COMMUN. MATH. SCI.
Vol. 15, No. 3, pp. 863–880

ON THE LINEARIZED LOG-KDV EQUATION*

DMITRY E. PELINOVSKY†

Abstract. The logarithmic KdV (log-KdV) equation admits global solutions in an energy space and exhibits Gaussian solitary waves. Orbital stability of Gaussian solitary waves is known to be an open problem. We address properties of solutions to the linearized log-KdV equation at the Gaussian solitary waves. By using the decomposition of solutions in the energy space in terms of Hermite functions, we show that the time evolution of the linearized log-KdV equation is related to a Jacobi difference operator with a limit circle at infinity. This exact reduction allows us to characterize both spectral and linear orbital stability of Gaussian solitary waves. We also introduce a convolution representation of solutions to the linearized log-KdV equation with the Gaussian weight and show that the time evolution in such a weighted space is dissipative with the exponential rate of decay.

Keywords. logarithmic KdV equation, Gaussian solitary waves; Hermite functions; Jacobi difference equation; semi-groups; linear orbital stability; spectral stability

AMS subject classifications. 35Q53; 35P10; 37K45; 39A70.

1. Introduction

We address the logarithmic Korteweg–de Vries (log-KdV) equation derived in the context of solitary waves in granular chains with Hertzian interaction forces [6–8]:

$$v_t + v_{xxx} + (v \log|v|)_x = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}. \tag{1.1}$$

The log-KdV equation (1.1) has a two-parameter family of Gaussian solitary waves

$$v(t,x) = e^c V(x - ct - a), \quad a, c \in \mathbb{R}, \tag{1.2}$$

where V is a symmetric standing wave given by

$$V(x) := e^{\frac{1}{2} - \frac{x^2}{4}}, \quad x \in \mathbb{R}. \tag{1.3}$$

Global solutions to the log-KdV equation (1.1) were constructed in [2] in the energy space

$$X := \left\{ v \in H^1(\mathbb{R}) : \quad v^2 \log|v| \in L^1(\mathbb{R}) \right\}, \tag{1.4}$$

by a modification of analytic methods available for the log-NLS equation [5] (also reviewed in Section 9.3 in [4]). In the energy space X, the following quantities for the momentum and energy,

$$Q(v) = \int_{\mathbb{R}} v^2 dx \tag{1.5}$$

and

$$E(v) = \int_{\mathbb{R}} \left[(\partial_x v)^2 - v^2 \log|v| + \frac{1}{2} v^2 \right] dx \tag{1.6}$$

are non-increasing functions of time t. Uniqueness, continuous dependence, and energy conservation are established in [2] under the additional condition $\partial_x \log|v| \in L^\infty(\mathbb{R} \times \mathbb{R})$,

*Received: July 8, 2016; accepted (in revised form): November 28, 2016. Communicated by Alexander Kiselev.

†Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, L8S 4K1, Canada, and Department of Applied Mathematics, Nizhny Novgorod State Technical University, Nizhny Novgorod 603950, Russia (dmpeli@math.mcmaster.ca).

which is not satisfied in the neighborhood of the family of Gaussian solitary waves (1.2) and (1.3). As a result, orbital stability of the Gaussian solitary waves was not established for the log-KdV equation (1.1), in a sharp contrast with that in the log-NLS equation, where orbital stability of the Gaussian solitary waves was proved in [3].

A possible path towards analysis of orbital stability of Gaussian solitary waves is to study their linear and spectral stability by using the linearized log-KdV equation

$$u_t = \partial_x L u, \tag{1.7}$$

where $L : H^2(\mathbb{R}) \cap L^2_2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Schrödinger operator with a harmonic potential given by the differential expression

$$L = -\partial_x^2 + \frac{1}{4}(x^2 - 6). \tag{1.8}$$

Here and in what follows, we denote with $H^s(\mathbb{R})$ the Sobolev space of s-times weakly differentiable functions on the real line whose derivatives up to order s are in $L^2(\mathbb{R})$. The norm $\|u\|_{H^s}$ for u in the Sobolev space $H^s(\mathbb{R})$ is equivalent to the norm $\|(I-\partial_x^2)^{s/2} u\|_{L^2}$ in the Lebesgue space $L^2(\mathbb{R})$. We denote with $L^2_s(\mathbb{R})$ the space of square integrable functions with the weight $<x>^s = (1+x^2)^{s/2}$.

The linearized log-KdV equation (1.7) arises at the formal linearization of the log-KdV equation (1.1) at the perturbation $u := v - V$. The Schrödinger operator L is the Hessian operator of the second variation of $E(v)$ at $v = V$. Although $E(v)$ is not a C^2 functional at $v = 0$, the second variation of $E(v)$ is well defined at $v = V$ by

$$E_c(u) = \langle Lu, u \rangle_{L^2}, \tag{1.9}$$

which is formally conserved in the time evolution of the linearized log-KdV equation (1.7).

The goal of this work is to obtain new estimates for the linearized log-KdV equation (1.7), which may be useful to control perturbations to the Gaussian solitary waves in the log-KdV equation (1.1). Indeed, if we substitute $v(t,x) = V(x) + w(t,x)$ to the log-KdV equation (1.1), we obtain an equivalent evolution equation

$$w_t = \partial_x L w - \partial_x N(w), \tag{1.10}$$

where the linearized part coincides with the linearized log-KdV equation (1.7) and the nonlinear term $N(w)$ is given by

$$N(w) = w \log\left(1 + \frac{w}{V}\right) + V\left[\log\left(1 + \frac{w}{V}\right) - \frac{w}{V}\right].$$

It is clear that the nonlinear term $N(w)$ does not behave uniformly in x unless w decays at least as fast as V in Equation (1.3). On the other hand, if $w(t,x) = V(x) h(t,x)$, where h is a bounded function in its variables, then $N(w) = V n(h)$, where $n(h) = h\log(1+h) + \log(1+h) - h$ is analytic in h for any $h \in (-1,1)$. Therefore, obtaining new estimates for the linearized log-KdV equation (1.7) in a function space with Gaussian weights may be useful in the nonlinear analysis of the log-KdV equation (1.10).

The spectrum of L in $L^2(\mathbb{R})$ consists of equally spaced simple eigenvalues

$$\sigma(L) = \{-1, 0, 1, 2, \dots\},$$

which include exactly one negative eigenvalue with the eigenvector V (defined without normalization). Therefore, $E(v)$ is not convex at V in X. Nevertheless, $E_c(u)$ is positive in the constrained space

$$X_c := \left\{ u \in H^1(\mathbb{R}) \cap L^2_1(\mathbb{R}) : \quad \langle V, u \rangle_{L^2} = 0 \right\}, \tag{1.11}$$

which corresponds to the fixed value $Q(v)=Q(V)$ in Equation (1.5) at the linearized approximation.

Several results were obtained for the linearized log-KdV equation (1.7). In [8], *linear orbital stability* of Gaussian solitary waves was obtained in the following sense: for every $u(0)\in X_c$, there exists a unique global solution $u(t)\in X_c$ of the linearized log-KdV equation (1.7) which satisfies the following bound

$$\|u(t)\|_{H^1\cap L^2_1}\leqslant C\|u(0)\|_{H^1\cap L^2_1},\quad t\in\mathbb{R}, \tag{1.12}$$

for some t-independent positive constant C. This result was obtained in [8] from the conservation of $E_c(u)$ in the time evolution of smooth solutions to the linearized log-KdV equation (1.7), the symplectic decomposition of the solution $u(t)\in X_c$, $t\in\mathbb{R}$ into the translational part and the residual part,

$$u(t)=b(t)\partial_x V+y(t),\quad \langle V,y(t)\rangle_{L^2}=\langle\partial_x^{-1}V,y(t)\rangle_{L^2}=0, \tag{1.13}$$

and the coercivity of $E_c(y)$ in the squared $H^1(\mathbb{R})\cap L^2_1(\mathbb{R})$ norm in the sense

$$\|y\|^2_{H^1\cap L^2_1}\leqslant CE_c(y),\quad \langle V,y\rangle_{L^2}=\langle\partial_x^{-1}V,y\rangle_{L^2}=0, \tag{1.14}$$

for some positive constant C. The first two facts are rather standard in energy methods for linear PDEs, whereas the last fact, that is, the inequality (1.14), should not be taken as granted. The coercivity of E_c in X_c was conjectured in the arguments of [8]. Here ∂_x^{-1} denotes a suitable anti-derivative, e.g.

$$(\partial_x^{-1}V)(x):=\int_{-\infty}^{x}V(x')dx'.$$

We note that the second constraint in (1.13) and (1.14) is well-defined if $y\in X_c$, since $L^2_1(\mathbb{R})$ is embedded into $L^1(\mathbb{R})$.

