_{j=0}^{n-1}\frac{t}{n}\int_0^b\int_0^b\int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\mathrm{sgn}_+(\lambda-k)\mathrm{sgn}_+(u_B^n(\frac{jt}{n},\mathbf{x})-\lambda)$$
$$\times\langle f'(\lambda),\vec{\nu}\rangle\psi(\frac{jt}{n})\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\lambda dk$$
$$=\frac{t}{n}\int\limits_{\mathbf{R}^+}\int_0^b\int_0^b\int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\mathrm{sgn}_+(\lambda-k)\sum_{j=0}^{n-1}\Big(\mathrm{sgn}_+(u_B^j(\mathbf{x})-\lambda)-\mathrm{sgn}_+(u_B(\tau,\mathbf{x})-\lambda)\Big)$$
$$\times\langle f'(\lambda),\vec{\nu}\rangle\psi(\frac{jt}{n})\kappa_{[jt/n,(j+1)t/n)}(\tau)\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\lambda dkd\tau$$
$$+\frac{t}{n}\int\limits_{\mathbf{R}^+}\int_0^b\int_0^b\int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\mathrm{sgn}_+(\lambda-k)\mathrm{sgn}_+(u_B(\tau,\mathbf{x})-\lambda)$$
$$\times\langle f'(\lambda),\vec{\nu}\rangle\sum_{j=0}^{n-1}\psi(\frac{jt}{n})\kappa_{[jt/n,(j+1)t/n)}(\tau)\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\lambda dkd\tau. \tag{3.21}$$

Since $\mathrm{sgn}_+(\lambda)$ is a BV-function whose BV norm equals one, it holds for $M=\|f'\|_{L^\infty((0,b))}\|\psi\|_{L^\infty([0,t))}$ as $n\to\infty$

$$\Big|\int_0^b\int_0^b\int\limits_{\mathbf{R}^+}\int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\mathrm{sgn}_+(\lambda-k)\sum_{j=0}^{n-1}\Big(\mathrm{sgn}_+(u_B^j(\mathbf{x})-\lambda)-\mathrm{sgn}_+(u_B(\tau,\mathbf{x})-\lambda)\Big)$$
$$\times\langle f'(\lambda),\vec{\nu}\rangle\psi(\frac{jt}{n})\kappa_{[jt/n,(j+1)t/n)}(\tau)\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\tau d\lambda dk\Big|$$
$$\leq M\Big|\int_0^b\int\limits_{\mathbf{R}^+}\int\limits_{\partial\Omega}\sum_{j=0}^{n-1}|u_B^j(\mathbf{x})-u_B(\tau,\mathbf{x})|\kappa_{[jt/n,(j+1)t/n)}(\tau)\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\tau dk\Big|$$
$$=M\Big|\int_0^b\int\limits_{\mathbf{R}^+}\int\limits_{\partial\Omega}|\sum_{j=0}^{n-1}u_B^j(\mathbf{x})\kappa_{[jt/n,(j+1)t/n)}(\tau)-u_B(\tau,\mathbf{x})|\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\tau dk\Big|\overset{(3.16)}{\to}0.$$

Therefore, since ψ is a continuous function, (3.21) converges as $n\to\infty$ toward

$$\int_0^b\int_0^b\int\limits_{\substack{\mathbf{R}^+\times\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}}\mathrm{sgn}_+(\lambda-k)\mathrm{sgn}_+(u_B(\tau,\mathbf{x})-\lambda)\langle f'(\lambda),\vec{\nu}\rangle\psi(\tau)\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\tau d\lambda dk. \tag{3.22}$$

