

LOCAL WELL-POSEDNESS FOR THE BÉNARD CONVECTION WITHOUT SURFACE TENSION*

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Abstract. We consider the Bénard convection in a three-dimensional domain bounded below by a fixed flatten boundary and above by a free moving surface. The domain is horizontally periodic. The fluid dynamics are governed by the Boussinesq approximation and the effect of surface tension is neglected on the free surface. Here we develop a local well-posedness theory for the equations of general case in the framework of the nonlinear energy method.

Key words. free boundary problems; incompressible viscous fluids; Bénard convection.

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1. Introduction

1.1. Formulation of the problem. In this paper, we consider the Bénard convection in a shallow horizontal layer of a fluid heated from below evolving in a moving domain

$$\Omega(t) = \{y \in \Sigma \times \mathbb{R} \mid -1 < y_3 < \eta(y_1, y_2, t)\}.$$

Here we assume that $\Sigma = (L_1\mathbb{T}) \times (L_2\mathbb{T})$ for $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the usual 1-torus and $L_1, L_2 > 0$ the periodicity lengths. Assuming the Boussinesq approximation [4], we obtain the basic hydrodynamic equations governing Bénard convection as

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \frac{1}{\rho_0} \nabla p &= \nu \Delta u + g\alpha\theta e_{y_3}, \quad \text{in } \Omega(t), \\ \partial_t \theta + u \cdot \nabla \theta &= \kappa \Delta \theta, \quad \text{in } \Omega(t), \\ u|_{t=0} &= u_0(y_1, y_2, y_3), \quad \theta|_{t=0} = \theta_0(y_1, y_2, y_3). \end{aligned}$$

Here, $u = (u_1, u_2, u_3)$ is the velocity field of the fluid satisfying $\operatorname{div} u = 0$, p the pressure, $g > 0$ the strength of gravity, $\nu > 0$ the kinematic viscosity, α the thermal expansion coefficient, $e_{y_3} = (0, 0, 1)$ the unit upward vector, θ the temperature field of the fluid, κ the thermal diffusivity coefficient, and ρ_0 the density at the temperature T_0 . Notice that we have made the shift of actual pressure \bar{p} by $p = \bar{p} + gy_3 - p_{atm}$ with the constant atmosphere pressure p_{atm} .

The boundary condition is

$$\begin{aligned} \partial_t \eta - u' \cdot \nabla \eta + u_3 &= 0, \quad \text{on } \{y_3 = \eta(t, y_1, y_2)\}, \\ (pI - \nu \mathbb{D}(u))n &= g\eta n + \sigma Hn + (\mathbf{t} \cdot \nabla)\sigma \mathbf{t}, \quad \text{on } \{y_3 = \eta(t, y_1, y_2)\}, \\ n \cdot \nabla \theta + Bi\theta &= -1, \quad \text{on } \{y_3 = \eta(t, y_1, y_2)\}, \\ u|_{y_3=-1} &= 0, \quad \theta|_{y_3=-1} = 0. \end{aligned}$$

Here, $u' = (u_1, u_2)$, I the 3×3 identity matrix, $\mathbb{D}(u)_{ij} = \partial_i u_j + \partial_j u_i$ the symmetric gradient of u , \mathcal{N} the upward normal vector of the free surface $\{y_3 = \eta\}$, $n = \mathcal{N}/|\mathcal{N}|$ the

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unit upward normal vector of the free surface $\{y_3 = \eta\}$ where $\mathcal{N} = (-\partial_1\eta, -\partial_2\eta, 1)$ is the upward normal vector of the free surface $\{y_3 = \eta\}$ and $|\mathcal{N}| = \sqrt{(\partial_1\eta)^2 + (\partial_2\eta)^2 + 1}$, \mathbf{t} the unit tangential vector of the free surface, $Bi \geq 0$ the Biot number and H the mean curvature of the free surface. For simplicity, we only consider the case without surface tension in this paper, i.e. $\sigma = 0$.

We will always assume the natural condition that there exists a positive number δ_0 such that $1 + \eta_0 \geq \delta_0 > 0$ on Σ , which means that the initial free surface is strictly separated from the bottom. And without loss of generality, we may assume that $\rho_0 = \mu = \kappa = \alpha = g = Bi = 1$. That is, we will consider the equations

$$\left\{ \begin{array}{ll} \partial_t u + u \cdot \nabla u + \nabla p - \Delta u - \theta e_{y_3} = 0 & \text{in } \Omega(t), \\ \operatorname{div} u = 0 & \text{in } \Omega(t), \\ \partial_t \theta + u \cdot \nabla \theta - \Delta \theta = 0 & \text{in } \Omega(t), \\ (pI - \mathbb{D}u)n = \eta n & \text{on } \{y_3 = \eta(t, y_1, y_2)\}, \\ \nabla \theta \cdot n + \theta = -1 & \text{on } \{y_3 = \eta(t, y_1, y_2)\}, \\ u = 0, \quad \theta = 0 & \text{on } \{y_3 = -1\}, \\ u|_{t=0} = u_0, \quad \theta|_{t=0} = \theta_0 & \text{in } \Omega(0), \\ \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \{y_3 = \eta(t, y_1, y_2)\}, \\ \eta|_{t=0} = \eta_0 & \text{on } \{y_3 = \eta(t, y_1, y_2)\}. \end{array} \right. \quad (1.1)$$

The discussion of fourth equation in the system (1.1) may be found in [14]. The eighth equation in the system (1.1) implies that the free surface is advected with the fluid.

1.2. Previous results. Traditionally, the Bénard convection problem has been studied in fixed upper boundary and in free boundary surface with surface tension.

For the problem with surface tension case, the existence and decay of global in-time solutions of the Bénard convection problem with free boundary surface was proved by T. Iohara, T. Nishida and Y. Teramoto in L^2 spaces. T. Iohara proved this in 2-D setting. T. Nishida and Y. Teramoto proved this in the 3-D background. They all utilized the framework of [3] in the Lagrangian coordinates.

1.3. Geometrical formulation. In the absence of surface tension effect, we will solve this problem in Eulerian coordinates. First, we straighten the time dependent domain $\Omega(t)$ to a time independent domain Ω . The idea was introduced by J. T. Beale in Section 5 of [3]. And in [6, 7] and [8], Y. Guo and I. Tice proved the local and global existence results for the incompressible Navier–Stokes equations with a deformable surface using this idea. In [6, 7] and [8], Guo and Tice assume that the surface function η in some norms is small, which means η is a small perturbation for the plane $\{y_3 = 0\}$. In order to study the free boundary problem of the incompressible Navier–Stokes equations with a general surface function η , L. Wu introduced the ε -Poisson integral method in [13]. In this paper, we will use the flattening transformation method introduced by L. Wu. We define $\bar{\eta}^\varepsilon$ by

$$\bar{\eta}^\varepsilon = \mathcal{P}^\varepsilon \eta = \text{the parametrized harmonic extension of } \eta.$$

The definition of $\mathcal{P}^\varepsilon \eta$ can be seen in the Section 1.3.1 of [13] for the periodic case. We introduce the mapping Φ^ε from Ω to $\Omega(t)$ as

$$\Phi^\varepsilon : (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 + (1 + x_3)\bar{\eta}^\varepsilon) = (y_1, y_2, y_3), \quad (1.2)$$

and its Jacobian matrix

$$\nabla\Phi^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ A^\varepsilon & B^\varepsilon & J^\varepsilon \end{pmatrix},$$

and the transform matrix

$$\mathcal{A}^\varepsilon = ((\nabla\Phi^\varepsilon)^{-1})^\top = \begin{pmatrix} 1 & 0 & -A^\varepsilon K^\varepsilon \\ 0 & 1 & -B^\varepsilon K^\varepsilon \\ 0 & 0 & K^\varepsilon \end{pmatrix},$$

where

$$A^\varepsilon = (1+x_3)\partial_1\bar{\eta}^\varepsilon, B^\varepsilon = (1+x_3)\partial_2\bar{\eta}^\varepsilon, J^\varepsilon = 1+\bar{\eta}^\varepsilon + (1+x_3)\partial_3\bar{\eta}^\varepsilon, K^\varepsilon = 1/J^\varepsilon. \tag{1.3}$$

According to Theorem 2.7 in [13] and the assumption that $1+\eta_0 > \delta_0 > 0$, there exists a $\delta > 0$ such that $J^\varepsilon(0) > \delta > 0$ for a sufficiently small ε depending on $\|\eta_0\|_{H^{5/2}}$. This implies that $\Phi^\varepsilon(0)$ is a homomorphism. Furthermore, $\Phi^\varepsilon(0)$ is a C^1 diffeomorphism deduced from Lemma 2.5 and 2.6 in [13]. For simplicity, in the following, we just write $\bar{\eta}$ instead of $\bar{\eta}^\varepsilon$, while the same fashion applies to \mathcal{A} , Φ , A , B , J and K . Then, we define some transformed operators. The differential operators $\nabla_{\mathcal{A}}$, $\text{div}_{\mathcal{A}}$ and $\Delta_{\mathcal{A}}$ are defined as follows.

$$\begin{aligned} (\nabla_{\mathcal{A}} f)_i &= \mathcal{A}_{ij} \partial_j f, \\ \text{div}_{\mathcal{A}} u &= \mathcal{A}_{ij} \partial_j u_i, \\ \Delta_{\mathcal{A}} f &= \nabla_{\mathcal{A}} \cdot \nabla_{\mathcal{A}} f. \end{aligned}$$

The symmetric \mathcal{A} -gradient $\mathbb{D}_{\mathcal{A}}$ is defined as $(\mathbb{D}_{\mathcal{A}} u)_{ij} = \mathcal{A}_{ik} \partial_k u_j + \mathcal{A}_{jk} \partial_k u_i$. And we write the stress tensor as $S_{\mathcal{A}}(p, u) = pI - \mathbb{D}_{\mathcal{A}} u$, where I is the 3×3 identity matrix. Then we note that $\text{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) = \nabla_{\mathcal{A}} p - \Delta_{\mathcal{A}} u$ for vector fields satisfying $\text{div}_{\mathcal{A}} u = 0$. We have also written $\mathcal{N} = (-\partial_1 \eta, -\partial_2 \eta, 1)$ for the nonunit normal to $\{y_3 = \eta(y_1, y_2, t)\}$. Then the original Equation (1.1) becomes

$$\begin{cases} \partial_t u - \partial_t \bar{\eta}(1+x_3)K \partial_3 u + u \cdot \nabla_{\mathcal{A}} u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p - \theta \nabla_{\mathcal{A}} y_3 = 0 & \text{in } \Omega, \\ \nabla_{\mathcal{A}} \cdot u = 0 & \text{in } \Omega, \\ \partial_t \theta - \partial_t \bar{\eta}(1+x_3)K \partial_3 \theta + u \cdot \nabla_{\mathcal{A}} \theta - \Delta_{\mathcal{A}} \theta = 0 & \text{in } \Omega, \\ (pI - \mathbb{D}_{\mathcal{A}} u) \cdot \mathcal{N} = \eta \mathcal{N} & \text{on } \Sigma, \\ \nabla_{\mathcal{A}} \theta \cdot \mathcal{N} + \theta |\mathcal{N}| = -|\mathcal{N}| & \text{on } \Sigma, \\ u = 0, \quad \theta = 0 & \text{on } \Sigma_b, \\ u(x, 0) = u_0, \quad \theta(x, 0) = \theta_0 & \text{in } \Omega, \\ \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma, \\ \eta(x', 0) = \eta_0(x') & \text{on } \Sigma, \end{cases} \tag{1.4}$$

where $e_3 = (0, 0, 1)$ and we can split the Equation (1.4) into a equation governing Bénard convection and a transport equation, i.e.

$$\left\{ \begin{array}{ll} \partial_t u - \partial_t \bar{\eta}(1+x_3)K\partial_3 u + u \cdot \nabla_{\mathcal{A}} u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p - \theta \nabla_{\mathcal{A}} y_3 = 0 & \text{in } \Omega, \\ \nabla_{\mathcal{A}} \cdot u = 0 & \text{in } \Omega, \\ \partial_t \theta - \partial_t \bar{\eta}(1+x_3)K\partial_3 \theta + u \cdot \nabla_{\mathcal{A}} \theta - \Delta_{\mathcal{A}} \theta = 0 & \text{in } \Omega, \\ (pI - \mathbb{D}_{\mathcal{A}} u) \cdot \mathcal{N} = \eta \mathcal{N} & \text{on } \Sigma, \\ \nabla_{\mathcal{A}} \theta \cdot \mathcal{N} + \theta |\mathcal{N}| = -|\mathcal{N}| & \text{on } \Sigma, \\ u = 0, \quad \theta = 0 & \text{on } \Sigma_b, \\ u(x, 0) = u_0, \quad \theta(x, 0) = \theta_0 & \text{in } \Omega, \end{array} \right. \tag{1.5}$$

and

$$\left\{ \begin{array}{ll} \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma, \\ \eta(x', 0) = \eta_0(x') & \text{on } \Sigma. \end{array} \right. \tag{1.6}$$

Clearly, all the quantities in these two above systems are related to η .

1.4. Main theorem. The main result of this paper is the local well-posedness of the Bénard convection. Before stating our result, we need to mention the issue of compatibility conditions for the initial data (u_0, θ_0, η_0) . We will study regularity for up to N temporal derivatives, for $N \geq 2$ an integer. This requires us to use u_0, θ_0 and η_0 to construct the initial data $\partial_t^j u(0), \partial_t^j \theta(0)$ and $\partial_t^j \eta(0)$ for $j=1, \dots, N$ and $\partial_t^j p(0)$ for $j=0, \dots, N-1$. These data must then satisfy various conditions, which we describe in detail in Section 5.1, and thus will not state here.

Now for stating our result, we need to explain the notation for spaces and norms. When we write $\|\partial_t^j u\|_{H^k}, \|\partial_t^j \theta\|_{H^k}$ and $\|\partial_t^j p\|_{H^k}$, we always mean that the space is $H^k(\Omega)$, and when we write $\|\partial_t^j \eta\|_{H^s}$, we always mean that the space is $H^s(\Sigma)$, where $H^k(\Omega)$ and $H^s(\Sigma)$ are the usual Sobolev spaces for $k, s \geq 0$.

THEOREM 1.1. *Let $N \geq 2$ be an integer. Assume that $\eta_0 + 1 \geq \delta > 0$, and that the initial data (u_0, θ_0, η_0) satisfies*

$$\mathcal{E}_0 := \|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \|\eta_0\|_{H^{2N+1/2}}^2 < \infty,$$

as well as the N -th compatibility conditions (5.28). Then there exists a $0 < T_0 < 1$ such that, for any $0 < T < T_0$, there exists a solution (u, p, θ, η) to Equation (1.4) on the interval $[0, T]$ that achieves the initial data. The solution obeys the estimate

$$\begin{aligned} & \sum_{j=0}^N \left(\sup_{0 \leq t \leq T} \|\partial_t^j u\|_{H^{2N-2j}}^2 + \|\partial_t^j u\|_{L^2 H^{2N-2j+1}}^2 \right) + \|\partial_t^{N+1} u\|_{(\mathcal{X}_T)^*} \\ & + \sum_{j=0}^{N-1} \left(\sup_{0 \leq t \leq T} \|\partial_t^j p\|_{H^{2N-2j-1}}^2 + \|\partial_t^j p\|_{L^2 H^{2N-2j}}^2 \right) \\ & + \sum_{j=0}^N \left(\sup_{0 \leq t \leq T} \|\partial_t^j \theta\|_{H^{2N-2j}}^2 + \|\partial_t^j \theta\|_{L^2 H^{2N-2j+1}}^2 \right) + \|\partial_t^{N+1} \theta\|_{(\mathcal{X}_T^*)^*} \\ & + \left(\sup_{0 \leq t \leq T} \|\eta\|_{H^{2N+1/2}(\Sigma)}^2 + \sum_{j=1}^N \sup_{0 \leq t \leq T} \|\partial_t^j \eta\|_{H^{2N-2j+3/2}}^2 + \sum_{j=2}^{N+1} \|\partial_t^j \eta\|_{L^2 H^{2N-2j+5/2}}^2 \right) \\ & \leq C(\Omega_0, \delta) P(\mathcal{E}_0), \end{aligned} \tag{1.7}$$

where $C(\Omega_0, \delta) > 0$ depends on the initial domain Ω_0 and δ , $P(\cdot)$ is a polynomial satisfying $P(0) = 0$, and the temporal norm L^2 is computed on $[0, T]$. The solution is unique among functions that achieve the initial data and for which the left-hand side of the inequality (1.7) is finite. Moreover, η is such that the mapping $\Phi(\cdot, t)$ defined by the expression (1.2) is a C^{2N-1} diffeomorphism for each $t \in [0, T]$.

REMARK 1.1. The space \mathcal{X}_T is defined in section 2 of [6].

REMARK 1.2. Since the mapping $\Phi(\cdot, t)$ is a C^{2N-1} diffeomorphism, we may change coordinates to produce solutions to Equation (1.1).

1.5. Notation and terminology. Now, we mention some definitions, notation and conventions that we will use throughout this paper.

- (1) Constants. The constant $C > 0$ will denote a universal constant that only depends on the parameters of the problem, N and Ω , but does not depend on the data, etc. They are allowed to change from line to line. We will write $C = C(z)$ to indicate that the constant C depends on z . And we will write $a \lesssim b$ to mean that $a \leq Cb$ for a universal constant $C > 0$.
- (2) Polynomials. We will write $P(\cdot)$ to denote polynomials in one variable, and they may change from one inequality or equality to another.
- (3) Norms. We will write H^k for $H^k(\Omega)$ for $k \geq 0$, and $H^s(\Sigma)$ with $s \in \mathbb{R}$ for usual Sobolev spaces. Typically, we will write $H^0 = L^2$, with the exception of using $L^2([0, T]; H^k)$ (or $L^2([0, T]; H^s(\Sigma))$) to denote the space of temporal square-integrable functions with values in H^k (or $H^s(\Sigma)$).

Sometimes we will write $\|\cdot\|_k$ instead of $\|\cdot\|_{H^k(\Omega)}$ or $\|\cdot\|_{H^k(\Sigma)}$. We assume that functions have natural spaces. For example, the functions u, p, θ and $\bar{\eta}$ live on Ω , while η lives on Σ . So we may write $\|\cdot\|_{H^k}$ for the norms of u, p, θ and $\bar{\eta}$ in Ω , and $\|\cdot\|_{H^s}$ for norms of η on Σ .

1.6. Plan of the paper. In Section 2, we develop the machinery of time-dependent function spaces based on [6]. In Section 3, we make some elliptic estimates for the linear steady equations of the system (1.8). In Section 4, we will study the local existence theory of the following linear problem for (u, p, θ) , where we think of η (and hence \mathcal{A}, \mathcal{N} , etc.) as given:

$$\begin{cases} \partial_t u - \Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} p - \theta \nabla_{\mathcal{A}} y_3 = F^1 & \text{in } \Omega, \\ \nabla_{\mathcal{A}} \cdot u = 0 & \text{in } \Omega, \\ \partial_t \theta - \Delta_{\mathcal{A}} \theta = F^3 & \text{in } \Omega, \\ (pI - \mathbb{D}_{\mathcal{A}} u) \cdot \mathcal{N} = F^4 & \text{on } \Sigma, \\ \nabla_{\mathcal{A}} \theta \cdot \mathcal{N} + \theta |\mathcal{N}| = F^5 & \text{on } \Sigma, \\ u = 0, \quad \theta = 0 & \text{on } \Sigma_b, \end{cases} \tag{1.8}$$

subject to the initial condition $u(0) = u_0$ and $\theta(0) = \theta_0$, with the time-dependent Galerkin method. In Section 5, we construct the initial data and do some estimates for the forcing terms. In Section 6, we construct solutions to the system (1.4) using iteration and contraction, and complete the proof of Theorem 1.1.

2. Functional setting

2.1. Function spaces. Throughout this paper, we utilize the functional spaces defined by Guo and Tice in Section 2 of [6]. The only modification is the definition of space $\mathcal{H}^1(t)$. For the vector-valued space $\mathcal{H}^1(t)$, its definition is the same as in [6]. The following is the definition for the scalar-valued space $\mathcal{H}^1(t)$:

$$\mathcal{H}^1(t) := \{\theta \mid \|\theta\|_{\mathcal{H}^1} < \infty, \theta|_{\Sigma_b} = 0\}$$

with the norm $\|\theta\|_{\mathcal{H}^1} := (\theta, \theta)_{\mathcal{H}^1}^{1/2}$, where the inner product $(\cdot, \cdot)_{\mathcal{H}^1}$ is defined as

$$(\theta, \phi)_{\mathcal{H}^1} := \int_{\Omega} (\nabla_{\mathcal{A}(t)} \theta \cdot \nabla_{\mathcal{A}(t)} \phi) J(t).$$

The following lemma implies that this space \mathcal{H}^1 is equivalent to the usual Sobolev space H^1 .

LEMMA 2.1. *Suppose that $0 < \varepsilon_0 < 1$ and $\|\eta - \eta_0\|_{H^{5/2}(\Sigma)} < \varepsilon_0$. Then it holds that*

$$\|\theta\|_{H^0}^2 \lesssim \int_{\Omega} J|\theta|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)}) \|\theta\|_{H^0}^2, \tag{2.1}$$

$$\frac{1}{(1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3} \|\theta\|_{H^1(\Omega)}^2 \lesssim \int_{\Omega} J|\nabla_{\mathcal{A}} \theta|^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}(\Sigma)})^3 \|\theta\|_{H^1(\Omega)}^2. \tag{2.2}$$

Proof. From the Poincaré inequality, we know that $\|\theta\|_{H^1}$ is equivalent to $\|\nabla \theta\|_{H^0}$. So in the following, we will use $\|\theta\|_{H^1}$ instead of $\|\nabla \theta\|_{H^0}$.

From the assumption and the Sobolev inequalities, we may derive that

$$\delta \lesssim \|J\|_{L^\infty} \lesssim 1 + \|\bar{\eta}\|_{L^\infty} + \|\nabla \bar{\eta}\|_{L^\infty} \lesssim 1 + \|\eta\|_{H^{5/2}} \lesssim 1 + \|\eta_0\|_{H^{5/2}},$$

and

$$\begin{aligned} \|\mathcal{A}\|_{L^\infty} &\lesssim \max\{1, \|AK\|_{L^\infty}^2, \|BK\|_{L^\infty}^2, \|K\|_{L^\infty}^2\} \\ &\lesssim 1 + (1 + \|\nabla \bar{\eta}\|_{L^\infty}^2) \|K\|_{L^\infty}^2 \lesssim (1 + \|\eta_0\|_{H^{5/2}})^2. \end{aligned}$$

Thus, the inequality (2.1) is clearly derived from the estimate of $\|J\|_{L^\infty}$ and we have that

$$\begin{aligned} \int_{\Omega} J|\nabla_{\mathcal{A}} \theta|^2 &\lesssim (1 + \|\eta_0\|_{H^{5/2}}) \int_{\Omega} |\nabla_{\mathcal{A}} \theta|^2 \\ &\lesssim (1 + \|\eta_0\|_{H^{5/2}}) \max\{1, \|AK\|_{L^\infty}^2, \|BK\|_{L^\infty}^2, \|K\|_{L^\infty}^2\} \|\theta\|_{H^1}^2 \\ &\lesssim (1 + \|\eta_0\|_{H^{5/2}})^3 \|\theta\|_{H^1}^2. \end{aligned}$$

Now we have proved the second of the inequalities (2.2).

To prove the first inequality, we rewrite the $\|\theta\|_{\mathcal{H}^1}$ as

$$\int_{\Omega} J|\nabla_{\mathcal{A}} \theta|^2 = \int_{\Omega} J|\nabla_{\mathcal{A}_0} \theta|^2 + \int_{\Omega} J(\nabla_{\mathcal{A}} \theta + \nabla_{\mathcal{A}_0} \theta) \cdot (\nabla_{\mathcal{A}} \theta - \nabla_{\mathcal{A}_0} \theta),$$

Here \mathcal{A}_0 is in terms of η_0 . By the estimates of $\|J\|_{L^\infty}$, we derive that

$$\int_{\Omega} J|\nabla_{\mathcal{A}_0} \theta|^2 \gtrsim \frac{1}{1 + \|\eta_0\|_{H^{5/2}}} \int_{\Omega} J_0 |\nabla_{\mathcal{A}_0} \theta|^2$$

$$\begin{aligned}
 &= \frac{1}{1 + \|\eta_0\|_{H^{5/2}}} \int_{\Omega_0} |\nabla(\theta \circ \Phi(0))|^2 \\
 &\gtrsim \frac{1}{(1 + \|\eta_0\|_{H^{5/2}})^3} \|\theta\|_{H^1},
 \end{aligned}$$

where in the last inequality, we have used the following Lemma 3.1, since $\Phi(0)$ is a diffeomorphism. Here J_0 is in terms of η_0 . Then, using the estimates of $\|\mathcal{A}\|_{L^\infty}$ and $\|J\|_{L^\infty}$, we have that

$$\begin{aligned}
 \left| \int_{\Omega} J(\nabla_{\mathcal{A}}\theta + \nabla_{\mathcal{A}_0}\theta) \cdot (\nabla_{\mathcal{A}}\theta - \nabla_{\mathcal{A}_0}\theta) \right| &\lesssim \|J\|_{L^\infty} \|\mathcal{A} + \mathcal{A}_0\|_{L^\infty} \|\mathcal{A} - \mathcal{A}_0\|_{L^\infty} \|\theta\|_{H^1} \\
 &\lesssim \varepsilon_0 (1 + \|\eta_0\|_{H^{5/2}})^3 \|\theta\|_{H^1}.
 \end{aligned}$$

Then taking ε_0 sufficiently small, we may derive that

$$\begin{aligned}
 \int_{\Omega} J|\nabla_{\mathcal{A}}\theta|^2 &\gtrsim \int_{\Omega} J|\nabla_{\mathcal{A}_0}\theta|^2 - \left| \int_{\Omega} J(\nabla_{\mathcal{A}}\theta + \nabla_{\mathcal{A}_0}\theta) \cdot (\nabla_{\mathcal{A}}\theta - \nabla_{\mathcal{A}_0}\theta) \right| \\
 &\gtrsim \frac{1}{(1 + \|\eta_0\|_{H^{5/2}})^3} \|\theta\|_{H^1}.
 \end{aligned}$$

This is the first of the inequalities (2.2). □

We define an operator \mathcal{K}_t by $\mathcal{K}_t\theta = K(t)\theta$, where $K(t) := K$ is defined as in Equation (1.3). Clearly, \mathcal{K}_t is invertible and $\mathcal{K}_t^{-1}\Theta = K(t)^{-1}\Theta = J(t)\Theta$, and $J(t) := J = 1/K$.

PROPOSITION 2.2. *For each $t \in [0, T]$, \mathcal{K}_t is a bounded linear isomorphism: from $H^k(\Omega)$ to $H^k(\Omega)$ for $k = 0, 1, 2$; from $L^2(\Omega)$ to $\mathcal{H}^0(t)$; and from ${}_0H^1(\Omega)$ to $\mathcal{H}^1(t)$. In each case, the norms of the operators $\mathcal{K}_t, \mathcal{K}_t^{-1}$ are bounded by a polynomial $P(\|\eta(t)\|_{H^{\frac{7}{2}}})$. The mapping \mathcal{K} defined by $\mathcal{K}\theta(t) := \mathcal{K}_t\theta(t)$ is a bounded linear isomorphism: from $L^2([0, T]; H^k(\Omega))$ to $L^2([0, T]; H^k(\Omega))$ for $k = 0, 1, 2$; from $L^2([0, T]; H^0(\Omega))$ to \mathcal{H}_T^0 and from ${}_0H^1(\Omega)$ to \mathcal{H}_T^1 . In each case, the operators \mathcal{K} and \mathcal{K}^{-1} are bounded by the polynomial $P(\sup_{0 \leq t \leq T} \|\eta(t)\|_{H^{\frac{7}{2}}})$.*

Proof. It is easy to see that for each $t \in [0, T]$,

$$\|\mathcal{K}_t\theta\|_{H^0} \lesssim \|\mathcal{K}_t\|_{C^0} \|\theta\|_{H^0} \lesssim P(\|\eta(t)\|_{H^{\frac{7}{2}}}) \|\theta\|_{H^0}, \tag{2.3}$$

$$\|\mathcal{K}_t\theta\|_{H^1} \lesssim \|\mathcal{K}_t\|_{C^1} \|\theta\|_{H^1} \lesssim P(\|\eta(t)\|_{H^{\frac{7}{2}}}) \|\theta\|_{H^1}, \tag{2.4}$$

$$\|\mathcal{K}_t\theta\|_{H^2} \lesssim \|\mathcal{K}_t\|_{C^1} \|\theta\|_{H^2} + \|\mathcal{K}_t\|_{H^2} \|\theta\|_{C^0} \lesssim P(\|\eta(t)\|_{H^{\frac{7}{2}}}) \|\theta\|_{H^2}. \tag{2.5}$$

These inequalities imply that \mathcal{K}_t is a bounded operator from H^k to H^k , for $k = 0, 1, 2$. Since \mathcal{K}_t is invertible, we have the estimate $\|\mathcal{K}_t^{-1}\Theta\|_{H^k} \lesssim P(\|\eta(t)\|_{H^{\frac{7}{2}}}) \|\Theta\|_{H^k}$. Thus, \mathcal{K}_t is an isomorphism of H^k to H^k , for $k = 0, 1, 2$. With this fact in hand, Lemma 2.1 implies that \mathcal{K}_t is an isomorphism of $L^2(\Omega)$ to $\mathcal{H}^0(t)$ and of ${}_0H^1(\Omega)$ to $\mathcal{H}^1(t)$.

The mapping properties of the operator \mathcal{K} on space-time functions may be established in a similar manner. □

2.2. Pressure as a Lagrange multiplier. The introduction of pressure functions has been studied by Guo and Tice in Section 2 of [6], of which the modification was given by L. Wu in Section 2.2 of [13]. So we omit the details here.

3. Elliptic estimates

3.1. Preliminary. Before studying the linear problem (1.8), we need some elliptic estimates. In order to study the elliptic problem, we transform the equations on the domain Ω into constant coefficient equations on the domain $\Omega' = \Phi(\Omega)$, where Φ is defined by the expression (1.2). The following lemma shows that the mapping Φ is an isomorphism between $H^k(\Omega')$ and $H^k(\Omega)$. Here, the Sobolev spaces are either vector-valued or scalar-valued.