In [2], the nonzero spectrum of the linear operator

$$\partial_x L: H^3(\mathbb{R})\cap H^1_2(\mathbb{R})\cap\dot{H}^{-1}(\mathbb{R})\to L^2(\mathbb{R}) \tag{1.15}$$

was studied by using the Fourier transform that maps the third-order differential operator in physical space into a second-order differential operator in Fourier space. Indeed, the Fourier transform $\hat{u}(k):=\mathcal{F}(u)(k)=\int_{\mathbb{R}}u(x)e^{-ikx}dx$ applied to the linearized log-KdV equation (1.7) yields the time evolution in the form

$$i\hat{u}_t=k\hat{L}\hat{u}, \tag{1.16}$$

where $\hat{L}: H^2(\mathbb{R})\cap L^2_2(\mathbb{R})\to L^2(\mathbb{R})$ is the Fourier image of operator $L: H^2(\mathbb{R})\cap L^2_2(\mathbb{R})\to L^2(\mathbb{R})$ given by

$$\hat{L}=-\frac{1}{4}\partial_k^2+k^2-\frac{3}{2}. \tag{1.17}$$

By reducing the eigenvalue problem for $k\hat{L}$ to the symmetric Sturm–Liouville form, it was found in [2] that the spectrum of $\partial_x L$ in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ such that $0<\omega_1<\omega_2<\ldots$ with $\omega_n\to\infty$ as $n\to\infty$. We use the convention that the set $\mathbb{N}$ includes only positive integers, while the set $\mathbb{N}_0$ consists of all non-negative integers.

The double zero eigenvalue corresponds to the Jordan block

$$\partial_x L\partial_x V=0,\quad \partial_x LV=-\partial_x V, \tag{1.18}$$

whereas the purely imaginary eigenvalues $\lambda = \pm i\omega_n$ correspond to the eigenfunctions $u = u_{\pm n}(x)$, which are smooth in x but decay algebraically as $|x| \to \infty$. The Fourier transform of $u_{\pm n}$ is supported on the half-line $\mathbb{R}^{\pm}$ and decays like a Gaussian function at infinity. It follows from the spectrum of $\partial_x L$ in $L^2(\mathbb{R})$ that the Gaussian solitary waves are *spectrally stable*.

The eigenfunctions of $\partial_x L$ were also used in [2] for spectral decompositions in the constrained space X_c in order to provide an alternative proof of the *linear orbital stability* of the Gaussian solitary waves. This alternative technique still relies on the conjecture of the coercivity of $E_c(y)$ in the squared $H^1(\mathbb{R}) \cap L^2_1(\mathbb{R})$ norm, that is, on the inequality (1.14).

Because of the algebraic decay of the eigenfunctions of $\partial_x L$, it is not clear if a function of x that decays like the Gaussian function as $|x| \to \infty$ can be represented as series of eigenfunctions. Numerical simulations were undertaken in [2] to illustrate that solutions to the linearized log-KdV equation (1.7) with Gaussian initial data did not spread out as the time variable evolves. Nevertheless, the solutions exhibited visible radiation at the left side of the bell-shaped profile. Therefore, it remains unclear in [2] if the solution of the linearized log-KdV equation (1.7) can be written in the form $u(t,x) = V(x)h(t,x)$ with a bounded $h(t,\cdot)$ for all $t \in \mathbb{R}$.

In this paper we obtain new estimates for the linearized log-KdV equation (1.7). In the first part, we rely on the basis of Hermite functions in L^2-based Sobolev spaces and analyze the discrete operators that replace the differential operators. In the second part, we obtain dissipative estimates on the evolution of the linearized log-KdV equation (1.7) by representing solutions in terms of a convolution with the Gaussian solitary wave V.

The paper is structured as follows. Section 2 sets up the basic formalism of the Hermite functions and reports useful technical estimates. Section 3 is devoted to the proof of the coercivity bound (1.14). As explained above, this coercivity bound implies *linear orbital stability* of the Gaussian solitary wave in the constrained space X_c and it is assumed to be granted in [2, 8]. The proof of coercivity relies on the decomposition of y in terms of the Hermite functions.

Section 4 is devoted to the analysis of linear evolution of the linearized log-KdV equation (1.7) expressed in terms of the Hermite functions. It is shown that this evolution reduces to the self-adjoint Jacobi difference operator with the limit circle behavior at infinity. As a result, a boundary condition is needed at infinity in order to define the spectrum of the Jacobi operator and to obtain the norm-preserving property of the associated semi-group. Both *linear orbital stability* and *spectral stability* of Gaussian solitary waves (1.3) is equivalently proven by using the Jacobi difference operator.

In Section 5, we give numerical approximations of eigenvalues and eigenvectors of the Jacobi difference equation. We show numerically that there exist subtle differences between the representation of eigenvectors of $\partial_x L$ in the physical space and the representation of these eigenvectors by using decomposition in terms of the Hermite functions.

Section 6 reports weighted estimates for solutions to the linearized log-KdV equation (1.7) by using a convolution representation with the Gaussian weight. We show that the convolution representation is invariant under the time evolution of the linearized log-KdV equation (1.7), which is expressed by a dissipative operator on a half-line. The semi-group of the fundamental solution in the $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ norm decays to zero exponentially fast as time goes to infinity.

Section 7 concludes the paper with discussions of further prospects.

Acknowledgements. The author thanks Gerald Teschl and Thierry Gallay for help on obtaining results reported in Sections 4 and 6, respectively. The research of the author is supported by the Ministry of Education and Science of Russian Federation (the base part of the State Task No. 2014/133, Project No. 2839).

2. Preliminaries

We recall definitions of the Hermite functions [1, Chapter 22]:

$$\varphi_n(z) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(z) e^{-\frac{z^2}{2}}, \qquad n \in \mathbb{N}_0, \tag{2.1}$$

where $\{H_n\}_{n\in\mathbb{N}_0}$ denote the set of Hermite polynomials, e.g.,

$$\begin{aligned} H_0 &= 1, \\ H_1 &= 2z, \\ H_2 &= 4z^2 - 2, \\ H_3 &= 8z^3 - 12z. \end{aligned}$$

Hermite functions satisfy the Schrödinger equation for a quantum harmonic oscillator:

$$-\varphi_n''(z) + z^2 \varphi_n(z) = (1+2n)\varphi_n(x), \quad n \in \mathbb{N}_0, \tag{2.2}$$

at equally spaced energy levels. By the Sturm–Liouville theory [12], the set of Hermite functions $\{\varphi_n\}_{n\in\mathbb{N}_0}$ forms an orthogonal and normalized basis in $L^2(\mathbb{R})$.

In connection to the self-adjoint operator $L : H^2(\mathbb{R}) \cap L^2_2(\mathbb{R}) \to L^2(\mathbb{R})$ given by the differential expression (1.8), we obtain the eigenfunctions of $Lu_n = (n-1)u_n$, $n \in \mathbb{N}_0$ from the correspondence $x = \sqrt{2}z$. With proper normalization, we define

$$u_n(x) = \frac{1}{\sqrt{2^n n! \sqrt{2\pi}}} H_n\left(\frac{x}{\sqrt{2}}\right) e^{-\frac{x^2}{4}}, \qquad n \in \mathbb{N}_0. \tag{2.3}$$

It follows from the well-known relations for Hermite polynomials

$$H_n'(z) = 2nH_{n-1}(z), \quad 2zH_n(z) = H_{n+1}(z) + 2nH_{n-1}(z), \quad n \in \mathbb{N}_0,$$

that functions in the sequence of eigenfunctions $\{u_n\}_{n\in\mathbb{N}_0}$ satisfy the differential relations

$$2u_n'(x) = -\sqrt{n+1}\,u_{n+1}(x) + \sqrt{n}\,u_{n-1}(x), \quad n \in \mathbb{N}_0. \tag{2.4}$$

The following elementary result is needed in further estimates.

