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INVARIANT MEASURES FOR SDES DRIVEN BY LÉVY NOISE: A CASE STUDY FOR DISSIPATIVE NONLINEAR DRIFT IN INFINITE DIMENSION*

SERGIO ALBEVERIO†, LUCA DI PERSIO‡, ELISA MASTROGIACOMO§, AND BOUBAKER SMII¶

Abstract. We study a class of nonlinear stochastic partial differential equations with dissipative nonlinear drift, driven by Lévy noise. We define a Hilbert–Banach setting in which we prove existence and uniqueness of solutions under general assumptions on the drift and the Lévy noise. We then prove a decomposition of the solution process into a stationary component, the law of which is identified with the unique invariant probability measure μ of the process, and a component which vanishes asymptotically for large times in the $L^p(\mu)$-sense, for all $1 \leq p < +\infty$.

Keywords. nonlinear SPDEs; dissipative nonlinear drift; Lévy noise; invariant measure.

AMS subject classifications. 35S05; 47H06; 37L40; 60J75; 60G10.

1. Introduction

Stochastic differential equations for processes with values in infinite dimensional spaces have been studied in the literature under different assumptions on the coefficients and the driving noise term. They are intimately related with stochastic partial differential equations (SPDEs), looking upon the processes as taking values in the infinite dimensional state space expressing their dependence on the space variable. Among the by now numerous books on these topics for Gaussian noise let us mention [46, 47, 52, 57, 77, 80], see also, e.g., [1–3,6,27,38,40,48,51,62,67,73]. For the case of non Gaussian noises see the books [30,66,75,80] and the papers [7,20,25,26,29,31,37,53,54,56,61,64,65,68–72,78,79].

In [21] a study was initiated concerning a class of non linear stochastic differential equations with Lévy noise and a drift term consisting of a linear unbounded space-dependent part (typically a Laplacian) and an unbounded non linear part of the dissipative type and of at most polynomial growth at infinity.

This class is of particular interest since it contains the case of FitzHugh–Nagumo equations with space dependence, on a bounded domain of $\mathbb{R}^n$ or on bounded networks with 1-dimensional edges. Such equations are of interest in a number of areas including neurobiology and physiology, see, e.g., [4, 5, 32, 35, 36, 55, 60, 63, 82–87] . It is also related with the stochastic quantization equation in quantum field theory, mostly studied with Gaussian noise, see [1, 11, 14–18, 23, 44, 45, 58, 59, 74] and references therein. See also [10, 12–17, 20, 26, 31, 35, 53, 54, 71, 72] for related equations with Lévy noise, and [8, 9] for related work in hydrodynamics and [19, 39, 43, 49, 50] for applications in mathematical finance.

For the SPDE equations of the FitzHugh–Nagumo type with Lévy noise studied in [21], existence and uniqueness of solutions was proven, as well as existence and

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†Dept. Appl. Mathematics, University of Bonn, HCM, BiBoS, IZKS (albeverio@iam.uni-bonn.de).

‡Dept. of Computer Science, University of Verona, Strada le Grazie, 15 - 37134 - Verona, Italy (luca.dipersio@univr.it).

§University of Milano Bicocca, Department of Statistics and Quantitive Methods, Piazza Ateneo Nuovo, 1 - 20126 - Milano, Italy (elisa.mastrogiacomo@unimib.it).

¶Address of the corresponding author: King Fahd University of Petroleum and Minerals, Department of Mathematics and Statistics, Dhahran 31261, Saudi Arabia (boubaker@kfupm.edu.sa).

uniqueness of invariant measures. Moreover, asymptotic small noise expansions for the solutions have been established (see [21] for the case of Lévy noise and [6] for the case of Gaussian noise). For other works on asymptotics for S(P)DEs see also, e.g. [18, 22, 24, 28, 41, 78]. In the present paper we further work in the framework set up in [21] and provide a decomposition of the solution process as the sum of a stationary component and a component which vanishes asymptotically for large times in the L^p-sense, for all $1 \le p < +\infty$, with respect to the underlying probability measure.
The law of the stationary component is then identified with the unique invariant measure of the solution process.
This is a considerable extension, using new methods, of a result that was proved before in [5] where the noise is of Gaussian type.
The structure of this paper is as follows. In Section 2 the setting for the infinite dimensional stochastic differential equations with linear drift and Lévy noise is described. Cylindrical Lévy processes are hereby introduced and the invariant measures for Ornstein–Uhlenbeck processes driven by the corresponding noises are discussed. In Section 3 the results on existence and uniqueness of solutions of such equations with non linear drift and of corresponding invariant probability measures are recalled. In Section 4 the main theorem giving the additive decomposition of the solution process in a stationary and an asymptotically small component is formulated and proven. Its proof uses a double indexed finite dimensional approximand, one index m, referring to a Yosida approximation of the non linear term (which we allow to be non globally Lipschitz, typically with polynomial growth at infinity!) and the other one, n, referring to n-dimensional approximations of the SPDE.

In the proof results on stochastic convolution (based on [75]) and Gronwall's type estimates are exploited. Technical details concerning the proof are presented in the Appendices A and B. Applications to the study of models described by PDE's of the FitzHugh–Nagumo type with Lévy noise will be presented in further work. Also, we intend to exhibit explicitly in some cases the form of the invariant measure (for first results in this direction we refer to [5] and [7]).

2. The infinite dimensional Ornstein–Uhlenbeck process driven by Lévy noise

The study of stochastic evolution equations driven by Lévy noise is of current interest. In this section we concentrate on the linear stochastic differential equation

$$\begin{aligned} \mathrm{d}X(t) &= AX(t)\mathrm{d}t + \mathrm{d}L(t), \qquad t \ge 0, \\ X(0) &= x \in \mathcal{H}, \end{aligned} \tag{2.1}$$

where $\mathcal{H}$ is a real separable Hilbert space, $(L(t))_{t\ge 0}$ is an infinite dimensional cylindrical symmetric Lévy process and A is a self-adjoint strictly negative operator generating a C_0-semigroup. In particular we are going to recall a few results concerning the well-posedness of the above equation (Section 2.1) and the existence and uniqueness of explicit invariant measures (Section 2.2), in the case where A satisfies Hypothesis 2.1 below, mainly following [76]. See also, eg., [30, 31, 42, 80].

2.1. Cylindrical Lévy process. Let us recall that a general Lévy process $(L(t))_{t\ge 0}$ with values in a real separable Hilbert space $\mathcal{H}$ is a stochastically continuous process on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, having stationary independent increments and such that $L(0) = 0$, $\mathbb{P}$-a.s. Without loss of generality, we can assume that $L(t)$ has càdlàg (i.e. right continuous with left limits) trajectories in $\mathcal{H}$, since every Lévy process has a càdlàg modification (see [75, Theorem 4.3]). Moreover, one has the following infinite dimensional Lévy-Khintchine formula:

$$\mathbb{E}[e^{i\langle L(t),u\rangle}]=\exp(-t\psi(u)), \qquad u\in\mathcal{H}, t\geq 0, \tag{2.2}$$

where $\langle,\rangle$ is the scalar product in $\mathcal{H}$ and the exponent ψ can be expressed as follows:

$$\psi(u)=\frac{1}{2}\langle Qu,u\rangle-i\langle a,u\rangle-\int_{\mathcal{H}}\left(e^{i\langle u,y\rangle}-1-\frac{i\langle u,y\rangle}{1+|y|^2}\right)\nu(\mathrm{d}y), \qquad u\in\mathcal{H}. \tag{2.3}$$

Here Q is a symmetric non-negative trace class operator on $\mathcal{H}$, $a\in\mathcal{H}$ and $\nu_{\mathbb{R}}$ is the Lévy measure or jump intensity measure associated to $(L(t))_{t\geq 0}$, i.e. ν is a σ-finite Borel measure on $\mathcal{H}$ such that $\nu(\{0\})=0$ and

$$\int_{\mathcal{H}}(|y|^2\wedge 1)\nu(\mathrm{d}y)<+\infty.$$

We shall say that $L(t)$ is generated by the triplet (Q,ν,a) (in analogy with, e.g., [80, Chapter 2, Section 11, pgg. 65-66]. Let us remark that the integral on the right hand side of (2.3) can also be written with the term $-\frac{i\langle u,y\rangle}{1+|y|^2}$ replaced by $-i\langle u,y\rangle\chi_D(y)$, with D the unit ball in $\mathcal{H}$, and a replaced by

$$\tilde{a}=a+\int_{\mathcal{H}}y\left(\frac{1}{1+|y|^2}-\chi_D(y)\right)\nu(\mathrm{d}y).$$

In this paper we shall mainly be concerned with a concretely given cylindrical Lévy process $L=(L(t))_{t\geq 0}$ defined by the orthogonal expansion

$$L(t)=\sum_{n=1}^{\infty}\beta_n L^n(t)e_n, \tag{2.4}$$

where e_n is an orthonormal basis in $\mathcal{H}$, $L^n=(L^n(t))_{t\geq 0}$, $L^n(0)=0$, are independent real valued, symmetric, identically distributed Lévy processes with neither a Gaussian part nor a deterministic one defined on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. Such a process L, whenever it exists as a Lévy process on $\mathcal{H}$, will have $Q\equiv 0$ and $a\equiv 0$. Symmetric means that the Lévy measure $\nu_{\mathbb{R}}$ associated with $L^n(t)$ is independent of n and satisfies $\nu_{\mathbb{R}}(A)=\nu_{\mathbb{R}}(-A)$ for any Borel subset A of $\mathbb{R}$. Moreover, $\beta_n, n\in\mathbb{N}$, is a given (possibly unbounded) sequence of positive real numbers. Below we shall discuss choices of β_n such that $L(t)$ will indeed belong, for every $t\geq 0$, to a Hilbert space.

We assume that $(e_n)_{n\in\mathbb{N}}$ is made of the eigenvectors of the operator A in Equation (2.1), A being assumed, in particular, to have purely-discrete spectrum. More precisely we make the following assumptions:

Hypothesis 2.1. *A is a self-adjoint, strictly negative operator with domain $D(A)$ such that there is a fixed orthonormal basis $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ verifying: $(e_n)_{n\in\mathbb{N}}\subset D(A)$, $Ae_n=-\lambda_n e_n$, with $\lambda_n>0$, for any $n\in\mathbb{N}$ and $\lambda_n\uparrow+\infty$.*

Remark 2.1. From the above assumptions it follows that:

(1) A generates a contraction symmetric C_0-semigroup e^{tA}, $t\geq 0$, on $\mathcal{H}$. The action of e^{tA} on any element $u\in\mathcal{H}$ can be written as

$$e^{tA}u=\sum_{k=1}^{+\infty}e^{-\lambda_k t}\langle u,e_k\rangle e_k, \qquad k\in\mathbb{N},\ t\geq 0,$$

with strong convergence in $\mathcal{H}$.

(2) Since the law of $L^n, n\in\mathbb{N}$ is assumed to be symmetric, independent of n we have, for any $n\in\mathbb{N}$, $t\geq 0$, for the characteristic function of L^n:

$$\mathbb{E}[e^{ihL^n(t)}]=e^{-t\psi_{\mathbb{R}}(h)},$$

with $\psi_{\mathbb{R}}$ being given by

$$\psi_{\mathbb{R}}(h)=\int_{\mathbb{R}}(1-\cos(hy))\nu_{\mathbb{R}}(\mathrm{d}y), \quad h\in\mathbb{R}. \tag{2.5}$$

The definition of cylindrical Lévy process in Equation (2.4) has of course to be supplied with suitable assumptions on the β_n. Following [76, Section 2], we can prove that under such assumptions it is then possible to find a suitable Hilbert space $\mathcal{U}$ (see Proposition 2.1 below) where $(L(t))_{t\geq 0}$ is a well-defined Lévy process (in the sense described at the beginning of this subsection with $\mathcal{H}$ replaced by $\mathcal{U}$). This is equivalent to giving conditions under which the series on the right hand side of Equation (2.4) converges in $\mathcal{H}$. To this end, we recall that any infinite dimensional separable Hilbert space $\mathcal{H}$ can be identified with the space ℓ^2, using a basis $(e_n)_{n\in\mathbb{N}}$. In general, for a given sequence $\rho=(\rho_n)_{n\in\mathbb{N}}$ of real numbers, we set

$$\ell^2_\rho := \left\{ (x_n)\in\mathbb{R}^{\mathbb{N}} : \sum_{n\geq 1} x_n^2\rho_n^2 < +\infty \right\}.$$

The space ℓ^2_ρ becomes a Hilbert space with the inner product $\langle x,y\rangle := \sum_{n\geq 1} x_n y_n \rho_n^2$ for $x=(x_n), y=(y_n)\in\ell^2_\rho$. For $\rho_n=1$, $\forall n\in\mathbb{N}$, we have $\ell^2_\rho=\ell^2$. We quote from [76, Proposition 2.4] the result which provides conditions on $\beta_n, \nu_{\mathbb{R}}$ such that the cylindrical Lévy process of the form given in Equation (2.4) is well-defined in some Hilbert space $\mathcal{U}$.

Proposition 2.1.