Next, we denote

$$p_n(\tau,\mathbf{x},k)=\frac{t}{n}\sum_{j=1}^{n}\mathrm{sgn}_+(u_n(\frac{(j-1)t}{n},\mathbf{x})-k)\delta(\tau-\frac{(j-1)t}{n}) \tag{3.23}$$

where δ is the Dirac distribution on $[0,t)$. We can rewrite (3.20) through p_n as follows

$$
\begin{aligned}
&-\langle p_n,(\psi'(\tau)\varphi(\mathbf{x})+f'(k)\nabla\varphi(\mathbf{x}))\rho(k)\rangle\\
&\quad-\int_\Omega\int_0^b \rho(k)\mathrm{sgn}_+(u_0(\mathbf{x})-k)\psi(0)\varphi(\mathbf{x})dkd\mathbf{x}+\mathcal{O}(t/n)\\
&=\sum_{j=0}^{n-1}\frac{t}{n}\int_0^b\int_0^b \int\limits_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}} \mathrm{sgn}_+(\lambda-k)\mathrm{sgn}_+(u_B^n(\frac{jt}{n},\mathbf{x})-\lambda)\langle f'(\lambda),\vec{\nu}\rangle\times\\
&\qquad\times\psi(\frac{jt}{n})\varphi(\mathbf{x})\rho'(k)d\gamma(\mathbf{x})d\lambda dk-\langle\rho'(k)\psi(\tau)\varphi(\mathbf{x}),m_+^n(\tau,\mathbf{x},k)\rangle. \qquad (3.24)
\end{aligned}
$$

The sequence (p_n) is bounded in $\mathcal{M}((0,t)\times\Omega\times\mathbf{R})$ and it contains weakly-$\star$ converging subsequence which converges toward a measure p_+. However, this measure is regular since the bounded sequence

$$
\tilde{p}_n(\tau,\mathbf{x},k)=\sum_{j=1}^n \mathrm{sgn}_+(u_n(\frac{(j-1)t}{n},\mathbf{x})-k)\kappa_{[\frac{(j-1)t}{n},\frac{jt}{n}])}(\tau),\quad \tau\in(0,t),
$$

where $\kappa_{[\frac{(j-1)t}{n},\frac{jt}{n}])}$ is the characteristic function of the interval $[\frac{(j-1)t}{n},\frac{jt}{n}])$, also converges toward p_+ in the sense of distributions along the same subsequence. Since we can assume that m_+^n converges toward a measure m_+ along the same subsequence, it is clear that the function p_+ represents the kinetic super-solution to (1.1), (1.2), (1.3) in the sense of Definition 3.1 (we simply let $n\to\infty$ in (3.24) and use the fact that (3.21) converges toward (3.22)).

In order to get wanted relation for $V(\lambda)=V_-(\lambda)=|\lambda-k|_-$, remark that the function w satisfying

$$
-b\le w=u-b\le 0
$$

represents the weak solution to

$$
\partial_t w+\mathrm{div}_{\mathbf{x}} f(w+b)=0,
$$

with the initial and boundary data

$$
-b\le w_0=u_0-b\le 0,\quad -b\le w_B=u_B-b\le 0.
$$

Remark that in this case $\chi(\lambda,w_0)=\mathrm{sgn}_-(w_0-\lambda)$ and that $T(t)w_0<0$.

If we apply the the transport-collapse procedure described in this section, then the corresponding sequence of approximate solutions has the form $(w_n)=(u_n-b)$ for (u_n) defined in (3.7) and (3.8). We can thus repeat the arguments from (3.11) to conclude (keep in mind that now $\chi(\lambda,v(\mathbf{x}))=\mathrm{sgn}_-(v(\mathbf{x})-\lambda)=\mathrm{sgn}_+(v(\mathbf{x})-\lambda)-1\le 0$; keep in mind definition of the kinetic solution)

$$
\begin{aligned}
&\int_\Omega (V_-(T(t)(v))(\mathbf{x})-V_-(v)(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}\\
&\le\int_{-b}^0 \int\limits_{(\Omega-f'(\lambda+b)t)\setminus\Omega} V_-'(\lambda)\chi(\lambda,u_B(0,\mathbf{x})-b)\varphi(\mathbf{x}+tf'(\lambda+b))d\mathbf{x}d\lambda
\end{aligned}
$$

$$-\int_{-b}^{0}\int_{\Omega\setminus(\Omega-f'(\lambda+b)t)} V'_-(\lambda)\chi(\lambda,v(\mathbf{x}))\varphi(\mathbf{x})d\mathbf{x}d\lambda$$

$$+\int_{-b}^{0}\int_{(\Omega-f'(\lambda+b)t)\cap\Omega} V'_-(\lambda)\chi(\lambda,v(\mathbf{x}))(\varphi(\mathbf{x}+tf'(\lambda+b))-\varphi(\mathbf{x}))d\mathbf{x}d\lambda. \tag{3.25}$$