Lemma 2.1. *Fix $a \in [0,1]$, $b \geqslant 0$, and let $\{f_m\}_{m\in\mathbb{N}_0}$ be given by*

$$f_m = \prod_{k=1}^{m} \frac{\sqrt{2k-a}}{\sqrt{2k+b}}.$$

There is a positive constant C that depends on a and b such that

$$f_m \leqslant C m^{-(a+b)/4} \quad m \in \mathbb{N}. \tag{2.5}$$

Proof. We write

$$f_m = \exp\left[\frac{1}{2}\sum_{k=1}^{m} \log\left(1 - \frac{a}{2k}\right) - \frac{1}{2}\sum_{k=1}^{m} \log\left(1 + \frac{b}{2k}\right)\right]. \tag{2.6}$$

By Taylor series, for every $a \in [0,1]$, there is $C > 0$ that depends on a such that

$$\left|\log\left(1 - \frac{a}{2k}\right) + \frac{a}{2k}\right| \leqslant \frac{C}{k^2}, \quad k \in \mathbb{N}. \tag{2.7}$$

Furthermore, we recall Euler's constant $\gamma \approx 0.577215$ given by the limit

$$\gamma := \lim_{m\to\infty} \left| \sum_{k=1}^{m-1} \frac{1}{k} - \log(m) \right|. \tag{2.8}$$

Since $\sum_{k=1}^{m} k^{-2}$ is bounded as $m\to\infty$, the estimate (2.7) and the limit (2.8) yield the bound

$$\left| \frac{1}{2}\sum_{k=1}^{m} \log\left(1-\frac{a}{2k}\right) + \frac{a}{4}\log(m) \right| \leqslant C, \quad \forall m \in \mathbb{N}, \tag{2.9}$$

for some positive constant C. Substituting inequality (2.9) into Equation (2.6) proves the desired bound (2.5). □

The following technical result is needed for the proof of coercivity of the energy function.

LEMMA 2.2. *Let $\{f_n\}_{n\in\mathbb{N}_0}$ by defined by $f_n := \langle \partial_x^{-1} u_0, u_n\rangle_{L^2}$. Then, there is a positive constant C such that*

$$0 < f_n \leqslant C(1+n)^{-1/4}, \quad n \in \mathbb{N}_0. \tag{2.10}$$

Proof. Multiplying the differential relation (2.4) by $\partial_x^{-1}u_0$ and integrating by parts, we obtain

$$\begin{aligned}\sqrt{n}\langle \partial_x^{-1}u_0, u_{n-1}\rangle_{L^2} - \sqrt{n+1}\langle \partial_x^{-1}u_0, u_{n+1}\rangle_{L^2} &= 2\langle \partial_x^{-1}u_0, u_n'\rangle_{L^2} \\ &= -2\langle u_0, u_n\rangle_{L^2}, \quad n\in\mathbb{N}_0.\end{aligned} \tag{2.11}$$

Integrating directly, we compute

$$f_0 = \langle \partial_x^{-1}u_0, u_0\rangle_{L^2} = \frac{1}{2}\|u_0\|_{L^1}^2 = \sqrt{2\pi}. \tag{2.12}$$

Furthermore, using Equation (2.11) at $n=0$, we also compute

$$f_1 = \langle \partial_x^{-1}u_0, u_1\rangle_{L^2} = 2\|u_0\|_{L^2}^2 = 2. \tag{2.13}$$

Thanks to orthogonality of Hermite functions, the right-hand side of Equation (2.11) is zero for $n\in\mathbb{N}$ and the numerical sequence $\{f_n\}_{n\in\mathbb{N}_0}$ satisfies the recurrence equation

$$f_{n+1} = \frac{\sqrt{n}}{\sqrt{n+1}} f_{n-1}, \quad n\in\mathbb{N}, \tag{2.14}$$

starting with the initial values for f_0 and f_1 in Equations (2.12) and (2.13). The recurrence equation (2.14) admits the exact solution

$$f_{2m} = \left(\prod_{k=1}^{m} \frac{\sqrt{2k-1}}{\sqrt{2k}}\right) f_0, \quad f_{2m+1} = \left(\prod_{k=1}^{m} \frac{\sqrt{2k}}{\sqrt{2k+1}}\right) f_1, \quad m\in\mathbb{N}. \tag{2.15}$$

Applying the bound (2.5) of Lemma 2.1 with $a=1$, $b=0$ or $a=0$, $b=1$ yields the bound (2.10). □

3. Coercivity of the energy function

In order to prove the coercivity bound (1.14), we define the L-compatible squared norm in space $H^1(\mathbb{R})\cap L^2_1(\mathbb{R})$,

$$\|u\|^2_{H^1\cap L^2_1} := \int_{\mathbb{R}} \left[u_x^2 + \frac{1}{4}x^2u^2 + \frac{1}{2}u^2 \right] dx. \tag{3.1}$$

The second variation $E_c(u)$ is defined by Equation (1.9). The following theorem yields the coercivity bound for the energy function, which was assumed in [2,8] without a proof.

THEOREM 3.1. *There exists a constant $C\in(0,1)$ such that for every $y\in H^1(\mathbb{R})\cap L^2_1(\mathbb{R})$ satisfying the constraints*

$$\langle u_0, y\rangle_{L^2} = \langle \partial_x^{-1}u_0, y\rangle_{L^2} = 0, \tag{3.2}$$

it is true that

$$C\|y\|^2_{H^1\cap L^2_1} \leqslant E_c(y) \leqslant \|y\|^2_{H^1\cap L^2_1}, \tag{3.3}$$

where $E_c(y) = \langle Ly, y\rangle_{L^2}$.

Proof. The upper bound in inequality (3.3) follows trivially from the identity

$$E_c(y) + 2\|y\|^2_{L^2} = \|y\|^2_{H^1\cap L^2_1},$$

whereas the lower bound holds if there is a constant $C>0$ such that for every $y\in H^1(\mathbb{R})\cap L^2_1(\mathbb{R})$ satisfying constraints (3.2), it is true that

$$\|y\|^2_{L^2} \leqslant C E_c(y). \tag{3.4}$$

By the spectral theorem, we represent every $y\in H^1(\mathbb{R})\cap L^2_1(\mathbb{R})$ by

$$y = \sum_{n\in\mathbb{N}_0} c_n u_n, \quad c_n = \langle u_n, y\rangle_{L^2}, \tag{3.5}$$

where the vector $c := (c_0, c_1, c_2, \ldots)$ belongs to $\ell^2_1(\mathbb{N}_0)$, the sequence space of squared summable sequences with the weight $<n> = (1+n^2)^{1/2}$.

It follows from the first constraint in (3.2) that $c_0 = 0$. Using the norm in (3.1), we obtain

$$E_c(y) = \sum_{n\in\mathbb{N}} (n-1)|c_n|^2 \geqslant \|y - c_1u_1\|^2_{L^2}.$$

Therefore,

$$\|y\|^2_{L^2} = |c_1|^2 + \|y - c_1u_1\|^2_{L^2} \leqslant |c_1|^2 + E_c(y),$$

and coercivity (3.4) is proved if we can show that $|c_1|^2$ is bounded by $E_c(y)$ up to a multiplicative constant. To show this, we use the second constraint in Equation (3.2). Since $\langle \partial_x^{-1}u_0, u_1\rangle_{L^2} = 2\|u_0\|^2_{L^2} = 2$, as it follows from Equation (2.13), we have

$$2c_1 = -\langle \partial_x^{-1}u_0, y - c_1u_1\rangle_{L^2} = -\sum_{n=2}^{\infty} c_n \langle \partial_x^{-1}u_0, u_n\rangle_{L^2}. \tag{3.6}$$

By Lemma 2.2, there is a positive constant $C_0 > 0$ such that

$$C_0 := \sum_{n=2}^{\infty} \frac{|\langle \partial_x^{-1}u_0, u_n\rangle_{L^2}|^2}{n-1} < \infty, \tag{3.7}$$

which follows from convergence of $\sum_{n\in\mathbb{N}} n^{-3/2}$. Hence, by using Cauchy–Schwarz inequality in (3.6), we obtain

$$4|c_1|^2 \leqslant C_0 \sum_{n=2}^{\infty}(n-1)|c_n|^2 = C_0 E_c(y), \tag{3.8}$$

so that the bound (3.4) follows. The statement of the theorem is proven. □

4. Time evolution of the linearized log-KdV equation

The time evolution of the linearized log-KdV equation (1.7) is considered in the constrained energy space X_c given by (1.11). For a vector $c := (c_0, c_1, c_2, \ldots) \in \ell_1^2(\mathbb{N}_0)$, we use the decomposition involving the Hermite functions,

$$u(t) = \sum_{n\in\mathbb{N}_0} c_n(t) u_n, \quad c_n(t) = \langle u_n, u(t)\rangle_{L^2}, \tag{4.1}$$