(1) The following conditions are equivalent:

$$(i) \qquad \sum_{n=1}^{\infty}(\beta_n L^n(t_0))^2 < +\infty \qquad \text{for some } t_0>0,\ \mathbb{P}-a.s.;$$

$$(ii) \qquad \sum_{n=1}^{\infty}(\beta_n L^n(t))^2 < +\infty \qquad \text{for every } t>0,\ \mathbb{P}-a.s.;$$

$$(iii) \qquad \sum_{n=1}^{\infty}\left(\beta_n^2\int_{|y|<1/\beta_n} y^2\nu_{\mathbb{R}}(\mathrm{d}y)+\int_{|y|\geq 1/\beta_n}\nu_{\mathbb{R}}(\mathrm{d}y)\right)<+\infty.$$

(2) The cylindrical Lévy process L given by Equation (2.4) *is a Lévy process taking values in the Hilbert space $\mathcal{U}:=\ell^2_\rho$, with any weight $\rho=(\rho_n)_{n\in\mathbb{N}}, \rho_n\geq 0$, such that*

$$\sum_{n=1}^{\infty}\left(\rho_n^2\beta_n^2\int_{\rho_n\beta_n|y|<1} y^2\nu_{\mathbb{R}}(\mathrm{d}y)+\int_{\rho_n\beta_n|y|\geq 1}\nu_{\mathbb{R}}(\mathrm{d}y)\right)<+\infty.$$

Now let us consider the Ornstein–Uhlenbeck process described by Equation (2.1), with L given by Equation (2.4), under Hypothesis 2.1. As a consequence of our assumptions and using the representation $x=\sum_{n=1}^{+\infty}x_n e_n$ for any $x\in\mathcal{H}$, $x_n\in\ell^2(\mathbb{R})$, we can decompose the solution process $X(t),\ t\geq 0$ of Equation (2.1) as

$$X(t)=\sum_{n=1}^{+\infty}X^n(t)e_n,$$

with $X^n(t),\ t\geq 0$, the solution of the one dimensional stochastic differential equation

$$\begin{cases}\mathrm{d}X^n(t)=-\lambda_n X^n(t)\mathrm{d}t+\beta_n\mathrm{d}L^n(t),\\ X^n(0)=x_n\in\mathbb{R},\ n\in\mathbb{N}.\end{cases}\tag{2.6}$$

The λ_n are the eigenvalues of A, as in Hypothesis 2.1. The condition $X(0)=x\in\mathcal{H}$ corresponds to having $X^n(0)=x_n$, with $(x_n)_{n\in\mathbb{N}}\in\ell^2(\mathbb{R})$. Note that $X(t),t\geq 0$ can also be looked upon as solution of Equation (2.1) in $\mathbb{R}^{\mathbb{N}}$. The solution of Equation (2.6) is the stochastic process

$$X^n(t)=e^{-\lambda_n t}x_n+\int_0^t e^{-\lambda_n(t-s)}\beta_n\mathrm{d}L^n(s),\qquad n\in\mathbb{N},\ t\geq 0.$$

The real-valued processes X^n, for $n\in\mathbb{N}$, can be assumed a.s. to be càdlàg (by a special finite dimensional case of the result in [62, Theorem 4.3, pag.39]). As follows from [76], if Hypothesis 2.1 holds and the L^n are as above, then from this we obtain

$$X(t)=e^{tA}x+L_A(t),\tag{2.7}$$

where

$$L_A(t):=\int_0^t e^{(t-s)A}\mathrm{d}L(s)=\sum_{n=1}^{\infty}\left(\int_0^t e^{-\lambda_n(t-s)}\beta_n\mathrm{d}L^n(s)\right)e_n.\tag{2.8}$$

Moreover if Hypothesis 2.1 and

$$\sum_{n=1}^{+\infty}\frac{1}{\lambda_n}\int_{1/\beta_n}^{1/\beta_n e^{\lambda_n}}\left(\frac{1}{u^3}\psi_0(u)+\frac{1}{u}\psi_1(u)\right)\mathrm{d}u<+\infty,$$

holds, for

$$\psi_0(u):=\int_{|y|\leq u}u^2\nu_{\mathbb{R}}(\mathrm{d}u)$$

$$\psi_1(u):=\int_{|y|>u}\nu_{\mathbb{R}}(\mathrm{d}u),$$

then we have that $(X(t))_{t\geq 0}$ is $\mathcal{F}_t$-adapted and Markovian. The convergence of the series in equality (2.8) is to be meant in probability (see [76, Theorem 2.8]). Using point (2) in Remark 2.1 and following [80, pag. 105], we can compute the characteristic function of the $X^n(t)$:

$$\mathbb{E}[e^{ihX^n(t)}]=\exp\left(ihe^{-\lambda_n t}x_n-\int_0^t\psi_{\mathbb{R}}(e^{-\lambda_n(t-s)}\beta_n h)\mathrm{d}s\right),\qquad h\in\mathbb{R},t\geq 0,n\in\mathbb{N},$$

where $\psi_{\mathbb{R}}$ is the function defined in Equation (2.5).

In the following we shall provide an application of this result to the construction of an invariant probability measure for Ornstein–Uhlenbeck processes driven by a quite general class of symmetric cylindrical Lévy noises, which we shall henceforth call, for simplicity, OU–Lévy driven processes.

2.2. Invariant measure for the infinite dimensional OU–Lévy driven processes. Following, e.g. [33] we say that a probability measure μ on a complete separable metric (i.e. Polish) space $\mathcal{H}$ is invariant with respect to a Markov semigroup $(P_t)_{t\geq 0}$, with transition probability kernel $P_t(x,\mathrm{d}y)$, $x,y\in\mathcal{H}$ on $\mathcal{H}$ if for any Borel subset $\Gamma\subset E$ and any $t\geq 0$ we have $\mu(\Gamma)=\int\mu(\mathrm{d}x)P_t(x,\Gamma)$. We say shortly that μ is an invariant measure for $(P_t)_{t\geq 0}$. See, e.g., [75, chapter 16] for equivalent formulations of this property, and, eg., [3] for a general discussion.

Proposition 2.2. *Assume Hypothesis 2.1. Moreover, assume that β_n, $n\in\mathbb{N}$, is a bounded sequence and that the symmetric Lévy measure $\nu_{\mathbb{R}}$ appearing in Equation* (2.5) *satisfies*

$$\int_1^{+\infty}\log(y)\nu_{\mathbb{R}}(\mathrm{d}y)<\infty.$$

Finally, assume that

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}<+\infty,$$

with the λ_n as in Hypothesis 2.1. Then the Lévy driven Ornstein–Uhlenbeck process $X=(X(t))_{t\geq 0}$ given by Equation (2.7) *admits a unique invariant measure.*

Proof. The proof follows basically [76, Proposition 2.11], adding however some explanations which we retain necessary for later use. To show that there exists an invariant measure we first notice that (according to [80, Theorem 17.5 pag. 108]) the one dimensional Ornstein–Uhlenbeck process $X^n(t)$ has an invariant measure μ_n which is the law of the random variable

$$\int_0^{\infty}e^{-\lambda_n u}\beta_n\mathrm{d}L^n(u),$$

having characteristic function

$$\hat{\mu}_n(h)=\exp\left(-\int_0^{\infty}\psi_{\mathbb{R}}(e^{-\lambda_n u}\beta_n h)\mathrm{d}u\right),\qquad h\in\mathbb{R}.$$

Let us consider the product measure $\mu=\Pi_{n\geq 1}\mu_n$ on $R^{\mathbb{N}}$. This is the law of the family $(\bar{\xi}_n)_{n\in\mathbb{N}}$ of independent random variables, where

$$\bar{\xi}_n:=\int_0^{\infty}e^{-\lambda_n u}\beta_n\mathrm{d}L^n(u),\quad n\geq 1.$$

We underline that $\bar{\xi}_n$ is an infinitely divisible real-valued random variable. Now define $\bar{\xi}=\sum_{n=1}^{\infty}\bar{\xi}_n e_n$. According to [76, Lemma 2.3] the random variable ξ takes values in $\mathcal{H}$ if and only if the Lévy measure ν_n of $\bar{\xi}_n$ satisfies

$$\sum_{n=1}^{\infty}\int_{\mathbb{R}}(1\wedge y^2)\nu_n(\mathrm{d}y)<+\infty. \tag{2.9}$$

Exploiting the result in [76, Proposition 2.11], we deduce that condition (2.9) is satisfied, hence $\bar{\xi}$ takes values in $\mathcal{H}$. This implies that $\mu(\mathcal{H})=1$.

In the following we are going to prove that μ is the unique invariant measure of X given by Equation (2.7), by showing that, for any $x\in\mathcal{H}$, $\lim_{t\to\infty}X(t)$ exists in probability and one has

$$\lim_{t\to+\infty}X(t)=\bar{\xi}\qquad\text{in probability.}\tag{2.10}$$

$\bar{\xi}$ is defined above and is also given, as seen from Equation (2.7) and Hypothesis 2.1 on A, by

$$L_A(+\infty)=\lim_{t\to+\infty}\int_0^t e^{(t-s)A}\mathrm{d}L(s).$$

In fact, let us consider first the case $x=0$ in Equation (2.7). In this case we have from Equations (2.7) and (2.8),

$$X(t)=\int_0^t e^{A(t-s)}\mathrm{d}L(s).$$

Set:

$$X^n(t)=\int_0^t e^{-\lambda_n(t-s)}\beta_n\mathrm{d}L^n(s),$$

so that

$$X(t)=\sum_{n=1}^{\infty}X^n(t)e_n.$$

We recall the following identity:

$$\mathbb{E}\left[e^{ih\int_s^t g(u)\mathrm{d}L^n(u)}\right]=\exp\left(-\int_s^t\psi_{\mathbb{R}}(g(u)h)\mathrm{d}u\right),h\in\mathbb{R},0\le s\le t,$$

which holds for any real continuous function g on $[s,t]$, see [80, pag. 105]. Let us define, for any $t\ge 0$, $\xi_n(t):=\int_0^t e^{-\lambda_n u}\beta_n\mathrm{d}L^n(u)$. Then if we compute the characteristic function of $\xi_n(t)$ we see, by a change of variables, that $X^n(t)$ and $\xi_n(t)$ have the same law. This fact allows us to estimate the following quantity:

$$a_\epsilon(t):=\mathbb{P}(|X(t)-\bar{\xi}|^2>\epsilon),\ \text{for any }\epsilon>0,t>0.$$

Recalling our temporary choice $x=0$, and using the definitions of $X(t)$ in Equations (2.7) and (2.8), and the fact that

$$\bar{\xi}=\sum_{n=1}^{\infty}\bar{\xi}_n e_n$$

with

$$\bar{\xi}_n=\int_0^{+\infty}e^{-\lambda_n u}\beta_n\mathrm{d}L^n(u),$$

we have, by Parseval's formula,

$$a_\epsilon(t) = \mathbb{P}\left(\left|\sum_{n=1}^{\infty} X^n(t)e_n - \bar{\xi}\right|^2 > \epsilon\right)$$
$$= \mathbb{P}\left(\left|\sum_{n=1}^{\infty} X^n(t)e_n - \sum_{n=1}^{\infty}\bar{\xi}_n e_n\right|^2 > \epsilon\right).$$

But using the fact that $X^n(t)$ and $\xi_n(t)$ have same law and that the joint laws of $(X^n(t), \bar{\xi})$ and $(\xi_n(t), \bar{\xi})$ also coincide, we have

$$a_\epsilon(t) = \mathbb{P}\left(\left|\sum_{n=1}^{\infty} \xi_n(t)e_n - \sum_{n=1}^{\infty}\bar{\xi}_n e_n\right|^2 > \epsilon\right), \quad t > 0.$$

Using the definitions of $\xi_n(t)$ and $\bar{\xi}_n$ we have, by Parseval's law:

$$a_\epsilon(t) = \mathbb{P}\left(\left|\sum_{n=1}^{\infty} \beta_n^2 \int_t^{\infty} e^{-\lambda_n u}\mathrm{d}L^n(u)\right|^2 > \epsilon\right).$$

Moreover, (see [76, pag. 11]) it is possible to prove that the law of the random variable in the latter term in the previous expression coincides with the one of

$$\sum_{n=1}^{\infty} e^{-2\lambda_n t}\xi_n(t)^2.$$

We then have, for any $t > 0$,

$$a_\epsilon(t) = \mathbb{P}\left(\sum_{n=1}^{\infty} e^{-2\lambda_n t}\xi_n^2(t) > \epsilon\right) \le \mathbb{P}\left(e^{-2\lambda_1 t}\sum_{n=1}^{\infty}\xi_n^2(t) > \epsilon\right) = \mathbb{P}(|\xi|^2 > e^{2\lambda_1 t}\epsilon). \tag{2.11}$$

Letting $t \to +\infty$ we find $\lim_{t\to+\infty} a_\varepsilon(t) = 0$, for any $\epsilon > 0$. From inequality (2.11) it follows then the claim (2.10) for the case $x = 0$. The general case, $x \neq 0$, can easily be obtained by translation. □

Remark 2.2. For further use (see Section 3.2), we emphasize that from the proof of the claim (2.10) we can also see that, for any $t > 0$, the random variables

$$\int_0^t e^{(t-s)A}\mathrm{d}L(s) \qquad \textit{and} \qquad \bar{\xi}(t) := \sum_{n=1}^{\infty}\int_0^t e^{-\lambda_n s}\beta_n \mathrm{d}L^n(s)e_n$$

have the same law. Hence we obtain that the random variable

$$L_A(+\infty) := \lim_{t\to+\infty}\int_0^t e^{(t-s)A}\mathrm{d}L(s)$$

is well-defined (the limit being in probability) and its law coincides with that of the random variable $\bar{\xi}$.