LEMMA 3.1. *Let $\Psi: \Omega \rightarrow \Omega'$ be a C^1 diffeomorphism satisfying $\Psi \in H_{loc}^{k+1}$, $\nabla \Psi - I \in H^k(\Omega)$ and the Jacobian $J = \det(\nabla \Psi) > \delta > 0$ almost everywhere in Ω for an integer $k \geq 3$. If $v \in H^m(\Omega')$, then $v \circ \Psi \in H^m(\Omega)$ for $m = 0, 1, \dots, k+1$, and*

$$\|v \circ \Psi\|_{H^m(\Omega)} \lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|v\|_{H^m(\Omega')},$$

where $C(\|\nabla \Psi - I\|_{H^k(\Omega)})$ is a constant depending on $\|\nabla \Psi - I\|_{H^k(\Omega)}$. Similarly, for $u \in H^m(\Omega)$, we have $u \circ \Psi^{-1} \in H^m(\Omega')$ for $m = 0, 1, \dots, k+1$, and

$$\|u \circ \Psi^{-1}\|_{H^m(\Omega')} \lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|u\|_{H^m(\Omega)}.$$

Let $\Sigma' = \Psi(\Sigma)$ be the top boundary of Ω' . If $v \in H^{m-\frac{1}{2}}(\Sigma')$ for $m = 1, \dots, k-1$, then $v \circ \Psi \in H^{m-\frac{1}{2}}(\Sigma)$, and

$$\|v \circ \Psi\|_{H^{m-\frac{1}{2}}(\Sigma)} \lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|v\|_{H^{m-\frac{1}{2}}(\Sigma')}.$$

If $u \in H^{m-\frac{1}{2}}(\Sigma)$ for $m = 1, \dots, k-1$, then $u \circ \Psi^{-1} \in H^{m-\frac{1}{2}}(\Sigma')$ and

$$\|u \circ \Psi^{-1}\|_{H^{m-\frac{1}{2}}(\Sigma')} \lesssim C(\|\nabla \Psi - I\|_{H^k(\Omega)}) \|u\|_{H^{m-\frac{1}{2}}(\Sigma)}.$$

Proof. The proof of this lemma is the same as Lemma 3.1 in [6], which has been proved by Y. Guo and I. Tice, so we omit the details here. \square

3.2. The \mathcal{A} -stationary convection problem. In this section, we consider the stationary equations

$$\left\{ \begin{array}{ll} \operatorname{div}_{\mathcal{A}} S_{\mathcal{A}}(p, u) - \theta \nabla_{\mathcal{A}} y_3 = F^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}} u = F^2 & \text{in } \Omega, \\ -\Delta_{\mathcal{A}} \theta = F^3 & \text{in } \Omega, \\ S_{\mathcal{A}}(p, u) \cdot \mathcal{N} = F^4 & \text{on } \Sigma, \\ \nabla_{\mathcal{A}} \theta \cdot \mathcal{N} + \theta |\mathcal{N}| = F^5 & \text{on } \Sigma, \\ u = 0, \quad \theta = 0 & \text{on } \Sigma_b \end{array} \right. \quad (3.1)$$

Before discussing the regularity for strong solution to (3.1), we need to define the weak solution of Equation (3.1). Suppose $F^1 \in (\mathcal{H}^1)^*$, $F^2 \in H^0$, $F^3 \in (\mathcal{H}^1)^*$, $F^4 \in H^{-\frac{1}{2}}(\Sigma)$ and $F^5 \in H^{-\frac{1}{2}}(\Sigma)$. Then (u, p, θ) is called a weak solution of Equation (3.1) if it satisfies $\nabla_{\mathcal{A}} \cdot u = F^2$,

$$(\nabla_{\mathcal{A}} \theta, \nabla_{\mathcal{A}} \phi)_{\mathcal{H}^0} + (\theta |\mathcal{N}|, \phi)_{H^0(\Sigma)} = \langle F^3, \phi \rangle_{(\mathcal{H}^1)^*} + \langle F^5, \phi \rangle_{H^{-\frac{1}{2}}(\Sigma)}, \quad (3.2)$$

and

$$\frac{1}{2} (\mathbb{D}_{\mathcal{A}} u, \mathbb{D}_{\mathcal{A}} \psi)_{\mathcal{H}^0} + (p, \nabla_{\mathcal{A}} \psi)_{\mathcal{H}^0} - (\theta \nabla_{\mathcal{A}} y_3, \psi)_{\mathcal{H}^0} = \langle F^1, \psi \rangle_{(\mathcal{H}^1)^*} - \langle F^4, \psi \rangle_{H^{-\frac{1}{2}}(\Sigma)}, \quad (3.3)$$

for any $\phi, \psi \in \mathcal{H}^1$.

LEMMA 3.2. *Suppose $F^1 \in (\mathcal{H}^1)^*$, $F^2 \in \mathcal{H}^0$, $F^3 \in (\mathcal{H}^1)^*$, $F^4 \in H^{-\frac{1}{2}}(\Sigma)$ and $F^5 \in H^{-\frac{1}{2}}(\Sigma)$. Then there exists a unique weak solution $(u, p, \theta) \in \mathcal{H}^1 \times \mathcal{H}^0 \times \mathcal{H}^1$ to Equation (3.1).*

Proof. For the Hilbert space \mathcal{H}^1 with inner product $(\theta, \phi) = (\nabla_{\mathcal{A}} \theta, \nabla_{\mathcal{A}} \phi)_{\mathcal{H}^0} + (\theta|_{\mathcal{N}}, \phi)_{H^0(\Sigma)}$, we can define a linear functional $\ell \in (\mathcal{H}^1)^*$ by

$$\ell(\phi) = \langle F^3, \phi \rangle_{(\mathcal{H}^1)^*} + \langle H^5, \phi \rangle_{H^{-\frac{1}{2}}(\Sigma)},$$

for all $\phi \in \mathcal{H}^1$. Then by using the Riesz representation theorem, there exists a unique $\theta \in \mathcal{H}^1$ such that

$$(\nabla_{\mathcal{A}} \theta, \nabla_{\mathcal{A}} \phi)_{\mathcal{H}^0} + (\theta|_{\mathcal{N}}, \phi)_{H^0(\Sigma)} = \langle F^3, \phi \rangle_{(\mathcal{H}^1)^*} + \langle H^5, \phi \rangle_{H^{-\frac{1}{2}}(\Sigma)},$$

for all $\phi \in \mathcal{H}^1$.

By Lemma 2.6 in [6], there exists a $\bar{u} \in \mathcal{H}^1$ such that $\text{div}_{\mathcal{A}} \bar{u} = F^2$. Then, we may restrict our test function to $\psi \in \mathcal{X}$. A straight application of the Riesz representation theorem to the Hilbert space \mathcal{X} with inner product defined as $(u, \psi) = (\mathbb{D}_{\mathcal{A}} u, \mathbb{D}_{\mathcal{A}} \psi)_{\mathcal{H}^0}$ provides a unique $w \in \mathcal{X}$ such that

$$\begin{aligned} & \frac{1}{2} (\mathbb{D}_{\mathcal{A}} w, \mathbb{D}_{\mathcal{A}} \psi)_{\mathcal{H}^0} \\ &= -\frac{1}{2} (\mathbb{D}_{\mathcal{A}} \bar{u}, \mathbb{D}_{\mathcal{A}} \psi)_{\mathcal{H}^0} + (\theta \nabla_{\mathcal{A}} y_3, \psi)_{\mathcal{H}^0} + \langle F^1, \psi \rangle_{(\mathcal{H}^1)^*} - \langle F^4, \psi \rangle_{H^{-\frac{1}{2}}(\Sigma)} \end{aligned} \tag{3.4}$$

for all $\psi \in \mathcal{X}$. Then we can find u satisfying

$$\frac{1}{2} (\mathbb{D}_{\mathcal{A}} u, \mathbb{D}_{\mathcal{A}} \psi)_{\mathcal{H}^0} - (\theta \nabla_{\mathcal{A}} y_3, \psi)_{\mathcal{H}^0} = \langle F^1, \psi \rangle_{(\mathcal{H}^1)^*} - \langle F^4, \psi \rangle_{H^{-\frac{1}{2}}(\Sigma)}, \tag{3.5}$$

by $u = w + \bar{u} \in \mathcal{H}^1$, with $\text{div}_{\mathcal{A}} u = F^2$.

It is easily seen that u is unique. Suppose that there exists another \tilde{u} that satisfies (3.5). Then we have $\text{div}_{\mathcal{A}} (u - \tilde{u}) = 0$, and $(\mathbb{D}_{\mathcal{A}} (u - \tilde{u}), \mathbb{D}_{\mathcal{A}} \psi)_{\mathcal{H}^0} = 0$ for any $\psi \in \mathcal{X}$. By taking $\psi = u - \tilde{u}$, and using the Korn's inequality, we know that $\|u - \tilde{u}\|_{H^0} = 0$, which implies $u = \tilde{u}$.

In order to introduce the pressure p , we can define $\lambda \in (\mathcal{H}^1)^*$ as the difference of the left- and right-hand sides of Equation (3.4). Then $\lambda(\psi) = 0$ for all $\psi \in \mathcal{X}$. According to Proposition 2.12 in [13], there exists a unique $p \in \mathcal{H}^0$ satisfying $(p, \text{div}_{\mathcal{A}} \psi)_{\mathcal{H}^0} = \lambda(\psi)$ for all $\psi \in \mathcal{H}^1$. \square

In the next result, we establish the strong solutions of Equation (3.1) and present some elliptic estimates.

LEMMA 3.3. *Suppose that $\eta \in H^{k+\frac{1}{2}}(\Sigma)$ for $k \geq 3$ such that the mapping Φ defined in the expression (1.2) is a C^1 diffeomorphism of Ω to $\Omega' = \Phi(\Omega)$. If $F^1 \in H^0$, $F^2 \in H^1$, $F^3 \in H^0$, $F^4 \in H^{\frac{1}{2}}$ and $F^5 \in H^{\frac{1}{2}}$, then the problem (3.1) admits a unique strong solution $(u, p, \theta) \in H^2(\Omega) \times H^1(\Omega) \times H^2(\Omega)$, i.e. (u, p, θ) satisfy the problem (3.1) a.e. in Ω and on Σ, Σ_b . Moreover, for $r = 2, \dots, k-1$, we have the estimate*

$$\begin{aligned} \|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} &\lesssim C(\eta) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} \right. \\ &\quad \left. + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right), \end{aligned} \tag{3.6}$$

whenever the right-hand side is finite, where $C(\eta)$ is a constant depending on $\|\eta\|_{H^{k+\frac{1}{2}}(\Sigma)}$.

Proof. First, we consider the problem

$$\begin{cases} -\Delta_{\mathcal{A}}\theta = F^3 & \text{in } \Omega, \\ \nabla_{\mathcal{A}}\theta \cdot \mathcal{N} + \theta|\mathcal{N}| = F^5 & \text{on } \Sigma, \\ \theta = 0 & \text{on } \Sigma_b. \end{cases}$$

Since the coefficients of this equation are not constants, We transform this problem to one on $\Omega' = \Phi(\Omega)$ by introducing the unknowns Θ according to $\theta = \Theta \circ \Phi$. Then Θ should be solutions to the usual problem on $\Omega' = \{-1 \leq y_3 \leq \eta(y_1, y_2)\}$ with upper boundary $\Sigma' = \{y_3 = \eta\}$

$$\begin{cases} -\Delta\Theta = F^3 \circ \Phi^{-1} = G^3 & \text{in } \Omega', \\ \nabla\Theta \cdot \mathcal{N} + \Theta|\mathcal{N}| = F^5 \circ \Phi^{-1} = G^5 & \text{on } \Sigma', \\ \Theta = 0 & \text{on } \Sigma'_b. \end{cases} \tag{3.7}$$

Note that, according to Lemma 3.1, $G^3 \in H^0(\Omega')$ and $G^5 \in H^{1/2}(\Sigma')$. Then we may argue as in Lemma 2.8 of [2] and use Theorem 10.5 in [1], to obtain that there exists a unique $\Theta \in H^2(\Omega')$, solving problem (3.7) with

$$\|\Theta\|_{H^2(\Omega')} \lesssim C(\eta) \left(\|G^3\|_{H^0(\Omega')} + \|G^5\|_{H^{\frac{1}{2}}(\Sigma')} \right),$$

for $C(\eta)$ a constant depending on $\|\eta\|_{H^{k+\frac{1}{2}}}$.

For the \mathcal{A} -Stokes equations, we introduce the unknowns v, q by $u = v \circ \Phi$ and $q = p \circ \Phi$. For the usual Stokes problem

$$\begin{cases} S(q, v) - \Theta e_3 = F^1 \circ \Phi^{-1} = G^1 & \text{in } \Omega' \\ \nabla \cdot v = F^2 \circ \Phi^{-1} = G^2 & \text{in } \Omega' \\ S(q, v)\mathcal{N} = F^4 \circ \Phi^{-1} = G^4 & \text{on } \Sigma' \\ v = 0 & \text{on } \Sigma_b, \end{cases} \tag{3.8}$$

we use the same argument as in the proof of Lemma 3.6 in [6] with $G^1 + \Theta e_3$ instead of G^1 . Then we have that there exist unique $v \in H^2(\Omega')$, $q \in H^1(\Omega')$ solving problem (3.8) with

$$\|v\|_{H^2(\Omega')} + \|q\|_{H^1(\Omega')} \lesssim C(\eta) \left(\|G^1\|_{H^0(\Omega')} + \|G^2\|_{H^1(\Omega')} + \|G^4\|_{H^{\frac{1}{2}}(\Sigma')} + \|\Theta\|_{H^0(\Omega')} \right),$$

for $C(\eta)$ a constant depending on $\|\eta\|_{H^{k+\frac{1}{2}}}$. So we have that

$$\begin{aligned} \|v\|_{H^2(\Omega')} + \|q\|_{H^1(\Omega')} + \|\Theta\|_{H^2(\Omega')} &\lesssim C(\eta) \left(\|G^1\|_{H^0(\Omega')} + \|G^2\|_{H^1(\Omega')} \right. \\ &\quad \left. + \|G^3\|_{H^0(\Omega')} + \|G^4\|_{H^{\frac{1}{2}}(\Sigma')} + \|G^5\|_{H^{\frac{1}{2}}(\Sigma')} \right), \end{aligned} \tag{3.9}$$

for $C(\eta)$ a constant depending on $\|\eta\|_{H^{k+\frac{1}{2}}}$. Then we may argue as in Lemma 3.6 of [6] to derive that, for $r = 2, \dots, k - 1$,

$$\|v\|_{H^r(\Omega')} + \|q\|_{H^{r-1}(\Omega')} + \|\Theta\|_{H^r(\Omega')}$$

$$\begin{aligned} &\lesssim C(\eta) \left(\|G^1\|_{H^{r-2}(\Omega')} + \|G^2\|_{H^{r-1}(\Omega')} + \|G^3\|_{H^{r-2}(\Omega')} \right. \\ &\quad \left. + \|G^4\|_{H^{r-\frac{3}{2}}(\Sigma')} + \|G^5\|_{H^{r-\frac{3}{2}}(\Sigma')} \right), \end{aligned} \tag{3.10}$$

for $C(\eta)$ a constant depending on $\|\eta\|_{H^{k+\frac{1}{2}}}$.

Now, we transform back to Ω with $u = v \circ \Phi$, $p = q \circ \Phi$ and $\theta = \Theta \circ \Phi$. It is readily verified that (u, p, T) are strong solutions of Equation (3.1). According to Lemma 3.1,

$$\begin{aligned} \|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} &\lesssim C(\eta) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} \right. \\ &\quad \left. + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right), \end{aligned}$$

whenever the right-hand side is finite, and where $C(\eta)$ is a constant depending on $\|\eta\|_{H^{k+\frac{1}{2}}(\Sigma)}$. This is what we want. \square

In the next lemma, we verify that the constant in the estimate (3.6) can actually only depend on the initial free surface.

LEMMA 3.4. *Let $k \geq 3$ be an integer and suppose that $\eta \in H^{k+\frac{1}{2}}(\Sigma)$ and $\eta_0 \in H^{k+\frac{1}{2}}(\Sigma)$. Then there exists a positive number $\varepsilon_0 < 1$ such that if $\|\eta - \eta_0\|_{H^{k-\frac{3}{2}}} \leq \varepsilon_0$, the solution to Equation (3.1) satisfies*

$$\begin{aligned} \|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} &\lesssim C(\eta_0) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} \right. \\ &\quad \left. + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right), \end{aligned} \tag{3.11}$$

for $r = 2, \dots, k - 1$, whenever the right-hand side is finite, and where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+\frac{1}{2}}}$.

Proof. Here, we use the same idea as in Lemma 2.17 of [13]. We rewrite Equation (3.1) with its coefficients determined by η_0 , i.e. it can be thought as a perturbation of equations of Equation (3.1) in terms of initial data

$$\left\{ \begin{array}{ll} \operatorname{div}_{\mathcal{A}_0} S_{\mathcal{A}_0}(p, u) - \theta \nabla_{\mathcal{A}_0} y_{3,0} = F^1 + F^{1,0} & \text{in } \Omega, \\ \nabla_{\mathcal{A}_0} \cdot u = F^2 + F^{2,0} & \text{in } \Omega, \\ -\Delta_{\mathcal{A}_0} \theta = F^3 + F^{3,0} & \text{in } \Omega, \\ S_{\mathcal{A}_0}(p, u) \mathcal{N}_0 = F^4 + F^{4,0} & \text{on } \Sigma, \\ \nabla_{\mathcal{A}_0} \theta \cdot \mathcal{N}_0 + \theta |\mathcal{N}_0| = F^5 + F^{5,0} & \text{on } \Sigma, \\ u = 0, \quad \theta = 0 & \text{on } \Sigma_b, \end{array} \right. \tag{3.12}$$

where

$$\begin{aligned} F^{1,0} &= \nabla_{\mathcal{A}_0 - \mathcal{A}} \cdot S_{\mathcal{A}}(p, u) + \nabla_{\mathcal{A}_0} \cdot S_{\mathcal{A}_0 - \mathcal{A}}(p, u) + \theta \nabla_{\mathcal{A}_0 - \mathcal{A}} y_3 + \theta \nabla_{\mathcal{A}_0} (y_{3,0} - y_3), \\ F^{2,0} &= \operatorname{div}_{\mathcal{A}_0 - \mathcal{A}} u, \\ F^{3,0} &= \nabla_{\mathcal{A}_0 - \mathcal{A}} \cdot \nabla_{\mathcal{A}} \theta + \nabla_{\mathcal{A}_0} \cdot \nabla_{\mathcal{A}_0 - \mathcal{A}} \theta, \\ F^{4,0} &= S_{\mathcal{A}_0}(p, u) (\mathcal{N}_0 - \mathcal{N}) + S_{\mathcal{A}_0 - \mathcal{A}}(p, u) \mathcal{N}, \\ F^{5,0} &= \nabla_{\mathcal{A}_0} \theta \cdot (\mathcal{N}_0 - \mathcal{N}) + \nabla_{\mathcal{A}_0 - \mathcal{A}} \theta \cdot \mathcal{N} + \theta (|\mathcal{N}_0| - |\mathcal{N}|). \end{aligned}$$

Here, $\mathcal{A}_0, \mathcal{N}_0$ and $y_{3,0}$ are quantities of \mathcal{A}, \mathcal{N} and y_3 in terms of η_0 . By the assumption, we know that $\eta - \eta_0 \in H^{k+\frac{1}{2}}(\Sigma)$ and $\|\eta - \eta_0\|_{H^{k-\frac{3}{2}}(\Sigma)}^\ell \leq \|\eta - \eta_0\|_{H^{k-\frac{3}{2}}(\Sigma)} < 1$ for any positive integer ℓ . By straightforward computation, we may derive that

$$\begin{aligned} \|F^{1,0}\|_{H^{r-2}} &\leq C \left(1 + \|\eta_0\|_{H^{k+\frac{1}{2}}}\right)^4 \|\eta - \eta_0\|_{H^{k-\frac{3}{2}}} (\|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^{r-2}}), \\ \|F^{2,0}\|_{H^{r-1}} &\leq C \left(1 + \|\eta_0\|_{H^{k+\frac{1}{2}}}\right)^2 \|\eta - \eta_0\|_{H^{k-\frac{3}{2}}} \|u\|_{H^r}, \\ \|F^{3,0}\|_{H^{r-2}} &\leq C \left(1 + \|\eta_0\|_{H^{k+\frac{1}{2}}}\right)^4 \|\eta - \eta_0\|_{H^{k-\frac{3}{2}}} \|\theta\|_{H^r}, \\ \|F^{4,0}\|_{H^{r-\frac{3}{2}}(\Sigma)} &\leq C \left(1 + \|\eta_0\|_{H^{k+\frac{1}{2}}}\right)^2 \|\eta - \eta_0\|_{H^{k-\frac{3}{2}}} (\|u\|_{H^r} + \|p\|_{H^{r-1}}), \\ \|F^{5,0}\|_{H^{r-\frac{3}{2}}(\Sigma)} &\leq C \left(1 + \|\eta_0\|_{H^{k+\frac{1}{2}}}\right)^2 \|\eta - \eta_0\|_{H^{k-\frac{3}{2}}} \|\theta\|_{H^r}, \end{aligned}$$

for $r = 2, \dots, k - 1$.

Based on Lemma 3.3, we have the estimate

$$\begin{aligned} &\|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} \\ &\lesssim C(\eta_0) \left(\|F^1 + F^{1,0}\|_{H^{r-2}} + \|F^2 + F^{2,0}\|_{H^{r-1}} + \|F^3 + F^{3,0}\|_{H^{r-2}} \right. \\ &\quad \left. + \|F^4 + F^{4,0}\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5 + F^{5,0}\|_{H^{r-\frac{3}{2}}(\Sigma)} \right), \end{aligned}$$

where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+\frac{1}{2}}}$. Combining the above estimates, we have

$$\begin{aligned} &\|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} \\ &\lesssim C(\eta_0) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right) \\ &\quad + C(\eta_0) \left(1 + \|\eta_0\|_{H^{k+\frac{1}{2}}}\right)^4 \|\eta - \eta_0\|_{H^{k-\frac{3}{2}}} (\|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r}), \end{aligned} \tag{3.13}$$

for $r = 2, \dots, k - 1$. Then, if $\|\eta - \eta_0\|_{H^{k-\frac{3}{2}}}$ is to be chosen small enough such that the second term of the above inequality on the right-hand side less than $\frac{1}{2}(\|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r})$, then it can be absorbed into the left-hand side, and we have that

$$\begin{aligned} &\|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} \\ &\lesssim C(\eta_0) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right), \end{aligned}$$

for $r = 2, \dots, k - 1$. □

Notice that the estimate (3.11) can only go up to order $k - 1$, which does not satisfy our requirement. In the next result, we can achieve two more orders with a bootstrap argument, where we use the idea of [13].

PROPOSITION 3.5. *Let $k \geq 3$ be an integer. Suppose that $\eta \in H^{k+\frac{1}{2}}(\Sigma)$ as well as $\eta_0 \in H^{k+\frac{1}{2}}(\Sigma)$ satisfying $\|\eta - \eta_0\|_{H^{k+\frac{1}{2}}(\Sigma)} \leq \varepsilon_0$. Then the solution to Equation (3.1) satisfies*

$$\begin{aligned} &\|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} \\ &\lesssim C(\eta_0) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right) \end{aligned} \tag{3.14}$$

for $r=2, \dots, k+1$, whenever the right hand side is finite, and where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+\frac{1}{2}}(\Sigma)}$.

Proof. Here, we only consider the case for $r=k$ and $r=k+1$, since the conclusion has been proved when $r \leq k-1$. For $m \in \mathbb{N}$, we define η^m by throwing away high frequencies:

$$\hat{\eta}^m(n) = \begin{cases} \hat{\eta}(n), & \text{for } |n| \leq m-1, \\ 0, & \text{for } |n| \geq m. \end{cases}$$

Then for each m , $\eta^m \in H^j(\Sigma)$ for arbitrary $j \geq 0$ and $\eta^m \rightarrow \eta$ in $H^{k+\frac{1}{2}}(\Sigma)$ as $m \rightarrow \infty$.

We consider the problem (3.1) with \mathcal{A} and \mathcal{N} replaced by \mathcal{A}^m and \mathcal{N}^m , and y_3 replaced by y_3^m . Since $\eta^m \in H^{\frac{5}{2}}$, we may apply Lemma 3.3 to deduce that there exists a unique (u^m, p^m, θ^m) which solves

$$\left\{ \begin{array}{l} \operatorname{div}_{\mathcal{A}^m} S_{\mathcal{A}^m}(p^m, u^m) - \theta^m \nabla_{\mathcal{A}^m} y_3^m = F^1 \quad \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}^m} u^m = F^2 \quad \text{in } \Omega, \\ -\Delta_{\mathcal{A}^m} \theta^m = F^3 \quad \text{in } \Omega, \\ S_{\mathcal{A}^m}(p^m, u^m) \mathcal{N}^m = F^4 \quad \text{on } \Sigma, \\ \nabla_{\mathcal{A}^m} \theta^m \cdot \mathcal{N}^m + \theta^m |\mathcal{N}^m| = F^5 \quad \text{on } \Sigma, \\ u^m = 0, \quad \theta^m = 0 \quad \text{on } \Sigma_b, \end{array} \right. \tag{3.15}$$

and satisfies

$$\begin{aligned} & \|u^m\|_{H^r} + \|p^m\|_{H^{r-1}} + \|\theta^m\|_{H^r} \\ & \lesssim C(\|\eta^m\|_{H^{k+\frac{5}{2}}}) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right) \end{aligned}$$

for $r=2, \dots, k+1$. In the following, we will prove that the constant $C(\|\eta^m\|_{H^{k+\frac{5}{2}}})$ can be improved only in terms of $\|\eta^m\|_{H^{k+\frac{1}{2}}}$.

For convenience, we define

$$\mathcal{Z} = C(\eta_0) P(\eta^m) \left(\|F^1\|_{H^{r-2}}^2 + \|F^2\|_{H^{r-1}}^2 + \|F^3\|_{H^{r-2}}^2 + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 \right),$$

where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+\frac{1}{2}}}$ and $P(\eta)$ is a polynomial of $\|\eta^m\|_{H^{k+\frac{1}{2}}}$. Then after the same computation as in the proof of Proposition 2.18 in [13], except for F being replaced by $F^1 + \theta^m \nabla_{\mathcal{A}^m} y_3^m$, we have

$$\|u^m\|_{H^r} + \|p^m\|_{H^{r-1}} \lesssim \mathcal{Z},$$

for $r=2, \dots, k+1$. That's because in the above estimate, we only need to consider the terms $\|\theta^m\|_{H^r}$, for $r=2, \dots, k-1$, but $\|\theta^m\|_{H^r} \lesssim \mathcal{Z}$ is assured by Lemma 3.11.

Then we consider the temperature θ^m . In the following of bootstrap argument, we may abuse notation using θ instead of θ^m and also $\eta, \mathcal{A}, \mathcal{N}$ instead of $\eta^m, \mathcal{A}^m, \mathcal{N}^m$. We write explicitly the equation of θ as

$$\begin{aligned} & \partial_{11}\theta + \partial_{22}\theta + (1+A^2+B^2)K^2\partial_{33}\theta - 2AK\partial_{13}\theta - 2BK\partial_{23}\theta \\ & + (AK\partial_3(AK) + BK\partial_3(BK) - \partial_1(AK) - \partial_2(BK) + K\partial_3K)\partial_3\theta = -F^3. \end{aligned} \tag{3.16}$$

Step 1. $r = k$ case. By Lemma 3.11,

$$\|\theta\|_{H^{k-1}}^2 \lesssim C(\eta_0) \left(\|F^3\|_{H^{k-3}}^2 + \|F^5\|_{H^{k-\frac{5}{2}}(\Sigma)}^2 \right) \lesssim \mathcal{L},$$

where the constant $C(\eta_0)$ only depends on $\|\eta_0\|_{H^{k+\frac{1}{2}}}$. For $i = 1, 2$, $\partial_i \theta$ satisfies the equation

$$\begin{cases} -\Delta_{\mathcal{A}} \partial_i \theta = \bar{F}^3 & \text{in } \Omega, \\ \nabla_{\mathcal{A}} \partial_i \theta \cdot \mathcal{N} + \partial_i \theta |_{\mathcal{N}} = \bar{F}^5 & \text{on } \Sigma, \\ \partial_i \theta = 0 & \text{on } \Sigma_b, \end{cases}$$

where

$$\begin{aligned} \bar{F}^3 &= \partial_i F^3 + \operatorname{div}_{\partial_i \mathcal{A}} \nabla_{\mathcal{A}} \theta + \operatorname{div}_{\mathcal{A}} \nabla_{\partial_i \mathcal{A}} \theta, \\ \bar{F}^5 &= \partial_i F^5 - \nabla_{\partial_i \mathcal{A}} \theta \cdot \mathcal{N} - \nabla_{\mathcal{A}} \theta \cdot \partial_i \mathcal{N} - \theta \partial_i |_{\mathcal{N}}. \end{aligned}$$

Applying Lemmas A.1–A.2 in [6], we have

$$\begin{aligned} \|\bar{F}^3\|_{H^{k-3}}^2 + \|\bar{F}^5\|_{H^{k-\frac{5}{2}}(\Sigma)}^2 &\lesssim \|F^3\|_{H^{k-2}}^2 + \|F^5\|_{H^{k-\frac{3}{2}}(\Sigma)}^2 + P(\eta) \|\theta\|_{H^{k-1}}^2 \\ &\lesssim \mathcal{L}. \end{aligned}$$

Employing the $k - 1$ order elliptic estimate, we have

$$\|\partial_i \theta\|_{H^{k-1}}^2 \lesssim C(\eta_0) \left(\|\bar{F}^3\|_{H^{k-3}}^2 + \|\bar{F}^5\|_{H^{k-\frac{5}{2}}(\Sigma)}^2 \right) \lesssim \mathcal{L}.$$

Then taking the derivative ∂_3^{k-2} on both sides of Equation (3.16) and focusing on the term $(1 + A^2 + B^2)K^2 \partial_3^k \theta$, the estimates of all the other terms in H^0 -norm implies that

$$\|\partial_3^k \theta\|_{H^0}^2 \lesssim \mathcal{L}.$$

Thus, we have proved that

$$\|\theta\|_{H^k}^2 \lesssim \mathcal{L}.$$

Step 2. $r = k + 1$ case.

For $i, j = 1, 2$, $\partial_{ij} \theta$ satisfies the equation

$$\begin{cases} -\Delta_{\mathcal{A}} \partial_{ij} \theta = \tilde{F}^3 & \text{in } \Omega, \\ \nabla_{\mathcal{A}} \partial_{ij} \theta \cdot \mathcal{N} + \partial_{ij} \theta |_{\mathcal{N}} = \tilde{F}^5 & \text{on } \Sigma, \\ \partial_{ij} \theta = 0 & \text{on } \Sigma_b, \end{cases}$$

where

$$\begin{aligned} \tilde{F}^3 &= \partial_{ij} F^3 + \operatorname{div}_{\partial_{ij} \mathcal{A}} \nabla_{\mathcal{A}} \theta + \operatorname{div}_{\mathcal{A}} \nabla_{\partial_{ij} \mathcal{A}} \theta + \operatorname{div}_{\partial_i \mathcal{A}} \nabla_{\partial_j \mathcal{A}} \theta + \operatorname{div}_{\partial_j \mathcal{A}} \nabla_{\partial_i \mathcal{A}} \theta \\ &\quad + \operatorname{div}_{\partial_i \mathcal{A}} \nabla_{\mathcal{A}} \partial_j \theta + \operatorname{div}_{\partial_j \mathcal{A}} \nabla_{\mathcal{A}} \partial_i \theta + \operatorname{div}_{\mathcal{A}} \nabla_{\partial_i \mathcal{A}} \partial_j \theta + \operatorname{div}_{\mathcal{A}} \nabla_{\partial_j \mathcal{A}} \partial_i \theta, \\ \tilde{F}^5 &= \partial_{ij} F^5 - \nabla_{\mathcal{A}} \theta \cdot \partial_{ij} \mathcal{N} - (\nabla_{\partial_i \mathcal{A}} \theta + \nabla_{\mathcal{A}} \partial_i \theta) \cdot \partial_j \mathcal{N} - (\nabla_{\partial_j \mathcal{A}} \theta + \nabla_{\mathcal{A}} \partial_j \theta) \cdot \partial_i \mathcal{N} \\ &\quad - (\nabla_{\partial_{ij} \mathcal{A}} \theta + \nabla_{\partial_i \mathcal{A}} \partial_j \theta - \nabla_{\partial_j \mathcal{A}} \partial_i \theta) \mathcal{N} - \theta \partial_{ij} |_{\mathcal{N}} - \partial_i \theta \partial_j |_{\mathcal{N}} - \partial_j \theta \partial_i |_{\mathcal{N}}. \end{aligned}$$

Applying Lemmas A.1–A.2 in [6] to the forcing terms, we have

$$\begin{aligned} \|\tilde{F}^3\|_{H^{k-3}}^2 + \|\tilde{F}^5\|_{H^{k-\frac{5}{2}}(\Sigma)}^2 &\lesssim \|F^3\|_{H^{k-1}}^2 + \|F^5\|_{H^{k-\frac{1}{2}}(\Sigma)}^2 + P(\eta)\|\theta\|_{H^k}^2 \\ &\lesssim \mathcal{L}. \end{aligned}$$

Then Lemma 3.11 implies that

$$\|\partial_{ij}\theta\|_{H^{k-1}}^2 \lesssim C(\eta_0) \left(\|\tilde{F}^3\|_{H^{k-3}}^2 + \|\tilde{F}^5\|_{H^{k-\frac{5}{2}}(\Sigma)}^2 \right) \lesssim \mathcal{L}.$$

Since we have proved the case $r = k$, we take the derivative $\partial_3^{k-2}\partial_i$ on both sides of Equation (3.16) for $i = 1, 2$ and focus on the term $(1 + A^2 + B^2)K^2\partial_3^k\partial_i\theta$. Utilizing the estimates of all the other terms in H^0 -norm, we have

$$\|\partial_3^k\partial_i\theta\|_{H^0}^2 \lesssim \mathcal{L}.$$

Then, taking derivative ∂_3^{k-1} on both sides of (3.16) and focusing on the term of $(1 + A^2 + B^2)K^2\partial_3^{k+1}\theta$, by all the estimates above, we have

$$\|\partial_3^{k+1}\theta\|_{H^0}^2 \lesssim \mathcal{L}.$$

Therefore, we have proved

$$\|\theta\|_{H^{k+1}}^2 \lesssim \mathcal{L}.$$

Now, we go back to the original notation. According to the convergence of η^m , we have

$$\begin{aligned} &\|u^m\|_{H^r}^2 + \|p^m\|_{H^{r-1}}^2 + \|\theta^m\|_{H^r}^2 \\ &\lesssim C(\eta_0)P(\eta^m) \left(\|F^1\|_{H^{r-2}}^2 + \|F^2\|_{H^{r-1}}^2 + \|F^3\|_{H^{r-2}}^2 + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 \right) \\ &\lesssim C(\eta_0)P(\eta) \left(\|F^1\|_{H^{r-2}}^2 + \|F^2\|_{H^{r-1}}^2 + \|F^3\|_{H^{r-2}}^2 + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 \right) \\ &\lesssim C(\eta_0) \left(\|F^1\|_{H^{r-2}}^2 + \|F^2\|_{H^{r-1}}^2 + \|F^3\|_{H^{r-2}}^2 + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)}^2 \right), \end{aligned} \tag{3.17}$$

for $r = 2, \dots, k + 1$, where in the last inequality we have used the assumption that $\|\eta - \eta_0\|_{H^{k+\frac{1}{2}}} \leq \varepsilon_0$ and the term $P(\eta_0)$ is absorbed by $C(\eta_0)$. Here $C(\eta_0)$ depends only on $\|\eta_0\|_{H^{k+\frac{1}{2}}}$.