By using $Lu_n = (n-1)u_n$ and the differential relations (2.4), the evolution problem for the vector $c \in \ell_1^2(\mathbb{N}_0)$ is written as the lattice differential equation

$$2\frac{dc_n}{dt} = n\sqrt{n+1}c_{n+1} - (n-2)\sqrt{n}c_{n-1}, \quad n \in \mathbb{N}_0. \tag{4.2}$$

It follows from Equation (4.2) for $n=0$ that if $u(0)\in X_c$ (so that $c_0(0)=0$), then $c_0(t)=0$ and $u(t)\in X_c$ for every t. If $c_0(t)=0$, then it follows from Equation (4.2) for $n=1$ that the time evolution of a projection of $u(t)$ to u_1 (which is proportional to the translational mode $\partial_x V$) is given by

$$\frac{dc_1}{dt} = \frac{1}{\sqrt{2}}c_2. \tag{4.3}$$

The projection $c_1(t)$ is decoupled from the rest of the system (4.2). Therefore, introducing $b_n = c_{n+1}$ for $n\in\mathbb{N}$, we close the evolution system (4.2) at the lattice differential equation

$$2\frac{db_n}{dt} = (n+1)\sqrt{n+2}b_{n+1} - (n-1)\sqrt{n+1}b_{n-1}, \quad n\in\mathbb{N}. \tag{4.4}$$

Since $c\in\ell_1^2(\mathbb{N})$, then $b\in\ell_1^2(\mathbb{N})$, so that we can introduce $a_n = \sqrt{n}b_n$, $n\in\mathbb{N}$ with the vector $a\in\ell^2(\mathbb{N})$. The sequence $\{a_n\}_{n\in\mathbb{N}}$ satisfies the evolution system in the skew-symmetric form

$$2\frac{da_n}{dt} = \sqrt{n(n+1)(n+2)}a_{n+1} - \sqrt{(n-1)n(n+1)}a_{n-1}, \quad n\in\mathbb{N}. \tag{4.5}$$

The evolution system (4.5) can be expressed in the symmetric form by using the transformation

$$a_n = i^n f_n, \quad n\in\mathbb{N}. \tag{4.6}$$

The new sequence $\{f_n\}_{n\in\mathbb{N}}$ satisfies the evolution system written in the operator form

$$\frac{df}{dt} = \frac{i}{2}Jf, \tag{4.7}$$

where J is the Jacobi operator defined by

$$(Jf)_n := \sqrt{n(n+1)(n+2)}f_{n+1} + \sqrt{(n-1)n(n+1)}f_{n-1}, \quad n\in\mathbb{N}. \tag{4.8}$$

The Jacobi difference equation is $Jf = zf$. According to the definition in Section 2.6 of [11], the Jacobi operator J is said to have a limit circle at infinity if a solution f of $Jf = zf$ with $f_1 = 1$ is in $\ell^2(\mathbb{N})$ for some $z \in \mathbb{C}$. By Lemma 2.15 in [11], this property remains true for all $z \in \mathbb{C}$. The following lemma shows that this is exactly our case.

LEMMA 4.1. *The Jacobi operator J defined by Equation (4.8) has a limit circle at infinity.*

Proof. Let us consider the case $z = 0$ and define a solution of $Jv = 0$ with $v_1 = 1$. The numerical sequence $\{v_n\}_{n\in\mathbb{N}}$ satisfies the recurrence relation

$$v_{n+1} = -\frac{\sqrt{n-1}}{\sqrt{n+2}} v_{n-1}, \quad n \in \mathbb{N},$$

starting with $v_1 = 1$. Then, $v_n = 0$ for even n, whereas v_n for odd n is given by the exact solution

$$v_{2m+1} = (-1)^m \prod_{k=1}^{m} \frac{\sqrt{2k-1}}{\sqrt{2k+2}}, \quad m \in \mathbb{N}.$$

By Lemma 2.1 with $a = 1$ and $b = 2$, there exists a positive constant C such that

$$|v_{2m-1}| \leqslant C m^{-3/4} \quad m \in \mathbb{N}. \tag{4.9}$$

This guarantees that $v \in \ell^2(\mathbb{N})$. ∎

By Lemma 2.16 in [11], the Jacobi operator $J_{\max} : D(J_{\max}) \to \ell^2(\mathbb{N})$ with the domain

$$D(J_{\max}) := \{f \in \ell^2(\mathbb{N}) : \quad Jf \in \ell^2(\mathbb{N})\} \tag{4.10}$$

is self-adjoint if $W_\infty(f,g) = 0$ for all $f, g \in D(J_{\max})$, where the discrete Wronskian is given by

$$W_n(f,g) := \sqrt{n(n+1)(n+2)}(f_n g_{n+1} - f_{n+1} g_n), \quad n \in \mathbb{N} \tag{4.11}$$

and $W_\infty(f,g) = \lim_{n\to\infty} W_n(f,g)$.

In order to define a self-adjoint extension of the Jacobi operator J with the limit circle at infinity, we need to define a boundary condition as follows:

$$BC(J) := \{v \in D(J_{\max}) : \quad W_\infty(v,f) = 0 \quad \text{for some } \ f \in D(J_{\max})\}, \tag{4.12}$$

where v is real. By Lemma 2.17 and Theorem 2.18 in [11], the operator $\mathcal{J} : \mathcal{D}(v) \to \ell^2(\mathbb{N})$, where $v \in BC(J)$ and

$$\mathcal{D}(v) := \{f \in D(J_{\max}) : \quad W_\infty(v,f) = 0\}, \tag{4.13}$$

represents a self-adjoint extension of the Jacobi operator J with the limit circle at infinity.

Moreover, by Lemma 2.19 in [11], the real spectrum of $\mathcal{J}$ in $\ell^2(\mathbb{N})$ is purely discrete. Since there is at most one linearly independent solution of the Jacobi difference equation $Jf = zf$ with J given by (4.8), each isolated eigenvalue of the real spectrum of $\mathcal{J}$ is simple.

By Lemma 2.20 in [11], all self-adjoint extensions of $J_{\min} : \ell_0(\mathbb{N}) \to \ell^2(\mathbb{N})$ are uniquely defined by the choice $v \in BC(J)$ in (4.12) spanned by a linear combination of two linearly independent solutions of $Jf = 0$, where the sequence space $\ell_0(\mathbb{N})$ contains finite (compactly supported) sequences. Since the value of f_0 plays no role for the Jacobi operator J given by (4.8) and the value f_1 can be uniquely normalized to $f_1 = 1$, we have a unique choice for v given by the solution of $Jv = 0$ with $v_1 = 1$. Combining these facts together, we have obtained the following result.

LEMMA 4.2. *Let $v \in BC(J)$ be the unique solution of $Jv=0$ with $v_1=1$ determined in the proof of Lemma 4.1. Then, $\mathcal{J}:\mathcal{D}(v)\to \ell^2(\mathbb{N})$ with the domain (4.13) is a unique self-adjoint extension of the Jacobi operator J given by (4.8). Moreover, the spectrum of $\mathcal{J}$ consists of a countable set of simple real isolated eigenvalues.*

The following theorem and corollary provide the *linear orbital stability* of Gaussian solitary waves (1.3) expressed by using the decomposition in terms of Hermite functions. The same result was obtained in [2,8] by using alternative techniques involving either the energy method [8] or the spectral decompositions [2].