3. The stochastic semilinear differential equation

3.1. The state equation: existence and uniqueness of solutions. In this section we study a semilinear SDE driven by Lévy noise, thus extending the setting of Section 2 to include a non linear drift term. Basically, as in the setting of [21] and [75], we consider a stochastic differential equation of the form

$$\begin{cases} \mathrm{d}X(t) = AX(t)\mathrm{d}t + F(X(t))\mathrm{d}t + B\mathrm{d}L(t), & t \geq 0 \\ X(0) = x \in D(F) \end{cases} \tag{3.1}$$

where the stochastic process $X = (X(t))_{t\geq 0}$ takes values in a real separable Hilbert space $\mathcal{H}$, A is (as in Section 2) a linear operator from a dense domain $D(A)$ in $\mathcal{H}$ into $\mathcal{H}$ which generates a C_0-semigroup $S(t) = e^{tA}$, $t \geq 0$, of strict negative type, in the sense that $\|e^{tA}\| \leq e^{-\omega t}$, for some $\omega > 0$ and all $t > 0$ (in Section 2.1, Hypothesis 2.1 we had $\omega = \lambda_1$).

B is a linear bounded operator from a suitable Hilbert space $\mathcal{U}$ (which is to be made more precise— see Remark 3.1 below) into $\mathcal{H}$. F is a mapping from its domain $D(F) \subset \mathcal{H}$ into $\mathcal{H}$, continuous, nonlinear, Fréchet differentiable and such that

$$\langle F(u) - F(v) - \eta(u-v), u-v \rangle < 0, \quad \text{for some} \quad \eta > 0 \tag{3.2}$$

and all $u, v \in D(F)$, where $\langle , \rangle$ is the scalar product in $\mathcal{H}$.

The connection between A and F consists in requiring that $\omega > \eta$, with ω defined at the beginning of the present section and η given by condition (3.2). By this $A+F$ is maximal dissipative (or m-dissipative) in the sense of [46, pag. 73], i.e. the range of $\lambda\mathbf{1} - (A+F)$ is $\mathcal{H}$, for some - and consequently for all - $\lambda > 0$.

We assume that L is a cylindrical Lévy process according to the description given in Section 2.1, i.e.

$$L(t) = \sum_{n=1}^{\infty} \beta_n L^n(t) e_n,$$

with $(\beta_n)_{n\in\mathbb{N}}$ a given (possibly unbounded) sequence of positive real numbers, $(L^n(t))_{n\in\mathbb{N}}, t \geq 0$ a sequence of real-valued, symmetric, i.i.d. Lévy processes without Gaussian part and $(e_n)_{n\in\mathbb{N}}$ an orthonormal basis of $\mathcal{H}$. Moreover, we will work under Hypothesis 2.1, hence assuming, in particular, that $(e_n)_{n\in\mathbb{N}}$ is made of eigenvectors of A.

Remark 3.1. According to Remark 2.1 and Proposition 2.1 and the subsequent paragraph, identifying $\mathcal{H}$ with the space ℓ^2, the cylindrical Lévy process $L(t)$ is a well-defined Lévy process taking values in the Hilbert space $\mathcal{U} := \ell^2_\rho$ for a suitable sequence $\rho = (\rho_n)_{n\in\mathbb{N}}$. This means that (see Proposition 2.1, 1.(i) or 1.(ii))

$$\sum_{n=1}^{\infty} (\beta_n^2 L^n(t)^2 \rho_n^2) < +\infty, \quad \text{for any } t > 0, \ \mathbb{P}-a.s.$$

We assume that B is a linear bounded operator acting on $\mathcal{U}$ with values in $\mathcal{H}$. Through the identification of $\mathcal{H}$ with ℓ^2 we can think of $BL(t)$ as written in the following form:

$$BL(t) = \sum_{n=1}^{\infty} b_n \beta_n L^n(t) e_k, \quad t > 0, \tag{3.3}$$

with $(b_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} \frac{b_n}{\rho_n} < +\infty$.

L can be assumed to have càdlàg paths (as follows e.g. by [30] and [75, p.39]). We assume as in Equation (2.4) that it is a pure jump process, in the sense that no Gaussian nor deterministic part is present. In what follows we introduce several objects and hypotheses, which we illustrate below (Remark 3.2 and Example 3.1): $\mathcal{B}$ is a reflexive Banach space continuously embedded into $\mathcal{H}$ as a dense Borel subset. We assume that $A_{|\mathcal{B}}$, $F_{|\mathcal{B}}$ are almost m-dissipative in $\mathcal{B}$, in the sense that $A_{|\mathcal{B}}+\omega\mathbf{1}_{|\mathcal{B}}$ and $F_{|\mathcal{B}}+\eta\mathbf{1}_{|\mathcal{B}}$ are m-dissipative in $\mathcal{B}$, where $|_{\mathcal{B}}$ stands for restriction to $\mathcal{B}$. Moreover we assume that $D(F)\supset\mathcal{B}$ and that F maps bounded subsets of $\mathcal{B}$ into bounded subsets of $\mathcal{H}$. We also assume that the stochastic convolution $L_A(t)$ of $B\,\mathrm{d}L(t)$ with $S(t)=e^{tA}$, i.e.

$$L_A(t) := \int_0^t S(t-s)B\,dL(s), \quad t\geq 0, \tag{3.4}$$

has a càdlàg version in $D((-A_{|\mathcal{B}})^{\alpha})$ for some $\alpha\in[0,1)$. Finally, we impose the following condition: for all $T>0$

$$\int_0^T |F(L_A(t))|_{\mathcal{B}}\mathrm{d}t < +\infty, \quad \mathbb{P}-a.s. \tag{3.5}$$

(where $\mathbb{P}$ is as in Section 2.1, see [75, p. 183] for more details). It follows from our assumptions that L_A is square-integrable, $\mathcal{F}_t$-adapted (where $\mathcal{F}_t$ is as in Section 2.1 the natural σ-algebra associated with L) (see [75, p. 163]).

Remark 3.2. We give two cases where the property (3.4) of $(L_A(t))_{t\geq 0}$ holds:

(1) $\mathcal{B}$ is a Hilbert space and $S(t)$ is a contraction semigroup in this space $\mathcal{B}$ and $BL(t)$ takes values in $\mathcal{B}$; see, e.g. [75, p.158, Theorem 9.20].

(2) $S(t)$ is analytic in $\mathcal{B}$ and BL has càdlàg trajectories in $D\left((-A_{|\mathcal{B}})^{\alpha}\right)$, for some $\alpha\in[0,1)$ (see, e.g. ([75, p.163, Prop. 9.28]).

Example 3.1. *Let us provide an example for the setting* $(\mathcal{H},\mathcal{U},\mathcal{B},L,A,F)$ *where both conditions* (3.4) *and* (3.5) *hold. It suffices to consider the case* $B\equiv 1$, *since the general case* $B\neq 1$ *can be covered by modifying the* ρ_k, $k\in\mathbb{N}$ *in the definition of* $\mathcal{U}$. *Let* $\Lambda\subset\mathbb{R}^d$, *bounded, open, with smooth boundary,* $d\in\mathbb{N}$, *and let* $\mathcal{H}=\mathcal{U}:=L^2(\Lambda)$. *Let* F *be of the form of a multinomial mapping of odd degree* $2n+1$, $n\in\mathbb{N}$, *i.e.* F *is a mapping of the form* $F(u)=g_{2n+1}(u)$, $u\in\mathbb{R}$, *where* $g_{2n+1}:\mathbb{R}\to\mathbb{R}$, *is a polynomial of degree* $2n+1$ *with first derivative absolutely bounded from above, see [21]. It follows that* $D(F)=L^{2(2n+1)}(\Lambda)\subsetneq L^2(\Lambda)$, *for any* $n\in\mathbb{N}$. *We take* $\mathcal{B}:=L^{2p}(\Lambda)$ *with* $p\geq 2n+1$. *Let* $A=\Delta$ *be the Laplacian in* $L^2(\Lambda)$ *with Dirichlet boundary conditions on the boundary* $\partial\Lambda$. *Then* A *satisfies Hypothesis 2.1. Let* L *be a stochastic process on* $\mathcal{H}=L^2(\Lambda)$ *of the type described in Remark 3.1, and such that the corresponding Lévy measure* ν *satisfies*

$$\int_{L^2(\Lambda)} |x|_{W^{\beta,2p(2n+1)}}\,\nu(\mathrm{d}x) < +\infty, \tag{3.6}$$

where $W^{\beta,2p(2n+1)}$ *is the fractional Sobolev space in* $L^2(\Lambda)$ *with given index* $\beta>0$; *moreover we require* $\int_{|x|\leq 1}|x|^2\nu(\mathrm{d}x)+\nu(x||x|\geq 1)<+\infty$. *From property* (3.6) *(similarly as in [75, Prop. 6.9]) it follows that* $L(t)\in D\left(-A^{\gamma}_{2p(2n+1)}\right)\subset L^{2p(2n+1)}(\Lambda)$ *for some* $\gamma>0$ *and any* $t>0$ *and has càdlàg trajectories in* $D\left((-A^{\gamma}_{2p(2n+1)})\right)$, *where* $A_{2p(2n+1)}$ *denotes* *the generator of the heat semigroup* $e^{tA_{2p(2n+1)}}, t \geq 0$, *with Diricklet boundary conditions operating in* $L^{2p(2n+1)}(\Lambda)$. *Then similarly as in [75, Prop. 9.28, p.163 and Theorem 10.15, pag.187],* L_A *is well-defined and satisfies conditions* (3.4) *and* (3.5).

Now we are ready to state the main result of this section, which concerns the existence and uniqueness of solutions to Equation (3.1). We refer to [21, Theorem 4.9] for the proof.

THEOREM 3.1. *Assume* A, Q, F, L *satisfy all previous assumptions and let* L_A *satisfy conditions* (3.4) *and* (3.5) *above. Then there exists a unique càdlàg mild solution to the SDE* (3.1) *(with* $B = 1$*) in the sense of being adapted, càdlàg in* $\mathcal{B}$, *and satisfying almost surely*

$$X(t) = S(t)x + \int_0^t S(t-s)F(X(s))\,\mathrm{d}s + L_A(t), \quad t \geq 0, \tag{3.7}$$

for any $x \in \mathcal{B} \subset D(F)$, *with* $X(t) \in D(F)$ *for all* $t \geq 0$.

For each $x \in \mathcal{H}$ *there exists a generalized solution of Equation* (3.1), *i.e. there exists* $(X_n(t))_{n \in \mathbb{N}}$, $t \geq 0$, $X_n(t) \in \mathcal{B}$, *where* X_n *are unique mild adapted solutions of Equation* (3.1) *with* $X_n(0) = x$ *and such that* $|X_n(t) - X(t)|_{\mathcal{H}} \to 0$ *on each bounded interval* $t \in [0, T]$ *for any* $T > 0$, *as* $n \to \infty$.

Moreover $X(t)$ *defines Feller families on* $\mathcal{B}$ *and on* $\mathcal{H}$, *in the sense that the Markov semigroup* P_t *associated with* $X(t)$ *maps, for any* $t \geq 0$, $C_b(\mathcal{H})$ *into* $C_b(\mathcal{H})$ *and* $C_b(\mathcal{B})$ *into* $C_b(\mathcal{B})$.

3.2. Existence and uniqueness of an invariant measure. In the following we deal with the asymptotic behavior of the Markov semigroup corresponding to Equation (3.1). We will see that in our case, in addition to existence and uniqueness of the invariant measure μ (whose notion has been recalled in Section 2), we can prove that it is also exponentially mixing. We quote from [75, chapter 16] the definition of exponentially mixing invariant measure. Let $Lip(\mathcal{H})$ be the space of all real-valued Lipschitz continuous functions $\psi: \mathcal{H} \to \mathbb{R}$ endowed with the norm $\|\psi\|_\infty + \|\psi\|_{Lip}$, with $\|\psi\|_\infty$ the sup-norm and

$$\|\psi\|_{Lip} := \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|_{\mathcal{H}}}; \tag{3.8}$$

$\|\psi\|_{Lip}$ is said to be the smallest Lipschitz constant for ψ, see [75, p.16]. We say that an invariant measure μ is exponentially mixing with exponent $\omega > 0$ and bound function $c: \mathcal{H} \to (0, +\infty)$ with respect to a Markov semigroup $(P_t)_{t \geq 0}$ if

$$\left| P_t\psi(x) - \int_{\mathcal{H}} \psi(y)\,\mathrm{d}\mu(y) \right| \leq c(x)\mathrm{e}^{-\omega t}\|\psi\|_{Lip}, \quad \forall x \in \mathcal{H},\ \forall t > 0,\ \psi \in Lip(E). \tag{3.9}$$

If μ is exponentially mixing, then $P_t(x, \Gamma) \to \mu(\Gamma)$ as $t \uparrow +\infty$ for any Borel subset Γ of E (cfr. [46] and [75], p. 288).