The inequality of boundedness (3.17) implies that the sequence $\{(u^m, p^m, \theta^m)\}$ is uniformly bounded in $H^r \times H^{r-1} \times H^r$, so we can extract a weakly convergent subsequence, which is still denoted by $\{(u^m, p^m, \theta^m)\}$. That is, $u^m \rightharpoonup u^0$ in $H^r(\Omega)$, $p^m \rightharpoonup p^0$ in $H^{r-1}(\Omega)$ and $\theta^m \rightharpoonup \theta^0$ in $H^r(\Omega)$. Since $\eta^m \rightarrow \eta$ in $H^{k+\frac{1}{2}}(\Sigma)$, we also have that $\mathcal{A}^m \rightarrow \mathcal{A}$, $J^m \rightarrow J$ in $H^k(\Omega)$, and $\mathcal{N}^m \rightarrow \mathcal{N}$ in $H^{k-\frac{1}{2}}(\Sigma)$.

After multiplying the equation $\operatorname{div}_{\mathcal{A}^m} u^m = F^2$ by wJ^m for $w \in C_c^\infty(\Omega)$ and integrating by parts, we see that

$$\begin{aligned} \int_{\Omega} F^2 w J^m &= \int_{\Omega} \operatorname{div}_{\mathcal{A}^m} (u^m) w J^m = - \int_{\Omega} u^m \cdot \nabla_{\mathcal{A}^m} w J^m \\ &\rightarrow - \int_{\Omega} u^0 \cdot \nabla_{\mathcal{A}} w J = \int_{\Omega} \operatorname{div}_{\mathcal{A}} (u^0) w J, \end{aligned}$$

from which we deduce that $\operatorname{div}_{\mathcal{A}} u^0 = F^2$. Then multiplying the third equation in the system (3.1) by wJ^m for $w \in {}_0H^1(\Omega)$ and integrating by parts, we have that

$$\int_{\Omega} \nabla_{\mathcal{A}^m} \theta^m \cdot \nabla_{\mathcal{A}^m} wJ^m + \int_{\Sigma} \theta^m w |\mathcal{N}^m| = \int_{\Omega} F^3 wJ^m + \int_{\Sigma} F^5 w,$$

which, by passing to the limit $m \rightarrow \infty$, reveals that

$$\int_{\Omega} \nabla_{\mathcal{A}} \theta^0 \cdot \nabla_{\mathcal{A}} wJ + \int_{\Sigma} \theta^0 w |\mathcal{N}| = \int_{\Omega} F^3 wJ + \int_{\Sigma} F^5 w.$$

Finally we multiply the first equation in the system (3.1) by wJ^m for $w \in {}_0H^1(\Omega)$ and integrate by parts to see that

$$\int_{\Omega} \frac{1}{2} \mathbb{D}_{\mathcal{A}^m} u^m : \mathbb{D}_{\mathcal{A}^m} wJ^m - p^m J^m - \theta^m \nabla_{\mathcal{A}^m} y_3^m \cdot wJ^m = \int_{\Omega} F^1 \cdot wJ^m - \int_{\Sigma} F^4 \cdot w.$$

Passing to the limit $m \rightarrow \infty$, we deduce that

$$- \int_{\Omega} \frac{1}{2} \mathbb{D}_{\mathcal{A}} u^0 : \mathbb{D}_{\mathcal{A}} wJ + p^0 \operatorname{div}_{\mathcal{A}}(w)J - \theta^0 \nabla_{\mathcal{A}} y_3 \cdot wJ = \int_{\Omega} F^1 \cdot wJ - \int_{\Sigma} F^4 \cdot w.$$

After integrating by parts again, we deduce that (u^0, p^0, θ^0) satisfies Equation (3.1). Since (u, p, θ) is the unique solution to Equation (3.1), we have that $u = u^0$, $p = p^0$ and $\theta = \theta^0$. Then, according to the weak lower semicontinuity and the uniform boundedness of the inequality (3.17), we have that

$$\begin{aligned} & \|u\|_{H^r} + \|p\|_{H^{r-1}} + \|\theta\|_{H^r} \\ & \lesssim C(\eta_0) \left(\|F^1\|_{H^{r-2}} + \|F^2\|_{H^{r-1}} + \|F^3\|_{H^{r-2}} + \|F^4\|_{H^{r-\frac{3}{2}}(\Sigma)} + \|F^5\|_{H^{r-\frac{3}{2}}(\Sigma)} \right), \end{aligned}$$

for $r = 2, \dots, k+1$, where $C(\eta_0)$ is a constant depending on $\|\eta_0\|_{H^{k+\frac{1}{2}}(\Sigma)}$. □

3.3. The \mathcal{A} -Poisson problem. Now we consider the elliptic problem

$$\begin{cases} \Delta_{\mathcal{A}} p = f^1 & \text{in } \Omega, \\ p = f^2 & \text{on } \Sigma, \\ \nabla_{\mathcal{A}} p \cdot \nu = f^3 & \text{on } \Sigma_b, \end{cases} \tag{3.18}$$

where ν is the outward-pointing normal on Σ_b . The details of the elliptic estimates of the problem (3.18) have appeared in [6] and [13], so we omit them here.

4. Linear estimates

Now we study the problem (1.8), following the path of [6]. First, we will employ two notions of solution: weak and strong.

4.1. The weak solution. Suppose that a smooth solution to the problem (1.8) exists. Then by integrating over Ω by parts, and in time from 0 to T , we see that

$$\begin{aligned} & (\partial_t u, \psi)_{L^2 \mathcal{H}^0} + \frac{1}{2} (u, \psi)_{L^2 \mathcal{H}^1} - (p, \operatorname{div}_{\mathcal{A}} \psi)_{L^2 \mathcal{H}^0} - (\theta \nabla_{\mathcal{A}} y_3, \psi)_{L^2 \mathcal{H}^0} \\ & = (F^1, \psi)_{L^2 \mathcal{H}^0} - (F^4, \psi)_{L^2 H^0(\Sigma)}, \\ & (\partial_t \theta, \phi)_{L^2 \mathcal{H}^0} + (\nabla_{\mathcal{A}} \theta, \nabla_{\mathcal{A}} \phi)_{L^2 \mathcal{H}^0} + (\theta |\mathcal{N}|, \phi)_{L^2 H^0(\Sigma)} \\ & = (F^3, \psi)_{L^2 \mathcal{H}^0} + (F^5, \psi)_{L^2 H^0(\Sigma)}, \end{aligned} \tag{4.1}$$

for $\phi, \psi \in \mathcal{H}_T^1$.

If we were to restrict the test function ψ to $\psi \in \mathcal{X}$, the term $(p, \operatorname{div}_{\mathcal{A}} \psi)_{L^2 \mathcal{H}^0}$ would vanish. Then we have a pressureless weak formulation:

$$\begin{aligned} & (\partial_t u, \psi)_{L^2 \mathcal{H}^0} + \frac{1}{2} (u, \psi)_{L^2 \mathcal{H}^1} - (\theta \nabla_{\mathcal{A}} y_3, \psi)_{L^2 \mathcal{H}^0} \\ &= (F^1, \psi)_{L^2 \mathcal{H}^0} - (F^4, \psi)_{L^2 H^0(\Sigma)}, \\ & (\partial_t \theta, \phi)_{L^2 \mathcal{H}^0} + (\nabla_{\mathcal{A}} \theta, \nabla_{\mathcal{A}} \phi)_{L^2 \mathcal{H}^0} + (\theta |\mathcal{N}|, \phi)_{L^2 H^0(\Sigma)} \\ &= (F^3, \psi)_{L^2 \mathcal{H}^0} + (F^5, \psi)_{L^2 H^0(\Sigma)}, \end{aligned} \tag{4.2}$$

This leads us to define a weak solution without pressure.

DEFINITION 4.1. *Suppose that $u_0 \in \mathcal{Y}(0)$, $\theta_0 \in H^0(\Omega)$, $F^1 - F^4 \in (\mathcal{X}_T)^*$ and $F^3 + F^5 \in (\mathcal{H}_T^1)^*$. If there exists a pair (u, θ) achieving the initial data u_0, θ_0 and satisfying $u \in \mathcal{H}_T^1$, $\theta \in \mathcal{H}_T^1$ and $\partial_t u \in (\mathcal{X}_T)^*$, $\partial_t \theta \in (\mathcal{H}_T^1)^*$, such that*

$$\begin{aligned} & \langle \partial_t u, \psi \rangle_{(\mathcal{X}_T)^*} + \frac{1}{2} (u, \psi)_{L^2 \mathcal{H}^1} - (\theta \nabla_{\mathcal{A}} y_3, \psi)_{L^2 \mathcal{H}^0} = (F^1 - F^4, \psi)_{(\mathcal{X}_T)^*}, \\ & \langle \partial_t \theta, \phi \rangle_{(\mathcal{H}_T^1)^*} + (\theta, \phi)_{L^2 \mathcal{H}^1} + (\theta |\mathcal{N}|, \phi)_{L^2 H^0(\Sigma)} = (F^3 + F^5, \psi)_{(\mathcal{H}_T^1)^*} \end{aligned} \tag{4.3}$$

holds for any $\psi \in \mathcal{X}_T$ and $\phi \in \mathcal{H}_T^1$, we call the pair (u, θ) a pressureless weak solution.

Since our aim is to construct solutions with high regularity to the problem (1.8), we will directly construct strong solutions to Equation (4.3). And it is easy to see that weak solutions will arise as a byproduct of the construction of strong solutions to the problem (1.8). Hence, we will not study the existence of weak solutions.

Now we derive some properties and uniqueness of weak solutions.

LEMMA 4.2. *Suppose that u, θ are weak solutions of Equation (4.3). Then, for almost every $t \in [0, T]$,*

$$\begin{aligned} & \frac{1}{2} \|u(t)\|_{\mathcal{H}^0(t)}^2 + \frac{1}{2} \int_0^t \|u(s)\|_{\mathcal{H}^1(s)}^2 \, ds \\ &= \frac{1}{2} \|u(0)\|_{\mathcal{H}^0(0)}^2 + (F^1 - F^4, u)_{(\mathcal{X}_t)^*} + \frac{1}{2} \int_0^t \int_{\Omega} |u(s)|^2 \partial_s J(s) \, ds + \int_0^t \int_{\Omega} \theta(s) \nabla_{\mathcal{A}} y_3 \cdot u(s) \, ds, \\ & \frac{1}{2} \|\theta(t)\|_{\mathcal{H}^0(t)}^2 + \int_0^t \|\theta(s)\|_{\mathcal{H}^1(s)}^2 \, ds + \int_0^t \int_{\Sigma} |\theta(s)|^2 |\mathcal{N}| \, ds \\ &= \frac{1}{2} \|\theta(0)\|_{\mathcal{H}^0(0)}^2 + (F^3 + F^5, \theta)_{(\mathcal{H}_t^1)^*} + \frac{1}{2} \int_0^t \int_{\Omega} |\theta(s)|^2 \partial_s J(s) \, ds. \end{aligned} \tag{4.4}$$

Also,

$$\sup_{0 \leq t \leq T} \|\theta(t)\|_{\mathcal{H}^0(t)}^2 + \|\theta\|_{\mathcal{H}_T^1}^2 \lesssim \exp(C_0(\eta)T) \left(\|\theta(0)\|_{\mathcal{H}^0(0)}^2 + \|F^3 + F^5\|_{(\mathcal{H}_T^1)^*}^2 \right), \tag{4.5}$$

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u(t)\|_{\mathcal{H}^0(t)}^2 + \|u\|_{\mathcal{H}_T^1}^2 &\lesssim \exp(CC_0(\eta)T) \left(\|u(0)\|_{\mathcal{H}^0(0)}^2 + \|\theta(0)\|_{\mathcal{H}^0(0)}^2 \right. \\ &\quad \left. + \|F^1 - F^4\|_{(\mathcal{X}_T)^*}^2 + \|F^3 + F^5\|_{(\mathcal{H}_T^1)^*}^2 \right), \end{aligned} \tag{4.6}$$

where $C_0(\eta) := \max\{\sup_{0 \leq t \leq T} \|\partial_t JK\|_{L^\infty}, \sup_{0 \leq t \leq T} \|\nabla_{\mathcal{A}} y_3\|_{L^\infty}\}$.

Proof. The identity (4.4) follows directly from Lemma 2.4 in [6] and Equation (4.3) by using the test function $\psi = u\chi_{[0,t]} \in \mathcal{X}_T$, and $\phi = \theta\chi_{[0,t]} \in \mathcal{H}_T^1$, where $\chi_{[0,t]}$ is a temporal indicator function to 1 on the interval $[0,t]$.

From Equation (4.4), we can directly derive the inequalities

$$\begin{aligned} & \frac{1}{2} \|\theta(t)\|_{\mathcal{H}^0(t)}^2 + \|\theta\|_{\mathcal{H}_t^1}^2 \\ & \leq \frac{1}{2} \|\theta(0)\|_{\mathcal{H}^0(0)}^2 + \|F^3 + F^5\|_{(\mathcal{H}_t^1)^*} \|\theta\|_{\mathcal{H}_t^1} + \frac{1}{2} C_0(\eta) \|\theta(t)\|_{\mathcal{H}_t^0}^2, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \frac{1}{2} \|u(t)\|_{\mathcal{H}^0(t)}^2 + \frac{1}{2} \|u\|_{\mathcal{H}_t^1}^2 & \leq \frac{1}{2} \|u(0)\|_{\mathcal{H}^0(0)}^2 + \|F^1 - F^4\|_{(\mathcal{X}_t)^*} \|u\|_{\mathcal{H}_t^1} \\ & \quad + \frac{1}{2} C_0(\eta) \|u(t)\|_{\mathcal{H}_t^0}^2 + CC_0(\eta) \|\theta\|_{\mathcal{H}_t^1} \|u\|_{\mathcal{H}_t^1}, \end{aligned} \tag{4.8}$$

where, for the inequality (4.8), we have used the Poincaré inequality in Lemma A.14 of [6], and

$$\|u\|_{\mathcal{H}_t^k}^2 = \int_0^t \|u(s)\|_{\mathcal{H}^k(s)}^2 ds \quad \text{for } k=0,1,$$

and similarly for $\|\theta\|_{\mathcal{H}_t^k}^2$, $\|F^1 - F^4\|_{(\mathcal{X}_t)^*}$, $\|F^3 + F^5\|_{(\mathcal{H}_t^1)^*}$. Inequalities (4.7), (4.8) and Cauchy’s inequality imply that

$$\begin{aligned} & \frac{1}{2} \|\theta(t)\|_{\mathcal{H}^0(t)}^2 + \frac{3}{4} \|\theta\|_{\mathcal{H}_t^1}^2 \\ & \leq \frac{1}{2} \|\theta(0)\|_{\mathcal{H}^0(0)}^2 + \|F^3 - F^5\|_{(\mathcal{H}_t^1)^*}^2 + \frac{1}{2} C_0(\eta) \|\theta(t)\|_{\mathcal{H}_t^0}^2, \end{aligned} \tag{4.9}$$

$$\begin{aligned} & \frac{1}{2} \|u(t)\|_{\mathcal{H}^0(t)}^2 + \frac{1}{8} \|u\|_{\mathcal{H}_t^1}^2 \\ & \leq \frac{1}{2} \|u(0)\|_{\mathcal{H}^0(0)}^2 + \|F^1 - F^4\|_{(\mathcal{X}_t)^*}^2 + \frac{1}{2} C_0(\eta) \|u(t)\|_{\mathcal{H}_t^0}^2 + CC_0(\eta) \|\theta\|_{\mathcal{H}_t^1}^2. \end{aligned}$$

Then the inequalities (4.5) and (4.6) follow from the integral inequality (4.9) and Gronwall’s lemma. □

PROPOSITION 4.3. *Weak solutions to Equation (4.3) are unique.*

Proof. Suppose that (u^1, θ^1) and (u^2, θ^2) are both weak solutions to Equation (4.3). Then (w, ϑ) , defined by $w = u^1 - u^2$ and $\vartheta = \theta^1 - \theta^2$, is a weak solution with $F^1 - F^4 = 0$, $F^3 + F^5 = 0$, $w(0) = u^1(0) - u^2(0) = 0$ and $\vartheta(0) = \theta^1(0) - \theta^2(0)$. Then the bounds (4.5) and (4.6) imply that $w = 0$ and $\vartheta = 0$. Hence, weak solutions to Equation (4.3) are unique. □

4.2. The strong solution. Before we define the strong solution, we need to define an operator D_t as

$$D_t u := \partial_t u - Ru \quad \text{for } R := \partial_t M M^{-1}, \tag{4.10}$$

with $M = K \nabla \Phi$, where K, Φ are as defined in the expressions (1.2) and (1.3). It is easily seen that D_t preserves the $\text{div}_{\mathcal{A}}$ -free condition, since

$$J \text{div}_{\mathcal{A}}(D_t v) = J \text{div}_{\mathcal{A}}(M \partial_t(M^{-1} v)) = \text{div}(\partial_t(M^{-1} v)) = \partial_t \text{div}(M^{-1} v) = \partial_t(J \text{div}_{\mathcal{A}} v),$$

where the equality $J \operatorname{div}_{\mathcal{A}} v = \operatorname{div}(M^{-1}v)$ can be found on page 299 of [6].

DEFINITION 4.4. *Suppose that the forcing functions satisfy*

$$\begin{aligned} F^1 &\in L^2([0, T]; H^1(\Omega)) \cap C^0([0, T]; H^0(\Omega)), \\ F^3 &\in L^2([0, T]; H^1(\Omega)) \cap C^0([0, T]; H^0(\Omega)), \\ F^4 &\in L^2([0, T]; H^{\frac{3}{2}}(\Sigma)) \cap C^0([0, T]; H^{\frac{1}{2}}(\Sigma)), \\ \partial_t(F^1 - F^4) &\in L^2([0, T]; ({}_0H^1(\Omega))^*), \quad \partial_t(F^3 + F^5) \in L^2([0, T]; ({}_0H^1(\Omega))^*). \end{aligned} \tag{4.11}$$

We also assume that $u_0 \in H^2 \cap \mathcal{X}(0)$ and $\theta_0 \in H^2 \cap \mathcal{H}^1(0)$. If there exists a pair (u, p, θ) achieving the initial data u_0, θ_0 and satisfying

$$\begin{aligned} u &\in L^2([0, T]; H^3) \cap C^0([0, T]; H^2) \cap \mathcal{X}_T, \quad \partial_t u \in L^2([0, T]; H^1) \cap C^0([0, T]; H^0), \\ D_t u &\in \mathcal{X}_T, \quad \partial_t^2 u \in \mathcal{X}_T^*, \quad p \in L^2([0, T]; H^2) \cap C^0([0, T]; H^1), \\ \theta &\in L^2([0, T]; H^3) \cap C^0([0, T]; H^2), \quad \partial_t \theta \in L^2([0, T]; H^1) \cap C^0([0, T]; H^0), \\ \partial_t^2 \theta &\in (\mathcal{H}_T^1)^*, \end{aligned} \tag{4.12}$$

such that they satisfy Equation (1.8) in the strong sense, we call it a strong solution.

Then, we have to prove the lower regularity of strong solutions.

THEOREM 4.5. *Suppose that the forcing terms and the initial data satisfy the condition in Definition 4.4, and that $u_0, F^4(0)$ satisfy the compatibility condition*

$$\Pi_0(F^4(0) + \mathbb{D}_{\mathcal{A}} u_0 \mathcal{N}_0) = 0, \quad \text{where } \mathcal{N}_0 = (-\partial_1 \eta_0, -\partial_2 \eta_0, 1), \tag{4.13}$$

and Π_0 is an orthogonal projection onto the tangent space of the surface $\{x_3 = \eta_0\}$ defined by

$$\Pi_0 v = v - (v \cdot \mathcal{N}_0) \mathcal{N}_0 |\mathcal{N}_0|^{-2}. \tag{4.14}$$

Then there exists a strong solution (u, p, θ) satisfying the condition (4.12). Moreover,

$$\begin{aligned} &\|u\|_{L^\infty H^2}^2 + \|u\|_{L^2 H^3}^2 + \|\partial_t u\|_{L^\infty H^0}^2 + \|\partial_t u\|_{L^2 H^1}^2 + \|\partial_t^2 u\|_{(\mathcal{X}_T)^*}^2 + \|p\|_{L^\infty H^1}^2 + \|p\|_{L^2 H^2}^2 \\ &\quad + \|\theta\|_{L^\infty H^2}^2 + \|\theta\|_{L^2 H^3}^2 + \|\partial_t \theta\|_{L^\infty H^0}^2 + \|\partial_t \theta\|_{L^2 H^1}^2 + \|\partial_t^2 \theta\|_{(\mathcal{H}_T^1)^*}^2 \\ &\lesssim P(\|\eta_0\|_{H^{5/2}}) (1 + \mathcal{K}(\eta)) \exp(C(1 + \mathcal{K}(\eta))T) \left(\|u_0\|_{H^2}^2 + \|\theta_0\|_{H^2}^2 + \|F^1(0)\|_{H^0}^2 \right. \\ &\quad + \|F^3(0)\|_{H^0}^2 + \|F^4(0)\|_{H^{1/2}(\Sigma)}^2 + \|F^1\|_{L^2 H^1}^2 + \|F^3\|_{L^2 H^1}^2 + \|F^4\|_{L^2 H^{3/2}(\Sigma)}^2 \\ &\quad \left. + \|F^5\|_{L^2 H^{3/2}(\Sigma)}^2 + \|\partial_t(F^1 - F^4)\|_{(\mathcal{X}_T)^*}^2 + \|\partial_t(F^3 + F^5)\|_{(\mathcal{H}_T^1)^*}^2 \right), \end{aligned} \tag{4.15}$$

where C is a constant independent of η and $\mathcal{K}(\eta)$ is defined as

$$\mathcal{K}(\eta) := \sup_{0 \leq t \leq T} (\|\eta\|_{H^{9/2}}^2 + \|\partial_t \eta\|_{H^{7/2}}^2 + \|\partial_t^2 \eta\|_{H^{5/2}}^2). \tag{4.16}$$

The initial pressure, $p(0) \in H^1(\Omega)$, is determined by the terms $u_0, \theta_0, F^1(0), F^4(0)$ as a weak solution to

$$\begin{cases} \operatorname{div}_{\mathcal{A}_0} (\nabla_{\mathcal{A}_0} p(0) - F^1(0) - \theta_0 \nabla_{\mathcal{A}_0} y_{3,0}) = -\operatorname{div}_{\mathcal{A}_0} (R(0)u_0) \in H^0(\Omega), \\ p(0) = (F^4(0) + \mathbb{D}_{\mathcal{A}_0} u_0 \mathcal{N}_0) \cdot \mathcal{N}_0 |\mathcal{N}_0|^{-2} \in H^{1/2}(\Sigma), \\ (\nabla_{\mathcal{A}_0} p(0) - F^1(0)) \cdot \nu = \Delta_{\mathcal{A}_0} u_0 \cdot \nu \in H^{-1/2}(\Sigma_b), \end{cases} \tag{4.17}$$

where $y_{3,0}$ is in terms of η_0 . Also, $\partial_t\theta(0)$ satisfies

$$\partial_t\theta(0) = \Delta_{\mathcal{A}_0}\theta_0 + F^3(0) \in H^0(\Omega), \tag{4.18}$$

and $D_tu(0) = \partial_tu(0) - R(0)u_0$ satisfies

$$D_tu(0) = \Delta_{\mathcal{A}_0}u_0 - \nabla_{\mathcal{A}_0}p(0) + F^1(0) + \theta_0e_3 - R(0)u_0 \in \mathcal{D}(0). \tag{4.19}$$

Moreover, $\partial_t\theta$ satisfies

$$\begin{cases} \partial_t(\partial_t\theta) - \Delta_{\mathcal{A}}(\partial_t\theta) = \partial_tF^3 + G^3 & \text{in } \Omega, \\ \nabla_{\mathcal{A}}(\partial_t\theta) \cdot \mathcal{N} + \partial_t\theta|\mathcal{N}| = \partial_tF^5 + G^5 & \text{on } \Sigma, \\ \partial_t\theta = 0 & \text{on } \Sigma_b, \end{cases} \tag{4.20}$$

and D_tu satisfies

$$\begin{cases} \partial_t(D_tu) - \Delta_{\mathcal{A}}(D_tu) + \nabla_{\mathcal{A}}(\partial_tp) - D_t(\theta\nabla_{\mathcal{A}}y_3) = D_tF^1 + G^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}}(D_tu) = 0 & \text{in } \Omega, \\ S_{\mathcal{A}}(\partial_tp, D_tu)\mathcal{N} = \partial_tF^4 + G^4 & \text{on } \Sigma, \\ D_tu = 0 & \text{on } \Sigma_b, \end{cases} \tag{4.21}$$

in the weak sense of Equation (4.3), where G^1 is defined by

$$G^1 = -(R + \partial_tJK)\Delta_{\mathcal{A}}u - \partial_tRu + (\partial_tJK + R + R^\top)\nabla_{\mathcal{A}}p + \operatorname{div}_{\mathcal{A}}(\mathbb{D}_{\mathcal{A}}(Ru) - R\mathbb{D}_{\mathcal{A}}u + \mathbb{D}_{\partial_t\mathcal{A}}u)$$

(R^\top denoting the matrix transpose of R), G^3 by

$$G^3 = -\partial_tJK\Delta_{\mathcal{A}}\theta + \operatorname{div}_{\mathcal{A}}(-R\nabla_{\mathcal{A}}\theta + \nabla_{\partial_t\mathcal{A}}\theta),$$

G^4 by

$$G^4 = \mathbb{D}_{\mathcal{A}}(Ru)\mathcal{N} - (pI - \mathbb{D}_{\mathcal{A}}u)\partial_t\mathcal{N} + \mathbb{D}_{\partial_t\mathcal{A}}u\mathcal{N},$$

and G^5 by

$$G^5 = -\nabla_{\mathcal{A}}\theta \cdot \partial_t\mathcal{N} - \nabla_{\partial_t\mathcal{A}}\theta \cdot \mathcal{N} - \theta\partial_t|\mathcal{N}|.$$

More precisely, Equations (4.20) and (4.21) hold in the weak sense of Equation (4.3) in that

$$\begin{aligned} & \langle \partial_t^2\theta, \phi \rangle_{(\mathcal{H}_T^1)^*} + (\partial_t\theta, \phi)_{\mathcal{H}_T^1} + (\partial_t\theta|\mathcal{N}|, \phi)_{L^2H^0(\Sigma)} \\ &= \langle \partial_t(F^3 + F^5) \rangle_{(\mathcal{H}_T^1)^*} + (\partial_tJKF^3, \phi)_{\mathcal{H}_T^0} - (\partial_tJK\partial_t\theta, \phi)_{\mathcal{H}_T^0} \\ & \quad - \int_0^T \int_{\Omega} (\partial_tJK\nabla_{\mathcal{A}}\theta \cdot \nabla_{\mathcal{A}}\phi + \nabla_{\partial_t\mathcal{A}}\theta \cdot \nabla_{\mathcal{A}}\phi + \nabla_{\mathcal{A}}\theta \cdot \nabla_{\partial_t\mathcal{A}}\phi) J \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} & \langle \partial_tD_tu, \psi \rangle_{(\mathcal{X}_T)^*} + \frac{1}{2}(\partial_tu, \psi)_{\mathcal{H}_T^1} \\ &= \langle \partial_t(F^1 - F^4), \psi \rangle_{(\mathcal{X}_T)^*} + (\partial_t(\theta\nabla_{\mathcal{A}}y_3), \psi)_{\mathcal{H}^0} - (\partial_tRu + R\partial_tu, \psi)_{\mathcal{H}_T^0} \\ & \quad + (\partial_tJKF^1, \psi)_{\mathcal{H}_T^0} - (\partial_tJK\theta e_3, \psi)_{\mathcal{H}_T^0} - (\partial_tJK\partial_tu, \psi)_{\mathcal{H}_T^0} - (p, \operatorname{div}_{\mathcal{A}}(R\psi))_{\mathcal{H}_T^0} \end{aligned}$$

$$-\frac{1}{2} \int_0^T \int_{\Omega} (\partial_t JK \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\mathcal{A}} \psi + \mathbb{D}_{\partial_t \mathcal{A}} u : \mathbb{D}_{\mathcal{A}} \psi + \mathbb{D}_{\mathcal{A}} u : \mathbb{D}_{\partial_t \mathcal{A}} \psi) J \tag{4.23}$$

for all $\phi \in \mathcal{H}_T^1$, $\psi \in \mathcal{X}_T$.

Proof. Here we will use the Galerkin method; the reader is referred to [5].

Step 1. The construction of approximate solutions for θ . Since the scalar-valued space $H^2(\Omega) \cap_0 H^1(\Omega)$ is separable, we can choose a countable basis $\{\tilde{w}^j\}_{j=1}^\infty$. Note that this basis is time-independent. Now, we need to construct a time-dependent basis for $H^2 \cap \mathcal{H}^1$. We define $\phi^j = \phi^j(t) := K(t)\tilde{w}^j$. According to the Proposition 2.2, $\phi^j(t) \in H^2(\Omega) \cap \mathcal{H}^1(t)$, and $\{\phi^j(t)\}_{j=1}^\infty$ is a basis of $H^2(\Omega) \cap \mathcal{H}^1(t)$ for each $t \in [0, T]$. Moreover,

$$\partial_t \phi^j(t) = \partial_t K(t)\tilde{w}^j = \partial_t K JK \tilde{w}^j = \partial_t K J \phi^j(t), \tag{4.24}$$

which allows us to express $\partial_t \phi^j$ in terms of ϕ^j . For any integer $m \geq 1$, we define the finite-dimensional space $\mathcal{H}_m^1(t) := \text{span} \{\phi^1(t), \dots, \phi^m(t)\} \subset H^2(\Omega) \cap \mathcal{H}^1(t)$ and we define $\mathcal{P}_t^m : H^2(\Omega) \rightarrow \mathcal{H}_m^1(t)$ for $H^2(\Omega)$ orthogonal projection onto $\mathcal{H}_m^1(t)$. Clearly, if $\theta \in H^2(\Omega) \cap \mathcal{H}^1(t)$, $\mathcal{P}_t^m \theta \rightarrow \theta$ as $m \rightarrow \infty$.