THEOREM 4.1. *For every $a(0)\in \ell^2(\mathbb{N})$, there exists a unique solution $a(t)\in \ell^2(\mathbb{N})$ to the evolution system (4.5) for every $t\in\mathbb{R}$ satisfying $\|a(t)\|_{\ell^2}=\|a(0)\|_{\ell^2}$.*

Proof. The semi-group property of the solution operator $e^{\frac{i}{2}Jt}$ both for $t\in\mathbb{R}^+$ and $t\in\mathbb{R}^-$ associated with the linear system (4.7) follows from the result of Lemma 4.2 and the classical semi-group theory [9]. The result is transferred to the sequence $a\in\ell^2(\mathbb{N})$ by using the transformation (4.6). ∎

COROLLARY 4.1. *For every $u(0)\in X_c$ given by (1.11), there exists a unique solution $u(t)\in X_c$ to the linearized log-KdV equation (1.7) for every $t\in\mathbb{R}$ satisfying $E_c(u(t))=E_c(u(0))$.*

Proof. By using the transformations $a_n=\sqrt{n}b_n$ and $b_n=c_{n+1}$ for $n\in\mathbb{N}$ and the decomposition (4.1), we obtain

$$\|a\|_{\ell^2}^2=\sum_{n\in\mathbb{N}} n|c_{n+1}|^2=E_c(u), \quad u\in X_c, \tag{4.14}$$

where $c_0=0$ is set uniquely in X_c. Recall that $c_0=0$ is an invariant reduction of the linearized log-KdV equation (1.7). The assertion of the corollary follows from Theorem 4.1 and the equivalence (4.14). ∎

REMARK 4.1. At the first glance, Equation (4.3) might imply that the projection $c_1(t)$ to the translational mode u_1 may grow at most linearly as $t\to\infty$ in the energy space X_c with conserved $E_c(u(t))=E_c(u(0))$. However, it follows directly from the linearized log-KdV equation (1.7) [2] for the solution $u\in C(\mathbb{R},X_c)$ that

$$\langle \partial_x^{-1}u_0,u(t)\rangle_{L^2}=\langle \partial_x^{-1}u_0,u(0)\rangle_{L^2}, \quad t\in\mathbb{R}.$$

By using Equations (2.13) and (4.1), we obtain

$$2c_1(t)=\langle \partial_x^{-1}u_0,u(0)\rangle_{L^2}-\sum_{n\in\mathbb{N}} c_{n+1}(t)\langle \partial_x^{-1}u_0,u_n\rangle_{L^2},$$

where the second term is globally bounded for all $t\in\mathbb{R}$, as in the estimates (3.6)–(3.8).

5. Numerical approximations of eigenvalues and eigenvectors

We discuss here numerical approximations of eigenvalues in the spectrum of the self-adjoint operator $\mathcal{J}:\mathcal{D}(v)\to\ell^2(\mathbb{N})$ constructed in Lemma 4.2. The real eigenvalues z of $\mathcal{J}$ transform to the purely imaginary eigenvalues λ of the spectral problem $\partial_x Lu=\lambda u$ by using the decomposition (4.1). Therefore, the result of Lemma 4.2 also provides an alternative proof of the *spectral stability* of the Gaussian solitary waves (1.3), which is also established in [2]. However, by comparing the eigenvectors obtained in the two alternative approaches, we will see some sharp differences in the definition of function spaces these eigenvectors belong to.

The following claim is based on the numerical approximations of eigenvalues and eigenvectors.

CLAIM 5.1. *The Jacobi difference equation $Jf = zf$ for $\mathcal{J}:\mathcal{D}(v)\to\ell^2(\mathbb{N})$ admits a bi-infinite sequence of simple real eigenvalues ordered as*

$$\cdots < -z_2 < -z_1 < z_0 = 0 < z_1 < z_2 < \cdots \tag{5.1}$$

Let $f^{(n)} \in \mathcal{D}(v)$ be an eigenvector of $Jf^{(n)} = z_n f^{(n)}$ for $n\in\mathbb{N}$ and denote $A_m = f^{(n)}_{2m-1}$ and $B_m = f^{(n)}_{2m}$ for $m\in\mathbb{N}$. Then, the decay rates of the two sequences are as follows:

$$A_m = \mathcal{O}(m^{-3/4}), \quad B_m = \mathcal{O}(m^{-5/4}) \quad \text{as} \quad m\to\infty. \tag{5.2}$$

To proceed with numerical approximations, we note that if $v \in D(J_{\max})$ is a solution of $Jv = 0$ like in the proof of Lemma 4.1, then $v_n = 0$ for even n. Let us denote $V_m = v_{2m-1}$ for $m\in\mathbb{N}$. From the bound (4.9), we note that $V_m = \mathcal{O}(m^{-3/4})$ as $m\to\infty$.

Let $f \in D(J_{\max})$ be a solution of $Jf = zf$ for $z\in\mathbb{R}^+$ and denote $A_m = f_{2m-1}$ and $B_m = f_{2m}$ for $m\in\mathbb{N}$. It follows from the definition (4.8) that A and B satisfy the coupled system of difference equations:

$$\begin{cases} B_m = -\frac{\sqrt{2m-2}}{\sqrt{2m+1}} B_{m-1} + \frac{z}{\sqrt{(2m-1)(2m)(2m+1)}} A_m, \\ A_{m+1} = -\frac{\sqrt{2m-1}}{\sqrt{2m+2}} A_m + \frac{z}{\sqrt{(2m)(2m+1)(2m+2)}} B_m, \end{cases} \quad m\in\mathbb{N}, \tag{5.3}$$

starting with $A_1 = 1$. The discrete Wronskian (4.11) is now explicitly computed as

$$W_n = \begin{cases} \sqrt{(2m-1)(2m)(2m+1)} B_m V_m, & n = 2m-1, \\ -\sqrt{(2m)(2m+1)(2m+2)} B_m V_{m+1}, & n = 2m. \end{cases} \tag{5.4}$$

Since generally $B_m = \mathcal{O}(m^{-3/4})$ as $m\to\infty$, by applying Lemma 2.1 with $a=0$ and $b=3$ to the first equation of system (5.3), the limit $W_\infty = \lim_{n\to\infty}|W_n|$ exists and is generally nonzero. Moreover, the sign alternation of $\{V_m\}_{m\in\mathbb{N}}$ and $\{B_m\}_{m\in\mathbb{N}}$ ensures that the sequence $\{W_n\}_{n\in\mathbb{N}}$ is sign-definite for large enough n, so that the limit is actually $W_\infty = \lim_{n\to\infty} W_n$. This is confirmed in Figure 5.1(a), which shows the Wronskian sequence $\{W_n\}_{n\in\mathbb{N}}$ given by Equation (5.4) for $z=1$.

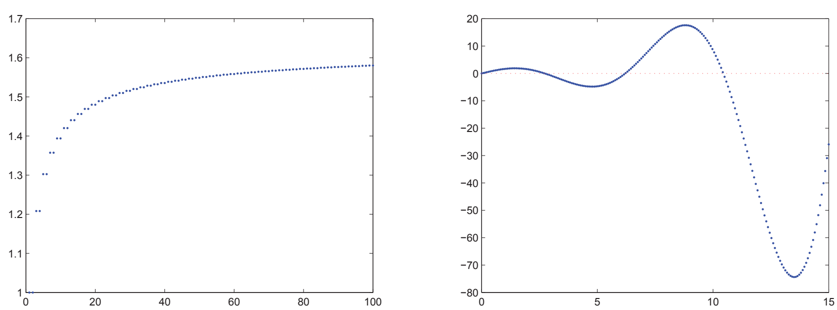

FIG. 5.1. *(a) Convergence of the sequence $\{W_n\}_{n\in\mathbb{N}}$ as $n\to\infty$ for $z=1$. (b) Oscillatory behavior of W_∞ versus z.*

Computing numerically W_∞ by truncation of $\{W_n\}_{n\in\mathbb{N}}$ at a sufficiently large n, e.g. at $n=1000$, we plot W_∞ versus z on Figure 5.1(b). Oscillations of W_∞ are observed and the first two zeros of W_∞ are located at

$$z_1 \approx 2.7054, \quad z_2 \approx 6.1540.$$

These values are nicely compared to the first two eigenvalues computed in [2] for $E_k = 2z_k$:

$$E_1 \approx 5.4109, \quad E_2 \approx 12.3080.$$

The numerical approximations confirm that the eigenvalues obtained by using the Jacobi difference equation are the same as the eigenvalues obtained in [2] from the Sturm–Liouville problem derived in the Fourier space. Numerically, we find for the first two zeros $z_{1,2}$ of the limiting Wronskian W_∞ that the decay rate of the sequence $\{A_m\}_{m\in\mathbb{N}}$ remains generic whereas the decay rate of the sequence $\{B_m\}_{m\in\mathbb{N}}$ becomes faster, in accordance with the decay rates claimed in (5.2).