We have the following from [75, Theorem 16.6 p.293]:

THEOREM 3.2. *Let us consider the SDE* (3.1) *under the assumptions on* A *and* F *given at the beginning of the Subsection 3.1 and under the assumptions of Theorem 3.1. Assume, in addition:*

$$\sup_{t \geq 0} \mathbb{E}(|L_A(t)|_{\mathcal{H}} + |F(L_A(t))|_{\mathcal{H}}) < +\infty.$$

Then there exists a unique invariant measure μ for the Markov semigroup $(P_t)_{t\geq 0}$ on $\mathcal{H}$ associated with the mild solution X of Equation (3.1) *(i.e. $(P_t)_{t\geq 0}$ gives the transition probabilities for X).*

μ is exponentially mixing with exponent $\omega+\eta$ (with ω as in Section 3.1 and η as in condition (3.2) *and a bound function c of linear growth in the sense that*

$$|c(x)|\leq C(|x|+1), \tag{3.10}$$

for some constant $C>0$ and for all $x\in\mathcal{H}$.

4. Decomposition of the solution process in a stationary and an asymptotically small component

Let us consider the cylindrical Lévy process $L=(L(t))_{t\geq 0}$, as in Sections 2 and 3. Following [75, p. 295], let $\overline{L}(t)$, $t\in\mathbb{R}$ be the corresponding double-sided process such that $\overline{L}(t)=L(t)$, $t\geq 0$ and $\overline{L}(-t)$, for $t\geq 0$ is a process independent of $L(t)$, $t\geq 0$ and such that all finite dimensional distributions of $\overline{L}(-t)$ coincide with those of $L(t)$, $t\geq 0$.

Our aim now is to split the solution $X(t)$ of the Equation (3.1) into the sum of a stationary process $r(t)$ and an asymptotically (for $t\to+\infty$) vanishing process $v(t)$. We will split the solution $X_m^{(n)}$, $n,m\in\mathbb{N}$, of an approximating equation (see (4.2) below) into the sum of a stationary process $r_m^{(n)}$ and an asymptotically vanishing process $v_m^{(n)}$ satisfying some suitable properties to be made precise below.

Let $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ be a sequence of finite dimensional subspaces of the Banach space $\mathcal{B}$ introduced in Section 3 and $\{\Pi_n\}$ a sequence of self-adjoint operators from $\mathcal{H}$ onto $\mathcal{B}_n$ such that $\Pi_n x\to x$ in $\mathcal{B}$, for arbitrary $x\in\mathcal{B}$. The existence of such spaces and operators can be proved as in [89, Proposition 3].

Moreover let $F_m, m\in\mathbb{N}$ be the m-th Yosida approximation of F (i.e. $F_m:=m(mI-F)^{-1}$). We know that F_m is Lipschitz continuous and it satisfies the following estimates:

$$\begin{aligned} &|F_m(x)-F(x)|_{\mathcal{H}}\to 0, && m\to\infty,\ x\in D(F)\\ &|F_m(x)|_{\mathcal{H}}\leq |F(x)|_{\mathcal{H}}, && x\in D(F),\ \forall m\in\mathbb{N}\end{aligned}$$

and

$$\langle F_m(x)-F_m(y),x-y\rangle\leq m|x-y|^2,\quad \forall m\in\mathbb{N}.$$

(see [46, pag.44]). For any $n,m\in\mathbb{N}$ we consider the following families of equations

$$\begin{cases} dX_m^{(n)}(t)=AX_m^{(n)}(t)dt+\Pi_n F_m(\Pi_n X_m^{(n)}(t))dt+\Pi_n B dL(t),\\ X_m^{(n)}(0)=\Pi_n x\in\mathcal{H}, \end{cases} \tag{4.1}$$

and

$$\begin{cases} dX^{(n)}(t)=AX^{(n)}(t)dt+\Pi_n F(\Pi_n X^{(n)}(t))dt+\Pi_n B dL(t),\\ X^{(n)}(0)=\Pi_n x\in\mathcal{H}, \end{cases} \tag{4.2}$$

which can be seen as approximating problems relative to Equation (3.1).

We are going to prove that the unique solution of Equation (2.7) admits a characterization in terms of a stationary process r and a process v which vanishes at $t\to+\infty$. To this end, we proceed by splitting the solution $X_m^{(n)}$ of the approximating problems into the sum of a stationary process $r_m^{(n)}$ and an asymptotically vanishing process $v_m^{(n)}$

satisfying suitable properties. We shall work under the following additional assumption:

Hypothesis 4.1. *$L(t)$ has finite moments up to a certain order $p=2a$, $a\in\mathbb{N}$. Equivalently, the Lévy measure ν satisfies the inequality*

$$\int_{|x|\geq 1}|x|^k\nu(\mathrm{d}x)<+\infty,\quad k=1,\dots,2a.$$

In particular, $L^n(t)$, $n\geq 1$ has finite moments up to a certain order $p=2a, a\in\mathbb{N}$. Equivalently, $\nu_{\mathbb{R}}$ satisfies the inequality

$$\int_{|x|\geq 1}|x|^k\nu_{\mathbb{R}}(\mathrm{d}x)<+\infty,\quad k=1,\dots,2a.$$

Let us define the two sequences of $\mathcal{F}_t$-adapted processes $r_m^{(n)}$ and $v_m^{(n)}$ respectively such that $v_m^{(n)}(t)=X_m^{(n)}(t)-r_m^{(n)}(t)$, with $r_m^{(n)}$ a solution of the equation

$$r_m^{(n)}(t)=\int_{-\infty}^t S(t-s)\Pi_nF_m\Pi_n(r_m^{(n)}(s))\mathrm{d}s+\bar{L}_A^{\infty,n}(t),\tag{4.3}$$

with $\bar{L}_A^{\infty,n}(t)$, $t\in\mathbb{R}$, being defined by $\bar{L}_A^{\infty,n}(t)=\lim_{a\longrightarrow+\infty}\bar{L}_A^{a,n}(t)$, with

$$\bar{L}_A^{a,n}(t):=\int_{-a}^t S(t-s)\Pi_nB\,\mathrm{d}\bar{L}(s),t\geq -a,\quad a\geq 0$$

and F_m,Π_n are as above.

Let us see first that the random variable $\bar{L}_A^{a,n}(t)$ is well-defined, for any $t\in\mathbb{R}$. In fact, computing first formally, we have

$$\begin{aligned}
&\mathbb{E}|\bar{L}_A^{a,n}(t)|^2\\
&=\mathbb{E}\left|\sum_{k=1}^n\left(\int_{-a}^t e^{-\lambda_k(t-s)}\beta_kb_k\,\mathrm{d}\bar{L}^k(s)\right)e_k\right|^2\\
&=\sum_{k=1}^n\mathbb{E}\left|\int_{-a}^t e^{-\lambda_k(t-s)}\beta_kb_k\,\mathrm{d}\bar{L}^k(s)\right|^2\\
&=\sum_{k=1}^n\mathbb{E}\left|\int_{-a}^t e^{-\lambda_k(t-s)}\mathrm{d}s\left(\int_{|x|\geq 1}\beta_kb_k|x|N(\mathrm{d}s,\mathrm{d}x)+\int_{|x|<1}\beta_kb_k|x|\tilde{N}(\mathrm{d}s,\mathrm{d}x)\right)\right|^2\\
&=\sum_{k=1}^n\int_{-a}^t e^{-2\lambda_k(t-s)}\mathrm{d}s\left(\int_{|x|\geq 1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)+\int_{|x|<1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)\right)\\
&=\sum_{k=1}^n\left[\frac{1}{2\lambda_k}(1-e^{-2\lambda_k(t+a)})\left(\int_{|x|\geq 1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)+\int_{|x|<1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)\right)\right]\\
&\leq\sum_{k=1}^n\left[\frac{1}{2\lambda_k}\left(\int_{|x|\geq 1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)+\int_{|x|<1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)\right)\right].
\end{aligned}$$

By assumption the latter term is bounded and, in particular, we have

$$\mathbb{E}|\bar{L}_A^{a,n}(t)|^2\leq\sum_{k=1}^n\left[\frac{1}{2\lambda_k}\left(\int_{|x|\geq 1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)+\int_{|x|<1}\beta_k^2b_k^2|x|^2\nu_{\mathbb{R}}(\mathrm{d}x)\right)\right]\leq C,$$

where C is independent of n and a. This implies that $\bar{L}_A^{a,n}(t)$ is indeed well-defined and also that

$$\bar{L}_A^{\infty,n}(t) = \lim_{a\to\infty} \bar{L}_A^{a,n}(t) = \int_{-\infty}^t S(t-s)\Pi_n B \,\mathrm{d}\bar{L}(s)$$

$$\bar{L}_A^{\infty}(t) = \lim_{a\to\infty}\lim_{n\to\infty} \bar{L}_A^{a,n}(t)) = \int_{-\infty}^t S(t-s) B \,\mathrm{d}\bar{L}(s)$$

are well-defined as mean-square integrable processes, (where we recall that $S(t) = e^{tA}$).

Similarly, we can also prove that the stochastic convolution

$$\int_{-\infty}^0 S(t-s)\Pi_n B \,\mathrm{d}\bar{L}(s)$$

introduced below in Equation (4.9) is well-defined as a mean square integrable process for any $t > 0$.

We give the following notion of solution for the differential form of Equation (4.3), namely

$$\mathrm{d}r_m^{(n)}(t) = Ar_m^{(n)}(t)\mathrm{d}t + \Pi_n F_m \Pi_n(r_m^{(n)}(t))\mathrm{d}t + \Pi_n B \mathrm{d}L(t), \quad t \in \mathbb{R}, \tag{4.4}$$

with $\lim_{t\to-\infty} r_m^{(n)}(t) = 0$ (in the sense explained after estimate (4.7) below).

Definition 4.1. *An $\mathcal{F}_t$-adapted process $r_m^{(n)}$ is said to be a mild solution to the differential form* (4.4) *of Equation* (4.3) *if it satisfies the integral equation* (4.3) *for any* $t \in \mathbb{R}$.

Theorem 4.1. *For any $n, m \in \mathbb{N}$, there exists a unique mild solution $r_m^{(n)}$ to the differential form of Equation* (4.3)*, such that*

$$\sup_{t\in\mathbb{R}} \mathbb{E}|r_m^{(n)}(t)|_{\mathcal{H}}^p \le C_p, \tag{4.5}$$

for every $2 \le p < +\infty$ and for some positive constant C_p (independent of n and m). Further, $r_m^{(n)}$ is a stationary process, that is, for every $h \in \mathbb{R}^+$, $k \in \mathbb{N}$, any $-\infty < t_1 \ldots \le t_k < +\infty$ and any $A_1, \ldots, A_k \in \mathcal{B}(\mathcal{H})$ we have

$$\mathbb{P}(r_m^{(n)}(t_1+h) \in A_1, \ldots, r_m^{(n)}(t_k+h) \in A_k) = \mathbb{P}(r_m^{(n)}(t_1) \in A_1, \ldots, r_m^{(n)}(t_n) \in A_k).$$

Proof. Let us first prove the uniqueness: Assume that $(x(t))_{t\in\mathbb{R}}$ and $(y(t))_{t\in\mathbb{R}}$ are two solutions of Equation (4.4). Dissipativity of $A + \Pi_n F_m(\Pi_n)$, which follows from our assumptions on A and F, implies

$$\begin{aligned} &\mathrm{d}|x(t)-y(t)|^2 \\ &= \langle A(x(t)-y(t)) + \Pi_n F_m(\Pi_n x(t)) - \Pi_n F_m(\Pi_n y(t)), x(t), x(t)-y(t)\rangle \mathrm{d}t, \qquad t \in \mathbb{R} \\ &\le -2(\omega-\eta)|x(t)-y(t)|^2 \mathrm{d}t. \end{aligned}$$

By using Gronwall's lemma, we deduce from this that for any $\xi > 0$ and $t \ge -\xi$ the following inequality holds

$$|x(t)-y(t)|^2 \le |x(-\xi)-y(-\xi)|^2 e^{-2(\omega-\eta)(t+\xi)}.$$

Letting $\xi \to +\infty$ we conclude that $x(t)=y(t)$ for any $t\in\mathbb{R}$ (here we use the requirement $\omega>\eta$ put after condition (3.2)). For the existence of a solution $r_m^{(n)}(t)$ to Equation (4.3) let $r_m^{(n)}(t,-\xi)$, $\xi>0$ be the unique solution of the equation:

$$\begin{cases} \mathrm{d}r_m^{(n)}(t,-\xi)=Ar_m^{(n)}(t)\mathrm{d}t+\Pi_n F_m \Pi_n (r_m^{(n)}(t,-\xi))\mathrm{d}t+\Pi_n B\mathrm{d}L(t), \quad t\geq -\xi\,, \\ r_m^{(n)}(-\xi,-\xi)=e^{\xi A}\Pi_n x, \qquad x\in\mathcal{H}. \end{cases} \tag{4.6}$$