For each $m \geq 1$, we define an approximate solution

$$\theta^m = d_j^m(t)\phi^j(t), \quad \text{with } d_j^m(t) : [0, T] \rightarrow \mathbb{R} \quad \text{for } j = 1, \dots, m,$$

where as usual we use the Einstein convention of summation of the repeated index j . We want to choose d_j^m such that

$$(\partial_t \theta^m, \phi)_{\mathcal{H}^0} + (\theta^m, \phi)_{\mathcal{H}^1} + (\theta^m |\mathcal{N}|, \phi)_{H^0(\Sigma)} = (F^3, \phi)_{\mathcal{H}^0} + (F^5, \phi)_{H^0(\Sigma)}, \tag{4.25}$$

with the initial data $\theta^m(0) = \mathcal{P}_0^m \theta_0 \in \mathcal{H}_m^1(0)$ for each $\phi \in \mathcal{H}_m^1(t)$. And Equation (4.25) is equivalent to the system of ODEs for d_j^m :

$$\begin{aligned} & d_j^m (\phi^j, \phi^k)_{\mathcal{H}^0} + d_j^m \left((\partial_t K J \phi^j, \phi^k)_{\mathcal{H}^0} + (\phi^j, \phi^k)_{\mathcal{H}^1} + (\phi^j |\mathcal{N}|, \phi^k)_{H^0(\Sigma)} \right) \\ &= (F^3, \phi^k)_{\mathcal{H}^0} + (F^5, \phi^k)_{H^0(\Sigma)} \end{aligned} \tag{4.26}$$

for $j, k = 1, \dots, m$. The $m \times m$ matrix with j, k entry $(\phi^j, \phi^k)_{\mathcal{H}^0}$ is invertible, the coefficients of the linear system (4.26) are $C^1([0, T])$, and the forcing terms are $C^0([0, T])$, so the usual well-posedness of ODEs guarantees that the existence of a unique solution $d_j^m \in C^1([0, T])$ to Equation (4.26) that satisfies the initial data. This provides the desired solution, θ^m , to Equation (4.25). Since F^3, F^5 satisfy the condition (4.11), Equation (4.26) may be differentiated in time to see that $d_j^m \in C^{1,1}([0, T])$, which means d_j^m is twice differentiable almost everywhere in $[0, T]$.

Step 2. The energy estimates for θ^m . Since $\theta^m(t) \in \mathcal{H}_m^1(t)$, we take $\phi = \theta^m$ as a test function in Equation (4.25). Using the Poincaré-type inequalities in Lemma A.14 of [6] and usual trace theory, we have

$$\partial_t \frac{1}{2} \|\theta^m\|_{\mathcal{H}^0}^2 + \|\theta^m\|_{\mathcal{H}^1}^2 \lesssim (\|F^3\|_{\mathcal{H}^0} + \|F^5\|_{H^{1/2}(\Sigma)}) \|\theta^m\|_{\mathcal{H}^1} - \frac{1}{2} \int_{\Omega} |\theta^m|^2 \partial_t J.$$

Then, applying Cauchy's inequality, we derive that

$$\partial_t \frac{1}{2} \|\theta^m\|_{\mathcal{H}^0}^2 + \frac{1}{4} \|\theta^m\|_{\mathcal{H}^1}^2 \lesssim \|F^3\|_{\mathcal{H}^0}^2 + \|F^5\|_{H^{1/2}(\Sigma)}^2 + C_0(\eta) \frac{1}{2} \|\theta^m\|_{\mathcal{H}^0}^2$$

with $C_0(\eta) := 1 + \sup_{0 \leq t \leq T} \|\partial_t JK\|_{L^\infty}$. Using Lemma 2.9 in [13], we have

$$\begin{aligned} \|\theta^m(0)\|_{\mathcal{H}^0} &\leq P(\|\eta_0\|_{H^{5/2}})\|\theta^m(0)\|_{H^0} \leq P(\|\eta_0\|_{H^{5/2}})\|\theta^m(0)\|_{H^2} \\ &= P(\|\eta_0\|_{H^{5/2}})\|\mathcal{P}_0^m \theta_0\|_{H^2} \leq P(\|\eta_0\|_{H^{5/2}})\|\theta_0\|_{H^2}. \end{aligned} \tag{4.27}$$

Now, we can utilize Gronwall’s lemma to deduce energy estimates for θ^m :

$$\begin{aligned} &\sup_{0 \leq t \leq T} \|\theta^m\|_{\mathcal{H}^0}^2 + \|\theta^m\|_{\mathcal{H}_T^1}^2 \\ &\lesssim P(\|\eta_0\|_{H^{5/2}}) \exp(C_0(\eta)T) (\|\theta_0\|_{H^2}^2 + \|F^3\|_{\mathcal{H}_T^0}^2 + \|F^5\|_{L^2 H^{1/2}(\Sigma)}^2). \end{aligned} \tag{4.28}$$

Step 3. Estimates for $\partial_t \theta^m(0)$. If $\theta \in H^2(\Omega) \cap \mathcal{H}^1(t)$, $\phi \in \mathcal{H}^1$, integration by parts reveals that

$$(\theta, \phi)_{\mathcal{H}^1} = \int_{\Omega} -\Delta_{\mathcal{A}} \theta \phi J + \int_{\Sigma} (\nabla_{\mathcal{A}} \theta \cdot \mathcal{N}) \phi = (-\Delta_{\mathcal{A}} \theta, \phi)_{\mathcal{H}^0} + (\nabla_{\mathcal{A}} \theta \cdot \mathcal{N}, \phi)_{H^0(\Sigma)}. \tag{4.29}$$

Evaluating Equation (4.25) at $t=0$ and employing Equation (4.29), we have that

$$(\partial_t \theta^m(0), \phi)_{\mathcal{H}^0} = (\Delta_{\mathcal{A}_0} \theta^m(0) + F^3(0), \phi)_{\mathcal{H}^0}, \tag{4.30}$$

for all $\phi \in \mathcal{H}_m^1(t)$.

By virtue of Equation (4.24), we have that

$$\partial_t \theta^m - \partial_t K(t)J(t)\theta^m(t) = \dot{d}_j^m(t)\phi^j(t) \in \mathcal{H}_m^1(t), \tag{4.31}$$

so that $\phi = \partial_t \theta^m(0) - \partial_t K(0)J(0)\theta^m(0) \in \mathcal{H}_m^1(0)$ is a choice for the test function in Equation (4.30). So using this test function in Equation (4.30), we have

$$\begin{aligned} \|\partial_t \theta^m(0)\|_{\mathcal{H}^0}^2 &\leq \|\partial_t K(0)J(0)\theta^m(0)\|_{\mathcal{H}^0} \|\partial_t \theta^m(0)\|_{\mathcal{H}^0} \\ &\quad + \|\partial_t \theta^m(0) - \partial_t K(0)J(0)\theta^m(0)\|_{\mathcal{H}^0} \|\Delta_{\mathcal{A}_0} \theta^m(0) + F^3(0)\|_{\mathcal{H}^0}. \end{aligned} \tag{4.32}$$

Then after using the inequality (4.27) and Cauchy’s inequality for the right-hand side of the inequality (4.32), we have the bound

$$\|\partial_t \theta^m(0)\|_{\mathcal{H}^0}^2 \lesssim C_1(\eta) (\|\theta_0\|_{H^2}^2 + \|F^3(0)\|_{\mathcal{H}^0}^2) \tag{4.33}$$

with $C_1(\eta) = P(\|\eta_0\|_{H^{5/2}}) (1 + \|\partial_t K(0)J(0)\|_{L^\infty}^2 + \|\mathcal{A}_0\|_{C^1}^2)$.

Step 4. Energy estimates for $\partial_t \theta^m$. Now, suppose that $\phi(t) = c_j^m(t)\phi^j$ for $c_j^m \in C^{0,1}([0, T])$, $j = 1, \dots, m$; it is proved as in Equation (4.31), that $\partial_t \phi - \partial_t K(t)J(t)\phi \in \mathcal{H}_m^1(t)$ as well. Then in Equation (4.25), using this ϕ , and temporally differentiating the resulting equation, and then subtracting from the result Equation (4.25) with test function $\partial_t \phi - \partial_t K(t)J(t)\phi$, we find that

$$\begin{aligned} &\langle \partial_t^2 \theta^m, \phi \rangle_{(\mathcal{H}^1)^*} + (\partial_t \theta^m, \phi)_{\mathcal{H}^1} + (\partial_t \theta^m | \mathcal{N} |, \phi)_{H^0(\Sigma)} \\ &= \langle \partial_t (F^3 + F^5), \phi \rangle_{(\mathcal{H}^1)^*} + (F^3, (\partial_t KJ + \partial_t JK)\phi)_{\mathcal{H}^0} + (F^5, \partial_t KJ\phi)_{H^0(\Sigma)} \\ &\quad - (\partial_t \theta^m, (\partial_t KJ + \partial_t JK)\phi)_{\mathcal{H}^0} - (\theta^m, \partial_t KJ\phi)_{\mathcal{H}^1} - (\theta^m, \partial_t KJ\phi)_{H^0(\Sigma)} \\ &\quad - \int_{\Omega} (\partial_t JK \nabla_{\mathcal{A}} \theta^m \cdot \nabla_{\mathcal{A}} \phi + \nabla_{\partial_t \mathcal{A}} \theta^m \cdot \nabla_{\mathcal{A}} \phi + \nabla_{\mathcal{A}} \theta^m \cdot \nabla_{\partial_t \mathcal{A}} \phi) J. \end{aligned} \tag{4.34}$$

According to Equation (4.31) and the fact that $d_j^m(t)$ is twice differentiable almost everywhere as we have pointed in the first step, we use $\phi = \partial_t \theta^m - \partial_t K J \theta^m$ as a test function in Equation (4.34). Utilizing Cauchy's inequality, trace theory and Remark 2.3 in [6], we have that

$$\begin{aligned} & \partial_t \left(\frac{1}{2} \|\partial_t \theta^m\|_{\mathcal{H}^0}^2 - (\partial_t \theta^m, \partial_t K J \theta^m)_{\mathcal{H}^0} \right) + \frac{1}{4} \|\partial_t \theta^m\|_{\mathcal{H}^1}^2 \\ & \leq C_0(\eta) \left(\frac{1}{2} \|\theta^m\|_{\mathcal{H}^0}^2 - (\partial_t \theta^m, \partial_t K J \theta^m)_{\mathcal{H}^0} \right) + C_2(\eta) \|\partial_t \theta^m\|_{\mathcal{H}^1}^2 \\ & \quad + C \left(\|F^3\|_{\mathcal{H}^0}^2 + \|F^5\|_{H^{1/2}(\Sigma)}^2 \right) + C \|\partial_t(F^3 + F^5)\|_{(\mathcal{H}^1)^*}, \end{aligned} \tag{4.35}$$

where $C_2(\eta)$ is defined as

$$\begin{aligned} C_2(\eta) := & \sup_{0 \leq t \leq T} \left[1 + \|\partial_t(\partial_t K J)\|_{L^\infty}^2 + \|\partial_t K J\|_{C^1}^2 + \|\partial_t \mathcal{A}\|_{L^\infty}^2 \right. \\ & \left. + (1 + \|\mathcal{A}\|_{L^\infty}^2)(1 + \|\partial_t J K\|_{L^\infty}^2) \right] (1 + \|\partial_t K J\|_{C^1}^2). \end{aligned}$$

Then according to Cauchy's inequality and Gronwall's lemma, the inequality (4.35) implies that

$$\begin{aligned} & \sup_{0 \leq t \leq T} (\|\partial_t \theta^m\|_{\mathcal{H}^0}^2 + \|\partial_t \theta^m\|_{\mathcal{H}^1}^2) \\ & \lesssim \exp(C_0(\eta)T) \left(\|\partial_t \theta^m(0)\|_{\mathcal{H}^0}^2 + C_1(\eta) \|\theta^m(0)\|_{\mathcal{H}^0}^2 + \|F^3\|_{\mathcal{H}_T^0}^2 \right. \\ & \quad \left. + \|F^5\|_{L^2 H^{1/2}}^2 + \|\partial_t(F^3 + F^5)\|_{(\mathcal{H}_T^1)^*}^2 \right) \\ & \quad + C_2(\eta) \left(\sup_{0 \leq t \leq T} \|\theta^m\|_{\mathcal{H}^0}^2 + \int_0^T \exp(C_0(\eta)(T-s)) \|\theta^m(s)\|_{\mathcal{H}^1}^2 ds \right). \end{aligned} \tag{4.36}$$

Now, the energy estimates for $\partial_t \theta^m$ are deduced by combining the estimate (4.36) with the estimates (4.27), (4.28) and (4.33),

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|\partial_t \theta^m\|_{\mathcal{H}^0}^2 + \|\partial_t \theta^m\|_{\mathcal{H}_T^1}^2 \\ & \lesssim (C_1(\eta) + C_2(\eta)) \exp(C_0(\eta)T) (\|\theta^m(0)\|_{\mathcal{H}^0}^2 + \|F^3(0)\|_{\mathcal{H}^0}^2) \\ & \quad + \exp(C_0(\eta)T) \left[C_2(\eta) \left(\|F^3\|_{\mathcal{H}_T^0}^2 + \|F^5\|_{L^2 H^{1/2}}^2 \right) + \|\partial_t(F^3 + F^5)\|_{(\mathcal{H}_T^1)^*}^2 \right]. \end{aligned} \tag{4.37}$$

Step 5. Improved estimates for θ^m . Using the $\phi = \partial_t \theta^m - \partial_t K J \theta^m \in \mathcal{H}_m^1(t)$ as a test function in Equation (4.25), we can improve the energy estimates for θ^m :

$$\begin{aligned} & \partial_t \frac{1}{2} \left(\|\theta^m\|_{\mathcal{H}^1}^2 + \|\theta^m\|_{H^0(\Sigma)}^2 \right) + \|\partial_t \theta^m\|_{\mathcal{H}^0}^2 \\ & = (\partial_t \theta^m, \partial_t K J \theta^m)_{\mathcal{H}^0} + (\theta^m, \partial_t K J \theta^m)_{\mathcal{H}^1} + (F^3, \partial_t \theta^m - \partial_t K J \theta^m)_{\mathcal{H}^0} \\ & \quad + (F^5, \partial_t \theta^m - \partial_t K J \theta^m)_{H^0(\Sigma)} + \int_{\Omega} \left(\nabla_{\mathcal{A}} \theta^m \cdot \nabla_{\partial_t \mathcal{A}} \theta^m + \partial_t J K \frac{|\nabla_{\mathcal{A}} \theta^m|^2}{2} J \right). \end{aligned} \tag{4.38}$$

Since we have already controlled $\|\theta^m\|_{\mathcal{H}_T^1}^2$ and $\|\partial_t \theta^m\|_{\mathcal{H}_T^1}^2$, integrating Equation (4.38) in time implies that

$$\sup_{0 \leq t \leq T} \|\theta^m\|_{\mathcal{H}^1}^2 + \|\partial_t \theta^m\|_{\mathcal{H}_T^0}^2$$

$$\begin{aligned} &\lesssim P(\|\eta_0\|_{H^{5/2}})(C_1(\eta) + C_2(\eta))\exp(C_0(\eta)T) (\|\theta_0\|_{H^0}^2 + \|F^3(0)\|_{\mathcal{H}^0}^2) \\ &\quad + P(\|\eta_0\|_{H^{5/2}})\exp(C_0(\eta)T) \left[C_2(\eta) \left(\|F^3\|_{\mathcal{H}_T^0}^2 + \|F^5\|_{L^2H^{1/2}}^2 \right) \right. \\ &\quad \left. + \|\partial_t(F^3 + F^5)\|_{(\mathcal{H}_T^1)^*}^2 \right]. \end{aligned} \tag{4.39}$$

Step 6. Uniform bounds for the estimates (4.37) and (4.39). Now, we seek to estimate the constants $C_i(\eta)$, $i = 0, 1, 2$ in terms of the quantity $\mathcal{K}(\eta)$. A direct computation combined with Lemma A.10 in [6] reveal that

$$C_0(\eta) + C_1(\eta) + C_2(\eta) \leq C(1 + \mathcal{K}(\eta)), \tag{4.40}$$

for a constant C independent of η .

Step 7. Passing to the limit. According to the energy estimates (4.37) and (4.39) and Lemma 2.1, we know that the sequence $\{\theta^m\}$ is uniformly bounded in $L^\infty H^1$ and $\{\partial_t \theta^m\}$ is uniformly bounded in $L^\infty H^0 \cap L^2 H^1$. Then, up to extracting a subsequence, we know that

$$\theta^m \overset{*}{\rightharpoonup} \theta \text{ weakly-}^* \text{ in } L^\infty H^1, \partial_t \theta^m \overset{*}{\rightharpoonup} \partial_t \theta \text{ in } L^\infty H^0, \partial_t \theta^m \rightharpoonup \partial_t \theta \text{ weakly in } L^2 H^1,$$

as $m \rightarrow \infty$. By lower semicontinuity, the energy estimates reveal that

$$\|\theta\|_{L^\infty H^1}^2 + \|\partial_t \theta\|_{L^\infty H^0}^2 + \|\partial_t \theta\|_{L^2 H^1}^2$$

is bounded from above by the right-hand side of (4.15).

According these convergence results, we can integrate (4.34) temporally from 0 to T and let $m \rightarrow \infty$ to deduce that $\partial_t^2 \theta^m \rightharpoonup \partial_t^2 \theta$ weakly in $(\mathcal{H}_T^1)^*$, with an action of $\partial_t^2 \theta$ on an element $\phi \in \mathcal{H}_T^1$ defined by replacing θ^m with θ everywhere in (4.34). From passing to the limit in (4.34), it is straightforward to show that $\|\partial_t^2 \theta\|_{(\mathcal{H}_T^1)^*}^2$ is bounded from above by the right-hand side of the estimate (4.15). This bound shows that $\partial_t \theta \in C^0 L^2$.

Step 8. In the limit, Equation (4.25) implies that for almost every t ,

$$(\partial_t \theta, \phi)_{\mathcal{H}^0} + (\theta, \phi)_{\mathcal{H}^1} + (\theta |\mathcal{N}|, \phi)_{H^0(\Sigma)} = (F^3, \phi)_{\mathcal{H}^0} + (F^5, \phi)_{H^0(\Sigma)} \quad \text{for every } \phi \in \mathcal{H}^1. \tag{4.41}$$

For almost every $t \in [0, T]$, $\theta(t)$ is the unique weak solution to the elliptic problem (3.1) in the sense of Equation (3.2), with F^3 replaced by $F^3(t) - \partial_t \theta(t)$ and F^5 replaced by $F^5(t)$. Since $F^3(t) - \partial_t \theta(t) \in H^0(\Omega)$ and $F^5(t) \in H^{1/2}(\Sigma)$, Lemma 3.3 shows that this elliptic problem admits a unique strong solution, which must coincide with the weak solution. Then applying Proposition 3.5, we have the bound

$$\|\theta(t)\|_{H^r}^2 \lesssim C(\eta_0) \left(\|\partial_t \theta(t)\|_{\mathcal{H}^{r-2}}^2 + \|F^3(t)\|_{\mathcal{H}^{r-2}}^2 + \|F^5(t)\|_{H^{r-3/2}(\Sigma)}^2 \right) \tag{4.42}$$

when $r = 2, 3$. When $r = 2$, we take the supremum of the inequality (4.42) over $t \in [0, T]$, and when $r = 3$, we integrate over $[0, T]$; the resulting inequalities imply that $\theta \in L^\infty H^2 \cap L^2 H^3$ with estimates as in the inequality (4.15).

Then for the linear Navier–Stokes equations, the process is exactly the same as in [6]. Then we know that (u, p, θ) is a strong solution of Equation (1.8) with the estimates as in the inequality (4.15).

Step 9. The weak solution satisfied by $\partial_t\theta$ and D_tu . We may integrate Equation (4.34) in time from 0 to T and pass the limit $m \rightarrow \infty$. For any $\phi \in \mathcal{H}^1$, we have $\partial_t KJ\phi \in \mathcal{H}^1$, so that we may substitute $\partial_t KJ\phi$ for ϕ in Equation (4.41); this yields

$$\begin{aligned} & \langle \partial_t^2 \theta, \phi \rangle_{(\mathcal{H}_T^1)^*} + (\partial_t \theta, \phi)_{\mathcal{H}_T^1} + (\partial_t \theta | \mathcal{N} |, \phi)_{L^2 H^0(\Sigma)} \\ &= \langle \partial_t (F^3 + F^5) \rangle_{(\mathcal{H}_T^1)^*} + (\partial_t JK F^3, \phi)_{\mathcal{H}_T^0} - (\partial_t JK \partial_t \theta, \phi)_{\mathcal{H}_T^0} \\ & \quad - \int_0^T \int_{\Omega} (\partial_t JK \nabla_{\mathcal{A}} \theta \cdot \nabla_{\mathcal{A}} \phi + \nabla_{\partial_t \mathcal{A}} \theta \cdot \nabla_{\mathcal{A}} \phi + \nabla_{\mathcal{A}} \theta \cdot \nabla_{\partial_t \mathcal{A}} \phi) J \end{aligned} \tag{4.43}$$

for all $\phi \in \mathcal{H}_T^1$. This is exactly Equation (4.22). To justify that Equation (4.22) implies Equation (4.20), we may integrate by parts for the equality

$$\begin{aligned} & - \int_0^T \int_{\Omega} (\partial_t JK \nabla_{\mathcal{A}} \theta \cdot \nabla_{\mathcal{A}} \phi + \nabla_{\partial_t \mathcal{A}} \theta \cdot \nabla_{\mathcal{A}} \phi + \nabla_{\mathcal{A}} \theta \cdot \nabla_{\partial_t \mathcal{A}} \phi) J \\ &= - \int_0^T \int_{\Omega} (-R \nabla_{\mathcal{A}} u + \nabla_{\partial_t \mathcal{A}} u) \cdot \nabla_{\mathcal{A}} \phi J \\ &= (\operatorname{div}_{\mathcal{A}} (-R \nabla_{\mathcal{A}} u + \nabla_{\partial_t \mathcal{A}} u), \phi)_{\mathcal{H}_T^0} - \langle \nabla_{\mathcal{A}} u \cdot \partial_t \mathcal{N} + \nabla_{\partial_t \mathcal{A}} u \cdot \mathcal{N}, \phi \rangle_{L^2 H^{-1/2}}. \end{aligned} \tag{4.44}$$

We then may deduce from Equation (4.22) that $\partial_t\theta$ is a weak solution of Equation (4.20) in the sense of Equation (4.3) with $\partial_t\theta(0) \in \mathcal{H}^0(0)$. Then we may appeal to the computation in [6] to deduce that $p(0)$ satisfies the Equation (4.17) and D_tu is a weak solution of Equation (4.21) in the sense of (4.3) with $D_tu(0) \in \mathcal{Y}(0)$. \square

4.3. Higher regularity. In order to state our higher regularity results for Equation (1.8), we need to construct the initial data and compatible conditions. First, we define the vector or scalar fields $\mathfrak{E}^{01}, \mathfrak{E}^{02}, \mathfrak{E}^1, \mathfrak{E}^3$ in Ω and $\mathfrak{E}^4, \mathfrak{E}^5$ on Σ by

$$\begin{aligned} \mathfrak{E}^{01}(G^1, v, q) &= \Delta_{\mathcal{A}} v - \nabla_{\mathcal{A}} q + G^1 - Rv, \\ \mathfrak{E}^{02}(G^3, \Theta) &= \Delta_{\mathcal{A}} \Theta + G^3, \\ \mathfrak{E}^1(v, q) &= -(R + \partial_t JK) \Delta_{\mathcal{A}} v - \partial_t Rv + (\partial_t JK + R + R^\top) \nabla_{\mathcal{A}} q \\ & \quad + \operatorname{div}_{\mathcal{A}} (\mathbb{D}_{\mathcal{A}}(Rv) - R \mathbb{D}_{\mathcal{A}} v + \mathbb{D}_{\partial_t \mathcal{A}} v), \\ \mathfrak{E}^3(\Theta) &= -\partial_t JK \Delta_{\mathcal{A}} \Theta + \operatorname{div}_{\mathcal{A}} (-R \nabla_{\mathcal{A}} \Theta + \nabla_{\partial_t \mathcal{A}} \Theta), \\ \mathfrak{E}^4(v, q) &= \mathbb{D}_{\mathcal{A}}(Rv) \mathcal{N} - (qI - \mathbb{D}_{\mathcal{A}} v) \partial_t \mathcal{N} + \mathbb{D}_{\partial_t \mathcal{A}} v \mathcal{N}, \\ \mathfrak{E}^5(\Theta) &= -\nabla_{\mathcal{A}} \Theta \cdot \partial_t \mathcal{N} - \nabla_{\partial_t \mathcal{A}} \Theta \cdot \mathcal{N} - \Theta \partial_t | \mathcal{N} |, \end{aligned} \tag{4.45}$$

and we define functions \mathfrak{f}^1 in Ω , \mathfrak{f}^2 on Σ and \mathfrak{f}^3 on Σ_b by

$$\begin{aligned} \mathfrak{f}^1(G^1, v) &= \operatorname{div}_{\mathcal{A}}(G^1 - Rv), \\ \mathfrak{f}^2(G^4, v) &= (G^4 + \mathbb{D}_{\mathcal{A}} v \mathcal{N}) \cdot \mathcal{N} | \mathcal{N} |^{-2}, \\ \mathfrak{f}^3(G^1, v) &= (G^1 + \Delta_{\mathcal{A}} v) \cdot \nu. \end{aligned} \tag{4.46}$$

We write $F^{1,0} = F^1 + \theta \nabla_{\mathcal{A}} y_3$, $F^{3,0} = F^3$, $F^{4,0} = F^4$ and $F^{5,0} = F^5$. When $F^1, F^3,$

$F^4, F^5, u, p,$ and θ are regular enough, we can recursively define

$$\begin{aligned}
 F^{1,j} &:= D_t F^{1,j-1} - \partial_t^{j-1}(\theta \nabla_{\mathcal{A}} y_3) + D_t^{j-1}(\theta \nabla_{\mathcal{A}} y_3) + \mathfrak{E}^1(D_t^{j-1} u, \partial_t^{j-1} p) \\
 &= D_t^j F^1 - \left(\partial_t^{j-1}(\theta \nabla_{\mathcal{A}} y_3) - D_t^{j-1}(\theta \nabla_{\mathcal{A}} y_3) \right) + \sum_{\ell=0}^{j-1} D_t^\ell \mathfrak{E}^1(D_t^{j-\ell-1} u, \partial_t^{j-\ell-1} p), \\
 F^{3,j} &:= \partial_t F^{3,j-1} + \mathfrak{E}^3(\partial_t^{j-1} \theta) = \partial_t^j F^3 + \sum_{\ell=0}^{j-1} \partial_t^\ell \mathfrak{E}^3(\partial_t^{j-\ell-1} \theta),
 \end{aligned} \tag{4.47}$$

in Ω and

$$\begin{aligned}
 F^{4,j} &:= \partial_t F^{4,j-1} + \mathfrak{E}^4(D_t^{j-1} u, \partial_t^{j-1} p) = \partial_t^j F^4 + \sum_{\ell=0}^{j-1} \partial_t^\ell \mathfrak{E}^4(D_t^{j-\ell-1} u, \partial_t^{j-\ell-1} p), \\
 F^{5,j} &:= \partial_t F^{5,j-1} + \mathfrak{E}^5(\partial_t^{j-1} \theta) = \partial_t^j F^5 + \sum_{\ell=0}^{j-1} \partial_t^\ell \mathfrak{E}^5(\partial_t^{j-\ell-1} \theta)
 \end{aligned} \tag{4.48}$$

on Σ , for $j = 1, \dots, N$.

Now, we define the sums of norms with F^1, F^3, F^4 and F^5 :

$$\begin{aligned}
 \mathfrak{F}(F^1, F^3, F^4, F^5) &:= \sum_{j=0}^{N-1} \left(\|\partial_t^j F^1\|_{L^2 H^{2N-2j-1}} + \|\partial_t^j F^3\|_{L^2 H^{2N-2j-1}} \right) \\
 &\quad + \|\partial_t^N F^1\|_{L^2({}_0H^1(\Omega))^*} + \|\partial_t^N F^3\|_{L^2({}_0H^1(\Omega))^*} \\
 &\quad + \sum_{j=0}^N \left(\|\partial_t^j F^4\|_{L^2 H^{2N-2j-1/2}} + \|\partial_t^j F^5\|_{L^2 H^{2N-2j-1/2}} \right) \\
 &\quad + \sum_{j=0}^{N-1} \left(\|\partial_t^j F^1\|_{L^\infty H^{2N-2j-2}} + \|\partial_t^j F^3\|_{L^\infty H^{2N-2j-2}} \right) \\
 &\quad + \sum_{j=0}^{N-1} \left(\|\partial_t^j F^4\|_{L^\infty H^{2N-2j-3/2}} + \|\partial_t^j F^5\|_{L^\infty H^{2N-2j-3/2}} \right), \tag{4.49} \\
 \mathfrak{F}_0(F^1, F^3, F^4, F^5) &:= \sum_{j=0}^{N-1} \left(\|\partial_t^j F^1(0)\|_{H^{2N-2j-2}} + \|\partial_t^j F^3(0)\|_{H^{2N-2j-2}} \right) \\
 &\quad + \sum_{j=0}^{N-1} \left(\|\partial_t^j F^4(0)\|_{H^{2N-2j-3/2}} + \|\partial_t^j F^5(0)\|_{H^{2N-2j-3/2}} \right).
 \end{aligned}$$

For simplicity, we will write \mathfrak{F} for $\mathfrak{F}(F^1, F^3, F^4, F^5)$ and \mathfrak{F}_0 for $\mathfrak{F}_0(F^1, F^3, F^4, F^5)$ throughout the rest of this paper. From Lemma A.4 and Lemma 2.4 of [6], we know that if $\mathfrak{F} < \infty$, then

$$\begin{aligned}
 \partial_t^j F^1 &\in C^0([0, T]; H^{2N-2j-2}(\Omega)), \quad \partial_t^j F^3 \in C^0([0, T]; H^{2N-2j-2}(\Omega)), \\
 \partial_t^j F^4 &\in C^0([0, T]; H^{2N-2j-3/2}(\Sigma)), \quad \text{and} \quad \partial_t^j F^5 \in C^0([0, T]; H^{2N-2j-3/2}(\Sigma))
 \end{aligned}$$

for $j=0, \dots, N-1$. For η , we define

$$\begin{aligned} \mathfrak{D}(\eta) &:= \sum_{j=2}^{N+1} \|\partial_t^j \eta\|_{L^2 H^{2N-2j+5/2}}^2, \\ \mathfrak{E}(\eta) &:= \|\eta\|_{L^\infty H^{2N+1/2}(\Sigma)}^2 + \sum_{j=1}^N \|\partial_t^j \eta\|_{L^\infty H^{2N-2j+3/2}(\Sigma)}^2, \\ \mathfrak{K}(\eta) &:= \mathfrak{D}(\eta) + \mathfrak{E}(\eta), \\ \mathfrak{E}_0(\eta) &:= \|\eta_0\|_{H^{2N+1/2}(\Sigma)}^2 + \sum_{j=1}^N \|\partial_t^j \eta(0)\|_{H^{2N-2j+3/2}(\Sigma)}^2. \end{aligned} \tag{4.50}$$

The following lemmas are similar to Lemmas 4.5, 4.6, 4.7 in [6] as well as the ideas of the proofs, so we omit these details here.