Let us now recall the correspondence of eigenvectors of the Jacobi difference equation $Jf = zf$ and eigenvectors of the linearized log-KdV operator $\partial_x L u = \lambda u$, where $\lambda = \frac{i}{2}z$. Recall the decomposition (4.1) and the notations $b_n = c_{n+1}$ for $n \in \mathbb{N}$. From the previous transformations, we obtain

$$y := u - c_1 u_1 = \sum_{n\in\mathbb{N}} b_n u_{n+1} = \sum_{n\in\mathbb{N}} \frac{i^n}{\sqrt{n}} f_n u_{n+1} = y_{\text{odd}} + i y_{\text{even}}, \tag{5.5}$$

where

$$y_{\text{odd}} = \sum_{m\in\mathbb{N}} \frac{(-1)^m}{\sqrt{2m}} B_m u_{2m+1} \quad \text{and} \quad y_{\text{even}} = \sum_{m\in\mathbb{N}} \frac{(-1)^{m-1}}{\sqrt{2m-1}} A_m u_{2m} \tag{5.6}$$

are respectively the odd and even components of the eigenvector with respect to x. Thanks to the decay of the sequences $\{A_m\}_{m\in\mathbb{N}}$ and $\{B_m\}_{m\in\mathbb{N}}$ in (5.2), we note that

$$y_{\text{odd}} \in H^2(\mathbb{R}) \cap L^2_2(\mathbb{R}), \quad y_{\text{even}} \in H^1(\mathbb{R}) \cap L^2_1(\mathbb{R}), \tag{5.7}$$

but that $y_{\text{even}} \notin H^2(\mathbb{R}) \cap L^2_2(\mathbb{R})$. Therefore, generally $y \notin \text{Dom}(\partial_x L)$ defined by (1.15). Thus, the eigenvector y given by the decomposition (5.5) does not solve the eigenvalue problem $\partial_x L y = \lambda y$ in the classical sense compared to the eigenvectors constructed in [2] with the Fourier transform.

The following claim clarifies the sense of the eigenvectors of $\partial_x L y = \lambda y$.

CLAIM 5.2. *Let $y = y_{\text{odd}} + i y_{\text{even}}$ be the eigenvector constructed numerically in Claim 5.1 by the decomposition (5.5). Then, the components*

$$y_{\text{odd}} \in H^2(\mathbb{R}) \cap L^2_2(\mathbb{R}) \cap H^{-1}(\mathbb{R}) \quad \text{and} \quad y_{\text{even}} \in H^1(\mathbb{R}) \cap L^2_1(\mathbb{R}) \cap H^{-1}(\mathbb{R}) \tag{5.8}$$

satisfy the coupled system

$$z L^{-1} \partial_x^{-1} y_{\text{odd}} = 2 y_{\text{even}}, \quad -z \partial_x^{-1} y_{\text{even}} = 2 L y_{\text{odd}}, \tag{5.9}$$

where each term is defined in $L^2(\mathbb{R})$.

Let us denote $\lambda = \frac{i}{2} z$ and project the eigenvalue problem $\partial_x L u = \lambda u$ to u_1 and y. The projection c_1 is uniquely found by

$$z c_1 = \sqrt{2} A_1, \tag{5.10}$$

which can also be obtained from the projection equation (4.3). The component y satisfies formally $\lambda y = \partial_x L y$. After separating the even and odd parts of the eigenvalue problem, we obtain the coupled system

$$z y_{\text{odd}} = 2 \partial_x L y_{\text{even}}, \quad -z y_{\text{even}} = 2 \partial_x L y_{\text{odd}}. \tag{5.11}$$

As we have indicated above, it is difficult to prove that each term of the coupled system (5.11) belongs to $L^2(\mathbb{R})$ if $y_{\rm even}$ and $y_{\rm odd}$ are given by (5.5) and (5.6) with the decay rates as in (5.2). However, it is easy to formulate each term of the coupled system (5.9) in $L^2(\mathbb{R})$, if the components $y_{\rm even}$ and $y_{\rm odd}$ belong in the function space (5.8), which is in agreement with the decay rates in (5.2).

To show this, we proceed as follows. According to (5.6) and (5.7), $y_{\rm odd} \in H^2(\mathbb{R}) \cap L^2_2(\mathbb{R})$ is odd, hence $\partial_x^{-1} y_{\rm odd} \in L^2(\mathbb{R})$ and the first constraint in (5.8) is satisfied. Since the kernel of L is spanned by the odd function, we have $\partial_x^{-1} y_{\rm odd} \in {\rm Range}(L)$ so that $L^{-1}\partial_x^{-1} y_{\rm odd} \in L^2(\mathbb{R})$ and $y_{\rm even} \in L^2(\mathbb{R})$ as is given by the first equation in (5.9). Similarly, from (5.6) and (5.7), we have $L y_{\rm odd} \in L^2(\mathbb{R})$ so that the second equation in (5.9) implies that $\partial_x^{-1} y_{\rm even} \in L^2(\mathbb{R})$. Hence, the second constraint in (5.8) is satisfied. Thus, the coupled system (5.9) represents the eigenvalue problem $\partial_z L y = \lambda y$ in the function space (5.4). Thus, the coupled system (5.9) represents the eigenvalue problem $\partial_z L y = \lambda y$ in the function space (5.4).

Remark 5.1. The formulation (5.9) in Claim 5.2 settles the issue of zero eigenvalue $z_0 = 0$, which should not be listed as an eigenvalue of the problem $\partial_z L y = \lambda y$. Indeed, the first Equation (5.9) with $z_0 = 0$ implies $y_{\rm even} = 0$, hence $V_m = v_{2m-1} = 0$, where v is a solution of $Jv = 0$. Thus, the existence of the eigenvector $v \in \ell^2(\mathbb{N})$ for the eigenvalue $z_0 = 0$ of the Jacobi difference equation does not imply the existence of the zero eigenvalue $\lambda = 0$ in the proper formulation (5.8)–(5.9) of the system $\partial_x L y = \lambda y$. The same result can be obtained from the projection equation (5.10). If $z_0 = 0$, then $V_1 \equiv A_1 = 0$, which corresponds to the zero solution $v = 0$ of the Jacobi difference equation $Jv = 0$.

Remark 5.2. Delicate analytical issues in the decomposition (5.5) involving Hermite functions are likely to be related to the fact that eigenvectors u of the eigenvalue problem $\partial_x L u = \lambda u$ decay algebraically as $|x| \to \infty$, while the decay of each Hermite function u_n in the decomposition (5.6) is given by a Gaussian function.

6. Dissipative properties of the linearized log-KdV equation

For the KdV equation with exponentially decaying solitary waves, the exponentially weighted spaces were used to introduce effective dissipation in the long-time behavior of perturbations to the solitary waves and to prove their asymptotic stability [10]. For the log-KdV equation with Gaussian solitary waves, it makes sense to introduce Gaussian weights in order to obtain a dissipative evolution of the linear perturbations. Here we show how the Gaussian weights can be introduced for the linearized log-KdV equation (1.7).

Let us represent a solution to the linearized log-KdV equation (1.7) in the following form

$$u(t,x) = a(t) u_0(x) + b(t) u_1(x) + y(t,x), \tag{6.1}$$

with

$$y(t,x) = \int_x^\infty u_0(x') w(t, x - x') dx = \int_{-\infty}^0 u_0(x - z) w(t,z) dz, \tag{6.2}$$

where (a,b,w) are new variables to be found. It is clear that the representation (6.2) imposes restrictions on the class of functions of y in the energy space X_c. We will show that these restrictions are invariant with respect to the time evolution of the linearized log-KdV equation (1.7).