From here, as for V_+, we obtain (3.10). Remark that at the final step we need to reintroduce the change $\lambda\mapsto\lambda+b$ to get exactly (3.10) with $p_-=p_+-1$ for p_+ given as the limit of (3.23). ∎

Now, notice that the functions p_+ and p_- defined in the proof of the previous theorem satisfy conditions from [21, Definition 3.1]. This follows from the following theorem providing relation between Definition 1.3 and Definition 1.1 (from which the kinetic formulation given in [21, Definition 3.1] is derived).

THEOREM 3.2. *A function u satisfying Definition 1.3 satisfies Definition 1.1.*

Proof. It is enough to notice

$$|\langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle|\leq L,$$

where L is the constant such that $\|f'\|_\infty\leq L$. Then, the last term in the left-hand side of (1.11) satisfies for every $t\geq 0$ (the other terms there are the same as the corresponding ones from (1.5)):

$$|\int_a^b \int_{\substack{\partial\Omega\\ \langle f'(\lambda),\vec{\nu}(\mathbf{x})\rangle<0}} \mathrm{sgn}_+(\lambda-k)\langle f'(\lambda-a),\vec{\nu}(\mathbf{x})\rangle \mathrm{sgn}_+(u_B(t,\mathbf{x})-\lambda)d\gamma(\mathbf{x})d\lambda|$$
$$\leq L\int_{\partial\Omega}\int_a^b \mathrm{sgn}_+(\lambda-k)\,\mathrm{sgn}_+(u_B(t,\mathbf{x})-\lambda)d\lambda d\gamma(\mathbf{x})$$
$$=L\int_{\partial\Omega}|u_B(t,\mathbf{x})-k|_+d\mathbf{x}$$

from where we conclude that u satisfies (1.5). The proof that u satisfies (1.6) is the same. ∎

Direct corollary of the previous theorem is existence and uniqueness of the function u satisfying Definition 1.3.

COROLLARY 3.1. *There exists a function u satisfying conditions of Definition 1.3 and it is unique.*

Proof. As we have already noticed (see the comments before Theorem 3.2), the functions p_+ and p_- constructed in Theorem 3.1 satisfy conditions of [21, Definition 3.1]. Therefore, according to [21, Corollary 4.2], the function p_+ has the form $p_+(t,\mathbf{x},k)=\mathrm{sgn}_+(u(t,\mathbf{x})-k)$ and therefore, $p_-(t,\mathbf{x},k)=\mathrm{sgn}_-(u(t,\mathbf{x})-k)$ for some $u\in L^\infty(\mathbf{R}^+\times\Omega)$. Now, it is a standard fare to conclude that u satisfies conditions of Definition 1.3 (it is the same as the proof of [21, Theorem 3.3]). According to Theorem 3.2 and the results from [24], we conclude that u is a unique solution to (1.1), (1.2), (1.3) in the sense of Definition 1.3. ∎

The fact that u satisfying Definition 1.3 is unique and that it satisfies Definition 1.1 at the same time actually, implies equivalence of the two definitions. The following corollary holds.

COROLLARY 3.2. *Definition 1.1 and Definition 1.3 are equivalent.*

Proof. The statement follows since there exists a unique function u satisfying Definition 1.1. Since Definition 1.3 implies Definition 1.1, it follows that Definition 1.3 determines the same admissible solution u as Definition 1.1 implying the statement of the corollary. □

Another consequence of Corollary 3.1 is convergence of the transport collapse scheme given by (3.7), (3.8). Indeed, according to Corollary 3.1, any weakly converging subsequence of (p_n) must converge to the same value which implies:

COROLLARY 3.3. *The transport-collapse scheme defined by* (3.7), (3.8) *is convergent.*