LEMMA 4.6. *If $k=0, \dots, 2N-1$ and v, Θ are sufficiently regular, then*

$$\|\partial_t v - D_t v\|_{L^2 H^k}^2 \lesssim P(\mathfrak{K}(\eta)) \|v\|_{L^2 H^k}^2, \tag{4.51}$$

$$\|\partial_t(\Theta \nabla_{\mathcal{A}} y_3) - D_t(\Theta \nabla_{\mathcal{A}} y_3)\|_{L^2 H^k}^2 \lesssim P(\mathfrak{K}(\eta)) \|\Theta\|_{L^2 H^k}^2, \tag{4.52}$$

and if $k=0, \dots, 2N-2$, then

$$\|\partial_t v - D_t v\|_{L^\infty H^k}^2 \lesssim P(\mathfrak{K}(\eta)) \|v\|_{L^\infty H^k}^2, \tag{4.53}$$

$$\|\partial_t(\Theta \nabla_{\mathcal{A}} y_3) - D_t(\Theta \nabla_{\mathcal{A}} y_3)\|_{L^\infty H^k}^2 \lesssim P(\mathfrak{K}(\eta)) \|\Theta\|_{L^\infty H^k}^2. \tag{4.54}$$

If $m=1, \dots, N-1, j=1, \dots, m$, and v, Θ are sufficiently regular, then

$$\|\partial_t^j v - D_t^j v\|_{L^2 H^{2m-2j+3}}^2 \lesssim P(\mathfrak{K}(\eta)) \sum_{\ell=0}^{j-1} (\|\partial_t^\ell v\|_{L^2 H^{2m-2j+3}}^2 + \|\partial_t^\ell v\|_{L^\infty H^{2m-2j+2}}^2), \tag{4.55}$$

$$\|\partial_t^j v - D_t^j v\|_{L^\infty H^{2m-2j+2}}^2 \lesssim P(\mathfrak{K}(\eta)) \sum_{\ell=0}^{j-1} \|\partial_t^\ell v\|_{L^\infty H^{2m-2j+2}}^2, \tag{4.56}$$

$$\begin{aligned} &\|\partial_t^j(\Theta \nabla_{\mathcal{A}} y_3) - D_t^j(\Theta \nabla_{\mathcal{A}} y_3)\|_{L^2 H^{2m-2j+2}}^2 \\ &\lesssim P(\mathfrak{K}(\eta)) \sum_{\ell=0}^{j-1} (\|\partial_t^\ell \Theta\|_{L^2 H^{2m-2j+3}}^2 + \|\partial_t^\ell \Theta\|_{L^\infty H^{2m-2j+2}}^2), \end{aligned} \tag{4.57}$$

$$\|\partial_t^j(\Theta \nabla_{\mathcal{A}} y_3) - D_t^j(\Theta \nabla_{\mathcal{A}} y_3)\|_{L^\infty H^{2m-2j+3}}^2 \lesssim P(\mathfrak{K}(\eta)) \sum_{\ell=0}^{j-1} \|\partial_t^\ell \Theta\|_{L^\infty H^{2m-2j+2}}^2, \tag{4.58}$$

and

$$\begin{aligned} &\|\partial_t D_t^m v - \partial_t^{m+1} v\|_{L^2 H^1}^2 + \|\partial_t^2 D_t^m v - \partial_t^{m+2} v\|_{(\mathcal{X}_T)^*}^2 \\ &\lesssim P(\mathfrak{K}(\eta)) \left(\|\partial_t^{m+1} v\|_{(\mathcal{X}_T)^*}^2 + \sum_{\ell=0}^m (\|\partial_t^\ell v\|_{L^2 H^1}^2 + \|\partial_t^\ell v\|_{L^\infty H^2}^2) \right). \end{aligned} \tag{4.59}$$

Also, if $j = 0, \dots, N$ and v is sufficiently regular, then

$$\|\partial_t^j v(0) - D_t^j v(0)\|_{H^{2N-2j}}^2 \lesssim P(\mathfrak{E}_0(\eta)) \sum_{\ell=0}^{j-1} \|\partial_t^\ell v(0)\|_{H^{2N-2j}}^2, \tag{4.60}$$

and if $j = 0, \dots, N - 1$ and Θ is sufficiently regular, then

$$\|\partial_t^j (\Theta(0) \nabla_{\mathcal{A}_0} y_{3,0}) - D_t^j (\Theta(0) \nabla_{\mathcal{A}_0} y_{3,0})\|_{H^{2N-2j-2}}^2 \lesssim P(\mathfrak{E}_0(\eta)) \sum_{\ell=0}^{j-1} \|\partial_t^\ell \Theta(0)\|_{H^{2N-2j-2}}^2. \tag{4.61}$$

Here all of the $P(\cdot)$ are polynomials, allowed to be changed from line to line.

LEMMA 4.7. For $m = 1, \dots, N - 1$ and $j = 1, \dots, m$, the following estimates hold whenever the right-hand sides are finite:

$$\begin{aligned} & \|F^{1,j}\|_{L^2 H^{2m-2j+1}}^2 + \|F^{3,j}\|_{L^2 H^{2m-2j+1}}^2 + \|F^{4,j}\|_{L^2 H^{2m-2j+3/2}}^2 + \|F^{5,j}\|_{L^2 H^{2m-2j+3/2}}^2 \\ & \lesssim P(\mathfrak{K}(\eta)) \left(\mathfrak{F} + \sum_{\ell=0}^{j-1} (\|\partial_t^\ell u\|_{L^2 H^{2m-2\ell+3}}^2 + \|\partial_t^\ell \theta\|_{L^2 H^{2m-2\ell+3}}^2) \right. \\ & \quad + \sum_{\ell=0}^{j-1} (\|\partial_t^\ell u\|_{L^\infty H^{2m-2\ell+2}}^2 + \|\partial_t^\ell \theta\|_{L^\infty H^{2m-2\ell+2}}^2 + \|\partial_t^\ell p\|_{L^2 H^{2m-2\ell+2}}^2 \\ & \quad \left. + \|\partial_t^\ell p\|_{L^\infty H^{2m-2\ell+1}}^2) \right), \end{aligned} \tag{4.62}$$

$$\begin{aligned} & \|F^{1,j}\|_{L^\infty H^{2m-2j}}^2 + \|F^{3,j}\|_{L^\infty H^{2m-2j}}^2 + \|F^{4,j}\|_{L^\infty H^{2m-2j+1/2}}^2 + \|F^{5,j}\|_{L^\infty H^{2m-2j+1/2}}^2 \\ & \lesssim P(\mathfrak{K}(\eta)) \left(\mathfrak{F} + \sum_{\ell=0}^{j-1} (\|\partial_t^\ell u\|_{L^\infty H^{2m-2\ell+2}}^2 + \|\partial_t^\ell \theta\|_{L^\infty H^{2m-2\ell+2}}^2 \right. \\ & \quad \left. + \|\partial_t^\ell p\|_{L^\infty H^{2m-2\ell+1}}^2) \right), \end{aligned} \tag{4.63}$$

$$\begin{aligned} & \|\partial_t(F^{1,m} - F^{4,m})\|_{L^2({}_0H^1(\Omega))^*}^2 + \|\partial_t(F^{3,m} + F^{5,m})\|_{L^2({}_0H^1(\Omega))^*}^2 \\ & \lesssim P(\mathfrak{K}(\eta)) \left(\mathfrak{F} + \|\partial_t^m u\|_{L^2 H^2}^2 + \|\partial_t^m \theta\|_{L^2 H^2}^2 + \|\partial_t^m p\|_{L^2 H^1}^2 \right. \\ & \quad + \sum_{\ell=0}^{m-1} (\|\partial_t^\ell u\|_{L^\infty H^2}^2 + \|\partial_t^\ell u\|_{L^2 H^2}^3 + \|\partial_t^\ell \theta\|_{L^\infty H^2}^2 + \|\partial_t^\ell \theta\|_{L^2 H^2}^3 \\ & \quad \left. + \|\partial_t^\ell p\|_{L^\infty H^1}^2 + \|\partial_t^\ell p\|_{L^2 H^2}^2) \right). \end{aligned} \tag{4.64}$$

Similarly, for $j = 1, \dots, N - 1$,

$$\begin{aligned} & \|F^{1,j}(0)\|_{H^{2N-2j-2}}^2 + \|F^{3,j}(0)\|_{H^{2N-2j-2}}^2 + \|F^{4,j}(0)\|_{H^{2N-2j-3/2}}^2 + \|F^{5,j}(0)\|_{H^{2N-2j-3/2}}^2 \\ & \lesssim P(\mathfrak{E}_0(\eta)) \left(\mathfrak{F}_0 + \|\partial_t^j \theta(0)\|_{H^{2N-2j}} + \sum_{\ell=0}^{j-1} (\|\partial_t^\ell u(0)\|_{H^{2N-2\ell}} \right. \end{aligned}$$

$$+ \|\partial_t^\ell \theta(0)\|_{H^{2N-2\ell}} + \|\partial_t^\ell p(0)\|_{H^{2N-2\ell-1}} \Big). \tag{4.65}$$

Here all of the $P(\cdot)$ are polynomials, allowed to be changed from line to line.

LEMMA 4.8. *Suppose that v, q, G^1, G^3 are evaluated at $t=0$ and are sufficiently regular for the right-hand sides of the following estimates to make sense. If $j=0, \dots, N-1$, then*

$$\begin{aligned} & \|\mathfrak{E}^{01}(G^1, v, q)\|_{H^{2N-2j-2}}^2 \\ & \lesssim P(\mathfrak{E}_0(\eta)) (\|v\|_{H^{2N-2j}}^2 + \|q\|_{H^{2N-2j-1}}^2 + \|G^1\|_{H^{2N-2j-2}}^2), \end{aligned} \tag{4.66}$$

$$\|\mathfrak{E}^{02}(G^3, \Theta)\|_{H^{2N-2j-2}}^2 \lesssim P(\mathfrak{E}_0(\eta)) (\|\Theta\|_{H^{2N-2j}}^2 + \|G^3\|_{H^{2N-2j-2}}^2). \tag{4.67}$$

If $j=0, \dots, N-2$, then

$$\begin{aligned} & \|\mathfrak{f}^1(G^1, v)\|_{H^{2N-2i-3}}^2 + \|\mathfrak{f}^2(G^4, v)\|_{H^{2N-2i-3/2}}^2 + \|\mathfrak{f}^3(G^1, v)\|_{H^{2N-2i-5/2}}^2 \\ & \lesssim P(\mathfrak{E}_0(\eta)) (\|G^1\|_{H^{2N-2j-2}}^2 + \|G^4\|_{H^{2N-2j-3/2}}^2 + \|v\|_{H^{2N-2j}}^2). \end{aligned} \tag{4.68}$$

For $j=N-1$, if $\operatorname{div}_{\mathcal{A}(0)} v(0) = 0$ in Ω , then

$$\|\mathfrak{f}^2(G^4, v)\|_{H^{1/2}}^2 + \|\mathfrak{f}^3(G^1, v)\|_{H^{-1/2}}^2 \lesssim P(\mathfrak{E}_0(\eta)) (\|G^1\|_{H^2}^2 + \|G^4\|_{H^{1/2}}^2 + \|v\|_{H^2}^2). \tag{4.69}$$

Here all of the $P(\cdot)$ are polynomials allowed to be changed from line to line.

Now we can construct the initial data and compatible conditions. We assume that $u_0 \in H^{2N}(\Omega)$, $\theta_0 \in H^{2N}$, $\eta_0 \in H^{2N+1/2}(\Sigma)$. Then we will iteratively construct the initial data $D_t^j u(0)$, $\partial_t^j \theta(0)$ for $j=1, \dots, N$ and $\partial_t^j p(0)$ for $j=1, \dots, N-1$. First, we denote $F^{1,0}(0) = F^1(0) \in H^{2N-2}$, $F^{3,0}(0) = F^3(0) \in H^{2N-2}$, $F^{4,0}(0) = F^4(0) \in H^{2N-3/2}$, $F^{5,0}(0) = F^5(0) \in H^{2N-3/2}$ and $D_t^0 u(0) = u_0 \in H^{2N}$, $\partial_t^0 \theta(0) = \theta_0 \in H^{2N}$. Suppose now that we have constructed $F^{1,\ell} \in H^{2N-2\ell-2}$, $F^{3,\ell} \in H^{2N-2\ell-2}$, $F^{4,\ell} \in H^{2N-2\ell-3/2}$, $F^{5,\ell} \in H^{2N-2\ell-3/2}$, and $D_t^j u(0) \in H^{2N-2\ell}$, $\partial_t^\ell \theta(0) \in H^{2N-2\ell}$ for $0 \leq \ell \leq j \leq N-2$; we will construct $\partial_t^j p(0) \in H^{2N-2j-1}$ as well as $D_t^{j+1} u(0) \in H^{2N-2j-2}$, $\partial_t^{j+1} \theta(0) \in H^{2N-2j-2}$, $F^{1,j+1}(0) \in H^{2N-2j-4}$, $F^{3,j+1}(0) \in H^{2N-2j-4}$, $F^{4,j+1}(0) \in H^{2N-2j-7/2}$ and $F^{5,j+1}(0) \in H^{2N-2j-7/2}$ as follows.

By virtue of estimates, we know that

$$\begin{aligned} f^1 &= \mathfrak{f}^1(F^{1,j}(0), D_t^j u(0)) \in H^{2N-2j-3}, \\ f^2 &= \mathfrak{f}^2(F^{4,j}(0), D_t^j u(0)) \in H^{2N-2j-3/2}, \\ f^3 &= \mathfrak{f}^3(F^{1,j}(0), D_t^j u(0)) \in H^{2N-2j-5/2} \end{aligned} \tag{4.70}$$

This allows us to define $\partial_t^j p(0)$ as the solution to Equation (3.18). The choice of f^1, f^2, f^3 implies that $\partial_t^j p(0) \in H^{2N-2j-1}$, according to Proposition 2.15 of [14]. Now the estimates (4.65), (4.60) and (4.66) allow us to define

$$\begin{aligned} D_t^{j+1} u(0) &:= \mathfrak{E}^{01} \left(F^{1,j}(0) + \partial_t^j(\theta(0)) \nabla_{\mathcal{A}_0} y_{3,0}, D_t^j u(0), \partial_t^j p(0) \right) \in H^{2N-2j-2}, \\ \partial_t^{j+1} \theta(0) &:= \mathfrak{E}^{02} \left(F^{3,j}(0), \partial_t^j \theta(0) \right) \in H^{2N-2j-2}, \\ F^{1,j+1}(0) &:= D_t^j F^{1,j}(0) - \partial_t^j(\theta(0)) \nabla_{\mathcal{A}_0} y_{3,0} + D_t^j(\theta(0)) \nabla_{\mathcal{A}_0} y_{3,0} \end{aligned}$$

$$\begin{aligned}
 & + \mathfrak{E}^1 \left(D_t^j u(0), \partial_t^j p(0) \right) \in H^{2N-2j-4}, \\
 F^{3,j+1}(0) & := \partial_t F^{3,j}(0) + \mathfrak{E}^3 \left(\partial_t^j \theta(0) \right) \in H^{2N-2j-4}, \\
 F^{4,j+1}(0) & := \partial_t F^{4,j}(0) + \mathfrak{E}^4 \left(D_t^j u(0), \partial_t^j p(0) \right) \in H^{2N-2j-7/2}, \\
 F^{5,j+1}(0) & := \partial_t F^{5,j}(0) + \mathfrak{E}^5 \left(\partial_t^j \theta(0) \right) \in H^{2N-2j-7/2}.
 \end{aligned}$$

Then, from the above analysis, we can iteratively construct all of the desired data except for $D_t^N u(0)$, $\partial_t^{N-1} p(0)$ and $\partial_t^N \theta(0)$.

By construction, the initial data $D_t^j u(0)$, $\partial_t^j p(0)$ and $\partial_t^j \theta(0)$ are determined in terms of u_0 , θ_0 as well as $\partial_t^\ell F^1(0)$, $\partial_t^\ell F^3(0)$, $\partial_t^\ell F^4(0)$ and $\partial_t^\ell F^5(0)$ for $\ell=0, \dots, N-1$. In order to use these in Theorem 4.5 and to construct $D_t^N u(0)$, $\partial_t^{N-1} p(0)$ and $\partial_t^N \theta(0)$, we must enforce compatibility conditions for $j=0, \dots, N-1$. We say that the j -th compatibility condition is satisfied if

$$\begin{cases} D_t^j u(0) \in \mathcal{X}(0) \cap H^2(\Omega), \\ \Pi_0 \left(F^{4,j}(0) + \mathbb{D}_{\mathcal{A}_0} D_t^j u(0) \mathcal{N}_0 \right) = 0. \end{cases} \tag{4.71}$$

The construction of $D_t^j u(0)$ and $\partial_t^j p(0)$ ensures that $D_t^j u(0) \in H^2(\Omega)$ and $\text{div}_{\mathcal{A}_0}(D_t^j u(0)) = 0$.

In the following, we define $\partial_t^N \theta(0) \in H^0$, $\partial_t^{N-1} p(0) \in H^1$ and $D_t^N u(0) \in H^0$. First, we can define

$$\partial_t^N \theta(0) = \mathfrak{E}^{02}(F^{3,N-1}(0), \partial_t^{N-1} \theta(0)) \in H^0(\Omega),$$

employing the inequality (4.67) for the inclusion in H^0 . Then using the same analysis as in [6], the data $\partial_t^{N-1} p(0) \in H^1$ can be defined as a weak solution to Equation (3.18). Then we define

$$D_t^N u(0) = \mathfrak{E}^{01} \left(F^{1,N-1}(0) + \partial_t^{N-1}(\theta(0) \nabla_{\mathcal{A}_0} y_{3,0}), D_t^{N-1} u(0), \partial_t^{N-1} p(0) \right) \in H^0,$$

employing the inequalities (4.66) and (4.61) for the inclusion in H^0 . And $D_t^N u(0) \in \mathcal{Y}(0)$ is guaranteed by the construction of $\partial_t^{N-1} p(0)$. Combining the inclusions above with the bounds (4.65), (4.68), (4.66) and (4.67) implies that

$$\begin{aligned}
 & \sum_{j=0}^N \|D_t^j u(0)\|_{H^{2N-2j}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j p(0)\|_{H^{2N-2j-1}}^2 + \sum_{j=0}^N \|\partial_t^j \theta(0)\|_{H^{2N-2j}}^2 \\
 & \lesssim P(\mathfrak{E}_0(\eta)) \left(\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 \right). \tag{4.72}
 \end{aligned}$$

Before stating the result on higher regularity for solutions to Equation (1.8), we define some quantities:

$$\begin{aligned}
 \mathfrak{D}(u, p, \theta) & := \sum_{j=0}^N \left(\|\partial_t^j u\|_{L^2 H^{2N-2j+1}}^2 + \|\partial_t^j \theta\|_{L^2 H^{2N-2j+1}}^2 \right) + \|\partial_t^{N+1} u\|_{(\mathcal{X}_T)^*} \\
 & \quad + \|\partial_t^{N+1} \theta\|_{(\mathcal{X}_T^1)^*} + \sum_{j=0}^{N-1} \|\partial_t^j p\|_{L^2 H^{2N-2j}}, \tag{4.73}
 \end{aligned}$$

$$\mathfrak{E}(u, p, \theta) := \sum_{j=0}^N \left(\|\partial_t^j u\|_{L^\infty H^{2N-2j}}^2 + \|\partial_t^j \theta\|_{L^\infty H^{2N-2j}}^2 \right) + \sum_{j=0}^{N-1} \|\partial_t^j p\|_{L^\infty H^{2N-2j-1}},$$

$$\mathfrak{K}(u, p, \theta) := \mathfrak{D}(u, p, \theta) + \mathfrak{E}(u, p, \theta).$$

THEOREM 4.9. *Suppose that $u_0 \in H^{2N}(\Omega)$, $\theta_0 \in H^{2N}(\Omega)$, $\eta_0 \in H^{2N+1/2}(\Sigma)$, and $\mathfrak{F} < \infty$. Let $D_t^j u(0) \in H^{2N-2j}(\Omega)$, $\partial_t^j \theta(0) \in H^{2N-2j}(\Omega)$ and $\partial_t^j p(0) \in H^{2N-2j-1}(\Omega)$, for $j = 1, \dots, N-1$, along with $D_t^N u(0) \in \mathcal{Y}(0)$ and $\partial_t^N \theta(0) \in H^0$, all be determined in terms of u_0 , θ_0 and $\partial_t^j F^1(0)$, $\partial_t^j F^3(0)$, $\partial_t^j F^4(0)$, $\partial_t^j F^5(0)$ for $j=0, \dots, N-1$.*

There exists a universal constant $T_0 > 0$ such that if $0 < T \leq T_0$, then there exists a unique strong solution (u, p, θ) on $[0, T]$ such that

$$\begin{aligned} \partial_t^j u &\in C^0([0, T]; H^{2N-2j}(\Omega)) \cap L^2([0, T]; H^{2N-2j+1}(\Omega)) \quad \text{for } j=0, \dots, N, \\ \partial_t^j p &\in C^0([0, T]; H^{2N-2j-1}(\Omega)) \cap L^2([0, T]; H^{2N-2j}(\Omega)) \quad \text{for } j=0, \dots, N-1, \\ \partial_t^j \theta &\in C^0([0, T]; H^{2N-2j}(\Omega)) \cap L^2([0, T]; H^{2N-2j+1}(\Omega)) \quad \text{for } j=0, \dots, N, \\ \partial_t^{N+1} u &\in (\mathcal{X}_T)^*, \quad \text{and} \quad \partial_t^{N+1} \theta \in (\mathcal{H}_T^1)^*. \end{aligned}$$

The pair $(D_t^j u, \partial_t^j p, \partial_t^j \theta)$ satisfies

$$\begin{cases} \partial_t(D_t^j u) - \Delta_{\mathcal{A}}(D_t^j u) + \nabla_{\mathcal{A}}(\partial_t^j p) - \partial_t^j(\theta \nabla_{\mathcal{A}} y_3) = F^{1,j} & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}}(D_t^j u) = 0 & \text{in } \Omega, \\ \partial_t(\partial_t^j \theta) - \Delta_{\mathcal{A}}(\partial_t^j \theta) = F^{3,j} & \text{in } \Omega, \\ S_{\mathcal{A}}(\partial_t^j p, D_t^j u)_{\mathcal{N}} = F^{4,j} & \text{on } \Sigma, \\ \nabla_{\mathcal{A}}(\partial_t^j \theta) \cdot \mathcal{N} + \partial_t^j \theta |_{\mathcal{N}} = F^{5,j} & \text{on } \Sigma, \\ D_t^j u = 0, \quad \partial_t^j \theta = 0 & \text{on } \Sigma_b, \end{cases} \tag{4.74}$$

in the strong sense with initial data $(D_t^j u(0), \partial_t^j p(0), \partial_t^j \theta(0))$ for $j=0, \dots, N-1$, and in the weak sense with initial data $D_t^N u(0) \in \mathcal{Y}(0)$ and $\partial_t^N \theta(0) \in H^0$. Here the forcing terms $F^{1,j}$, $F^{3,j}$, $F^{4,j}$ and $F^{5,j}$ are as defined by Equations (4.47) and (4.48). Moreover, the solution satisfies the estimate

$$\mathfrak{K}(u, p, \theta) \lesssim P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(TP(\mathfrak{E}(\eta))) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}), \tag{4.75}$$

where the constant $C > 0$, is independent of η .

Proof. First, notice that $P(\cdot, \cdot)$ and $P(\cdot)$ throughout this proof are allowed to change from line to line. Theorem 4.5 guarantees the existence of (u, p, θ) satisfying the inclusions (4.12). The $(D_t^j u, \partial_t^j p, \partial_t^j \theta)$ are solutions of Equation (4.74) in the strong sense when $j=0$ and in the weak sense when $j=1$. Finally, the estimate (4.15) holds.

For an integer $m \geq 0$, let \mathbb{P}_m denote the proposition asserting the following three statements. First, $(D_t^j u, \partial_t^j p, \partial_t^j \theta)$ are solutions of Equation (4.74) in the strong sense for $j=0, \dots, m$ and in the weak sense when $j=m+1$. Second,

$$\partial_t^j u \in L^\infty H^{2m-2j+2} \cap L^2 H^{2m-2j+3}, \quad \partial_t^j \theta \in L^\infty H^{2m-2j+2} \cap L^2 H^{2m-2j+3}$$

for $j=0, 1, \dots, m+1$, $\partial_t^{m+2} u \in (\mathcal{X}_T)^*$, $\partial_t^{m+2} \theta \in (\mathcal{H}_T^1)^*$ and

$$\partial_t^j p \in L^\infty H^{2m-2j+1} \cap L^2 H^{2m-2j+2}$$

for $j=0, 1, \dots, m$. Third, the estimate

$$\sum_{j=0}^{m+1} \left(\|\partial_t^j u\|_{L^\infty H^{2m-2j+2}}^2 + \|\partial_t^j u\|_{L^2 H^{2m-2j+3}}^2 + \|\partial_t^j \theta\|_{L^\infty H^{2m-2j+2}}^2 + \|\partial_t^j \theta\|_{L^2 H^{2m-2j+3}}^2 \right)$$

$$\begin{aligned}
 & + \|\partial_t^{m+2}u\|_{\mathcal{X}_T}^2 + \|\partial_t^{m+2}\theta\|_{\mathcal{X}_T^1}^2 + \sum_{j=0}^m \left(\|\partial_t^j p\|_{L^\infty H^{2m-2j+1}}^2 + \|\partial_t^j p\|_{L^2 H^{2m-2j+2}}^2 \right) \\
 & \lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(TP(\mathfrak{E}(\eta))) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F})
 \end{aligned} \tag{4.76}$$

holds.

We will use a finite induction method to prove that \mathbb{P}_m holds. Theorem 4.5 implies that \mathbb{P}_0 holds. Then, in the rest of this proof, we will divide the proof into two steps.

Step 1. Proving the first assertion. Suppose that \mathbb{P}_m holds for $m=0, \dots, N-2$.

From parts (4.62)–(4.64) of Lemma 4.7, we have that

$$\begin{aligned}
 & \|F^{1,m+1}(v, q)\|_{L^2 H^1}^2 + \|F^{3,m+1}(\Theta)\|_{L^2 H^1}^2 + \|F^{4,m+1}(v, q)\|_{L^2 H^{3/2}}^2 \\
 & \quad + \|F^{5,m+1}(\Theta)\|_{L^2 H^{3/2}}^2 \\
 & \lesssim P(\mathfrak{R}(\eta)) \left(\mathfrak{F} + \sum_{\ell=0}^m (\|\partial_t^\ell v\|_{L^2 H^3}^2 + \|\partial_t^\ell \Theta\|_{L^2 H^3}^2) \right. \\
 & \quad \left. + \sum_{\ell=0}^m \left(\|\partial_t^\ell v\|_{L^\infty H^2}^2 + \|\partial_t^\ell \Theta\|_{L^\infty H^2}^2 + \|\partial_t^\ell q\|_{L^2 H^2}^2 + \|\partial_t^\ell q\|_{L^\infty H^1}^2 \right) \right),
 \end{aligned} \tag{4.77}$$

$$\begin{aligned}
 & \|F^{1,m+1}(v, q)\|_{L^\infty H^0}^2 + \|F^{3,m+1}(\Theta)\|_{L^\infty H^0}^2 + \|F^{4,j}(v, q)\|_{L^\infty H^{1/2}}^2 \\
 & \quad + \|F^{5,j}(\Theta)\|_{L^\infty H^{1/2}}^2 \\
 & \lesssim P(\mathfrak{R}(\eta)) \left(\mathfrak{F} + \sum_{\ell=0}^m \left(\|\partial_t^\ell v\|_{L^\infty H^2}^2 + \|\partial_t^\ell \Theta\|_{L^\infty H^2}^2 + \|\partial_t^\ell q\|_{L^\infty H^1}^2 \right) \right),
 \end{aligned} \tag{4.78}$$

$$\begin{aligned}
 & \|\partial_t(F^{1,m+1}(v, q) - F^{4,m+1}(v, q))\|_{L^2({}_0H^1(\Omega))^*}^2 \\
 & \quad + \|\partial_t(F^{3,m+1}(\Theta) - F^{5,m+1}(\Theta))\|_{L^2({}_0H^1(\Omega))^*}^2 \\
 & \lesssim P(\mathfrak{R}(\eta)) \left(\mathfrak{F} + \|\partial_t^{m+1}v\|_{L^2 H^2}^2 + \|\partial_t^{m+1}\Theta\|_{L^2 H^2}^2 + \|\partial_t^{m+1}q\|_{L^2 H^1}^2 \right. \\
 & \quad + \sum_{\ell=0}^m \left(\|\partial_t^\ell v\|_{L^\infty H^2}^2 + \|\partial_t^\ell v\|_{L^2 H^2}^3 + \|\partial_t^\ell \Theta\|_{L^\infty H^2}^2 + \|\partial_t^\ell \Theta\|_{L^2 H^2}^3 \right. \\
 & \quad \left. \left. + \|\partial_t^\ell q\|_{L^\infty H^1}^2 + \|\partial_t^\ell q\|_{L^2 H^2}^2 \right) \right).
 \end{aligned} \tag{4.79}$$

Now we will use the iteration method. We let u^0 be the extension of the initial data $\partial_t^j u(0)$, $j=1, \dots, N$, given by Lemma A.5 in [6], which also gives θ^0 , the extension of the initial data $\partial_t^j \theta(0)$, $j=1, \dots, N$; similarly, let p^0 be the extension of $\partial_t^j p(0)$, $j=1, \dots, N-1$, given by Lemma A.6 in [6]. By the estimate (4.72) and the estimates given in Lemmas A.5 and A.6 of [6], we have

$$\begin{aligned}
 & \sum_{j=0}^N \left(\|\partial_t^j u^0\|_{L^2 H^{2N-2j+1}}^2 + \|\partial_t^j u^0\|_{L^\infty H^{2N-2j}}^2 + \|\partial_t^j \theta^0\|_{L^2 H^{2N-2j+1}}^2 + \|\partial_t^j \theta^0\|_{L^\infty H^{2N-2j}}^2 \right) \\
 & \quad + \sum_{j=0}^{N-1} \left(\|\partial_t^j p^0\|_{L^2 H^{2N-2j}}^2 + \|\partial_t^j p^0\|_{L^\infty H^{2N-2j-1}}^2 \right)
 \end{aligned}$$

$$\begin{aligned} &\lesssim \sum_{j=0}^N \|D_t^j u(0)\|_{H^{2N-2j}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j p(0)\|_{H^{2N-2j-1}}^2 + \sum_{j=0}^N \|\partial_t^j \theta(0)\|_{H^{2N-2j}}^2 \\ &\lesssim P(\mathfrak{E}_0(\eta)) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0). \end{aligned} \tag{4.80}$$

According to the estimates (4.77)–(4.80), we may derive that $F^{1,m+1}(u^0, p^0)$, $F^{3,m+1}(\theta^0)$, $F^{4,m+1}(u^0, p^0)$ and $F^{5,m+1}(\theta^0)$ satisfy the condition (4.11). Also the compatibility condition (4.13) with F^4 replaced by $F^{4,m+1}(u^0, p^0)$ and u_0 replaced by $D_t^{m+1}u(0)$ holds by the condition (4.71) since u^0 and p^0 achieve the initial data. Then we can apply Theorem 4.5 to find a pair (v^1, q^1, Θ^1) satisfying the conclusions of the theorem. For simplicity, we abbreviate Equation (1.8) as $\mathcal{L}(v, q, \Theta) = \mathbb{F} = (F^1, F^3, F^4, F^5)$. Then

$$\begin{aligned} \mathcal{L}(v^1, q^1, \Theta^1) &= \mathbb{F}^{m+1} := (F^{1,m+1}(u^0, p^0), F^{3,m+1}(\theta^0), F^{4,m+1}(u^0, p^0), F^{5,m+1}(\theta^0)), \\ v^1(0) &= D_t^{m+1}u(0), \quad q^1(0) = \partial_t^{m+1}p(0), \quad \Theta^1(0) = \partial_t^{m+1}\theta(0). \end{aligned}$$

If we denote the left-hand side of the inequality (4.15) as $\mathfrak{B}(u, p, \theta)$, then we may combine the inequalities (4.15), (4.65), (4.77), (4.79) and (4.80) to derive that

$$\mathfrak{B}(v^1, q^1, \Theta^1) \lesssim P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}).$$

Now, suppose that (v^n, q^n, Θ^n) is given and satisfies $\mathfrak{B}(v^n, q^n, \Theta^n) < \infty$. We define (u^n, p^n, θ^n) which satisfies the ODEs:

$$\begin{cases} D_t^{m+1}u^n = v^n, \\ \partial_t^j u^n(0) = v^n(0) \quad \text{for } j=0, \dots, m, \end{cases} \tag{4.81}$$

$$\begin{cases} \partial_t^{m+1}p^n = q^n, \\ \partial_t^j p^n(0) = q^n(0) \quad \text{for } j=0, \dots, m, \end{cases} \tag{4.82}$$

$$\begin{cases} \partial_t^{m+1}\theta^n = \Theta^n, \\ \partial_t^j \theta^n(0) = \Theta^n(0) \quad \text{for } j=0, \dots, m. \end{cases} \tag{4.83}$$

From the wellposedness theory of linear ODEs, we know that these ODEs have unique solutions. If we define $\mathfrak{K}(v, q, \Theta)$ by

$$\begin{aligned} \mathfrak{K}(v, q, \Theta) &:= \|\partial_t^{m+1}v\|_{L^2 H^2}^2 + \|\partial_t^{m+1}q\|_{L^2 H^1}^2 + \|\partial_t^{m+1}\Theta\|_{L^2 H^2}^2 + \sum_{\ell=0}^m \left(\|\partial_t^\ell v\|_{L^2 H^3}^2 \right. \\ &\quad \left. + \|\partial_t^\ell v\|_{L^\infty H^2}^2 + \|\partial_t^\ell \Theta\|_{L^2 H^3}^2 + \|\partial_t^\ell \Theta\|_{L^\infty H^2}^2 + \|\partial_t^\ell q\|_{L^2 H^2}^2 + \|\partial_t^\ell q\|_{L^\infty H^1}^2 \right), \end{aligned}$$

then the solutions of Equations (4.81)–(4.83) satisfy the estimate

$$\begin{aligned} \mathfrak{K}(u^n, p^n, \theta^n) &\lesssim P(T)P(\mathfrak{K}(\eta)) \left(\sum_{j=0}^m \|\partial_t^j u(0)\|_{H^3}^2 + \|\partial_t^j p(0)\|_{H^2}^2 \right. \\ &\quad \left. + \|\partial_t^j \theta(0)\|_{H^3}^2 + T\mathfrak{B}(v^n, q^n, \Theta^n) \right) < \infty, \end{aligned} \tag{4.84}$$

where $P(T)$ is a polynomial in T .