We assume sufficient smoothness and decay of the variable w. By using the explicit computation with $L u_0 = -u_0$ and $x u_0(x) = -2 u_0'(x)$, we obtain

$$Ly = \int_{-\infty}^0 w(t,z) \left[(L u_0)(x-z) + \frac{1}{2} z (x-z) u_0(x-z) + \frac{1}{4} z^2 u_0(x-z) \right] dz$$

$$= \int_{-\infty}^{0} w(t,z) \left[-u_0(x-z) - z u_0'(x-z) + \frac{1}{4} z^2 u_0(x-z) \right] dz.$$

Integrating by parts, we further obtain

$$\partial_x L y = 2w(t,0) u_0(x) + \int_{-\infty}^{0} u_0(x-z) \left[-w_z - (zw)_{zz} + \frac{1}{4}(z^2 w)_z \right] dz.$$

Also recall that $\partial_x L u_0 = -u_0'(x) = \frac{1}{2} u_1(x)$ and $\partial_x L u_1 = 0$. Bringing together the left-side and the right-side of the linearized log-KdV equation (1.7) under the decomposition (6.1)–(6.2), we obtain the system of modulation equations

$$\begin{cases} \dot{a}(t) = 2w(t,0), \\ \dot{b}(t) = \frac{1}{2} a(t), \end{cases} \tag{6.3}$$

and the evolution problem

$$w_t = Hw, \tag{6.4}$$

where the linear operator $H : \mathrm{Dom}(H) \to L^2(\mathbb{R}^-)$ with the domain

$$\mathrm{Dom}(H) = \{ w \in L^2(\mathbb{R}^-), \quad Hw \in L^2(\mathbb{R}^-) \}$$

is given by

$$(Hw)(z) = -z w_{zz} - 3 w_z + \frac{1}{4}(z^2 w)_z, \quad z < 0. \tag{6.5}$$

Since $z = 0$ is a regular singular point of the differential operator $H : \mathrm{Dom}(H) \to L^2(\mathbb{R}^-)$, no boundary condition is needed to be set at $z = 0$. We show that the differential operator H is dissipative in $L^2(\mathbb{R}^-)$.

LEMMA 6.1. *For every $w \in \mathrm{Dom}(H) \subset L^2(\mathbb{R}^-)$, we have*

$$\langle Hw, w \rangle_{L^2(\mathbb{R}^-)} = -[w(0)]^2 + \int_{-\infty}^{0} \left[z(\partial_z w)^2 + \frac{1}{4} z w^2 \right] dz \leqslant -\frac{1}{2} \|w\|^2_{L^2(\mathbb{R}^-)}. \tag{6.6}$$

Proof. We obtain from integration by parts for every $w \in \mathrm{Dom}(H)$ that

$$\begin{aligned} &\int_{-\infty}^{0} w \left[-z w_{zz} - 3 w_z + \frac{1}{4}(z^2 w)_z \right] dz \\ &= \left[-z w w_z - w^2 + \frac{1}{8} z^2 w^2 \right] \Bigg|_{z \to -\infty}^{z=0} + \int_{-\infty}^{0} \left[z(\partial_z w)^2 + \frac{1}{4} z w^2 \right] dz \\ &= -[w(0)]^2 + \int_{-\infty}^{0} \left[z(\partial_z w)^2 + \frac{1}{4} z w^2 \right] dz. \end{aligned}$$

This yields the equality in (6.6). The inequality in (6.6) is proved from the Young's inequality

$$\|w\|^2_{L^2(\mathbb{R}^-)} = -\int_{-\infty}^{0} 2 z w w_z dz \leqslant \alpha^2 \int_{-\infty}^{0} |z| (\partial_z w)^2 dz + \alpha^{-2} \int_{-\infty}^{0} |z| w^2 dz,$$

where $\alpha > 0$ is at our disposal. Picking $\alpha^2 = 2$ yields the inequality in (6.6). ∎

The semi-group theory for dissipative operators is fairly standard [9], so we assume existence of a strong solution to the evolution problem (6.4) for every $t>0$. The next result shows that this solution decays exponentially fast in the $L^2(\mathbb{R}^-)$ norm.

COROLLARY 6.1. *Let $w\in C(\mathbb{R}^+,\mathrm{Dom}(H))\cap C^1(\mathbb{R}^+,L^2(\mathbb{R}^-))$ be a solution of the evolution problem (6.4). Then, the solution satisfies*

$$\|w(t)\|^2_{L^2(\mathbb{R}^-)}\leqslant \|w(0)\|^2_{L^2(\mathbb{R}^-)}e^{-t}. \tag{6.7}$$

Proof. The decay behavior (6.7) is obtained from a priori energy estimates. Indeed, it follows from (6.6) that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\mathbb{R}^-)}=\langle Hw,w\rangle_{L^2(\mathbb{R}^-)}\leqslant -\frac{1}{2}\|w(t)\|^2_{L^2(\mathbb{R}^-)}.$$

Gronwall's inequality yields the bound (6.7). □

We recall that the solution $u\in X_c$ needs to satisfy the constraint $\langle u_0,u\rangle_{L^2}=0$. The constraint is invariant with respect to the time evolution of the linearized log-KdV equation (1.7). These properties are equivalently represented in the decomposition (6.1)–(6.2), according to the following lemma.

LEMMA 6.2. *For every $w\in \mathrm{Dom}(H)\subset L^2(\mathbb{R}^-)$, we have*

$$a(t)+\int_{-\infty}^0 e^{-\frac{1}{8}z^2}w(t,z)dz=A, \quad t\in\mathbb{R}^+, \tag{6.8}$$

where A is constant in t. Moreover, if $u(0)\in X_c$, then $A=0$ and

$$|a(t)|^2\leqslant \sqrt{\pi}\|w(0)\|^2_{L^2(\mathbb{R}^-)}e^{-t}. \tag{6.9}$$

Proof. We compute directly

$$\begin{aligned}\langle u_0,u(t)\rangle_{L^2}&=a(t)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{4}x^2}\left(\int_{-\infty}^0 e^{-\frac{1}{4}(x-z)^2}w(t,z)dz\right)dx\\&=a(t)+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{-\frac{1}{8}z^2}w(t,z)\left(\int_{-\infty}^{\infty}e^{-\frac{1}{2}\left(x-\frac{1}{2}z\right)^2}dx\right)dz\\&=a(t)+\int_{-\infty}^0 e^{-\frac{1}{8}z^2}w(t,z)dz.\end{aligned}$$

Furthermore, $\langle u_0,u(t)\rangle_{L^2}=A$ is constant in t. As an alternative derivation, one can compute from the evolution problem (6.4) that

$$\frac{d}{dt}\int_{-\infty}^0 e^{-\frac{1}{8}z^2}w(t,z)dz=-2w(t,0)$$

and combine it with the first modulation equation in system (6.3) to obtain Equation (6.8).

If $u(0)\in X_c$, then $u(t)\in X_c$ and $A=0$. This yields $a(t)$ uniquely by

$$a(t)=-\int_{-\infty}^0 e^{-\frac{1}{8}z^2}w(t,z)dz.$$

Applying the decay bound (6.7) and the Cauchy–Schwarz inequality, we obtain the decay bound (6.9). □

COROLLARY 6.2. *If $u(0)\in X_c$, then there is $b_\infty \in \mathbb{R}$ such that $b(t)\to b_\infty$ as $t\to\infty$.*

Proof. It follows from the bound (6.9) that if $u(0)\in X_c$, then $a(t)\to 0$ as $t\to\infty$. Since $a(t)$ decays to zero exponentially fast, the assertion of the corollary follows from integration of the second modulation equation in system (6.3). ∎

Besides scattering to zero in the $L^2(\mathbb{R}^-)$ norm, the global solution of the evolution problem (6.4) also scatters to zero in the $L^\infty(\mathbb{R}^-)$ norm. The following lemma gives the relevant result based on a priori energy estimates.

LEMMA 6.3. *Let w be a smooth solution of the evolution problem (6.4) in a subset of $H^1(\mathbb{R}^-)$. Then, there exist positive constants γ and C such that*

$$\|w(t)\|_{L^\infty(\mathbb{R}^-)} \leqslant C\|w(0)\|_{H^1(\mathbb{R}^-)} e^{-\gamma t}. \tag{6.10}$$

Proof. The proof is developed similarly to the estimates in Lemma 6.1 and Corollary 6.1 but the estimates are extended for $\|\partial_z w(t)\|_{L^2(\mathbb{R}^-)}$. Differentiating Equation (6.5) in z, multiplying by w_z, and integrating by parts, we obtain for smooth solution w:

$$\begin{aligned}\langle w_z,(Hw)_z\rangle_{L^2(\mathbb{R}^-)} &= \left[-zw_z w_{zz} - \frac{3}{2}w_z^2 + \frac{1}{8}z^2 w_z^2 + \frac{1}{4}w^2\right]\Bigg|_{z\to-\infty}^{z=0} + \int_{-\infty}^0 z w_{zz}^2 dz + \frac{3}{4}\int_{-\infty}^0 z w_z^2 dz \\ &= -\frac{3}{2}[\partial_z w(0)]^2 + \frac{1}{4}[w(0)]^2 + \int_{-\infty}^0 z w_{zz}^2 dz + \frac{3}{4}\int_{-\infty}^0 z w_z^2 dz.\end{aligned}$$