Proof. As we have already noticed, the sequence (p_n) (and thus $(\tilde{p}_n)$ also) weakly converges toward the function $\text{sgn}_+(u(t,\mathbf{x})-k)$. This implies that the Young measure corresponding to the sequence $(u_n(\tau,\mathbf{x}))=(\int_0^b \tilde{p}_n(\tau,\mathbf{x},k)dk)$ has the form $\delta(u(t,\mathbf{x})-k)$ which in turn implies strong L^1_{loc}-convergence of (u_n) toward u. □

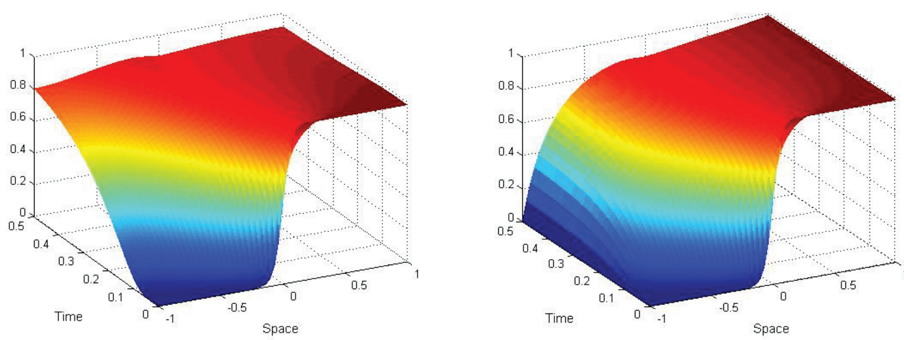

FIG. 3.1. *Cauchy problem (left) and boundary problem (right) with the initial condition* $u_0(x)=H_\varepsilon(x)$.

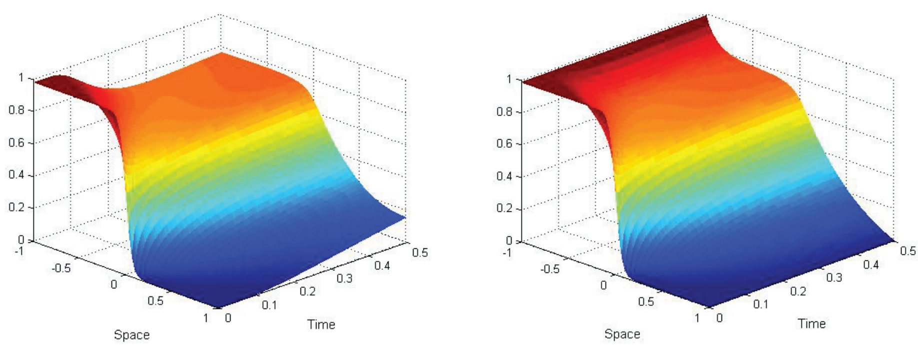

FIG. 3.2. *Cauchy problem (left) and boundary problem (right) with the initial condition* $u_0(x)=H_\varepsilon(-x)$.

Finally, remark that the procedure can be also applied in the case of conservation laws with a flux depending on $\mathbf{x}$. In this case, we would use the kinetic formulation derived in [8] but the proofs would be much more cumbersome and technical. However, when it comes to the numerics, the situation is equally demanding. Therefore, we provide examples of the scheme for one-dimensional scalar conservation law defined on $[0,0.5]\times[-1,1]$ with the flux $f(x,u)=H_\varepsilon(x)(1-u)(u+1)+4H_\varepsilon(-x)(1-u)(u+1)$, where H_ε is a standard regularization of the Heaviside function with $\varepsilon=10^{-4}$.

We present simulations for initial and initial-boundary value problems. In the first simulation, boundary conditions are $u|_{x=-1}=0$, $u|_{x=1}=1$ and the initial condition is $u|_{t=0}=H_\varepsilon(x)$. In the second simulation boundary conditions are $u|_{x=-1}=1$, $u|_{x=1}=0$ and the initial condition is $u|_{t=0}=H_\varepsilon(-x)$.

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