Applying Theorem 4.5 iteratively, we can obtain sequences $\{(v^n, q^n, \Theta^n)\}_{n=1}^\infty$ and $\{u^n, p^n, \theta^n\}_{n=1}^\infty$ satisfying Equations (4.81)–(4.83) and

$$\begin{aligned} \mathcal{L}(v^n, q^n, \Theta^n) &= \mathbb{F}^{m+1}(u^{n-1}, p^{n-1}, \theta^{n-1}), \\ v^n(0) &= D_t^{m+1}u(0), \quad q^n(0) = \partial_t^{m+1}p(0), \quad \Theta^n(0) = \partial_t^{m+1}\theta(0). \end{aligned} \tag{4.85}$$

Then

$$\begin{aligned} \mathcal{L}(v^{n+1} - v^n, q^{n+1} - q^n, \Theta^{n+1} - \Theta^n) &= \mathbb{F}^{m+1}(u^n - u^{n-1}, p^n - p^{n-1}, \theta^n - \theta^{n-1}), \\ v^{n+1}(0) - v^n(0) &= 0, \quad q^{n+1}(0) - q^n(0) = 0, \quad \Theta^{n+1}(0) - \Theta^n(0) = 0. \end{aligned}$$

Since the terms involving F^1, F^3, F^4 and F^5 are canceled in $\mathbb{F}^{m+1}(u^n - u^{n-1}, p^n - p^{n-1}, \theta^n - \theta^{n-1})$, we can use the estimates (4.77) and (4.79) to derive that

$$\begin{aligned} &\|F^{1,m+1}(u^n - u^{n-1}, p^n - p^{n-1})\|_{L^2H^1}^2 + \|F^{3,m+1}(\theta^n - \theta^{n-1})\|_{L^2H^1}^2 \\ &\quad + \|F^{4,m+1}(u^n - u^{n-1}, p^n - p^{n-1})\|_{L^2H^{3/2}}^2 + \|F^{5,m+1}(\theta^n - \theta^{n-1})\|_{L^2H^{3/2}}^2 \\ &\quad + \|\partial_t(F^{1,m+1}(u^n - u^{n-1}, p^n - p^{n-1}) - F^{4,m+1}(u^n - u^{n-1}, p^n - p^{n-1}))\|_{L^2({}_0H^1(\Omega))^*}^2 \\ &\quad + \|\partial_t(F^{3,m+1}(\theta^n - \theta^{n-1}) - F^{5,m+1}(\theta^n - \theta^{n-1}))\|_{L^2({}_0H^1(\Omega))^*}^2 \\ &\lesssim P(\mathfrak{K}(\eta))\mathfrak{K}(u^n - u^{n-1}, p^n - p^{n-1}, \theta^n - \theta^{n-1}). \end{aligned}$$

Since, for each n , (u^n, p^n, θ^n) achieves the same initial data, similar to the ODEs (4.81)–(4.83), we have that

$$\mathfrak{K}(u^n - u^{n-1}, p^n - p^{n-1}, \theta^n - \theta^{n-1}) \lesssim P(\mathfrak{K}(\eta))TP(T)\mathfrak{B}(v^n - v^{n-1}, q^n - q^{n-1}, \Theta^n - \Theta^{n-1}). \tag{4.86}$$

The above two estimates with the estimate (4.15) imply that

$$\begin{aligned} &\mathfrak{B}(v^{n+1} - v^n, q^{n+1} - q^n, \Theta^{n+1} - \Theta^n) \\ &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta))\exp(P(\mathfrak{E}(\eta))T) \times TP(T)\mathfrak{B}(v^n - v^{n-1}, q^n - q^{n-1}, \Theta^n - \Theta^{n-1}), \end{aligned} \tag{4.87}$$

which implies that there exists a universal $T_0 > 0$ such that if $T \leq T_0$, then the sequence $\{(v^n, q^n, \Theta^n)\}_{n=1}^\infty$ converges to (v, q, Θ) in the norm $\sqrt{\mathfrak{B}(\cdot, \cdot)}$, which reveals that $\{(u^n, p^n, \theta^n)\}_{n=1}^\infty$ converges to (u, p, θ) in the norm $\sqrt{\mathfrak{K}(\cdot, \cdot)}$.

By passing to the limit in Equations (4.81)–(4.83), we have that $v = D_t^{m+1}u$, $q = \partial_t^{m+1}p$ and $\Theta = \partial_t^{m+1}\theta$. Then, passing to the limit in Equation (4.85), we have that

$$\mathcal{L}(D_t^{m+1}u, \partial_t^{m+1}p, \partial_t^{m+1}\theta) = \mathbb{F}^{m+1}(u, p, \theta).$$

Then Theorem 4.5 with the assumption of \mathbb{P}_m , which provides that $(D_t^{m+1}u, \partial_t^{m+1}p, \partial_t^{m+1}\theta)$ are solutions of Equation (4.74) in the strong sense for $j = 0, \dots, m$, enables us to deduce the first assertion of \mathbb{P}_{m+1} .

Theorem 4.5, together with the estimates (4.62), (4.79) and (4.76), gives us that

$$\begin{aligned} &\mathfrak{B}(D_t^{m+1}u, \partial_t^{m+1}p, \partial_t^{m+1}\theta) \\ &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta))\exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 \\ &\quad + \mathfrak{F}_0 + \mathfrak{F} + \|\partial_t^{m+1}u\|_{L^2H^2}^2 + \|\partial_t^{m+1}p\|_{L^2H^1}^2 + \|\partial_t^{m+1}\theta\|_{L^2H^2}^2). \end{aligned} \tag{4.88}$$

On the other hand, the estimate (4.56) implies that

$$\|\partial_t^{m+1}u\|_{L^2H^2}^2 + \|\partial_t^{m+1}p\|_{L^2H^1}^2 + \|\partial_t^{m+1}\theta\|_{L^2H^2}^2$$

$$\begin{aligned}
 &\leq T \left(\|\partial_t^{m+1} u\|_{L^\infty H^2}^2 + \|\partial_t^{m+1} p\|_{L^\infty H^1}^2 + \|\partial_t^{m+1} \theta\|_{L^\infty H^2}^2 \right) \\
 &\lesssim T \left(\|\partial_t^{m+1} u - D_t^{m+1} u\|_{L^\infty H^2}^2 + \|D_t^{m+1} u\|_{L^\infty H^2}^2 + \|\partial_t^{m+1} p\|_{L^\infty H^1}^2 + \|\partial_t^{m+1} \theta\|_{L^\infty H^2}^2 \right) \\
 &\lesssim T \left(P(\mathfrak{K}(\eta)) \sum_{\ell=0}^m \|\partial_t^\ell u\|_{L^\infty H^2}^2 + \mathfrak{B}(D_t^{m+1} u, \partial_t^{m+1} p, \partial_t^{m+1} \theta) \right) \\
 &\lesssim T \left(P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}) \right. \\
 &\quad \left. + \mathfrak{B}(D_t^{m+1} u, \partial_t^{m+1} p, \partial_t^{m+1} \theta) \right), \tag{4.89}
 \end{aligned}$$

where in the last inequality, we have used the estimate (4.76) again. Combining the above two estimates, we may further restrict the size of universal $T_0 > 0$ such that if $T \leq T_0$, then

$$\begin{aligned}
 &\mathfrak{B}(D_t^{m+1} u, \partial_t^{m+1} p, \partial_t^{m+1} \theta) \\
 &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \tag{4.90}
 \end{aligned}$$

Step 2. Proving the second and third assertions. In the following, the second and third assertions will be derived simultaneously. The estimate (4.90), Lemma 4.6, and the estimate (4.76) imply that

$$\begin{aligned}
 &\|\partial_t^{m+1} u\|_{L^2 H^3}^2 + \|\partial_t^{m+2} u\|_{L^2 H^1}^2 + \|\partial_t^{m+3} u\|_{(\mathcal{X}_T)^*}^2 + \|\partial_t^{m+1} u\|_{L^\infty H^2}^2 + \|\partial_t^{m+2} u\|_{L^\infty H^0}^2 \\
 &\lesssim P(\mathfrak{K}(\eta)) \left(\sum_{\ell=0}^{m+2} \|\partial_t^\ell u\|_{L^2 H^{2m-2\ell+3}}^2 + \|\partial_t^\ell u\|_{L^\infty H^{2m-2\ell+2}}^2 \right) \\
 &\quad + P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}) \\
 &\lesssim P(\mathfrak{K}(\eta)) P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(p(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}) \\
 &\quad + P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}) \\
 &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \tag{4.91}
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\sum_{j=m+1}^{m+2} \left(\|\partial_t^j u\|_{L^2 H^{2(m+1)-2j+3}}^2 + \|\partial_t^j u\|_{L^\infty H^{2(m+1)-2j+2}}^2 \right) + \|\partial_t^{m+3} u\|_{(\mathcal{X}_T)^*}^2 \\
 &\quad + \sum_{j=m+1}^{m+2} \left(\|\partial_t^j p\|_{L^2 H^{2(m+1)-2j+2}}^2 + \|\partial_t^j p\|_{L^\infty H^{2(m+1)-2j+1}}^2 \right) \\
 &\quad + \sum_{j=m+1}^{m+2} \left(\|\partial_t^j \theta\|_{L^2 H^{2(m+1)-2j+3}}^2 + \|\partial_t^j \theta\|_{L^\infty H^{2(m+1)-2j+2}}^2 \right) + \|\partial_t^{m+3} \theta\|_{(\mathcal{X}_T)^*}^2 \\
 &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{K}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \tag{4.92}
 \end{aligned}$$

Thus, in order to derive the second and third assertions of \mathbb{P}_{m+1} , it suffices to prove that

$$\begin{aligned}
 &\sum_{j=0}^m \left(\|\partial_t^j u\|_{L^2 H^{2(m+1)-2j+3}}^2 + \|\partial_t^j p\|_{L^2 H^{2(m+1)-2j+2}}^2 + \|\partial_t^j \theta\|_{L^2 H^{2(m+1)-2j+3}}^2 \right) \\
 &\quad + \sum_{j=0}^m \left(\|\partial_t^j u\|_{L^\infty H^{2(m+1)-2j+2}}^2 + \|\partial_t^j p\|_{L^\infty H^{2(m+1)-2j+1}}^2 + \|\partial_t^j \theta\|_{L^\infty H^{2(m+1)-2j+2}}^2 \right)
 \end{aligned}$$

$$\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \tag{4.93}$$

In order to prove this estimate, we will use the elliptic regularity of Proposition 3.5 with $k = 2N$ and iteration argument. As the first step, we need the estimates for the forcing terms. Combining the estimate (4.76) with the estimates (4.62) and (4.63) of Lemma 4.7 implies that

$$\begin{aligned} & \sum_{j=1}^{m+1} \left(\|F^{1,j}\|_{L^2 H^{2m-2j+1}}^2 + \|F^{3,j}\|_{L^2 H^{2m-2j+1}}^2 + \|F^{4,j}\|_{L^2 H^{2m-2j+3/2}}^2 \right. \\ & \quad + \|F^{5,j}\|_{L^2 H^{2m-2j+3/2}}^2 + \|F^{1,j}\|_{L^\infty H^{2m-2j}}^2 + \|F^{3,j}\|_{L^\infty H^{2m-2j}}^2 \\ & \quad \left. + \|F^{4,j}\|_{L^\infty H^{2m-2j+1/2}}^2 + \|F^{5,j}\|_{L^\infty H^{2m-2j+1/2}}^2 \right) \\ & \lesssim P(\mathfrak{R}(\eta)) \left(\mathfrak{F} + \sum_{\ell=0}^{j-1} (\|\partial_t^\ell u\|_{L^2 H^{2m-2\ell+3}}^2 + \|\partial_t^\ell \theta\|_{L^2 H^{2m-2\ell+3}}^2) \right. \\ & \quad + \sum_{\ell=0}^{j-1} \left(\|\partial_t^\ell u\|_{L^\infty H^{2m-2\ell+2}}^2 + \|\partial_t^\ell \theta\|_{L^\infty H^{2m-2\ell+2}}^2 + \|\partial_t^\ell p\|_{L^2 H^{2m-2\ell+2}}^2 \right. \\ & \quad \left. \left. + \|\partial_t^\ell p\|_{L^\infty H^{2m-2\ell+1}}^2 \right) \right) \\ & \lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}), \end{aligned} \tag{4.94}$$

The estimates (4.90), (4.76) as well as parts (4.51), (4.53) of Lemma 4.6, allow us to deduce that

$$\begin{aligned} & \|\partial_t D_t^m u\|_{L^\infty H^2}^2 + \|\partial_t D_t^m u\|_{L^2 H^3}^2 \\ & \lesssim \|\partial_t D_t^m u - D_t^{m+1} u\|_{L^\infty H^2}^2 + \|\partial_t D_t^m u - D_t^{m+1} u\|_{L^2 H^3}^2 \\ & \quad + \|D_t^{m+1} u\|_{L^\infty H^2}^2 + \|D_t^{m+1} u\|_{L^2 H^3}^2 \\ & \lesssim P(\mathfrak{R}(\eta)) (\|D_t^m u\|_{L^\infty H^2}^2 + \|D_t^m u\|_{L^2 H^3}^2) + \|D_t^{m+1} u\|_{L^\infty H^2}^2 + \|D_t^{m+1} u\|_{L^2 H^3}^2 \\ & \lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \end{aligned} \tag{4.95}$$

Since Equation (4.74) is satisfied in the strong sense for $j = m$, for almost all $t \in [0, T]$, $(D_t^m u, \partial_t^m p, \partial_t^m \theta)$ solves elliptic system (3.1) with F^1 replaced by $F^{1,m} - \partial_t D_t^m u$, $F^2 = 0$, F^3 replaced by $F^{3,m} - \partial_t (\partial_t^m \theta)$ and F^4, F^5 replaced by $F^{4,m}, F^{5,m}$, respectively. Then, we apply Proposition 3.5 with $r = 5$, square the resulting estimate, and integrate over $[0, T]$, to deduce that

$$\begin{aligned} & \|D_t^m u\|_{L^2 H^5}^2 + \|\partial_t^m p\|_{L^2 H^4}^2 + \|\partial_t^m \theta\|_{L^2 H^5}^2 \\ & \lesssim \|F^{1,m} - \partial_t D_t^m u\|_{L^2 H^3}^2 + \|F^{3,m} - \partial_t (\partial_t^m \theta)\|_{L^2 H^3}^2 \\ & \quad + \|F^{4,m}\|_{L^2 H^{7/2}}^2 + \|F^{5,m}\|_{L^2 H^{7/2}}^2 \\ & \lesssim \|F^{1,m}\|_{L^2 H^3}^2 + \|\partial_t D_t^m u\|_{L^2 H^3}^2 + \|F^{3,m}\|_{L^2 H^3}^2 + \|\partial_t (\partial_t^m \theta)\|_{L^2 H^3}^2 \\ & \quad + \|F^{4,m}\|_{L^2 H^{7/2}}^2 + \|F^{5,m}\|_{L^2 H^{7/2}}^2 \\ & \lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}), \end{aligned} \tag{4.96}$$

where in the last inequality, we have used the estimates (4.90), (4.94) and (4.95). Similarly, Proposition 3.5 with $r = 4$ reveals that

$$\|D_t^m u\|_{L^\infty H^4}^2 + \|\partial_t^m p\|_{L^\infty H^3}^2 + \|\partial_t^m \theta\|_{L^\infty H^4}^2$$

$$\begin{aligned} &\lesssim \|F^{1,m} - \partial_t D_t^m u\|_{L^\infty H^2}^2 + \|F^{3,m} - \partial_t(\partial_t^m \theta)\|_{L^\infty H^2}^2 \\ &\quad + \|F^{4,m}\|_{L^\infty H^{5/2}}^2 + \|F^{5,m}\|_{L^\infty H^{5/2}}^2 \\ &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \end{aligned} \tag{4.97}$$

By iterating to estimate $\partial_t^j u$, $\partial_t^j p$ and $\partial_t^j \theta$ for $j = 1, \dots, m$, as well as the above two estimates, we have that

$$\begin{aligned} &\|\partial_t^m u\|_{L^\infty H^4}^2 + \|\partial_t^m u\|_{L^2 H^5}^2 \\ &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \end{aligned}$$

Thus, we have that

$$\begin{aligned} &\sum_{j=1}^m \left(\|\partial_t^j u\|_{L^2 H^{2(m+1)-2j+3}}^2 + \|\partial_t^j p\|_{L^2 H^{2(m+1)-2j+2}}^2 + \|\partial_t^j \theta\|_{L^2 H^{2(m+1)-2j+3}}^2 \right) \\ &\quad + \sum_{j=1}^m \left(\|\partial_t^j u\|_{L^\infty H^{2(m+1)-2j+2}}^2 + \|\partial_t^j p\|_{L^\infty H^{2(m+1)-2j+1}}^2 + \|\partial_t^j \theta\|_{L^\infty H^{2(m+1)-2j+2}}^2 \right) \\ &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \end{aligned} \tag{4.98}$$

Then we apply Proposition 3.5 with $r = 2(m+1) + 3 \leq 2N + 1$, square the resulting estimate, and integrate over $[0, T]$, to see that

$$\begin{aligned} &\|u\|_{L^2 H^{2(m+1)+3}}^2 + \|p\|_{L^2 H^{2(m+1)+2}}^2 + \|\theta\|_{L^2 H^{2(m+1)+3}}^2 \\ &\lesssim \|F^1 - \partial_t u\|_{L^2 H^{2(m+1)+1}}^2 + \|F^3 - \partial_t \theta\|_{L^2 H^{2(m+1)+1}}^2 \\ &\quad + \|F^4\|_{L^2 H^{2(m+1)+3/2}}^2 + \|F^5\|_{L^2 H^{2(m+1)+3/2}}^2 \\ &\lesssim \|F^1\|_{L^2 H^{2(m+1)+1}}^2 + \|\partial_t u\|_{L^2 H^{2(m+1)+1}}^2 + \|F^3\|_{L^2 H^{2(m+1)+1}}^2 + \|\partial_t \theta\|_{L^2 H^{2(m+1)+1}}^2 \\ &\quad + \|F^4\|_{L^2 H^{2(m+1)+3/2}}^2 + \|F^5\|_{L^2 H^{2(m+1)+3/2}}^2 \\ &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}), \end{aligned} \tag{4.99}$$

and then again with $r = 2(m+1) + 2 \leq 2N$ to see that

$$\begin{aligned} &\|u\|_{L^\infty H^{2(m+1)+2}}^2 + \|p\|_{L^\infty H^{2(m+1)+1}}^2 + \|\theta\|_{L^\infty H^{2(m+1)+2}}^2 \\ &\lesssim \|F^1 - \partial_t u\|_{L^\infty H^{2(m+1)}}^2 + \|F^3 - \partial_t \theta\|_{L^\infty H^{2(m+1)}}^2 \\ &\quad + \|F^4\|_{L^\infty H^{2(m+1)+1/2}}^2 + \|F^5\|_{L^\infty H^{2(m+1)+1/2}}^2 \\ &\lesssim \|F^1\|_{L^\infty H^{2(m+1)}}^2 + \|\partial_t u\|_{L^\infty H^{2(m+1)}}^2 + \|F^3\|_{L^\infty H^{2(m+1)}}^2 + \|\partial_t \theta\|_{L^\infty H^{2(m+1)}}^2 \\ &\quad + \|F^4\|_{L^\infty H^{2(m+1)+1/2}}^2 + \|F^5\|_{L^\infty H^{2(m+1)+1/2}}^2 \\ &\lesssim P(\mathfrak{E}_0(\eta), \mathfrak{R}(\eta)) \exp(P(\mathfrak{E}(\eta))T) (\|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \mathfrak{F}_0 + \mathfrak{F}). \end{aligned} \tag{4.100}$$

Thus, the estimate (4.93) is obtained by summing the estimates (4.98)–(4.100). This completes the proof. \square

5. Preliminaries for the nonlinear problem

In order to use linear theory for the problem (1.8) to solve the nonlinear problem (1.5), we have to define forcing terms F^1, F^3, F^4, F^5 to be used in the linear estimates. Given u, θ, η , we define

$$\begin{aligned} F^1(u, \theta, \eta) &= \partial_t \bar{\eta}(1 + x_3) K \partial_3 u - u \cdot \nabla_{\mathcal{A}} u & \text{and} & \quad F^4(u, \theta, \eta) = \eta \mathcal{N}, \\ F^3(u, \theta, \eta) &= \partial_t \bar{\eta}(1 + x_3) K \partial_3 \theta - u \cdot \nabla_{\mathcal{A}} \theta & \text{and} & \quad F^5(u, \theta, \eta) = -|\mathcal{N}|, \end{aligned} \tag{5.1}$$

where \mathcal{A} , \mathcal{N} , K are determined as before by η . Then we define the quantities $\mathfrak{K}_N(u, \theta)$ and $\mathfrak{K}_N(u)$ as

$$\begin{aligned} \mathfrak{K}_N(u, \theta) = & \sum_{j=0}^N \left(\|\partial_t^j u\|_{L^2 H^{2N-2j+1}}^2 + \|\partial_t^j u\|_{L^\infty H^{2N-2j}}^2 \right. \\ & \left. + \|\partial_t^j \theta\|_{L^2 H^{2N-2j+1}}^2 + \|\partial_t^j \theta\|_{L^\infty H^{2N-2j}}^2 \right), \end{aligned} \tag{5.2}$$

and

$$\mathfrak{K}_N(u) = \sum_{j=0}^N \left(\|\partial_t^j u\|_{L^2 H^{2N-2j+1}}^2 + \|\partial_t^j u\|_{L^\infty H^{2N-2j}}^2 \right). \tag{5.3}$$

5.1. Initial data estimates. Since η is unknown for the full nonlinear problem, and its evolution is coupled to that of u , p and θ , we must reconstruct the initial data to contain this coupling, only with u_0 , θ_0 and η_0 . Here we will define some quantities which have minor difference from [6]:

$$\mathcal{E}_0 := \|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2 + \|\eta_0\|_{H^{2N+1/2}}^2, \tag{5.4}$$

and

$$\mathfrak{E}_0(u, p, \theta) := \sum_{j=0}^N \|\partial_t^j u(0)\|_{H^{2N-2j}}^2 + \sum_{j=0}^{N-1} \|\partial_t^j p(0)\|_{H^{2N-2j-1}}^2 + \sum_{j=0}^N \|\partial_t^j \theta(0)\|_{H^{2N-2j}}^2. \tag{5.5}$$

For $j = 0, \dots, N - 1$,

$$\begin{aligned} & \mathfrak{F}_0^j(F^1(u, p, \theta), F^3(u, p, \theta), F^4(u, p, \theta), F^5(u, p, \theta)) \\ & := \sum_{\ell=0}^j \left(\|\partial_t^\ell F^1(0)\|_{H^{2N-2\ell-2}}^2 + \|\partial_t^\ell F^3(0)\|_{H^{2N-2\ell-2}}^2 + \|\partial_t^\ell F^4(0)\|_{H^{2N-2\ell-3/2}}^2 \right. \\ & \quad \left. + \|\partial_t^\ell F^5(0)\|_{H^{2N-2\ell-3/2}}^2 \right). \end{aligned} \tag{5.6}$$

$$\mathfrak{E}_0^0(\eta) := \|\eta_0\|_{H^{2N+1/2}}^2, \tag{5.7}$$

and for $j = 1, \dots, N$,

$$\mathfrak{E}_0^j(\eta) := \|\eta_0\|_{H^{2N+1/2}}^2 + \sum_{\ell=1}^j \|\partial_t^\ell \eta(0)\|_{H^{2N-2\ell+3/2}}^2. \tag{5.8}$$

$$\mathfrak{E}_0^0(u, p, \theta) := \|u_0\|_{H^{2N}}^2 + \|\theta_0\|_{H^{2N}}^2, \tag{5.9}$$

and for $j = 1, \dots, N$,

$$\mathfrak{E}_0^j(u, p, \theta) := \sum_{\ell=0}^j \|\partial_t^\ell u(0)\|_{H^{2N-2\ell}}^2 + \sum_{\ell=0}^{j-1} \|\partial_t^\ell p(0)\|_{H^{2N-2\ell-1}}^2 + \sum_{\ell=0}^j \|\partial_t^\ell \theta(0)\|_{H^{2N-2\ell}}^2. \tag{5.10}$$

The following lemma is a minor modification of Lemma 5.2 in [6], so we omit the details of proof.

LEMMA 5.1. For $j=0, \dots, N$,

$$\|\partial_t^j u(0) - D_t^j u(0)\|_{H^{2N-2j}}^2 \leq P_j(\mathfrak{E}_0^j(\eta), \mathfrak{E}_0^j(u, p, \theta)) \tag{5.11}$$

and

$$\|\partial_t^j(\theta(0)\nabla_{\mathcal{A}_0} y_{3,0}) - D_t^j(\theta(0)\nabla_{\mathcal{A}_0} y_{3,0})\|_{H^{2N-2j}}^2 \leq P_j(\mathfrak{E}_0^j(\eta), \mathfrak{E}_0^j(u, p, \theta)) \tag{5.12}$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$.

For $F^1(u, \theta, \eta)$, $F^3(u, \theta, \eta)$, $F^4(u, \theta, \eta)$ and $F^5(u, \theta, \eta)$ defined by Equation (5.1) and $j=0, \dots, N-1$, we have that

$$\mathfrak{F}_0^j(F^1(u, p, \theta), F^3(u, p, \theta), F^4(u, p, \theta), F^5(u, p, \theta)) \leq P_j(\mathfrak{E}_0^{j+1}(\eta), \mathfrak{E}_0^j(u, p, \theta)) \tag{5.13}$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$.

For $j=1, \dots, N-1$, let $F^{1,j}(0)$, $F^{3,j}(0)$, $F^{4,j}(0)$ and $F^{5,j}(0)$ are determined by Equations (4.47), (4.48) and (5.1). Then

$$\begin{aligned} & \|F^{1,j}(0)\|_{H^{2N-2j-2}}^2 + \|F^{3,j}(0)\|_{H^{2N-2j-2}}^2 + \|F^{4,j}(0)\|_{H^{2N-2j-3/2}}^2 + \|F^{5,j}(0)\|_{H^{2N-2j-3/2}}^2 \\ & \leq P_j(\mathfrak{E}_0^{j+1}(\eta), \mathfrak{E}_0^j(u, p, \theta)) \end{aligned} \tag{5.14}$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$.

For $j=1, \dots, N-1$,

$$\left\| \sum_{\ell=0}^j \binom{j}{\ell} \partial_t^\ell \mathcal{N}(0) \cdot \partial_t^{j-\ell} u(0) \right\|_{H^{2N-2j+3/2}}^2 \leq P_j(\mathfrak{E}_0^j(\eta), \mathfrak{E}_0^j(u, p, \theta)) \tag{5.15}$$

for $P_j(\cdot, \cdot)$ a polynomial such that $P_j(0, 0) = 0$. Also,

$$\|u_0 \cdot \mathcal{N}_0\|_{H^{2N-1/2}(\Sigma)}^2 \leq \|u_0\|_{H^{2N}}^2 (1 + \|\eta_0\|_{H^{2N+1/2}}^2). \tag{5.16}$$

This lemma allows us to construct all of the initial data $\partial_t^j u(0)$, $\partial_t^j \theta(0)$, $\partial_t^j \eta(0)$ for $j=0, \dots, N$ and $\partial_t^j p(0)$ for $j=0, \dots, N-1$.

Assume that $\mathcal{E}_0 < \infty$. As before, we will iteratively construct the initial data, but this time we will use Lemma 5.1. We define $\partial_t \eta(0) = u_0 \cdot \mathcal{N}_0$, where $u_0 \in H^{2N-1/2}(\Sigma)$, and \mathcal{N}_0 is determined by η_0 . The inequality (5.16) implies that $\|\partial_t \eta(0)\|_{H^{2N-1/2}}^2 \lesssim P(\mathcal{E}_0)$ for a polynomial $P(\cdot)$ such that $P(0) = 0$, and hence that $\mathfrak{E}_0^0(u, p, \theta) + \mathfrak{E}_0^1(\eta) \lesssim P(\mathcal{E}_0)$. Then the inequality (5.13) with $j=0$ implies that

$$\mathfrak{F}_0^0(F^1(u, p, \theta), F^3(u, p, \theta), F^4(u, p, \theta), F^5(u, p, \theta)) \leq P_0(\mathfrak{E}_0^0(\eta), \mathfrak{E}_0^0(u, p, \theta)) \lesssim P(\mathcal{E}_0) \tag{5.17}$$

for a polynomial $P(\cdot)$ such that $P(0) = 0$. Note that in these estimates and in the estimates below, the polynomials $P(\cdot)$ of \mathcal{E}_0 are allowed to change from line to line, but they always satisfy $P(0) = 0$.