As a result, smooth solutions to the evolution problem (6.4) satisfy the differential inequality

$$\frac{d}{dt}\frac{1}{2}\|w_z\|^2_{L^2(\mathbb{R}^-)} = \langle w_z,(Hw)_z\rangle_{L^2(\mathbb{R}^-)} \leqslant \frac{1}{4}[w(t,0)]^2 + \int_{-\infty}^0 z w_{zz}^2 dz + \frac{3}{4}\int_{-\infty}^0 z w_z^2 dz.$$

By using Young's inequality, we estimate

$$[w(t,0)]^2 = 2\int_{-\infty}^0 w w_z dz \leqslant \beta^2 \|w_z\|^2_{L^2(\mathbb{R}^-)} + \beta^{-2}\|w\|^2_{L^2(\mathbb{R}^-)}$$

and

$$\|w_z\|^2_{L^2(\mathbb{R}^-)} = -2\int_{-\infty}^0 z w_z w_{zz} dz \leqslant \alpha^2 \int_{-\infty}^0 |z| w_{zz}^2 dz + \alpha^{-2}\int_{-\infty}^0 |z| w_z^2 dz,$$

where α and β are to our disposal. Picking $\alpha^2 = 2$ and assuming $\beta^2 < 2$, we close the differential inequality as follows

$$\frac{d}{dt}\|w_z\|^2_{L^2(\mathbb{R}^-)} \leqslant -\left(1 - \frac{\beta^2}{2}\right)\|w_z\|^2_{L^2(\mathbb{R}^-)} + \frac{1}{2\beta^2}\|w\|^2_{L^2(\mathbb{R}^-)}.$$

Thanks to the exponential decay in the bound (6.7), we can rewrite the differential inequality in the form

$$\frac{d}{dt}\left[\|w_z\|^2_{L^2(\mathbb{R}^-)} e^{\left(1-\frac{\beta^2}{2}\right)t}\right] \leqslant \frac{1}{2\beta^2}\|w(0)\|^2_{L^2(\mathbb{R}^-)} e^{-\frac{\beta^2}{2}t}.$$

Integrating over time, we finally obtain

$$\|\partial_z w(t)\|^2_{L^2(\mathbb{R}^-)} \leqslant \left(\|\partial_z w(0)\|^2_{L^2(\mathbb{R}^-)} + \beta^{-4}\|w(0)\|^2_{L^2(\mathbb{R}^-)}\right) e^{-\left(1-\frac{\beta^2}{2}\right)t},$$

where $\beta^2 < 2$ is fixed arbitrarily. Thus, the $H^1(\mathbb{R}^-)$ norm of the smooth solution to the evolution problem (6.4) decays to zero exponentially fast as $t \to \infty$. The bound (6.10) follows by the Sobolev embedding of $H^1(\mathbb{R}^-)$ to $L^\infty(\mathbb{R}^-)$. □

Combining the results of this section, we summarize the main result on the dissipative properties of the solutions to the linearized log-KdV equation (1.7) represented in the convolution form (6.1)–(6.2).

THEOREM 6.1. *Assume that the initial data $u(0) \in X_c$ is represented by the convolution form (6.1)–(6.2) with some $a(0)$, $b(0)$, and $w(0) \in \mathrm{Dom}(H)$. There exists a solution of the linearized log-KdV equation (1.7) represented in the convolution form (6.1)–(6.2) with unique $(a,b) \in C^1(\mathbb{R}^+,\mathbb{R}^2)$ and $w \in C(\mathbb{R}^+,\mathrm{Dom}(H)) \cap C^1(\mathbb{R}^+,L^2(\mathbb{R}^-))$. Moreover, there is a $b_\infty \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \|u(t) - b_\infty u_1\|_{L^2 \cap L^\infty} = 0. \tag{6.11}$$

Proof. The existence result follows from the existence of the semi-group to the evolution problem (6.4) and the ODE theory for the system of modulation equations (6.3). Since $u_0 \in L^1(\mathbb{R})$, the scattering result (6.11) follows from the generalized Younge inequality, as well as the results of Corollary 6.1, Lemma 6.2, Corollary 6.2, and Lemma 6.3. □

7. Conclusion

We have obtained new results for the linearized log-KdV equation. By using Hermite function decompositions in Section 4, we have shown analytically how the semi-group properties of the linear evolution in the energy space can be recovered with the Jacobi difference operator. We have also established numerically in Section 5 the equivalence between computing the spectrum of the linearized operator with the Jacobi difference equation and that with the differential equation. Finally, we have used in Section 6 the convolution representation with the Gaussian weight to show that the solution to the linearized log–KdV equation can decay to zero in the $L^2 \cap L^\infty$ norms.

We finish this paper with two observations.

REMARK 7.1. From analysis of eigenfunctions of the spectral problem $\partial_x L u = \lambda u$, it is shown in [2] that the eigenfunctions are supported on a half-line in the Fourier space. The decomposition (4.1) in terms of the Hermite functions in the physical space can be rewritten in the Fourier space as another decomposition in terms of the Hermite functions. The Jacobi difference equation representing the spectral problem does not imply generally that the decomposition in the Fourier space returns an eigenfunction supported on a half-line. This property is not explicitly seen in the numerical computation of eigenvectors with the Jacobi difference operator.

REMARK 7.2. The linear evolution of the linearized log-KdV equation in the Fourier space (1.16) can be analyzed separately for $k \in \mathbb{R}^+$ and $k \in \mathbb{R}^-$. Since the time evolution is given by the linear Schrödinger-type equation, the fundamental solution is norm-preserving in the energy space. If the Gaussian weight is introduced on the positive half-line as follows:

$$\hat{u}(t,k) = e^{-k^2} \hat{w}(t,k), \quad k > 0,$$

then the time evolution is defined in the Fourier space by $i\hat{u}_t = \hat{H}\hat{u}$, where the linear operator $\hat{H} : \mathrm{Dom}(\hat{H}) \to L^2(\mathbb{R}^+)$ is given by

$$\hat{H} = \frac{1}{4} k \partial_k^2 - k^2 \partial_k + k. \tag{7.1}$$

If H and $\hat{H}$ in (6.5) and (7.1) are extended on the entire lines, then H and $\hat{H}$ are Fourier images of each other. Thus, a very similar introduction of the Gaussian weights (except, of course, the domains in the physical and Fourier space) may result in either dissipative or norm-preserving solutions of the linearized log-KdV equation.

Although the results obtained in this work give new estimates and new tools for analysis of the linearized log-KdV equation, it is unclear in the present time how to deal with the main problem of proving orbital stability of the Gaussian solitary waves in the nonlinear log-KdV equation (1.1). This challenging problem will remain open to new researchers.

REFERENCES

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover, New York, 1965.

[2] R. Carles and D. Pelinovsky, *On the orbital stability of Gaussian solitary waves in the log-KdV equation*, Nonlinearity, 27:3185–3202, 2014.

[3] T. Cazenave, *Stable solutions of the logarithmic Schrödinger equation*, Nonlinear Anal., 7:1127–1140, 1983.

[4] T. Cazenave, *Semilinear Schrödinger Equations*, Courant Institute of Mathematical Sciences, New York, 2003.

[5] T. Cazenave and A. Haraux, *Equations d'évolution avec non-linéarité logarithmique*, Ann. Fac. Sci. Toulouse Math., 2:21–51, 1980.

[6] A. Chatterjee, *Asymptotic solution for solitary waves in a chain of elastic spheres*, Phys. Rev. E, 59:5912–5919, 1999.

[7] E. Dumas and D.E. Pelinovsky, *Justification of the log-KdV equation in granular chains: the case of precompression*, SIAM J. Math. Anal., 46:4075–4103, 2014.

[8] G. James and D. Pelinovsky, *Gaussian solitary waves and compactons in Fermi–Pasta–Ulam lattices with Hertzian potentials*, Proc. Roy. Soc. A, 470:20130465, 2014.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer–Verlag, New York, 1983.

[10] R.L. Pego and M.I. Weinstein, *Asymptotic stability of solitary waves*, Commun. Math. Phys., 164:305–349, 1994.

[11] G. Teschl, *Jacobi Operators and Completely Integrable Nonlinear Lattices*, American Mathematical Society, Providence, 2000.

[12] G. Teschl, *Ordinary Differential Equations and Dynamical Systems*, American Mathematical Society, Providence, 2012.