Let us note that Equation (4.6) has indeed a unique solution being a finite dimensional Cauchy problem with Lipschitz and dissipative coefficients, see, e.g., [75, Prop. 9.8, pag.146] for details. From the smoothness properties of the stochastic convolution, one can assume that $r_m^{(n)}(\cdot;-\xi)$ admits a $\mathcal{B}$- càdlàg version (with $\mathcal{B}$ as in Section 3), which we still denote by $r_m^{(n)}(\cdot;-\xi)$. Consequently, for any $t\in\mathbb{R}$, we can define $r_m^{(n)}(t)$ as the limit of $r_m^{(n)}(t;-\xi)$ for $\xi\to+\infty$ and this turns out to be the solution of Equation (4.3). We will first prove that, for any $t\geq-\xi$, $\xi>0$ and $p\geq 2$ the following estimate holds

$$\sup_{-\xi\leq t\leq 0} \mathbb{E}|r_m^{(n)}(t;-\xi)|_{\mathcal{H}}^p < C_p\,,$$

where C_p is a positive constant independent of n,m and ξ, but possibly depending on p. For simplicity, and without loss of generality, we consider the case $p=2a$, $a\in\mathbb{N}$. We want to apply Itô's formula to the processes $|r_m^{(n)}(t,-\xi)|_{\mathcal{H}}^{2a}$. To this end, we refer to Appendices A and B, where the details are given for the finite dimensional case, but the same procedure considered there also works in infinite dimensions. If we repeat the arguments in the Appendix B, we see that there exists suitable constants C_i, $i=1,2$ and $0<\epsilon<1$ such that, for any $\xi>0$, $a,m,n\in\mathbb{N}$:

$$\begin{aligned} &\sup_{t\in[-\xi,0]} \mathbb{E}|r_m^{(n)}(t;-\xi)|^{2a} \\ &\leq e^{-2a\xi\omega}|x|^{2a}+C_2T-\left(\omega-\eta-\epsilon^{\frac{1}{n-1}}C_1\right)\int_{-\xi}^0 \sup_{s\in[-\xi,t]}\mathbb{E}|r_m^{(n)}(s;-\xi)|_{\mathcal{H}}^{2a}\mathrm{d}t. \end{aligned}$$

Then applying Gronwall's lemma we get

$$\begin{aligned} \sup_{t\in[-\xi,0]} \mathbb{E}|r_m^{(n)}(t;-\xi)|^{2a} &\leq [e^{-2a\xi\omega}|x|^{2a}+C_2T] \\ &\leq C_{2a}, \end{aligned}$$

where C_{2a} is a suitable constant independent of m,n and ξ.

Moreover, in a similar way we can prove that, for any fixed $t\in\mathbb{R}$, the sequence $\{r_m^{(n)}(t;-\xi)\}_{-\xi\leq t\leq 0}$ is a Cauchy sequence, in the sense of the Banach space $L^{2a}(\Omega,\mathcal{F},\mathbb{P})$ over the probability space, uniformly in t.

Now let $0\leq\gamma\leq\xi$. We shall estimate the norm:

$$\sup_{0\geq t\geq -\gamma} \mathbb{E}|r_m^{(n)}(t;-\xi)-r_m^{(n)}(t;-\gamma)|^{2a}.$$

To this end we notice that the process $y_{\xi,\gamma}(t):=r_m^{(n)}(t;-\xi)-r_m^{(n)}(t;-\gamma)$ can be written as:

$$y_{\xi,\gamma}(t)=(e^{\xi A}-e^{\gamma A})\Pi_n x+e^{tA}\int_{-\xi}^{-\gamma} e^{(\gamma-s)A}\Pi_n F_m(\Pi_n r_m^{(n)}(s;-\xi))ds$$
$$+e^{tA}\int_{-\xi}^{-\gamma} e^{(\gamma-s)A}B\mathrm{d}L(s)$$
$$+\int_{-\gamma}^{t} e^{(t-s)A}[\Pi_n F_m(\Pi_n r_m^{(n)}(s;-\xi))-\Pi_n F_m(\Pi_n r_m^{(n)}(s;-\gamma))]ds.$$

Moreover, as seen from Equation (4.6), the stochastic differential of $y_{\xi,\gamma}(t)$ is given by

$$\mathrm{d}y_{\xi,\gamma}(t)=Ay_{\xi,\gamma}\mathrm{d}t+[F_{n,m}(r_m^{(n)}(t;-\xi))-F_{n,m}(r_m^{(n)}(t;-\gamma))]\mathrm{d}t, \quad t>-\xi,\xi>0,$$

since the first three terms in inequality (4.7) below do not depend on t. Differentiating $|y_{\xi,\gamma}|^{2a}$ and reasoning as above, using Appendix B, we get

$$\mathbb{E}|y_{\xi,\gamma}(t)|^{2a}\le|e^{\xi A}x|^{2a}+\mathbb{E}|e^{tA}y_{\xi,\gamma}(-\gamma)|^{2a}-2(\omega-\eta)\int_{-\xi}^{t}\mathbb{E}|y_{\xi,\gamma}(r)|^{2a}\mathrm{d}s.$$

Gronwall's lemma then implies

$$\begin{aligned}\sup_{0\ge t\ge-\gamma}\mathbb{E}|y_{\xi,\gamma}(t)|^{2a}&\le(\mathbb{E}|e^{tA}y_{\xi,\gamma}(-\gamma)|^{2a}+|e^{\xi A}x|^{2a})e^{-2(\omega-\eta)(\xi+t)}\\&\le(e^{-2at\omega}\mathbb{E}|y_{\xi,\gamma}(-\gamma)|^{2a}+e^{-2a\xi\omega}|x|^{2a})e^{-2(\omega-\eta)(\xi+t)}\\&\le C,\end{aligned}\tag{4.7}$$

where C is a constant independent on ξ. We then conclude that for any $t\in\mathbb{R}$ the limit $r_m^{(n)}(t):=\lim_{\xi\to+\infty}r_m^{(n)}(t;-\xi)$ exists in $L^p(\Omega,\mathbb{P};\mathcal{B})$, for all $1\le p<+\infty$ and moreover,

$$\sup_{t\ge0}\mathbb{E}(|r_m^{(n)}(t)|)^{2a}\le C_{2a},$$

for some constant C_{2a} depending on a. In addition, by the condition at $t=-\xi$, $\xi>0$, in Equation (4.6), we deduce that $\lim_{t\to-\infty}r_m^{(n)}(t)=0$, if $x\neq0$, since $\lim_{\xi\to+\infty}e^{\xi A}x=0$, for any $x\in H, A$ being strictly negative.

Finally, for any $n,m\in\mathbb{N}$ we are going to show that the stochastic process $r_m^{(n)}$ is stationary. In order to prove this statement, we adapt to our case the argument given in [67]. In particular, we introduce the following Picard iteration:

$$\begin{cases} r_m^{(n,0)}(t)=x\\ r_m^{(n,k+1)}(t)=\int_{-\infty}^t e^{(t-s)A}\Pi_n F_m(\Pi_n r_m^{(n,k)}(s))\mathrm{d}s+\bar{L}_A^{\infty}(t),\end{cases}\tag{4.8}$$

where $\bar{L}_A^{\infty}(t)$ is as in Equation (4.3). We notice that $\lim_{k\to\infty}r^{(n,k)}(t)=\tilde{r}_m^{(n)}(t)$ exists (as seen by following corresponding arguments to those used in the Gaussian case in, e.g, [46]) and it is a stationary process. The crucial point is that $\tilde{r}_m^{(n)}$ and $r_m^{(n)}$ coincide. In fact, if we pass to the limit in Equation (4.8), we see that $r_m^{(n)}$ solves Equation (4.3) so that, by uniqueness of solutions of Equation (4.3) (which holds, the system being finite dimensional with globally Lipschitz coefficients), we get $\tilde{r}_m^{(n)}\equiv r_m^{(n)}$. Consequently, $r_m^{(n)}$ is stationary. This completes the proof of Theorem 4.1. □

Now, the process $v_m^{(n)}$ is given by

$$v_m^{(n)}(t) = e^{tA}\Pi_n x - \int_{-\infty}^{0} e^{(t-s)A}\Pi_n F_m \Pi_n(r_m^{(n)}(s))\mathrm{d}s - \int_{-\infty}^{0} e^{(t-s)A}\Pi_n B \mathrm{d}\bar{L}(s) + \int_0^t e^{(t-s)A}[\Pi_n F_m(\Pi_n X_m^{(n)}(s))\mathrm{d}s - \Pi_n F_m(\Pi_n r_m^{(n)}(s))]\mathrm{d}s. \tag{4.9}$$

Taking into account that the convolution process $\bar{L}_A^{\infty}(t)$ is well-defined (as we remarked just before Definition 4.1), we will discuss existence and uniqueness of a mild solution for the equation for the process $v_m^{(n)}$ and we will show that it vanishes, in a suitable sense, as $t \to +\infty$.

PROPOSITION 4.1. *Under the assumptions given in Theorem 3.1 and in Theorem 3.2 we have that for any $n,m \in \mathbb{N}$ there exists a unique mild solution $(v_m^{(n)}(t))_{t\geq 0}$ of Equation (4.9). Moreover, for any $1 \leq p < +\infty$, we have the following bound:*

$$\sup_{t\geq 0} \mathbb{E}|v_m^{(n)}(t)|^p \leq C_p,$$

where C_p is a positive constant independent of n and m. In addition, we have the following limit

$$\lim_{t\to+\infty} \mathbb{E}|v_m^{(n)}(t)|^p = 0, \ \textit{for any} \qquad 1 \leq p < +\infty.$$

Proof. The existence and uniqueness of a mild solution for the process $(v_m^{(n)})$ given by Equation (4.9) holds in a straightforward manner under our assumptions by results in finite dimensions, see, e.g, [30].

Without loss of generality we can assume that $p = 2a$ for $a \in \mathbb{N}$. By the dissipativity of the mapping $\Pi_n F_m \Pi_n$, which follows from the one of F and the fact that $A \leq -\omega$, we get, similarly as in [5]

$$\begin{aligned}
& d|v_m^{(n)}(t)|^{2a} \\
=& \mathrm{d}|X_m^{(n)}(t) - r_m^{(n)}(t)|^{2a} \\
=& 2a\langle A v_m^{(n)}(t) + \Pi_n F_m(\Pi_n X_m^{(n)}(t)) - \Pi_n F_m(\Pi_n r_m^{(n)}(t)), v_m^{(n)}(t)\rangle |v_m^{(n)}(t)|^{2a-2}\mathrm{d}t \\
& -2a\omega|v_m^{(n)}(t)|^{2a} + 2a\eta|v_m^{(n)}(t)|^{2a},
\end{aligned}$$

so that integrating on $[0,t]$, $0 \leq t \leq T$, and applying Gronwall's lemma, we obtain

$$\begin{aligned}
& \sup_{t\in[0,T]} \mathbb{E}|v_m^{(n)}(t)|^{2a} \\
\leq & e^{-(\omega-\eta)T}\mathbb{E}|v_m^{(n)}(0)|^{2a} \\
\leq & C_a e^{-(\omega-\eta)T}\left[|x|_{\mathcal{H}}^{2a} + \mathbb{E}\int_{-\infty}^{0} |e^{-sA}\Pi_n F_m(\Pi_n r^{(n)}(s))|_{\mathcal{H}}^{2a}\mathrm{d}s + \left|\int_{-\infty}^{0} e^{-sA}\Pi_n B \mathrm{d}L(s)\right|^{2a}\right] \\
\leq & C_a e^{-(\omega-\eta)T}\left[K_a + \sup\nolimits_{t\geq 0}\mathbb{E}\left(|L_A(t)|_{\mathcal{H}} + |F(L_A(t))|_{\mathcal{H}}\right)\right],
\end{aligned}$$

where C_a, K_a are positive constants depending only on a and $L_A(t)$ has been defined in Equation (2.8). Now from this inequality and the assumptions in Theorem 3.2, we deduce (similarly as in [5]) that

$$\sup_{t\in[0,T]} \mathbb{E}|v_m^{(n)}(t)|^{2a} \leq C_a e^{-(\omega-\eta)T}(K_a + C), \tag{4.10}$$

with $\omega > \eta > 0$. The result now follows by letting $T \to +\infty$. □

The next result states that $r_m^{(n)}$ and $v_m^{(n)}$ converge respectively to stochastic processes r and v in $L^p(\Omega, C([0,T];\mathcal{B}))$, $p \geq 1$, $T > 0$, where $\mathcal{B}$ is as in Section 3, moreover it also shows additional properties of r, v.

Proposition 4.2. *There exist a stationary process r and a process v in $L^p(\Omega; C([0,T];\mathcal{B}))$, $1 \leq p < +\infty$, such that the solution process $X(t)$ of Equation (3.1) in Theorem 3.1 can be written as $X(t) = r(t) + v(t)$, with*

$$\lim_{n,m\to+\infty} r_m^{(n)}(t) = r(t)$$
$$\lim_{n,m\to+\infty} v_m^{(n)}(t) = v(t),$$

$t \in [0,T]$, for any $T > 0$. Further, for any $1 \leq p < +\infty$, $\lim_{t\to+\infty} \mathbb{E}|v(t)|^p = 0$.