In this paragraph, we will give the iterative definition of $\partial_t^j p(0)$, $\partial_t^{j+1} u(0)$, $\partial_t^{j+1} \theta(0)$ and $\partial_t^{j+2} \eta(0)$ for $0 \leq j \leq N-2$. Now suppose that $\partial_t^\ell u(0)$, $\partial_t^\ell \theta(0)$ are known for $\ell=0, \dots, j$, $\partial_t^\ell \eta(0)$ is known for $\ell=0, \dots, j+1$, $\partial_t^\ell p(0)$ is known for $\ell=0, \dots, j-1$ (with the exception for $p(0)$ when $j=0$) and

$$\mathfrak{E}_0^j(u, p, \theta) + \mathfrak{E}_0^{j+1}(\eta) + \mathfrak{F}_0^j(F^1(u, p, \theta), F^3(u, p, \theta), F^4(u, p, \theta), F^5(u, p, \theta)) \lesssim P(\mathcal{E}_0). \tag{5.18}$$

And according to the inequalities (5.14) and (5.11), we know that

$$\begin{aligned} & \|D_t^j u(0)\|_{H^{2N-2j}}^2 + \|F^{1,j}(0)\|_{H^{2N-2j-2}}^2 + \|F^{3,j}(0)\|_{H^{2N-2j-2}}^2 \\ & \quad + \|F^{4,j}(0)\|_{H^{2N-2j-3/2}}^2 + \|F^{5,j}(0)\|_{H^{2N-2j-3/2}}^2 \\ & \lesssim P(\mathcal{E}_0). \end{aligned} \tag{5.19}$$

By virtue of estimates (4.68)

$$\begin{aligned} & \|\mathfrak{f}^1(F^{1,j}(0), D_t^j u(0))\|_{H^{2N-2i-3}}^2 + \|\mathfrak{f}^2(F^{3,j}(0), D_t^j u(0))\|_{H^{2N-2i-3/2}}^2 \\ & \quad + \|\mathfrak{f}^3(F^{1,j}(0), D_t^j u(0))\|_{H^{2N-2i-5/2}}^2 \\ & \lesssim P(\mathcal{E}_0). \end{aligned} \tag{5.20}$$

This allows us to define $\partial_t^j p(0)$ as the solution to the problem (3.18) with f^1, f^2, f^3 replaced by $\mathfrak{f}^1, \mathfrak{f}^2, \mathfrak{f}^3$. Proposition 2.15 in [13] with $k=2N$ and $r=2N-2j-1$ implies that

$$\|\partial_t^j p(0)\|_{H^{2N-2j-1}}^2 \lesssim P(\mathcal{E}_0). \tag{5.21}$$

Now we define

$$\partial_t^{j+1} \theta(0) = \mathfrak{E}^{02}(\partial_t^j \theta(0), F^{3,j}(0)) \in H^{2N-2j-2}. \tag{5.22}$$

Then according to the estimates (5.18) and (5.19), we have that

$$\|\partial_t^{j+1} \theta(0)\|_{H^{2N-2j-2}}^2 \lesssim P(\mathcal{E}_0). \tag{5.23}$$

Now the estimates (4.66), (5.18) and (5.19) allow us to define

$$D_t^{j+1} u(0) := \mathfrak{E}^{01} \left(F^{1,j}(0) + \partial_t^j(\theta(0) \nabla_{\mathcal{A}_0} y_{3,0}), D_t^j u(0), \partial_t^j p(0) \right) \in H^{2N-2j-2}, \tag{5.24}$$

and then according to the estimate (5.11), we have

$$\|\partial_t^{j+1} u(0)\|_{H^{2N-2j-2}}^2 \leq P(\mathcal{E}_0). \tag{5.25}$$

Now the estimates (5.16), (5.18) and (5.25) allow us to define

$$\partial_t^{j+2} \eta(0) = \sum_{\ell=0}^{j+1} \binom{j+1}{\ell} \partial_t^\ell \mathcal{N}(0) \cdot \partial_t^{j+1-\ell} u(0),$$

and imply the estimate

$$\|\partial_t^{j+2} \eta(0)\|_{H^{2N-2j-5/2}}^2 \leq P(\mathcal{E}_0). \tag{5.26}$$

Thus, the inequality (5.18) together with the conditions (5.21)–(5.26) imply that

$$\mathfrak{E}_0^{j+1}(u, p, \theta) + \mathfrak{E}_0^{j+2}(\eta) \leq P(\mathcal{E}_0),$$

and then the inequality (5.13) implies that

$$\mathfrak{F}_0^{j+1}(F^1(u, p, \theta), F^3(u, p, \theta), F^4(u, p, \theta), F^5(u, p, \theta)) \leq P(\mathcal{E}_0).$$

Hence that we can deduce the estimate

$$\mathfrak{E}_0^{j+1}(u, p, \theta) + \mathfrak{E}_0^{j+2}(\eta) + \mathfrak{F}_0^{j+1}(F^1(u, p, \theta), F^3(u, p, \theta), F^4(u, p, \theta), F^5(u, p, \theta)) \leq P(\mathcal{E}_0).$$

For $j = N - 2$, we have

$$\begin{aligned} &\mathfrak{E}_0^{N-1}(u, p, \theta) + \mathfrak{E}_0^N(\eta) + \mathfrak{F}_0^{N-1}(F^1(u, p, \theta), F^3(u, p, \theta), F^4(u, p, \theta), F^5(u, p, \theta)) \\ &\leq P(\mathcal{E}_0). \end{aligned} \tag{5.27}$$

Then, we only need to define $\partial_t^{N-1}p(0)$, $\partial_t^N\theta(0)$ and $\partial_t^N u(0)$. Like the construction after Lemma 4.8, we need the compatibility conditions on u_0 and η_0 . Now we have constructed $\partial_t^j p(0)$ for $j=0, \dots, N-2$, $\partial_t^j u(0)$, $\partial_t^j \theta(0)$, $F^{1:j}(0)$, $F^{3:j}(0)$, $F^{4:j}(0)$, $F^{5:j}(0)$ for $j=0, \dots, N-1$, and $\partial_t^j \eta(0)$ for $j=0, \dots, N$. We say that u_0 and η_0 satisfy the N -th order compatibility conditions if

$$\begin{cases} \nabla_{\mathcal{A}_0} \cdot (D_t^j u(0)) = 0 & \text{in } \Omega, \\ D_t^j u(0) = 0 & \text{on } \Sigma_b, \\ \Pi_0 \left(F^{4:j}(0) + \mathbb{D}_{\mathcal{A}_0} D_t^j u(0) \cdot \mathcal{N}_0 \right) = 0 & \text{on } \Sigma, \end{cases} \tag{5.28}$$

for $j=0, \dots, N-1$, where Π_0 is the projection defined as in Equation (4.14) and D_t is the operator defined by Equation (4.10). Note that if u_0 and η_0 satisfy the conditions (5.28), then the j -th compatibility condition (4.71) is satisfied for $j=0, \dots, N-1$. Then the construction of $\partial_t^{N-1}p(0)$ is the same as [6] using the compatibility condition (5.28) and the elliptic theory of \mathcal{A} -Poisson Equation (3.18) derived by Y. Guo and I. Tice in [6] and L. Wu in [13]. And

$$\|\partial_t^{N-1}p(0)\|_{H^1}^2 \leq P(\mathcal{E}_0). \tag{5.29}$$

Then we set $\partial_t^N \theta(0) = \mathfrak{E}^{02}(\partial_t^{N-1}\theta(0), F^{3,N-1}(0)) \in H^0$ due to the estimates (4.67) and (5.14), and set $D_t^N u(0) = \mathfrak{E}^{01}(F^{1,N-1}(0) + \partial_t^{N-1}(\theta \nabla_{\mathcal{A}_0} y_{3,0}), D_t^{N-1}u(0), \partial_t^{N-1}p(0)) \in H^0$ due to the estimate (4.66) and Lemma 5.1. And $D_t^N u(0) \in \mathcal{Y}(0)$ is guaranteed by the construction of $\partial_t^{N-1}p(0)$. As before, we have

$$\|\partial_t^N u(0)\|_{H^0}^2 + \|\partial_t^N \theta(0)\|_{H^0}^2 \lesssim P(\mathcal{E}_0). \tag{5.30}$$

This completes the construction of initial data. Then summing the estimates (5.27), (5.29) and (5.30), we directly have the following proposition.

PROPOSITION 5.2. *Suppose that u_0 , θ_0 and η_0 satisfy $\mathcal{E}_0 < \infty$. Let the initial data $\partial_t^j u(0)$, $\partial_t^j \theta(0)$, $\partial_t^j \eta(0)$ for $j=0, \dots, N$ and $\partial_t^j p(0)$ for $j=0, \dots, N-1$ be given as above. Then*

$$\mathcal{E}_0 \leq \mathfrak{E}_0(u, p, \theta) + \mathfrak{E}_0(\eta) \lesssim P(\mathcal{E}_0). \tag{5.31}$$

Here $\mathfrak{E}_0(\eta) = \mathfrak{E}_0^N(\eta)$, which is defined in Equation (4.50).

5.2. Transport equation. Here we consider the equation

$$\begin{cases} \partial_t \eta + u_1 \partial_1 \eta + u_2 \partial_2 \eta = u_3 & \text{on } \Sigma, \\ \eta(0) = \eta_0. \end{cases} \tag{5.32}$$

The local well-posedness of Equation (5.32) has been proved by L. Wu, which is Theorem 2.17 in [13]. The idea of his proof is similar to the proof of Theorem 5.4 in [6]. In [13], L. Wu has proved in Lemma 2.18, that the difference of η and η_0 in a small time period is also small.

5.3. Forcing estimates. In the next section for the estimates of full nonlinear problem, we need some forcing quantities. Besides \mathfrak{F} and \mathfrak{F}_0 , which have been defined in Equation (4.49), we define the following quantities:

$$\begin{aligned} \mathcal{F} &:= \sum_{j=0}^{N-1} \left(\|\partial_t^j F^1\|_{L^2 H^{2N-2j-1}}^2 + \|\partial_t^j F^3\|_{L^2 H^{2N-2j-1}}^2 \right) + \|\partial_t^N F^1\|_{L^2 H^0}^2 + \|\partial_t^N F^3\|_{L^2 H^0}^2 \\ &\quad + \sum_{j=0}^N \left(\|\partial_t^j F^4\|_{L^\infty H^{2N-2j-1/2}(\Sigma)}^2 + \|\partial_t^j F^5\|_{L^\infty H^{2N-2j-1/2}(\Sigma)}^2 \right), \\ \mathcal{H} &:= \sum_{j=0}^{N-1} \left(\|\partial_t^j F^1\|_{L^2 H^{2N-2j-1}}^2 + \|\partial_t^j F^3\|_{L^2 H^{2N-2j-1}}^2 \right) \\ &\quad + \sum_{j=0}^{N-1} \left(\|\partial_t^j F^4\|_{L^2 H^{2N-2j-1/2}(\Sigma)}^2 + \|\partial_t^j F^5\|_{L^2 H^{2N-2j-1/2}(\Sigma)}^2 \right), \end{aligned}$$

The following theorem is similar to Theorem 2.21 in [13] with obvious modification.

THEOREM 5.3. *The forcing terms satisfy the estimates*

$$\mathfrak{F} \lesssim P(\mathfrak{K}(\eta)) + P(\mathfrak{K}_N(u, \theta)), \tag{5.33}$$

$$\mathfrak{F}_0 \lesssim P(\mathcal{E}_0), \tag{5.34}$$

$$\mathcal{F} \lesssim P(\mathfrak{K}(\eta)) + P(\mathfrak{K}_N(u, \theta)), \tag{5.35}$$

$$\mathcal{H} \lesssim T(P(\mathfrak{K}(\eta)) + P(\mathfrak{K}_N(u, \theta))). \tag{5.36}$$

Proof. The proof of this theorem is the same as the proof of Theorem 2.21 in [13], so we omit the details here. □

6. Local well-posedness for the nonlinear problem

6.1. Construction of approximate solutions. In order to solve the problem (1.4), we will construct a sequence of approximate solutions $(u^m, p^m, \theta^m, \eta^m)$, then take the limit $m \rightarrow \infty$. First, we construct an initial pair (u^0, θ^0, η^0) as a start point, then we iteratively define all sequences $(u^m, p^m, \theta^m, \eta^m)$ for $m \geq 1$.

Suppose that the initial data (u_0, θ_0, η_0) has been given. According to Lemma A.5 in [6], there exist u^0 and θ^0 defined in $\Omega \times [0, \infty)$ with $\partial_t^j u^0(0) = \partial_t^j u(0)$, $\partial_t^j \theta^0(0) = \partial_t^j \theta(0)$, for $j = 0, \dots, N$, satisfying

$$\mathfrak{K}_N(u^0, \theta^0) \lesssim P(\mathcal{E}_0). \tag{6.1}$$

Then we consider Equation (3.18) with u replaced by u^0 . From Theorem 2.17 in [13], the hypothesis of which is satisfied by the inequalities (5.31) and (6.1), there exists a η^0 defined in $\Omega \times [0, T_0)$, which satisfies $\partial_t^j \eta^0(0) = \partial_t^j \eta(0)$ for $j = 0, \dots, N$ as well as

$$\mathfrak{K}(\eta^0) \lesssim P(\mathcal{E}_0).$$

Then for any integer $m \geq 1$, we formally define the sequence $(u^m, p^m, \theta^m, \eta^m)$ on the

time interval $[0, T_m)$ as the solutions of system

$$\left\{ \begin{array}{ll} \begin{array}{l} \partial_t u^m - \Delta_{\mathcal{A}^{m-1}} u^m + \nabla_{\mathcal{A}^{m-1}} p^m + \theta^m \nabla_{\mathcal{A}^{m-1}} y_3^{m-1} \\ = \partial_t \bar{\eta}^{m-1} (1 + x_3) K^{m-1} \partial_3 u^{m-1} - u^{m-1} \cdot \nabla_{\mathcal{A}^{m-1}} u^{m-1} \end{array} & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}^{m-1}} u^m = 0 & \text{in } \Omega, \\ \begin{array}{l} \partial_t \theta^m - \Delta_{\mathcal{A}^{m-1}} \theta^m \\ = \partial_t \bar{\eta}^{m-1} (1 + x_3) K^{m-1} \partial_3 \theta^{m-1} - u^{m-1} \cdot \nabla_{\mathcal{A}^{m-1}} \theta^{m-1} \end{array} & \text{in } \Omega, \\ S_{\mathcal{A}^{m-1}}(p^m, u^m) \mathcal{N}^{m-1} = \eta^{m-1} \mathcal{N}^{m-1} & \text{on } \Sigma, \\ \nabla_{\mathcal{A}^{m-1}} \theta^m \cdot \mathcal{N}^{m-1} + \theta^m |\mathcal{N}^{m-1}| = -|\mathcal{N}^{m-1}| & \text{on } \Sigma, \\ u^m = 0, \quad \theta^m = 0 & \text{on } \Sigma_b, \end{array} \right. \tag{6.2}$$

and

$$\partial_t \eta^m = u^m \cdot \mathcal{N}^m \quad \text{on } \Sigma, \tag{6.3}$$

where \mathcal{A}^{m-1} , \mathcal{N}^{m-1} , K^{m-1} are determined in terms of η^{m-1} and \mathcal{N}^m is in terms of η^m , with the initial data $(u^m(0), \theta^m(0), \eta^m(0)) = (u_0, \theta_0, \eta_0)$.

In the following, we will prove that these sequences can be defined for any integer $m \geq 1$ and the existence time T_m does not shrink to 0 as $m \rightarrow \infty$. The following theorem is a modified version of Theorem 2.24 in [13], which improves the estimate (4.75) using the energy structure and elliptic estimates.

THEOREM 6.1. *Suppose $J(0) > \delta > 0$. Assume that the initial data (u_0, θ_0, η_0) satisfy $\mathcal{E}_0 < \infty$ and $\partial_t^j u(0), \partial_t^j \theta(0), \partial_t^j \eta(0)$, for $j = 0, \dots, N$, are given as above from the Proposition 5.2. Then there exists a positive constant $\mathcal{L} < \infty$ and $0 < \bar{T} < 1$ depending on \mathcal{E}_0 , such that if $0 < T < \bar{T}$, then there exists a sequence $\{(u^m, p^m, \theta^m, \eta^m)\}_{m=0}^\infty$ (when $m = 0$, the sequence should be considered as (u^0, θ^0, η^0)) satisfying the iteration Equation (6.2) within the time interval $[0, T)$ and the following properties:*

(1) *The iteration sequence satisfies*

$$\mathfrak{K}_N(u^m, \theta^m) + \mathfrak{K}(\eta^m) \leq \mathcal{L} \tag{6.4}$$

for any integer $m \geq 0$, where the temporal norm is taken with respect to $[0, T)$.

(2) *$J^m(t) \geq \delta/2$ with $0 \leq t \leq T$, for any integer $m \geq 0$.*

Proof. In this proof, we will follow the path of proof of Theorem 2.24 in [13]. We will use an infinite induction to prove this theorem. Let us denote the above two assertions as statement \mathbb{P}_m .

Step 1. \mathbb{P}_0 case. The only modification here is that the construction of u^0 and θ^0 reveals that $\mathfrak{K}_N(u^0, \theta^0) \lesssim P(\mathcal{E}_0)$. Then the rest proof of this case is the same as the proof of Theorem 2.24 in [13]. Hence, \mathbb{P}_0 holds. That is $\mathfrak{K}_N(u^0, \theta^0) + \mathfrak{K}(\eta^0) \leq \mathcal{L}$ with the temporal norm taken with respect to $[0, T)$ and $J^0(t) \geq \delta/2$ for $0 \leq t \leq T$.

In the following, we suppose that \mathbb{P}_{m-1} holds for $m \geq 1$. Then we will prove that \mathbb{P}_m also holds.

Step 2. \mathbb{P}_m case: energy estimates of θ^m and u^m . By Theorem 4.9, the triple

$(D_t^N u^m, \partial_t^N p^m, \partial_t^N \theta^m)$ satisfies the equation

$$\left\{ \begin{array}{ll} \partial_t(D_t^N u^m) - \Delta_{\mathcal{A}^{m-1}}(D_t^N u^m) + \nabla_{\mathcal{A}^{m-1}}(\partial_t^N p^m) \\ \quad - \partial_t^N(\theta^m \nabla_{\mathcal{A}^{m-1}} y_3^{m-1}) = F^{1,N} & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}^{m-1}}(D_t^N u^m) = 0 & \text{in } \Omega, \\ \partial_t(\partial_t^N \theta^m) - \Delta_{\mathcal{A}^{m-1}}(\partial_t^N \theta^m) = F^{3,N} & \text{in } \Omega, \\ S_{\mathcal{A}^{m-1}}(\partial_t^N p^m, D_t^N u^m) \cdot \mathcal{N}^{m-1} = F^{4,N} & \text{on } \Sigma, \\ \nabla_{\mathcal{A}^{m-1}}(\partial_t^N \theta^m) \cdot \mathcal{N}^{m-1} + \partial_t^j \theta^m |_{\mathcal{N}^{m-1}} = F^{5,N} & \text{on } \Sigma, \\ D_t^N u^m = 0, \quad \partial_t^N \theta^m = 0 & \text{on } \Sigma_b, \end{array} \right. \quad (6.5)$$

in the weak sense, where $F^{1,N}$, $F^{3,N}$, $F^{4,N}$ and $F^{5,N}$ are given in terms of u^m , p^m , θ^m , and u^{m-1} , p^{m-1} , θ^{m-1} , η^{m-1} . Then for any test function $\phi \in (\mathcal{H}_T^1)^{m-1}$, where $(\mathcal{H}_T^1)^{m-1}$ is the space \mathcal{H}_T^1 with η replaced by η^{m-1} , the following holds:

$$\begin{aligned} & \langle \partial_t(\partial_t^N \theta^m), \phi \rangle_* + (\partial_t^N \theta^m, \phi)_{\mathcal{H}_T^1} + (\partial_t^N \theta^m |_{\mathcal{N}^{m-1}}, \phi)_{L^2 H^0(\Sigma)} \\ &= (F^{3,N}, \phi)_{\mathcal{H}_T^0} + (F^{5,N}, \phi)_{L^2 H^0(\Sigma)}. \end{aligned}$$

Therefore, when taking the test function $\phi = \partial_t^N \theta^m$, we have the energy structure

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} J^{m-1} |\partial_t^N \theta^m|^2 + \int_0^t \int_{\Omega} J^{m-1} |\nabla_{\mathcal{A}^{m-1}}(\partial_t^N \theta^m)|^2 + \int_0^t \int_{\Sigma} |\partial_t^N \theta^m|^2 |_{\mathcal{N}^{m-1}} \\ &= \frac{1}{2} \int_{\Omega} J^{m-1}(0) |\partial_t^N \theta^m(0)|^2 + \frac{1}{2} \int_0^t \int_{\Omega} \partial_t J^{m-1} |\partial_t^N \theta^m|^2 \\ & \quad + \int_0^t \int_{\Omega} J^{m-1} F^{3,N} \partial_t^N \theta^m + \int_0^t \int_{\Sigma} F^{5,N} \partial_t^N \theta^m. \end{aligned} \quad (6.6)$$

By induction hypothesis, the inequality (5.31), trace theory and Cauchy’s inequality, we have

$$\begin{aligned} & \|\partial_t^N \theta^m\|_{L^\infty H^0}^2 + \|\partial_t^N \theta^m\|_{L^2 H^1}^2 \\ & \lesssim \sup_{0 \leq t \leq T} \left(\frac{1}{2} \int_{\Omega} J^{m-1} |\partial_t^N \theta^m|^2 + \int_0^t \int_{\Omega} J^{m-1} |\nabla_{\mathcal{A}^{m-1}}(\partial_t^N \theta^m)|^2 + \int_0^t \int_{\Sigma} |\partial_t^N \theta^m|^2 \right) \\ & \lesssim \frac{1}{2} \int_{\Omega} J^{m-1}(0) |\partial_t^N \theta^m(0)|^2 + \frac{1}{2} \int_0^T \int_{\Omega} \partial_t J^{m-1} |\partial_t^N \theta^m|^2 + \int_0^T \int_{\Omega} J^{m-1} F^{3,N} \partial_t^N \theta^m \\ & \quad + \int_0^T \int_{\Sigma} F^{5,N} \partial_t^N \theta^m \\ & \lesssim P(\mathcal{E}_0) + T \mathcal{Z} \|\partial_t^N \theta^m\|_{L^\infty H^0}^2 + \sqrt{T} \mathcal{Z} \|F^{3,N}\|_{L^2 H^0} \|\partial_t^N \theta^m\|_{L^\infty H^0} \\ & \quad + \sqrt{T} \|F^{5,N}\|_{L^\infty H^{-1/2}(\Sigma)} \|\partial_t^N \theta^m\|_{L^2 H^{1/2}(\Sigma)} \\ & \lesssim P(\mathcal{E}_0) + T \mathcal{Z} \|\partial_t^N \theta^m\|_{L^\infty H^0}^2 + \sqrt{T} \|F^{3,N}\|_{L^2 H^0}^2 \\ & \quad + \sqrt{T} \mathcal{Z}^2 \|\partial_t^N \theta^m\|_{L^\infty H^0}^2 + \sqrt{T} \|F^{5,N}\|_{L^\infty H^{-1/2}(\Sigma)}^2 + \sqrt{T} \|\partial_t^N \theta^m\|_{L^2 H^{1/2}(\Sigma)}^2 \end{aligned} \quad (6.7)$$

for a polynomial $P(0)=0$. Taking $T \leq \min\{1/4, 1/(16\mathcal{Z}^4)\}$ and absorbing the extra terms on the right-hand side into left-hand side imply

$$\|\partial_t^N \theta^m\|_{L^\infty H^0}^2 + \|\partial_t^N \theta^m\|_{L^2 H^1}^2 \lesssim P(\mathcal{E}_0) + \sqrt{T} \|F^{3,N}\|_{L^2 H^0}^2 + \sqrt{T} \|F^{5,N}\|_{L^\infty H^{-1/2}(\Sigma)}^2. \tag{6.8}$$

By induction hypothesis, we have

$$\begin{aligned} \|F^{3,N}\|_{L^2 H^0}^2 &\lesssim P(\mathfrak{K}(\eta^{m-1})) \left(\sum_{j=0}^{N-1} \|\partial_t^j u^m\|_{L^2 H^2}^2 + \|\partial_t^j \theta^m\|_{L^2 H^2}^2 \right) + \mathcal{F} \\ &\lesssim P(\mathcal{E}_0 + \mathcal{Z}) + \mathcal{F}, \end{aligned}$$

$$\begin{aligned} \|F^{5,N}\|_{L^\infty H^{-1/2}(\Sigma)}^2 &\lesssim P(\mathfrak{K}(\eta^{m-1})) \left(\sum_{j=0}^{N-1} \|\partial_t^j u^m\|_{L^\infty H^2}^2 + \|\partial_t^j \theta^m\|_{L^\infty H^2}^2 \right) + \mathcal{F} \\ &\lesssim P(\mathcal{E}_0 + \mathcal{Z}) + \mathcal{F}. \end{aligned}$$

And, the energy estimates about u^m is the same as the proof of Theorem 2.24 in [13]. Therefore, we have

$$\|\partial_t^N u^m\|_{L^2 H^1}^2 + \|\partial_t^N \theta^m\|_{L^2 H^1}^2 \lesssim P(\mathcal{E}_0) + \sqrt{T} P(\mathcal{E}_0 + \mathcal{Z}) + \sqrt{T} \mathcal{F}. \tag{6.9}$$

Step 3. \mathbb{P}_m case: elliptic estimates for θ^m, u^m . For $0 \leq n \leq N-1$, the n -th order heat equation is

$$\begin{cases} \partial_t(\partial_t^n \theta^m) - \Delta_{\mathcal{A}^{m-1}} \partial_t^n \theta^m = F^{3,n} & \text{in } \Omega, \\ \nabla_{\mathcal{A}^{m-1}} \partial_t^n \theta^m \cdot \mathcal{N}^{m-1} + \partial_t^n \theta^m | \mathcal{N}^{m-1} | = F^{5,n} & \text{on } \Sigma, \\ \partial_t^n \theta^m = 0 & \text{on } \Sigma_b. \end{cases} \tag{6.10}$$

The elliptic estimate in the proof of Lemma 3.3 reveals that

$$\|\partial_t^n \theta^m\|_{L^2 H^{2N-2n+1}}^2 \lesssim \|F^{3,n}\|_{L^2 H^{2N-2n-1}}^2 + \|\partial_t^{n+1} \theta^m\|_{L^2 H^{2N-2n-1}}^2 + \|F^{5,n}\|_{L^2 H^{2N-2n-1/2}}^2. \tag{6.11}$$

As what we did before,

$$\begin{aligned} &\|F^{3,n}\|_{L^2 H^{2N-2n-1}}^2 \\ &\lesssim TP(\mathfrak{K}(\eta^{m-1})) \left(\sum_{j=0}^{N-2} \|\partial_t^j \theta^m\|_{L^\infty H^{2N-2j-1}}^2 + \|\partial_t^j u^m\|_{L^\infty H^{2N-2j-1}}^2 \right) + \mathcal{H} \\ &\lesssim TP(\mathcal{E}_0 + \mathcal{Z}) + \mathcal{H}. \end{aligned}$$

$$\begin{aligned} &\|F^{5,n}\|_{L^2 H^{2N-2n-1}}^2 \\ &\lesssim TP(\mathfrak{K}(\eta^{m-1})) \left(\sum_{j=0}^{N-2} \|\partial_t^j \theta^m\|_{L^\infty H^{2N-2j-1}}^2 + \|\partial_t^j u^m\|_{L^\infty H^{2N-2j-1}}^2 \right) + \mathcal{H} \\ &\lesssim TP(\mathcal{E}_0 + \mathcal{Z}) + \mathcal{H}. \end{aligned}$$

But for the term $\|\partial_t^{n+1} \theta^m\|_{L^2 H^{2N-2n-1}}^2$, we estimate backward from $N-1$ to 0. First, when $n=N-1$, this is the case of energy estimate of θ^m . Then we iteratively

use the elliptic estimates (6.11) from $n = N - 2$ to $n = 0$ to obtain all the control of $\|\partial_t^{n+1}\theta^m\|_{L^2H^{2N-2n-1}}^2$.

And the elliptic estimate for u^m is the same as the proof of Theorem 2.24 in [13]. Therefore, we have that

$$\begin{aligned} & \sum_{n=0}^{N-1} (\|\partial_t^n u^m\|_{L^2H^{2N-2n+1}}^2 + \|\partial_t^n \theta^m\|_{L^2H^{2N-2n+1}}^2) \\ & \lesssim P(\mathcal{E}_0) + \sqrt{T}P(1 + \mathcal{E}_0 + \mathcal{Z}) + \sqrt{T}\mathcal{F} + \mathcal{H}. \end{aligned} \tag{6.12}$$

Step 4. \mathbb{P}_m case: synthesis of estimates for u^m and θ^m . Combining the estimates (6.8) and (6.12) and Lemma 2.19 in [13], we deduce that

$$\mathfrak{K}_N(u^m, \theta^m) \lesssim P(\mathcal{E}_0) + \sqrt{T}P(\mathcal{E}_0 + \mathcal{Z}) + \sqrt{T}\mathcal{F} + \mathcal{H}. \tag{6.13}$$

Then by the induction hypothesis and the forcing estimates of Lemma 5.3, we have that

$$\mathcal{F} \lesssim P(\mathfrak{K}(\eta^{m-1})) + P(\mathfrak{K}_N(u^{m-1}, \theta^{m-1})) \lesssim P(\mathcal{Z}),$$

$$\mathcal{H} \lesssim T(P(\mathfrak{K}(\eta^{m-1})) + P(\mathfrak{K}_N(u^{m-1}, \theta^{m-1}))) \lesssim TP(\mathcal{Z}).$$

Hence we obtain the estimate

$$\mathfrak{K}_N(u^m, \theta^m) \leq C(P(\mathcal{E}_0) + \sqrt{T}P(\mathcal{E}_0 + \mathcal{Z})) \tag{6.14}$$

for some universal constant $C > 0$. Taking $\mathcal{Z} \geq 2CP(\mathcal{E}_0)$ and then taking T sufficiently small which depends on \mathcal{Z} , we can achieve that $\mathfrak{K}_N(u^m, \theta^m) \leq 2CP(\mathcal{E}_0) \leq \mathcal{Z}$.

Step 5. \mathbb{P}_m case: estimate for η^m and $J^m(t)$. These estimates are exactly the same as the proof of Theorem 2.24 in [13]. So we omit the details here.

Thus, we can take $\mathcal{Z} = P(\mathcal{E}_0)$ for some polynomial $P(\cdot)$ and T small enough, depending on \mathcal{Z} , to deduce that

$$\mathfrak{K}_N(u^m, \theta^m) \leq \mathcal{Z} \tag{6.15}$$

and

$$J^m(t) \geq \delta/2 \quad \text{for } t \in [0, T]. \tag{6.16}$$

Hence \mathbb{P}_m holds. By induction, \mathbb{P}_n holds for any integer $n \geq 0$. □

THEOREM 6.2. *Assume the same conditions as Theorem 6.1. Then*

$$\mathfrak{K}(u^m, p^m, \theta^m) + \mathfrak{K}(\eta^m) \lesssim P(\mathcal{E}_0) \tag{6.17}$$

for a polynomial $P(\cdot)$ satisfying $P(0) = 0$.

Proof. From the estimates (4.75), (5.31), Lemma 5.3 as well as Theorem 2.17 in [13], we directly have that

$$\mathfrak{K}(u^m, p^m, \theta^m) + \mathfrak{K}(\eta^m) \lesssim P(\mathcal{E}_0) + P(\mathfrak{K}_N(u^m, \theta^m) + \mathfrak{K}(\eta^m)).$$

Then, applying Theorem 6.1, we have that

$$\mathfrak{K}(u^m, p^m, \theta^m) + \mathfrak{K}(\eta^m) \lesssim P(\mathcal{E}_0). \tag{6.18} \quad \square$$

6.2. Contraction. According to Theorem 6.2, we may extract weakly converging subsequences from $\{(u^m, p^m, \theta^m, \eta^m)\}_{m=0}^\infty$. Unfortunately, the original sequence $\{(u^m, p^m, \theta^m, \eta^m)\}_{m=0}^\infty$ is not guaranteed to converge to the same limit. In order to obtain the desired solution to Equation (1.4) by passing to the limit in Equations (6.2) and (6.3), we need to study its contraction in some norm.

For $T > 0$, we define the norms

$$\begin{aligned} \mathfrak{N}(v, q, \Theta; T) &= \|v\|_{L^\infty H^2}^2 + \|v\|_{L^2 H^3}^2 + \|\partial_t v\|_{L^\infty H^0}^2 + \|\partial_t v\|_{L^2 H^1}^2 + \|q\|_{L^\infty H^1}^2 + \|q\|_{L^2 H^2}^2 \\ &\quad + \|\Theta\|_{L^\infty H^2}^2 + \|\Theta\|_{L^2 H^3}^2 + \|\partial_t \Theta\|_{L^\infty H^0}^2 + \|\partial_t \Theta\|_{L^2 H^1}^2, \\ \mathfrak{M}(\zeta; T) &= \|\zeta\|_{L^\infty H^{5/2}}^2 + \|\partial_t \zeta\|_{L^\infty H^{3/2}}^2 + \|\partial_t^2 \zeta\|_{L^2 H^{1/2}}^2, \end{aligned} \tag{6.18}$$

where the norm $L^p H^k$ is $L^p([0, T]; H^k(\Omega))$ in \mathfrak{N} , and is $L^p([0, T]; H^k(\Sigma))$ in \mathfrak{M} .

The next theorem is not only used to prove the contraction of approximate solutions, but also used to verify the uniqueness of solutions to Equation (1.4). To avoid confusion with $\{(u^m, p^m, \theta^m, \eta^m)\}$, we refer to velocities as v^j , w^j , pressures as q^j , temperatures as Θ^j , ϑ^j , and surface functions as ζ^j for $j = 1, 2$.