Proof. Again without loss of generality, we assume that $p = 2a$, $a \in \mathbb{N}$. For the convergence of the sequence $r_m^{(n)}$ and $v_m^{(n)}$ the proof is by contradiction. Assume that there exists $\varepsilon > 0$ such that, for all $m, n \in \mathbb{N}$:

$$\sup_{\substack{k,k'>n,\\ j,j'>m}} \mathbb{E}|r_j^{(k)}(t) - r_{j'}^{(k')}(t)|_{\mathcal{H}}^{2a} > 2\varepsilon, \qquad t \in [0,T].$$

Since the difference of two stationary processes is stationary, the expression on the left hand side is independent of time t. By choosing t positive large enough, thus by Proposition 4.1 making $\mathbb{E}|v_j^{(k)}(t)|^{2a}$ and $\mathbb{E}|v_{j'}^{(k')}(t)|^{2a}$ sufficiently small (smaller than $\varepsilon/2$), and using $X_j^{(k)}(t) = r_j^{(k)}(t) + v_j^{(k)}(t)$ it is easy to show that, using the triangle inequality,

$$\sup_{t\geq 0} \sup_{\substack{k,k'>n,\\ j,j'>m}} \mathbb{E}|r_j^{(k)}(t) - r_{j'}^{(k')}(t)|^{2a} = \sup_{t\geq 0} \sup_{\substack{k,k'>n,\\ j,j'>m}} \mathbb{E}|X_j^{(k)}(t) - X_{j'}^{(k')}(t)|^{2a} > \varepsilon,$$

for ε as above. But this contradicts the fact that $\lim_{n,m\to\infty} X_m^{(n)}(t)$ exists pointwise. As a consequence of the convergence of the sequences $r_m^{(n)}(t)$ and $X_m^{(n)}(t)$ to $r(t)$ (respectively $X(t)$), as $n, m \to +\infty$, we obtain the convergence of $v_m^{(n)}(t)$ for any $t \in [0,T]$, as $n, m \to +\infty$.
Now let us show that:

$$\lim_{t\to+\infty} \mathbb{E}|v(t)|^{2a} = 0, \tag{4.11}$$

where $v(t) := X(t) - r(t)$, $t \geq 0$. We have, using $X(t) = r(t) + v(t)$, that:

$$\mathbb{E}|v(t)|^{2a} = \mathbb{E}|X(t) - r(t)|^{2a} \leq c_a \mathbb{E}|X(t) - X_m^{(n)}(t)|^{2a} + c_a \mathbb{E}|X_m^{(n)}(t) - r_m^{(n)}(t)|^{2a} + c_a \mathbb{E}|r_m^{(n)}(t) - r(t)|^{2a}, \tag{4.12}$$

for some strictly positive constant c_a depending only on a. If n, m are large enough, then the first and third terms are less than εc_a, uniformly in $t \geq 1$. The second term is less than εc_a, for $t > T(\varepsilon)$, for some $T(\varepsilon)$ independent of n, m, for all n, m large enough. Combining the estimates we obtained on the three terms on the right hand side of inequality (4.12) we see that we have shown that $\mathbb{E}|v(t)|^{2a} < \varepsilon$ for sufficient large positive t, which completes the proof. ∎

From Theorem 3.2 we had already the existence of the invariant measure for the process X. We shall now prove that it is given by the law of the stationary process r:

THEOREM 4.2. *The invariant measure for the process $(X(t))_{t\geq 0}$, is given by the law $\mathcal{L}(r(t))$ of the stationary process r.*

Proof. It suffices to prove that the law $\mathcal{L}(r(t))$ is an invariant measure for X. For this it is sufficient to show that the dual $P_t^\star$ (acting on measures) of the transition semigroup for X satisfies $P_t^\star\mathcal{L}(r(t)) = \mathcal{L}(r(t))$. This is so by the stationarity of $r(t)$. Then, exploiting the uniqueness of invariant measures for X, see Theorem 3.2, we have that $\mathcal{L}(r(t))$ is the unique invariant measure for X. ∎

Appendix A. For further use, we want to prove the following property of the binomial formula:

LEMMA A.1. *Let a,b be two real positive numbers and $n\in\mathbb{N}$. Then, there exists a (sufficiently small) positive number $0<\epsilon<1$ and $c_n>1$ such that*

$$(a+b)^n - a^n \leq c_n\left(\epsilon^{\frac{1}{n-1}} a^n b + \frac{b^{n+1}+b^n+b}{\epsilon}\right) \tag{A.1}$$

$$(a+b)^n - a^n - na^{n-1}b \leq c_n\left(\epsilon^{\frac{1}{n-1}} a^n b + \frac{b^{n+1}+b^n+b}{\epsilon}\right) \tag{A.2}$$

Proof. We start by proving formula (A.1). We are going to take into account the following facts.

(1) binomial formula states that, for given real numbers $a,b>0$ and $n\in\mathbb{N}$ the following equaliity holds:

$$(a+b)^n = a^n + \binom{n}{1} ab^{n-1} + \dots \binom{n}{n-1} a^{n-1}b + b^n. \tag{A.3}$$

(2) For any $a_1,\dots,a_n>0$ and $n\in\mathbb{N}$ we have the Arithmetic Mean-Geometric Mean Inequality

$$\sqrt[n]{a_1\cdot\dots\cdot a_n} \leq \frac{a_1+\dots+a_n}{n}. \tag{A.4}$$

Using inequality (A.4) for

$$a_1 = \epsilon^{\frac{n}{n-1}-1}a^n, \dots, a_{n-1} = \epsilon^{\frac{n}{n-1}-1}a^n, a_n = \frac{1}{\epsilon}$$

we obtain

$$a^{n-1} = \sqrt[n]{a_1\cdot\dots\cdot a_n} \leq \frac{n-1}{n}\epsilon^{\frac{n}{n-1}-1}a^n + \frac{1}{n\epsilon}; \tag{A.5}$$

while, for $k=1,\dots,n-2$, using inequality (A.4) for

$$a_1 = \epsilon^{\frac{n}{k}-1}a^n, \quad \dots, \quad a_k = \epsilon^{\frac{n}{k}-1}a^n,$$
$$a_{k+1} = \frac{1}{\epsilon}b^n, \quad \dots, \quad a_{n-1} = \frac{1}{\epsilon}b^n, \quad a_n = \frac{1}{\epsilon}$$

we obtain

$$a^k b^{n-k-1} = \sqrt[n]{a_1 \cdot \ldots \cdot a_n} \leq \frac{k}{n} \epsilon^{\frac{n}{k}-1} a^n + \frac{n-k-1}{n\epsilon} b^n + \frac{1}{n\epsilon} \leq \epsilon^{\frac{n}{k}-1} a^n + \frac{b^n}{\epsilon} + \frac{1}{n\epsilon}; \quad \text{(A.6)}$$

Hence, for given $1 > \epsilon > 0$, using inequalities (A.5) and (A.6) we have the following estimate:

$$\begin{aligned}
(a+b)^n - a^n &= \binom{n}{1} ab^{n-1} + \ldots \binom{n}{n-1} a^{n-1} b + b^n \\
&= b\left[\binom{n}{1} ab^{n-2} + \ldots \binom{n}{n-1} a^{n-1} + b^{n-1}\right] \\
&= b\left[\sum_{k=1}^{n-2} \binom{n}{k} a^k b^{n-k-1} + n a^{n-1} + b^{n-1}\right] \\
&\leq b\left[\left(\epsilon^{\frac{n}{n-1}-1} a^n + \frac{b^n}{\epsilon} + \frac{1}{n\epsilon}\right) \sum_{k=1}^{n-2} \binom{n}{k} + (n-1)\epsilon^{\frac{n}{n-1}-1} a^n + \frac{1}{\epsilon} + b^{n-1}\right] \\
&\leq b\left[(2^n + n - 1)\epsilon^{\frac{n}{n-1}-1} a^n + \frac{2^n}{\epsilon} b^n + b^{n-1} + \frac{2^n}{n\epsilon} + \frac{1}{\epsilon}\right] \\
&\leq 2^{n+1} \epsilon^{\frac{1}{n-1}} a^n b + \frac{2^n}{\epsilon} b^{n+1} + \frac{2^n}{\epsilon} b^n + \frac{b}{\epsilon} \\
&\leq c_n \left(\epsilon^{\frac{1}{n-1}} a^n b + b^{n+1} + b^n + b\right),
\end{aligned}$$

which proves the first inequality.

Now, with analogous calculation, we can also derive formula (A.2). □

Appendix B. Let $Y(t)$ be the real-valued process defined by the SDE

$$\begin{aligned}
\mathrm{d}Y(t) &= [AY(t) + F(Y(t))]\mathrm{d}t + \mathrm{d}L(t), \\
Y(0) &= x \in \mathbb{R}
\end{aligned}$$

where the drift term gives a dissipative component and $L(t)$ is a pure jump real-valued Lévy process, i.e. $L(t)$ can be written as

$$L(t) = \int_0^t \int_{|x| \geq 1} x N(\mathrm{d}t, \mathrm{d}x) + \int_0^t \int_{|x|<1} x \tilde{N}(\mathrm{d}t, \mathrm{d}x). \quad \text{(B.1)}$$

Here, for a set $A \subset \mathbb{R}$ such that $0 \notin A$, $N(t, A)$ is a Poisson process with σ-finite (see [30] pg.89) jump intensity measure $\nu_{\mathbb{R}}$, while $\tilde{N}(t, A)$ (defined by $\tilde{N}(t, A) = N(t, A) - t\nu_{\mathbb{R}}(A)$) is the compensated Poisson process. By Corollary 2.4.12 in [30], $\nu_{\mathbb{R}}$ is a Lévy measure.

Hypothesis B.1. *$L(t)$ has finite moments up to a certain order $p = 2a+1, a \in \mathbb{N}$. Equivalently, $\nu_{\mathbb{R}}$ satisfies the inequality*

$$\int_{|x| \geq 1} |x|^k \nu_{\mathbb{R}}(\mathrm{d}x) < +\infty, \quad k = 1, \ldots, 2a.$$

It turns out that $\tilde{N}(t, A)$ is finite almost surely for all $t \geq 0$ (cf. Lemma 2.3.4 in [30]).

In the following, we want to apply Itô's formula in [30, Theorem 4.4.7] to the real-valued function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^{2a}$ for some $a \in \mathbb{N}$. More precisely, we want to prove the following result:

PROPOSITION B.1. *Let $Y(t)$ be the real-valued process defined by the SDE*

$$\begin{aligned} \mathrm{d}Y(t) &= [AY(t)+F(Y(t))]\mathrm{d}t+\mathrm{d}L(t), \\ Y(0) &= x\in\mathbb{R}, \end{aligned} \tag{B.2}$$

where $A\in\mathbb{R}$ and F is a continuous real-valued function. Assume that $A<-\omega$ and $F(x)<-\eta x$ for every $x\in\mathbb{R}$, for ω,η positive numbers in $\mathbb{R}$ such that $\omega-\eta>0$. Finally, $(L(t))_{t\geq 0}$ as in Equation (B.1)*. Then, for any $T>0$, the unique solution of the SDE* (B.2) *satisfies the following estimate:*

$$\sup_{t\in[0,T]} \mathbb{E}|Y(t)|^{2a} \leq C_{2a}, \tag{B.3}$$

where C_{2a} is a positive constant depending only on a, a being the positive value considered in Hypothesis B.1.

Proof. Existence and uniqueness for the solution of Equation (B.2) are well-known. Let us prove estimate (B.3). Since $f'(x)=2ax^{2a-1}$, we have:

$$Y(t)^{2a} = x^{2a} + \int_0^T 2a(AY(t)+F(Y(t)))Y(t^-)^{2a-1}\mathrm{d}t \tag{B.4}$$

$$+\int_0^T\int_{|x|\geq 1}((Y(t^-)+x)^{2a}-Y(t^-)^{2a})N(\mathrm{d}t,\mathrm{d}x) \tag{B.5}$$

$$+\int_0^T\int_{|x|<1}((Y(t^-)+x)^{2a}-Y(t^-)^{2a})\tilde{N}(\mathrm{d}t,\mathrm{d}x) \tag{B.6}$$

$$+\int_0^t\int_{|x|<1}((Y(t^-)+x)^{2a}-Y(t^-)^{2a}-2axY(t-)^{2a-1})\nu_{\mathbb{R}}(\mathrm{d}x)\ \mathrm{d}t. \tag{B.7}$$

In order to estimate $Y(t)^{2a}$, let us consider separately the integrals (B.4)–(B.7). The dissipativity of $A+F$ implies that the integral (B.4) is estimated by

$$\int_0^T 2a(AY(t)+F(Y(t)))Y(t^-)^{2a-1}\mathrm{d}t \leq -(\omega-\eta)Y(t^-)^{2a}.$$

Now let us consider $(Y(t^-)+x)^{2a}-Y(t^-)^{2a}$. Notice that

$$(Y(t^-)+x)^{2a} = Y(t^-)^{2a}+\binom{2a}{1}Y(t^-)^{2a-1}x+\cdots+\binom{2a}{2a-1}Y(t^-)x^{2a-1}+x^{2a}; \tag{B.8}$$

hence

$$\begin{aligned} &\mathbb{E}[(Y(t^-)+x)^{2a}]-\mathbb{E}[Y(t^-)^{2a}] \\ &=\binom{2a}{1}\mathbb{E}[Y(t^-)^{2a-1}]x+\cdots+\binom{2a}{2a-1}\mathbb{E}[Y(t^-)]x^{2a-1}+x^{2a} \\ &\leq\binom{2a}{1}\mathbb{E}[Y(t^-)^{2a-1}]|x|+\cdots+\binom{2a}{2a-1}\mathbb{E}[Y(t^-)]|x|^{2a-1}+|x|^{2a}. \end{aligned} \tag{B.9}$$

Following the lines of Lemma A.1 in Appendix A we obtain

$$\mathbb{E}[(Y(t^-)+x)^{2a}]-\mathbb{E}[Y(t^-)^{2a}]$$

$$\leq c_n \epsilon^{\frac{1}{n-1}} \mathbb{E}[Y(t^-)^{2a}]|x| + c_n \frac{|x|^{2a+1}+|x|^{2a}+|x|}{\epsilon}, \tag{B.10}$$

so that, for some $0<\epsilon<1$

$$\begin{aligned} &\sup_{t\in[0,T]} \left\{\mathbb{E}[(Y(t^-)+x)^{2a}] - \mathbb{E}[Y(t^-)^{2a}]\right\} \\ &\leq \sup_{t\in[0,T]} \left\{c_n \epsilon^{\frac{1}{n-1}} \mathbb{E}[Y(t^-)^{2a}]|x| + c_n \frac{|x|^{2a+1}+|x|^{2a}+|x|}{\epsilon}\right\}. \end{aligned} \tag{B.11}$$

Concerning the integrand in (B.7), again by proceeding as in Lemma A.1 we have that there exists $0<\epsilon<1$ such that

$$\begin{aligned} &\sup_{t\in[0,T]} \left\{(Y(t^-)+x)^{2a} - Y(t^-)^{2a} - 2axY_{t-}^{2a-1}\right\} \\ &\leq \sup_{t\in[0,T]} \left\{c_n \epsilon^{\frac{1}{n-1}} \mathbb{E}[Y(t^-)^{2a}]|x| + c_n \frac{|x|^{2a+1}+|x|^{2a}+|x|}{\epsilon}\right\}. \end{aligned} \tag{B.12}$$

Now, let us consider integral (B.5). We know by [30, Theorem 2.3.5] that, given $B:=\{|x|\geq 1\}$, $N(t,B)$ is a Poisson process with intensity $\nu_{\mathbb{R}}(B)$. Moreover, the process

$$\tilde{N}(t,B) = N(t,B) - \lambda \nu_{\mathbb{R}}(B)$$

is a martingale (see [30, pg. 90]). As function on a general set $K\subset\mathbb{R}$, $\tilde{N}(t,\mathrm{d}x)$ is a martingale-valued measure. In particular, the stochastic integral with respect to $\tilde{N}(t,\mathrm{d}x)$ is well defined (see [30, Section 4.2]). More precisely, if $H(s,x)$ is any predictable process (cf. [30, pag. 206] for more details) satisfying

$$\mathbb{E}\left[\int_0^T \int_A |H(s,x)|^2 \nu_{\mathbb{R}}(\mathrm{d}x)\mathrm{d}t\right] < +\infty, \qquad A\in\mathcal{B}(\mathbb{R}),$$

then $I_T(H) := \int_0^T \int_A H(s,x)\mathrm{d}\tilde{N}(t,\mathrm{d}x)$ is well defined as a real-valued random variable and

$$\mathbb{E}[I_T(H)] = 0, \qquad \mathbb{E}[I_T^2(H)] = \left[\int_0^T \int_A \mathbb{E}|H(s,x)|^2 \nu_{\mathbb{R}}(\mathrm{d}x)\mathrm{d}t\right].$$

Hence, using the relation between N and $\tilde{N}$, taking into account equality (B.8) and the fact that $\tilde{N}$ is a martingale, by adding and subtracting $-\mathrm{d}t\nu_{\mathbb{R}}(\mathrm{d}x)$ in integral (B.5) we obtain

$$\begin{aligned} &\mathbb{E}\left[\int_0^T \int_{|x|\geq 1} ((Y(t^-)+x)^{2a} - Y(t^-)^{2a}) N(\mathrm{d}t,\mathrm{d}x)\right] \\ &= \mathbb{E}\left[\int_0^T \int_{|x|\geq 1} ((Y(t^-)+x)^{2a} - Y(t^-)^{2a}) N(\mathrm{d}t,\mathrm{d}x) - \int_0^T \int_{|x|\geq 1} ((Y(t^-)+x)^{2a} - Y(t^-)^{2a}) \mathrm{d}t\nu_{\mathbb{R}}(\mathrm{d}x)\right] \\ &\quad + \mathbb{E}\int_0^T \int_{|x|\geq 1} ((Y(t^-)+x)^{2a} - Y(t^-)^{2a}) \mathrm{d}t\nu_{\mathbb{R}}(\mathrm{d}x) \\ &\leq \int_0^T \int_{|x|\geq 1} \sup_{s\in[0,t]} \mathbb{E}((Y(s^-)+x)^{2a} - Y(s^-)^{2a}) \mathrm{d}t\nu_{\mathbb{R}}(\mathrm{d}x) \\ &\leq c_n \epsilon^{\frac{1}{n-1}} \int_0^T \sup_{s\in[0,t]} \mathbb{E}|Y(s^-)|^{2a} \mathrm{d}t \int_{|x|\geq 1} |x|\nu(\mathrm{d}x) + c_n \frac{1}{\epsilon} T \int_{|x|\geq 1} (|x|^{2a+1}+|x|^{2a}+|x|)\nu_{\mathbb{R}}(\mathrm{d}x). \end{aligned}$$

Reasoning in the same way for integral (B.6) (recalling that $\tilde{N}$ is a martingale), we obtain that

$$\mathbb{E}\left[\int_0^T\int_{|x|<1}((Y(t^-)+x)^{2a}-Y(t^-)^{2a})\tilde{N}(\mathrm{d}t,\mathrm{d}x)\right]=0.$$

Finally, concerning integral (B.7), using estimate (B.12) we obtain:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\int_{|x|<1}((Y(t^-)+x)^{2a}-Y(t^-)^{2a}-2axY_{t-}^{2a-1})\nu_{\mathbb{R}}(\mathrm{d}x)\mathrm{d}t\right]\\&\le c_n\epsilon^{\frac{1}{n-1}}\int_0^T\sup_{s\in[0,t]}\mathbb{E}|Y(s^-)|^{2a}\mathrm{d}t\int_{|x|<1}|x|\nu_{\mathbb{R}}(\mathrm{d}x)+c_n\frac{1}{\epsilon}T\int_{|x|<1}(|x|^{2a+1}+|x|^{2a}+|x|)\nu_{\mathbb{R}}(\mathrm{d}x).\end{aligned}$$

Now, notice that $\nu_{\mathbb{R}}(\{|x|\ge 1\})$ is finite (see Remark 1 after Theorem 2.3.5 in [30]). Moreover, since $\nu_{\mathbb{R}}$ is a Lévy measure, we have

$$\int_{|x|<1}|x|^k\nu_{\mathbb{R}}(\mathrm{d}x)<+\infty,\qquad k\in\mathbb{N},\tag{B.13}$$

see ([30, pg.110]).

On the other hand,

$$\int_{|x|\ge 1}|x|^k\nu_{\mathbb{R}}(\mathrm{d}x)<+\infty$$

is not automatically satisfied (e.g. consider the case of an α-stable process). This motivates our assumptions on $\nu_{\mathbb{R}}$ in Hypothesis B.1. From this hypothesis and inequality (B.13) we see that $L(t)$ has finite moments up to order $2a+1$.
We have the following inequality:

$$\begin{aligned}\sup_{t\in[0,T]}\mathbb{E}|Y(t^-)|^{2a}\le|x|^{2a}-\left(\omega-\eta-c_n\epsilon^{\frac{1}{n-1}}\int_{\mathbb{R}}|x|\nu_{\mathbb{R}}(\mathrm{d}x)\right)\int_0^T\sup_{s\in[0,t]}\mathbb{E}|Y(s^-)|^{2a}\mathrm{d}t\\+c_n\frac{T}{\epsilon}\int_{\mathbb{R}}(|x|^{2a+1}+|x|^{2a}+|x|)\nu_{\mathbb{R}}(\mathrm{d}x).\end{aligned}$$

Set

$$\begin{aligned}C_{n,1}&:=c_n\int_{\mathbb{R}}|x|\nu_{\mathbb{R}}(\mathrm{d}x)\\C_{n,2}&:=c_n\frac{1}{\epsilon}\int_{\mathbb{R}}(|x|^{2a+1}+|x|^{2a}+|x|)\nu_{\mathbb{R}}(\mathrm{d}x)\end{aligned}$$

(notice that both constants are finite in virtue of Hypothesis B.1 and above considerations). By applying Gronwall's lemma we then get

$$\sup_{t\in[0,T]}\mathbb{E}|Y(t^-)|^{2a}\le(|x|^{2a}+C_2T)\exp((-\omega+\eta+\epsilon^{\frac{1}{n-1}}C_1)T),$$

for a suitable choice of $0<\epsilon<1$. From this it follows that

$$\sup_{t\in[0,T]}\mathbb{E}|Y(t^-)|^{2a}<C_{\epsilon,n},$$

for any $T>0$, with $C_{\epsilon,n}$ being a constant depending only on ϵ,n. □

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REFERENCES

[1] S. Albeverio, *Theory of Dirichlet forms and applications*, Lectures on Probability Theory and Statistics, (Saint-Flour, 2000), 1-106, Lecture Notes in Math., 1816, Springer, Berlin, 2003.

[2] S. Albeverio *Wiener and Feynman-path integrals and their applications*, Proceedings of the Norbert Wiener Centenary Congress, (1994), East Lansing, MI, Proc. Sympos. Appl. Math., Amer. Math. Soc., Providence, RI, 52:153–194, 1997.

[3] S. Albeverio, V. Bogachev, and M. Röckner, *On uniqueness of invariant measures for finite and infinite-dimensional diffusions*, Comm. Pure Appl. Math., 52(3):325–362, 2010.

[4] S. Albeverio and L. Di Persio, *Some stochastic dynamical models in neurobiology: recent developments*, European Communications in Mathematical and Theoretical Biology, 14:44–53, 2011.

[5] S. Albeverio, L. Di Persio, and E. Mastrogiacomo, *Invariant measures for stochastic differential equations on networks*, Proocedings of Symp. in Pure Mathematics, AMS, H. Holden et all.(ed.), 87:1–34, 2013.

[6] S. Albeverio, L. Di Persio, and E. Mastrogiacomo, *Small noise asymptotic expansions for stochastic PDE's I. The case of a dissipative polynomially bounded nonlinearity*, Tohôku. Math. J., 63:877–898, 2011.

[7] S. Albeverio, L. Di Persio, E. Mastrogiacomo, and B. Smii, *A class of Lévy driven SDEs and their explicit invariant measures*, Pot. Anal., 45:229–259, 2016.

[8] S. Albeverio and B. Ferrario, *Some methods of infinite dimensional analysis in hydrodynamics: recent progress and prospects*, Lecture Notes in Math., Springer, Berlin, 1942:1-50, 2008.

[9] S. Albeverio, F. Flandoli, and Y.G. Sinai, *SPDE in hydrodynamic: recent progress and prospects*, Lectures given at the C.I.M.E. Summer School held in Cetraro, G. Da Prato and M. Röckner (eds.), August 29–September 3, 2005, Lecture Notes in Mathematics, 1942, Springer-Verlag, Berlin, 2008.

[10] S. Albeverio, H. Gottschalk, and M.W. Yoshida, *Systems of classical particles in the grand canonical ensemble, scaling limits and quantum field theory*, Rev. Math. Phys., 17(2):175–226, 2005.

[11] S. Albeverio, H. Kawabi, S. Mihalache, and M. Röckner, paper in preparation.

[12] S. Albeverio, H. Gottschalk, and J-L. Wu, *Convoluted Generalized White noise, Schwinger Functions and their Analytic continuation to Wightman Functions*, Rev. Math. Phys., 8(6):763–817, 1996.

[13] S. Albeverio, H. Gottschalk, and M.W. Yoshida, *System of classical particles in the grand canonical ensemble, scaling limits and quantum field theory*, Rev. Math. Phys., 17(02):175–226, 2005.

[14] S. Albeverio and R. Hϕegh-Krohn, *Quasi invariant measures, symmetric diffusion processes and quantum fields*, Les Méthodes Mathématiques de la Théorie Quantique des Champs, Colloq. Internat. CNRS, Marseille, 248:11C59, 1975.

[15] S. Albeverio and R. Hϕegh-Krohn, *Dirichlet forms and diffusion processes on rigged Hilbert spaces*, Z. Wahr. Theor. Verw. Geb., 40:1–57, 1977.

[16] S. Albeverio, T. Hida, J. Potthoff, M. Röckner, and L. Streit, *Dirichlet forms in terms of white noise analysis. I. Construction and QFT examples*, Rev. Math. Phys., 1(2-3):291–312, 1989.

[17] S. Albeverio, H. Kawabi, and M. Röckner, *Strong uniqueness for both Dirichlet operators and stochastic dynamics to Gibbs measures on a path space with exponential interactions*, J.