THEOREM 6.3. *For $j = 1, 2$, suppose that $v^j, q^j, \Theta^j, w^j, \vartheta^j$ and ζ^j satisfy the initial data $\partial_t^k v^1(0) = \partial_t^k v^2(0), \partial_t^k \Theta^1(0) = \partial_t^k \Theta^2(0)$, for $k = 0, 1, q^1(0) = q^2(0)$ and $\zeta^1(0) = \zeta^2(0)$, and that the following system holds:*

$$\left\{ \begin{array}{ll} \partial_t v^j - \Delta_{\mathcal{A}^j} v^j + \nabla_{\mathcal{A}^j} q^j - \Theta^j \nabla_{\mathcal{A}^j} \vartheta^j \\ = \partial_t \bar{\zeta}^j (1 + x_3) K^j \partial_3 w^j - w^j \cdot \nabla_{\mathcal{A}^j} w^j & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}^j} v^j = 0 & \text{in } \Omega, \\ \partial_t \Theta^j - \Delta_{\mathcal{A}^j} \Theta^j = \partial_t \bar{\zeta}^j (1 + x_3) K^j \partial_3 \vartheta^j - w^j \cdot \nabla_{\mathcal{A}^j} \vartheta^j & \text{in } \Omega, \\ S_{\mathcal{A}^j}(q^j, v^j) \mathcal{N}^j = \zeta^j \mathcal{N}^j & \text{on } \Sigma, \\ \nabla_{\mathcal{A}^j} \Theta^j \cdot \mathcal{N}^j + \Theta^j |\mathcal{N}^j| = -|\mathcal{N}^j| & \text{on } \Sigma, \\ v^j = 0, \quad \Theta^j = 0 & \text{on } \Sigma_b, \\ \partial_t \zeta^j = w^j \cdot \mathcal{N}^j & \text{on } \Sigma, \end{array} \right. \tag{6.19}$$

where $\mathcal{A}^j, \mathcal{N}^j, K^j$ are determined by ζ^j . Assume that $\mathfrak{K}(v^j, q^j, \Theta^j), \mathfrak{K}(w^j, 0, \vartheta^j)$ and $\mathfrak{K}(\zeta^j)$ are bounded by \mathcal{L} .

Then there exists $0 < T_1 < 1$ such that for any $0 < T < T_1$, we have

$$\mathfrak{N}(v^1 - v^2, q^1 - q^2, \Theta^1 - \Theta^2; T) \leq \frac{1}{2} \mathfrak{N}(w^1 - w^2, 0, \vartheta^1 - \vartheta^2; T), \tag{6.20}$$

$$\mathfrak{M}(\zeta^1 - \zeta^2; T) \lesssim \mathfrak{M}(w^1 - w^2, 0, \vartheta^1 - \vartheta^2; T). \tag{6.21}$$

Proof. This proof follows the path of Theorem 6.2 in [6]. First, we define $v = v^1 - v^2, w = w^1 - w^2, \Theta = \Theta^1 - \Theta^2, \vartheta = \vartheta^1 - \vartheta^2, q = q^1 - q^2$.

Step 1. Energy evolution for differences. Like the proof of Theorem 6.2 in [6], we

can derive the PDE satisfied by v , q and Θ :

$$\left\{ \begin{array}{ll} \partial_t v + \operatorname{div}_{\mathcal{A}^1} S_{\mathcal{A}^1}(q, v) - \Theta \nabla_{\mathcal{A}^1} y_3^1 = \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2) + H^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}^1} v = H^2 & \text{in } \Omega, \\ \partial_t \Theta - \Delta_{\mathcal{A}^1} \Theta = \operatorname{div}_{\mathcal{A}^1} (\nabla_{(\mathcal{A}^1 - \mathcal{A}^2)} \Theta^2) + H^3 & \text{in } \Omega, \\ S_{\mathcal{A}^1}(q, v) \mathcal{N}^1 = \mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2 \mathcal{N}^1 + H^4 & \text{on } \Sigma, \\ \nabla_{\mathcal{A}^1} \Theta \cdot \mathcal{N}^1 + \Theta |\mathcal{N}^1| = -\nabla_{(\mathcal{A}^1 - \mathcal{A}^2)} \Theta^2 \cdot \mathcal{N}^1 + H^5 & \text{on } \Sigma, \\ v = 0, \quad \Theta = 0 & \text{on } \Sigma_b, \\ v(t=0) = 0, \quad \Theta(t=0) = 0, & \end{array} \right. \quad (6.22)$$

and the PDE satisfied by $\partial_t v$, $\partial_t q$, $\partial_t \Theta$ from taking the temporal derivative for the above system:

$$\left\{ \begin{array}{ll} \partial_t(\partial_t v) + \operatorname{div}_{\mathcal{A}^1} S_{\mathcal{A}^1}(\partial_t q, \partial_t v) - \partial_t(\Theta \nabla_{\mathcal{A}^1} y_3^1) \\ = \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{\partial_t(\mathcal{A}^1 - \mathcal{A}^2)} v^2) + \tilde{H}^1 & \text{in } \Omega, \\ \operatorname{div}_{\mathcal{A}^1} \partial_t v = \tilde{H}^2 & \text{in } \Omega, \\ \partial_t(\partial_t \Theta) - \Delta_{\mathcal{A}^1} \partial_t \Theta = \operatorname{div}_{\mathcal{A}^1} (\nabla_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} \Theta^2) + \tilde{H}^3 & \text{in } \Omega, \\ S_{\mathcal{A}^1}(\partial_t q, \partial_t v) \mathcal{N}^1 = \mathbb{D}_{(\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2)} v^2 \mathcal{N}^1 + \tilde{H}^4 & \text{on } \Sigma, \\ \nabla_{\mathcal{A}^1} \partial_t \Theta \cdot \mathcal{N}^1 + \partial_t \Theta |\mathcal{N}^1| = -\nabla_{\partial_t(\mathcal{A}^1 - \mathcal{A}^2)} \Theta^2 \cdot \mathcal{N}^1 + \tilde{H}^5 & \text{on } \Sigma, \\ \partial_t v = 0, \quad \partial_t \Theta = 0 & \text{on } \Sigma_b, \\ \partial_t v(t=0) = 0, \quad \partial_t \Theta(t=0) = 0, & \end{array} \right. \quad (6.23)$$

where H^2 , H^4 , \tilde{H}^2 and \tilde{H}^4 have been given by Y. Guo and I. Tice in [6],

$$\begin{aligned} H^1 &= \Theta^2 \nabla_{\mathcal{A}^1 - \mathcal{A}^2} y_3^1 + \Theta^2 \nabla_{\mathcal{A}^2} (y_3^1 - y_3^2) + \operatorname{div}_{\mathcal{A}^1 - \mathcal{A}^2} (\mathbb{D}_{\mathcal{A}^2} v^2) - \nabla_{\mathcal{A}^1 - \mathcal{A}^2} q^2 \\ &\quad + \partial_t \bar{\zeta}^1 (1 + x_3) K^1 (\partial_3 w^1 - \partial_3 w^2) + (\partial_t \bar{\zeta}^1 - \partial_t \bar{\zeta}^2) (1 + x_3) K^1 \partial_3 w^2 \\ &\quad + \partial_t \bar{\zeta}^1 (1 + x_3) (K^1 - K^2) \partial_3 w^2 - (w^1 - w^2) \cdot \nabla_{\mathcal{A}^1} w^1 - w^2 \cdot \nabla_{\mathcal{A}^1} (w^1 - w^2) \\ &\quad - w^2 \cdot \nabla_{\mathcal{A}^1 - \mathcal{A}^2} w^2, \\ H^3 &= \operatorname{div}_{\mathcal{A}^1 - \mathcal{A}^2} (\nabla_{\mathcal{A}^2} \Theta^2) + \partial_t \bar{\zeta}^1 (1 + x_3) K^1 (\partial_3 \vartheta^1 - \partial_3 \vartheta^2) \\ &\quad + (\partial_t \bar{\zeta}^1 - \partial_t \bar{\zeta}^2) (1 + x_3) K^1 \partial_3 \vartheta^2 + \partial_t \bar{\zeta}^1 (K^1 - K^2) \partial_3 w^2 - (w^1 - w^2) \cdot \nabla_{\mathcal{A}^1} \vartheta^1 \\ &\quad - w^2 \cdot \nabla_{\mathcal{A}^1} (\vartheta^1 - \vartheta^2) - w^2 \cdot \nabla_{\mathcal{A}^1 - \mathcal{A}^2} \vartheta^2, \\ H^5 &= -\nabla_{\mathcal{A}^2} \Theta^2 \cdot (\mathcal{N}^1 - \mathcal{N}^2) - \Theta^2 (|\mathcal{N}^1| - |\mathcal{N}^2|), \\ \tilde{H}^1 &= \partial_t H^1 + \operatorname{div}_{\partial_t \mathcal{A}^1} (\mathbb{D}_{\mathcal{A}^1 - \mathcal{A}^2} v^2) + \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{\mathcal{A}^1 - \mathcal{A}^2} \partial_t v^2) + \operatorname{div}_{\partial_t \mathcal{A}^1} (\mathbb{D}_{\mathcal{A}^1} v) \\ &\quad + \operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{\partial_t \mathcal{A}^1} v) - \nabla_{\partial_t \mathcal{A}^1} q, \\ \tilde{H}^3 &= \partial_t H^3 + \operatorname{div}_{\partial_t \mathcal{A}^1} (\nabla_{(\mathcal{A}^1 - \mathcal{A}^2)} \Theta^2) + \operatorname{div}_{\mathcal{A}^1} (\nabla_{(\mathcal{A}^1 - \mathcal{A}^2)} \partial_t \Theta^2) + \operatorname{div}_{\partial_t \mathcal{A}^1} \nabla_{\mathcal{A}^1} \Theta \\ &\quad + \operatorname{div}_{\mathcal{A}^1} \nabla_{\partial_t \mathcal{A}^1} \Theta, \\ \tilde{H}^5 &= \partial_t H^5 - \nabla_{(\mathcal{A}^1 - \mathcal{A}^2)} \partial_t \Theta^2 \cdot \mathcal{N}^1 - \nabla_{(\mathcal{A}^1 - \mathcal{A}^2)} \Theta^2 \cdot \partial_t \mathcal{N}^1 - \nabla_{\mathcal{A}^1} \Theta \cdot \partial_t \mathcal{N}^1 \\ &\quad - \nabla_{\partial_t \mathcal{A}^1} \Theta \cdot \mathcal{N}^1 - \Theta \partial_t |\mathcal{N}^1|. \end{aligned}$$

Then we can deduce the equations

$$\frac{1}{2} \int_{\Omega} |\partial_t v|^2 J^1(t) + \frac{1}{2} \int_0^t \int_{\Omega} |\mathbb{D}_{\mathcal{A}^1} \partial_t v|^2 J^1$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^t \int_{\Omega} |\partial_t v|^2 (\partial_t J^1 K^1) J^1 + \int_0^t \int_{\Omega} \partial_t (\Theta \nabla_{\mathcal{A}^1} y_3^1) \cdot \partial_t v J^1 + \int_0^t \int_{\Omega} J^1 (\tilde{H}^1 \cdot \partial_t v + \tilde{H}^2 \partial_t q) \\
 &\quad - \frac{1}{2} \int_0^t \int_{\Omega} J^1 \mathbb{D}_{\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2} v^2 : \mathbb{D}_{\mathcal{A}^1} \partial_t v - \int_0^t \int_{\Sigma} \tilde{H}^3 \cdot \partial_t v, \\
 &\quad \frac{1}{2} \int_{\Omega} |\partial_t \Theta|^2 J^1(t) + \int_0^t \int_{\Omega} |\nabla_{\mathcal{A}^1} \partial_t \Theta|^2 J^1 + \int_0^t \int_{\Sigma} |\partial_t \Theta|^2 |\mathcal{N}^1| \\
 &= \frac{1}{2} \int_0^t \int_{\Omega} |\partial_t \Theta|^2 (\partial_t J^1 K^1) J^1 + \int_0^t \int_{\Omega} J^1 \tilde{H}^3 \cdot \partial_t \Theta \\
 &\quad - \int_0^t \int_{\Omega} J^1 \nabla_{\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2} \Theta^2 \cdot \nabla_{\mathcal{A}^1} \partial_t \Theta + \int_0^t \int_{\Sigma} \tilde{H}^5 \cdot \partial_t \Theta. \tag{6.24}
 \end{aligned}$$

Step 2. Estimates for the forcing terms. Now we need to estimate the forcing terms that appear on the right-hand sides of Equation (6.24). Throughout this section, $P(\cdot)$ is written as a polynomial such that $P(0) = 0$, which is allowed to change from line to line. The estimates for $\|\tilde{H}^1\|_0$, $\|\tilde{H}^2\|_0$, $\|\partial_t \tilde{H}^2\|_0$, $\|\tilde{H}^4\|_{-1/2}$, $\|H^1\|_r$, $\|H^2\|_{r+1}$, $\|H^4\|_{r+1/2}$, $\|\operatorname{div}_{\mathcal{A}^1} (\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2)\|_r$ and $\|\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2 \mathcal{N}^1\|_{r+1/2}$ have been done by Guo and Tice in [6]. So we can directly use them only after replacing ε by \mathcal{Z} . By the same method, we can also deduce that

$$\begin{aligned}
 \|\tilde{H}^3\|_0 &\lesssim P(\sqrt{\mathcal{Z}}) (\|\Theta\|_2 + \|\zeta^1 - \zeta^2\|_{3/2} + \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{1/2} + \|\partial_t^2 \zeta^1 - \partial_t^2 \zeta^2\|_{1/2} \\
 &\quad + \|w^1 - w^2\|_0 + \|\partial_t w^1 - \partial_t w^2\|_0 + \|\vartheta^1 - \vartheta^2\|_1 + \|\partial_t \vartheta^1 - \partial_t \vartheta^2\|_1), \tag{6.25}
 \end{aligned}$$

$$\|\tilde{H}^5\|_{-1/2} \lesssim P(\sqrt{\mathcal{Z}}) (\|\zeta^1 - \zeta^2\|_{1/2} + \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{1/2} + \|\Theta\|_2), \tag{6.26}$$

and for $r = 0, 1$,

$$\|H^3\|_r \lesssim P(\sqrt{\mathcal{Z}}) (\|\zeta^1 - \zeta^2\|_{r+1/2} + \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{r-1/2} + \|w^1 - w^2\|_r + \|\vartheta^1 - \vartheta^2\|_{r+1}), \tag{6.27}$$

$$\|H^5\|_{r+1/2} \lesssim P(\sqrt{\mathcal{Z}}) \|\zeta^1 - \zeta^2\|_{r+3/2}, \tag{6.28}$$

$$\|\operatorname{div}_{\mathcal{A}^1} (\nabla_{\mathcal{A}^1 - \mathcal{A}^2} \Theta^2)\|_r \lesssim P(\sqrt{\mathcal{Z}}) \|\zeta^1 - \zeta^2\|_{r+3/2}, \tag{6.29}$$

$$\|\nabla_{\mathcal{A}^1 - \mathcal{A}^2} \Theta^2 \cdot \mathcal{N}^1\|_{r+1/2} \lesssim P(\sqrt{\mathcal{Z}}) \|\zeta^1 - \zeta^2\|_{r+3/2}. \tag{6.30}$$

Step 3. Energy estimates of $\partial_t v$ and $\partial_t \Theta$. First, owing to the assumption and Sobolev embeddings, we obtain that

$$\|J^1\|_{L^\infty} + \|K^1\|_{L^\infty} \lesssim 1 + P(\sqrt{\mathcal{Z}}) \quad \text{and} \quad \|\partial_t J^1\|_{L^\infty} \lesssim P(\sqrt{\mathcal{Z}}). \tag{6.31}$$

The bounds (6.31) reveal that

$$\frac{1}{2} \int_0^t \int_{\Omega} |\partial_t \Theta|^2 (\partial_t J^1 K^1) J^1 \lesssim P(\sqrt{\mathcal{Z}}) \frac{1}{2} \int_0^t \int_{\Omega} |\partial_t \Theta|^2 J^1. \tag{6.32}$$

In addition, estimates (6.25), (6.26) together with trace theory and the Poincaré inequality reveals that

$$\int_0^t \int_{\Omega} J^1 \tilde{H}^3 \cdot \partial_t \Theta - \int_0^t \int_{\Omega} J^1 \nabla_{\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2} \Theta^2 \cdot \nabla_{\mathcal{A}^1} \partial_t \Theta - \int_0^t \int_{\Sigma} \tilde{H}^5 \cdot \partial_t \Theta$$

$$\begin{aligned}
 &\leq \int_0^t \int_{\Omega} \|J^1\|_{L^\infty} \left(\|J^1\|_{L^\infty} \|\tilde{H}^3\|_0 \|\partial_t \Theta\|_0 + \|\nabla_{\partial_t \mathcal{A}^1 - \partial_t \mathcal{A}^2} \Theta^2\|_0 \|\nabla_{\mathcal{A}^1} \partial_t \Theta\|_0 \right) \\
 &\quad + \int_0^t \|\tilde{H}^5\|_{-1/2} \|\partial_t \Theta\|_{1/2} \\
 &\lesssim \int_0^t P(\sqrt{\mathcal{F}}) \sqrt{\mathcal{Z}},
 \end{aligned} \tag{6.33}$$

where we have written

$$\begin{aligned}
 \mathcal{Z} := &\|\zeta^1 - \zeta^2\|_{3/2}^2 + \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{1/2}^2 + \|\partial_t^2 \zeta^1 - \partial_t^2 \zeta^2\|_{1/2}^2 \\
 &+ \|w^1 - w^2\|_1^2 + \|\partial_t w^1 - \partial_t w^2\|_1^2 + \|\vartheta^1 - \vartheta^2\|_1^2 + \|\partial_t \vartheta^1 - \partial_t \vartheta^2\|_1^2 \\
 &+ \|v\|_2^2 + \|q\|_1^2 + \|\Theta\|_2^2.
 \end{aligned} \tag{6.34}$$

Combining the inequalities (6.32), (6.33), Equation (6.24), the Poincaré inequality of Lemma A.14 in [6] and Lemma 2.9 in [13], and utilizing Cauchy’s inequality to absorb $\|\partial_t \Theta\|_1$ into the left-hand side, yield that

$$\begin{aligned}
 &\frac{1}{2} \int_{\Omega} |\partial_t \Theta|^2 J^1(t) + \frac{1}{2} \int_0^t \|\partial_t \Theta\|_1^2 \\
 &\leq P(\sqrt{\mathcal{F}}) \frac{1}{2} \int_{\Omega} |\partial_t \Theta|^2 J^1(t) + \int_0^t P(\sqrt{\mathcal{F}}) \mathcal{Z}.
 \end{aligned} \tag{6.35}$$

Then Gronwall’s lemma and Lemma 2.9 in [13] imply that

$$\|\partial_t \Theta\|_{L^\infty H^0}^2 + \|\partial_t \Theta\|_{L^2 H^1}^2 \leq \exp\{P(\sqrt{\mathcal{F}})T\} \int_0^T P(\sqrt{\mathcal{F}}) \mathcal{Z}. \tag{6.36}$$

Then energy estimates for $\partial_t v$ are likely the same as what Guo and Tice did in [6], so we omit the details. The energy estimates for $\partial_t v$ and $\partial_t \Theta$ allow us to deduce that

$$\begin{aligned}
 &\|\partial_t v\|_{L^\infty H^0}^2 + \|\partial_t v\|_{L^2 H^1}^2 + \|\partial_t \Theta\|_{L^\infty H^0}^2 + \|\partial_t \Theta\|_{L^2 H^1}^2 \\
 &\leq \exp\{P(\sqrt{\mathcal{F}})T\} \left[P(\sqrt{\mathcal{F}}) \|q\|_{L^2 H^0}^2 + C \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{L^2 H^{-1/2}}^2 + \int_0^T P(\sqrt{\mathcal{F}}) \mathcal{Z} \right. \\
 &\quad + P(\sqrt{\mathcal{F}}) \|q\|_{L^\infty H^0} \left(\sum_{j=0}^1 \|\partial_t^j \zeta^1 - \partial_t^j \zeta^2\|_{L^\infty H^{1/2}} + \|v\|_{L^\infty H^1} \right) \\
 &\quad \left. + P(\sqrt{\mathcal{F}}) \|q\|_{L^2 H^0}^2 \left(\sum_{j=0}^2 \|\partial_t^j \zeta^1 - \partial_t^j \zeta^2\|_{L^2 H^{1/2}} + \|v\|_{L^2 H^1} \right) \right],
 \end{aligned} \tag{6.37}$$

where the temporal norm of L^∞ and L^2 are computed over $[0, T]$.

Step 4. Elliptic estimates for v , q and Θ . For $r=0,1$, we combine Proposition (3.5) with estimates (6.27)–(6.30) as well as the bounds of $\|H^1\|_r$, $\|H^2\|_{r+1}$, $\|H^4\|_{r+1/2}$ $\|\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2)\|_r$, $\|\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2 \mathcal{N}^1\|_{r+1/2}$ done in the proof of Theorem 6.2 in [6] to deduce that

$$\begin{aligned}
 &\|v\|_{r+2}^2 + \|q\|_{r+1}^2 + \|\Theta\|_{r+2}^2 \\
 &\lesssim C(\eta_0) \left(\|\partial_t v\|_r^2 + \|\operatorname{div}_{\mathcal{A}^1}(\mathbb{D}_{(\mathcal{A}^1 - \mathcal{A}^2)} v^2)\|_r^2 + \|H^1\|_r^2 + \|H^2\|_{r+1}^2 + \|\partial_t \Theta\|_r^2 \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \|H^3\|_r^2 + \|\operatorname{div}_{\mathcal{A}^1}(\nabla_{\mathcal{A}^1-\mathcal{A}^2}\Theta^2)\|_r^2 + \|\mathbb{D}_{(\mathcal{A}^1-\mathcal{A}^2)}v^2\mathcal{N}^1\|_{r+1/2}^2 + \|H^4\|_{r+1/2}^2 \\
 & + \|\nabla_{\mathcal{A}^1-\mathcal{A}^2}\Theta^2 \cdot \mathcal{N}^1\|_{r+1/2}^2 + \|H^5\|_{r+1/2}^2 \Big) \\
 \lesssim & C(\eta_0) \left(\|\partial_t v\|_r^2 + \|\partial_t \Theta\|_r^2 + \|\zeta^1 - \zeta^2\|_{r+1/2}^2 \right. \\
 & + P(\sqrt{\mathcal{E}})(\|\zeta^1 - \zeta^2\|_{r+3/2}^2 + \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{r-1/2}^2 \\
 & \left. + \|w^1 - w^2\|_{r+1}^2 + \|\vartheta^1 - \vartheta^2\|_{r+1}^2 \right). \tag{6.38}
 \end{aligned}$$

Then we take the supremum in time over $[0, T]$, when $r = 0$, to deduce

$$\begin{aligned}
 & \|v\|_{L^\infty H^2}^2 + \|q\|_{L^\infty H^1}^2 + \|\Theta\|_{L^\infty H^2}^2 \\
 \lesssim & C(\eta_0) \left(\|\partial_t v\|_{L^\infty H^0}^2 + \|\partial_t \Theta\|_{L^\infty H^0}^2 + \|\zeta^1 - \zeta^2\|_{L^\infty H^{1/2}}^2 \right. \\
 & + P(\sqrt{\mathcal{E}})(\|\zeta^1 - \zeta^2\|_{L^\infty H^{3/2}}^2 + \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{L^\infty H^{-1/2}}^2 \\
 & \left. + \|w^1 - w^2\|_{L^\infty H^1}^2 + \|\vartheta^1 - \vartheta^2\|_{L^\infty H^1}^2 \right). \tag{6.39}
 \end{aligned}$$

Then we integrate over $[0, T]$ when $r = 1$ to find

$$\begin{aligned}
 & \|v\|_{L^2 H^3}^2 + \|q\|_{L^2 H^2}^2 + \|\Theta\|_{L^2 H^3}^2 \\
 \lesssim & C(\eta_0) \left(\|\partial_t v\|_{L^2 H^1}^2 + \|\partial_t \Theta\|_{L^2 H^1}^2 + \|\zeta^1 - \zeta^2\|_{L^2 H^{3/2}}^2 \right. \\
 & + P(\sqrt{\mathcal{E}})(\|\zeta^1 - \zeta^2\|_{L^2 H^{5/2}}^2 + \|\partial_t \zeta^1 - \partial_t \zeta^2\|_{L^2 H^{1/2}}^2 \\
 & \left. + \|w^1 - w^2\|_{L^2 H^2}^2 + \|\vartheta^1 - \vartheta^2\|_{L^2 H^2}^2 \right). \tag{6.40}
 \end{aligned}$$

Step 5. Estimates of $\zeta^1 - \zeta^2$ and contraction. After making preparations in the above steps, we can derive the contraction results. Since this step follows in exactly the same manner as in the proof of Theorem 6.2 in [6], we omit the details here. Hence, we get the estimates (6.20) and (6.21). \square

6.3. Proof of Theorem 1.1. Now we can combine Theorem 6.2 and Theorem 6.3 to produce a unique strong solution to Equation (1.4). It is notable that Theorem 1.1 can be directly derived from the following theorem, which will be proved in the same manner as the proof of Theorem 6.3 in [6].

THEOREM 6.4. *Assume that u_0, θ_0, η_0 satisfy $\mathcal{E}_0 < \infty$ and that the initial data $\partial_t^j u(0)$, etc. are constructed in Section 5.1 and satisfy the N -th compatibility conditions (5.28). Then there exists $0 < T_0 < 1$ such that if $0 < T \leq T_0$, then there exists a solution (u, p, θ, η) to the problem (1.4) on the time interval $[0, T]$ that achieves the initial data and satisfies*

$$\mathfrak{K}(u, p, \theta) + \mathfrak{K}(\eta) \leq CP(\mathcal{E}_0), \tag{6.41}$$

for a universal constant $C > 0$. The solution is unique through functions that achieve the initial data. Moreover, η is such that the mapping $\Phi(\cdot, t)$, defined by the expression (1.2), is a C^{2N-1} diffeomorphism for each $t \in [0, T]$.

Proof.

Step 1. The sequences of approximate solutions. From the assumptions, we know that the hypothesis of Theorems 6.1 and 6.2 is satisfied. These two theorems allow us to produce a sequence $\{(u^m, p^m, \theta^m, \eta^m)\}_{m=1}^\infty$, which achieves the initial data, satisfies the systems (6.2), and obeys the uniform bound

$$\sup_{m \geq 1} (\mathfrak{K}(u^m, p^m, \theta^m) + \mathfrak{K}(\eta^m)) \leq CP(\mathcal{E}_0). \tag{6.42}$$

The uniform bound allows us to take weak and weak-* limits, up to the extraction of a subsequence:

$$\begin{aligned} \partial_t^j u^m &\rightharpoonup \partial_t^j u && \text{weakly in } L^2([0, T]; H^{2N-2j+1}(\Omega)) \text{ for } j=0, \dots, N, \\ \partial_t^{N+1} u^m &\rightharpoonup \partial_t^{N+1} u && \text{weakly in } (\mathcal{X}_T)^*, \\ \partial_t^j u^m &\overset{*}{\rightharpoonup} \partial_t^j u && \text{weakly } * \text{ in } L^\infty([0, T]; H^{2N-2j}(\Omega)) \text{ for } j=0, \dots, N, \\ \partial_t^j p^m &\rightharpoonup \partial_t^j p && \text{weakly in } L^2([0, T]; H^{2N-2j}(\Omega)) \text{ for } j=0, \dots, N, \\ \partial_t^j p^m &\overset{*}{\rightharpoonup} \partial_t^j p && \text{weakly } * \text{ in } L^\infty([0, T]; H^{2N-2j-1}(\Omega)) \text{ for } j=0, \dots, N, \\ \partial_t^j \theta^m &\rightharpoonup \partial_t^j \theta && \text{weakly in } L^2([0, T]; H^{2N-2j+1}(\Omega)) \text{ for } j=0, \dots, N, \\ \partial_t^{N+1} \theta^m &\rightharpoonup \partial_t^{N+1} \theta && \text{weakly in } (\mathcal{H}_T^1)^*, \\ \partial_t^j \theta^m &\overset{*}{\rightharpoonup} \partial_t^j \theta && \text{weakly } * \text{ in } L^\infty([0, T]; H^{2N-2j}(\Omega)) \text{ for } j=0, \dots, N, \end{aligned}$$

and

$$\begin{aligned} \partial_t^j \eta^m &\rightharpoonup \partial_t^j \eta && \text{weakly in } L^2([0, T]; H^{2N-2j+5/2}(\Sigma)) \text{ for } j=2, \dots, N+1, \\ \eta^m &\overset{*}{\rightharpoonup} \eta && \text{weakly } * \text{ in } L^\infty([0, T]; H^{2N+1/2}(\Sigma)), \\ \partial_t^j \eta^m &\overset{*}{\rightharpoonup} \partial_t^j \eta && \text{weakly } * \text{ in } L^\infty([0, T]; H^{2N-2j+3/2}(\Sigma)) \text{ for } j=1, \dots, N. \end{aligned}$$

The collection (v, q, Θ, ζ) achieving the initial data, that is, $\partial_t^j v(0) = \partial_t^j u(0)$, $\partial_t^j \Theta(0) = \partial_t^j \theta(0)$, $\partial_t^j \zeta(0) = \partial_t^j \eta(0)$ for $j=0, \dots, N$ and $\partial_t^j q(0) = \partial_t^j p(0)$ for $j=0, \dots, N-1$, is closed in the above weak topology by Lemma A.4 in [6]. Hence the limit (u, p, θ, η) achieves the initial data, since each $(u^m, p^m, \theta^m, \eta^m)$ is in the above collection.

Step 2. Contraction. For $m \geq 1$, we set $v^1 = u^{m+2}$, $v^2 = u^{m+1}$, $w^1 = u^{m+1}$, $w^2 = u^m$, $q^1 = p^{m+2}$, $q^2 = p^{m+1}$, $\Theta^1 = \theta^{m+2}$, $\Theta^2 = \theta^{m+1}$, $\vartheta^1 = \theta^{m+1}$, $\vartheta^2 = \theta^m$, $\zeta^1 = \eta^{m+1}$, $\zeta^2 = \eta^m$. Then from the construction of initial data, the initial data of v^j , w^j , q^j , Θ^j , ϑ^j , ζ^j match the hypothesis of Theorem 6.3. Because of Equation (6.2), Equation (6.19) holds. In addition, the bound (6.42) holds. Thus, all hypotheses of Theorem 6.3 are satisfied. Then

$$\begin{aligned} &\mathfrak{N}(u^{m+2} - u^{m+1}, p^{m+2} - p^{m+1}, \theta^{m+2} - \theta^{m+1}; T) \\ &\leq \frac{1}{2} \mathfrak{N}(u^{m+1} - u^m, p^{m+1} - p^m, \theta^{m+1} - \theta^m; T), \end{aligned} \tag{6.43}$$

$$\mathfrak{M}(\eta^{m+1} - \eta^m; T) \lesssim \mathfrak{N}(u^{m+1} - u^m, p^{m+1} - p^m, \theta^{m+1} - \theta^m; T). \tag{6.44}$$

The bound (6.43) implies that the sequence $\{(u^m, p^m, \theta^m)\}_{m=0}^\infty$ is Cauchy in the norm $\sqrt{\mathfrak{M}(\cdot, \cdot; T)}$. Thus

$$\left\{ \begin{array}{ll} u^m \rightarrow u & \text{in } L^\infty([0, T]; H^2(\Omega)) \cap L^2([0, T]; H^3(\Omega)), \\ \partial_t u^m \rightarrow \partial_t u & \text{in } L^\infty([0, T]; H^0(\Omega)) \cap L^2([0, T]; H^1(\Omega)), \\ p^m \rightarrow p & \text{in } L^\infty([0, T]; H^1(\Omega)) \cap L^2([0, T]; H^2(\Omega)), \\ \theta^m \rightarrow \theta & \text{in } L^\infty([0, T]; H^2(\Omega)) \cap L^2([0, T]; H^3(\Omega)), \\ \partial_t \theta^m \rightarrow \partial_t \theta & \text{in } L^\infty([0, T]; H^0(\Omega)) \cap L^2([0, T]; H^1(\Omega)), \end{array} \right. \quad (6.45)$$

as $m \rightarrow \infty$. Because of (6.44), we deduce that the sequence $\{\eta^m\}_{m=1}^\infty$ is Cauchy in the norm $\sqrt{\mathfrak{M}(\cdot; T)}$. Thus,

$$\left\{ \begin{array}{ll} \eta^m \rightarrow \eta & \text{in } L^\infty([0, T]; H^{5/2}(\Sigma)), \\ \partial_t \eta^m \rightarrow \partial_t \eta & \text{in } L^\infty([0, T]; H^{3/2}(\Sigma)), \\ \partial_t^2 \eta^m \rightarrow \partial_t^2 \eta & \text{in } L^2([0, T]; H^{1/2}(\Sigma)), \end{array} \right. \quad (6.46)$$

as $m \rightarrow \infty$.

Step 3. Interpolation and passing to the limit. This section is exactly the same as the proof of Theorem 6.3 in [6], which gives the existence of solutions and the estimate (6.41).

Step 4. Uniqueness and diffeomorphism. This section is similar to the proof of Theorem 6.3 in [6]. □

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