Funct. Anal., 262(2):602–638, 2012.
[18] S. Albeverio and S. Liang, *Asymptotic expansions for the Laplace approximations of sums of Banach space-valued random variables*, Ann. Probab., 33:300–336, 2005.
[19] S. Albeverio, E. Lytvynov, and A. Mahnig, *A model of the term structure of interest rates based on Lévy fields*, Stoch. Proc. Appl., 114:251–263, 2004.
[20] S. Albeverio, V. Mandrekar, and B. Rüdiger, *Existence of mild solutions for stochastic differential equations and semilinear equations with non-Gaussian Lévy noise*, Stoch. Proc. Appl., 119(3):835–863, 2009.
[21] S. Albeverio, E. Mastrogiacomo, and B. Smii, *Small noise asymptotic expansions for stochastic PDE's driven by dissipative nonlinearity and Lévy noise*, Stoch. Proc. Appl., 123:2084–2109, 2013.
[22] S. Albeverio and S. Mazzucchi, *The trace formula for the heat semigroup with polynomial potential*, Proc. Seminar Stochastic Analysis, Random Fields and Applications VI, Ascona 2008, R. Dalang, M. Dozzi, F. Russo (eds.) Birkhäuser, Basel, 3–22, 2011.
[23] S. Albeverio and M. Röckner, *Stochastic differential equations in infinite dimensions: solutions via Dirichlet forms*, Prob. Theo. Relat. Fields, 89:347–386, 1991.
[24] S. Albeverio, H. Röckle, and V. Steblovskaya, *Asymptotic expansions for Ornstein-Uhlenbeck semigroups perturbed by potentials over Banach spaces*, Stochastics Stochastics Rep., 69:195–238, 2000.
[25] S. Albeverio and B. Rüdiger, *Stochastic integrals and the Lévy–Itô decomposition theorem on separable Banach spaces*, Stoch. Anal. Appl., 23(2):217–253, 2005.
[26] S. Albeverio, B. Rüdiger, and J.L. Wu, *Invariant measures and symmetry property of Lévy type operators*, Pot. Ana., 13:147–168, 2000.
[27] S. Albeverio and B. Smii, *Asymptotic expansions for SDE's with small multiplicative noise*, Stoch. Proc. Appl., 125:1009–1031, 2014.
[28] S. Albeverio and V. Steblovskaya, *Asymptotics of infinite-dimensional integrals with respect to smooth measures. I*, Infin. Dimens. Anal. Quantum Prob. Relat. Top., 2:529–556, 1999.
[29] S. Albeverio, J.L. Wu, and T.S. Zhang, *Parabolic SPDEs driven by Poisson white noise*, Stoch. Proc. Appl., 74:21–36, 1998.
[30] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Second Edition, Cambridge U.P, 2009.
[31] D. Applebaum and J.L. Wu, *Stochastic partial differential equations driven by Lévy space time white noise*, Random Ops. and Stochastic Equations, 8:245–61, 2000.
[32] V. Barbu, F. Cordoni, and L. Di Persio, *Optimal control of stochastic FitzHugh–Nagumo equation*, International J. of Control, 89(4):746–756, 2016.
[33] H. Bauer, *Probability Theory*, Integration Theory, Springer Berlin, 2001.
[34] V.I. Bogachev, M. Röckner, and B. Schmuland , *Generalized Mehler semigroups and applications*, Probability Theory and Related Fields, 105(2):193–225, 1996.
[35] S. Bonaccorsi, C. Marinelli, and G. Ziglio, *Stochastic FitzHugh–Nagumo equations on networks with impulsive noise*, Electr. J. Prob., 13:1362–1379, 2008.
[36] S. Bonaccorsi and E. Mastrogiacomo, *Analysis of the stochastic FitzHugh-Nagumo system*, Infin. Dimens. Anal. Quantum Probab. Relat. Top., 11:427–446, 2008.
[37] Z. Brzeźniak and E. Hausenblas, *Uniqueness in law of the Itô integral with respect to Lévy noise*, in Seminar Stoch. Anal., Random Fields and Appl., Birkhauser, Basel VI, 37–57, 2011.
[38] Z. Brzezńiak and S. Peszat, *Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process*, Studia Mathematica, 137(3):261–299, 1999.
[39] R.A. Carmona and M.R. Tehranchi, *Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective*, Springer Finance, 2006.
[40] S. Cerrai, *Differentiability of Markov semigroups for stochastic reaction-diffusion equations and applications to control*, Stoch. Proc. Appl., 83(1):15–37, 1999.
[41] S. Cerrai and M. Freidlin, *Smoluchowski-Kramers approximation for a general class of SPDEs*, J. Evol. Equ., 6(4):657–689, 2006.
[42] A. Chojnowska-Michalik, and B. Goldys, *Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces*, Probab. Theory Related Fields, 102(3):331–356, 1995.
[43] F. Cordoni and L. Di Persio, *Invariant measure for the Vasicek interest rate model in the Heath-Jarrow-Morton-Musiela framework*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, 18(3), 1550022, 2015.
[44] G. Da Prato and A. Debussche, *Strong solutions to the stochastic quantization equations*, Ann. Probab., 31(4):1900–1916, 2003.
[45] G. Da Prato and L. Tubaro, *Self-adjointness of some infinite-dimensional elliptic operators and application to stochastic quantization*, Probab. Theory Related Fields, 118(1):131–145, 2000.
[46] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, London Mathematical

Society Lecture Note Series, Cambridge University Press, 229, 1996.

[47] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and its Applications, 44:xviii+454, 1992.

[48] R.C. Dalang and C. Mueller, *Some non-linear SPDE's that are second order in time*, Electronic J. Probab., 8(1):1–21, 2003.

[49] O. Fehmi and T. Schmidt, *Credit risk with infinite dimensional Lévy processes*, Stat. and Dec., 23:281–299, 2005.

[50] B. Forster, E. Lütkebohmert, and J. Teichmann, *Absolutely continuous laws of jump-diffusions in finite and infinite dimensions with applications to mathematical finance*, SIAM J. Math. Anal., 40(5):2132–2153, 2008/09.

[51] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Second revised and extended edition, de Gruter Studies in Mathematics, 19. Walter de Gruyter and Co., Bertlin, 2011.

[52] L. Gawarecki and V. Mandrekar, *Stochastic Differential Equations in Infinite Dimensions: with Applications*, Stochastic Partial Differential Equations, Springer, 2010.

[53] H. Gottschalk, B. Smii, and H. Thaler, *The Feynman graph representation of general convolution semigroups and its applications to Lévy statistics*, J. Bern. Soc., 14(2):322–351, 2008.

[54] H. Gottschalk and B. Smii, *How to determine the law of the solution to a SPDE driven by a Lévy space-time noise*, J. Math. Phys., 43:1–22, 2007.

[55] M. Hairer, *A theory of regularity structures*, Inventiones Mathematicae, 198(2):269–504, 2014.

[56] E. Hausenblas, *Burkholder–Davis–Gundy type inequalities of the Itô stochastic integral with respect to Lévy noise on Banach spaces*, arXiv:0902.2114 [math.PR], 2009.

[57] H. Holden, B. Oksendal, J. Uböe, and T. Zhang, *Stochastic partial differential equations. A modeling, white noise functional approach*, Second Edition, Universitext. Springer, New York, 2010.

[58] G. Jona-Lasinio and P.K. Mitter, *Large deviation estimates in the stochastic quantization of* ϕ_2^4, Comm. Math. Phys., 130(1):111–121, 1990.

[59] G. Jona-Lasinio and P.K. Mitter, *On the stochastic quantization of field theory*, Comm. Math. Phys., 101(3):409–436, 1985.

[60] G. Kallianpur and R.L. Wolpert, *Weak convergence of stochastic neuronal models*, in Stochastic Methods in Biology, (Nagoya, 1985), Lecture Notes in Biomathematics, Springer, Berlin, 70:116–145, 1987.

[61] P. Kotelenez, *Stochastic ordinary and stochastic partial differential equations: transition from microscopic to macroscopic equations*, Springer Science & Business Media, 58, 2007.

[62] G. Kallianpur and J. Xiong, *Stochastic Differential Equations on Infinite Dimensional Spaces*, IMS Lecture notes-monograph series, 26, 1995.

[63] K. Lehnertz, J. Arnhold, P. Grassberger, and C.E. Elger, *Chaos in Brain*, World Scientific, Singapore, 2000.

[64] A. Lökka, B. Oksendal, and F. Proske, *Stochastic partial differential equations driven by Lévy space-time white noise*, Ann. Appl. Prob., 14:1506–1528, 2004.

[65] V. Mandrekar and B. Rüdiger, *Lévy noises and stochastic integrals on Banach spaces*, Stochastic Partial Differential Equations and Applications VII, Lect. Notes Pure Appl. Math., Chapman & Hall/CRC, Boca Raton, FL, 245:193–213, 2006.

[66] V. Mandrekar and B. Rüdiger, *Stochastic Integration in Banach Spaces*, Theory and Applications. Springer, 2015.

[67] R. Marcus, *Parabolic Itô equations with monotone nonlinearities*, J. Funct. Anal., 29(3):275–286, 1978.

[68] Z.M. Ma and M. Röckner, *Introduction to the Theory of (Nonsymmetric) Dirichlet Forms*, Universitext. Springer-Verlag, Berlin, vi+209, 1992. ISBN:3-540-55848-9

[69] C. Marinelli,*Local well-posedness of Musiela's SPDE with Lévy noise*, Math. Finance, 20(3):341–363, 2010.

[70] C. Marinelli, L. Di Persio, and G. Ziglio, *Approximation and convergence of solutions to semilinear stochastic evolution equations with jumps*, J. Fun. Anal., 264(12):2784–2816, 2013.

[71] C. Marinelli and M. Röckner, *Uniqueness of mild solutions for dissipative stochastic reaction-diffusion equations with multiplicative Poisson noise*, Electron. J. Prob., 15:1528–1555, 2010.

[72] T. Meyer-Brandis and F. Proske, *Explicit representation of strong solutions of SDEs driven by infinite dimensional Lévy processes*, J. Theor. Prob., 23:301–314, 2010.

[73] R. Mikulevicius and B. Rozovskii, *Linear parabolic stochastic PDEs and Wiener chaos*, SIAM J. Math. Anal., 29(2):452–480, 1998.

[74] Sanjoy K. Mitter, *Stochastic quantization. Modeling and control of systems in engineering, quantum mechanics*, Economics and Biosciences (Sophia–Antipolis, 1988), Lecture Notes in Control and Inform. Sci., Springer, Berlin, 121:151–159, 1989.

[75] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 113, 2007.
[76] Enrico Priola and Jerzy Zabczyk, *On linear evolution equations for a class of cylindrical Lévy noises*, *Stochastic partial differential equations and applications*, Quad. Mat., Dept. Math., Seconda Univ. Napoli, Caserta, 25:223–242, 2010.
[77] C. Prévôt and Röckner Michael, *A Concise Course on Stochastic Partial Differential Equations*, Springer Berlin Heidelberg, 2008.
[78] C. Rovira and S. Tindel, *Sharp Laplace asymptotics for a parabolic SPDE*, Stochastics Stochastics Rep., 69(1-2):11–30, 2000.
[79] B Rüdiger and G. Ziglio, *Itô formula for stochastic integrals w.r.t compensated Poisson random measures on separable Banach spaces*, Stochastics, 78:377–410, 2006.
[80] K. Sato, *Lévy Processes and Infinite Divisibility*, Cambridge University Press, 1999.
[81] B. Smii, *A large diffusion expansion for the transition function of Lévy Ornstein–Uhlenbeck processes*, Appl. Math. Inf. Sci., 10(4):1–8, 2016.
[82] H.C. Tuckwell, *Analytical and simulation results for the stochastic spatial FitzHugh–Nagumo model neuron*, Neural Comput., 20(12), 3003–3033, 2008.
[83] H.C. Tuckwell, *Introduction to Theoretical Neurobiology. Vol. 1, Linear Cable theory and Dendritic Structure*, Cambridge Studies in Mathematical Biology, Cambridge University Press, Cambridge, 8:xii+291, 1988.
[84] H.C. Tuckwell, *Introduction to Theoretical Neurobiology. Vol. 2, Nonlinear and Stochastic Theories*, Cambridge Studies in Mathematical Biology, Cambridge University Press, Cambridge, 8:xii+265, 1988.
[85] H.C. Tuckwell, *Random perturbations of the reduced FitzHugh–Nagumo equation*, Phys. Scripta, 46(6):481–484, 1992.
[86] H.C. Tuckwell and J. Jost, *Moment analysis of the Hodgkin–Huxley system with additive noise*, Physica A, 388:4115–4125, 2009.
[87] H.C. Tuckwell, J. Jost, and B.S. Gutkin, *Inhibition and modulation of rhythmic neuronal spiking by noise*, Physical Review E, 80(3):031907, 2009.
[88] J.B. Walsh, *An introduction to stochastic partial differentail equations*, iIn Ecole d'été de probabilités de Saint-Flour, XIV-1984, Lecture Notes in Mathematics, Springer, Berlin, 1180:265–439, 1986.
[89] J. Zabczyk, *Symmetric solution of semilinear stochastic equations*, Proceedings of a Conference on Stochastic Partial Differential Equations, Trento, Italy, 1987, Lecture Notes in Mathematics, 1390:237–256, 1989.