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TIME PERIODIC SOLUTIONS TO THE COMPRESSIBLE NAVIER–STOKES–POISSON SYSTEM WITH DAMPING*

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Abstract. In this paper, we establish the existence of time periodic solutions to the two and three dimensional compressible Navier–Stokes–Poisson equations with the linear damping, under some smallness and symmetry assumptions on the time periodic external force. Based on the uniform estimates and the topological degree theory, we prove the existence of a time periodic solution in a bounded domain. Finally, the existence result in the whole space is obtained by a limiting process.

Keywords. Navier–Stokes–Poisson equations; time periodic solution; uniform estimates; topological degree theory

AMS subject classifications. 35M10; 35Q35; 35B10

1. Introduction

For a given constant density $\bar{\rho}$ of positively charged ions, the time evolution of the electron density $\rho=\rho(x,t)$ and the electron velocity $u=u(x,t)$ governed by the compressible Navier–Stokes–Poisson equations with the linear damping reads as

$$\begin{cases}\rho_t+\operatorname{div}(\rho u)=0,\\ \rho[u_t+(u\cdot\nabla)u]+\nabla P(\rho)=\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u+\rho\nabla\Phi-\delta\rho u+\rho f,\\ \Delta\Phi=\rho-\bar{\rho}.\end{cases}\tag{1.1}$$

Here $\Phi(x,t)$ represents the electric potential of the electrons at time $t\geq 0$ and position $x\in\mathbb{R}^N$ for $N=2,3$. The pressure function $P=P(\rho)$ is assumed to be a smooth function in a neighborhood of $\bar{\rho}$ satisfying $P'(\bar{\rho})>0$. The constants μ,λ are the viscosity coefficients with the usual physical conditions

$$\mu>0,\quad \lambda+\frac{2}{N}\mu\geq 0;$$

$\delta>0$ represents a friction coefficient. Moreover, $f(x,t)$ is a given external force, which is assumed to be periodic in time with period T.

Our main purpose of this paper is to investigate the existence of a time periodic solution in $\mathbb{R}^N, N=2,3$ for the problem (1.1), which has the same period as f. First, we combine the uniform estimates and the topological degree theory to show the problem admits a time periodic solution in a sequence of bounded domains. Then by several uniform bounds and the limiting process on the approximate solutions, we show the existence in the whole space immediately.

Our main result in this paper is stated as follows.

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Theorem 1.1. *Assume the time periodic external force* $f(x,t)\in W_2^{1,1}((0,T)\times\mathbb{R}^N), N=2,3$, *with* $f(-x,t)=-f(x,t)$. *If*

$$\|f(t)\|^2_{W_2^{1,1}}\le h,$$

for some small constant $h>0$, *then the problem* (1.1) *admits a time periodic solution* $(\rho,u,\nabla\Phi)$ *with the same period as the external force* $f(x,t)$, *satisfying* $(\rho-\bar{\rho},u,\nabla\Phi)\in X_{a_0}$, *where* X_{a_0} *is defined later.*

The Equations (1.1) with $\delta=0$ indeed calls the compressible Navier–Stokes–Poisson system is used to model and simulate the charge transport in semiconductor devices [11]. Recently, many interesting researchers have been devoted to many topics of this system, cf.[**?**, 4, 9, **?**, **?**, 16, 17, 18, 19, 20] and references therein. Here we recall some previous works related to our topic. The local and global existence of the multi-dimension renormalized solution was obtained in [3, 19]. Hao and Li [4] established the global strong solutions of the initial value problem for the multi-dimensional in the Besov space. Later, Tan and Wu [**?**] extended these results to the non-isentropic case in hybrid Besov space. The global existence and the asymptotics of the global solution near a constant equilibrium state for the isentropic and non-isentropic cases in $\mathbb{R}^3$ were achieved by Li and his collaborators in [9, 18]. Zheng [20] proved the global well-posedness in the L^p framework with initial data close to a stable equilibrium for the multi-dimension. However, at our best knowledge, it seems that there is no work on the periodic solutions whether $\delta=0$ or $\delta>0$.

When there is no electric field and $\delta=0$, system (1.1) reduces to the compressible Navier–Stokes equations, cf. [1, 2, **?**, 6, 7, 8, 10, 12, 15] and references therein. Here we only mention some of them related to our paper. In [10], Ma, Ukai and Yang combined the linear decay analysis and the contraction mapping theorem to obtain the existence and stability of time periodic solutions when the space dimension $N\ge 5$. For recent works, in [7], Jin and Yang considered the existence of time periodic solutions to the whole space $\mathbb{R}^3$ through the topological degree theory, provided

$$\int_0^T\left(\|f(t)\|^2_{L^{\frac{6}{5}}}+\|f(t)\|^4_{H^1}\right)dt+\|f(t)\|^2_{W_2^{1,1}}$$

is small enough and the pressure is given as $P(\rho)=\rho^\gamma,\gamma>1$. Also, by the spectral properties, the author in [8] obtained a time periodic solution for sufficiently small and symmetry condition on the time periodic external force when the space dimension is greater than or equal to 3. Thus, the case when $N=2$ is still unknown. Based on the idea in [7], in this paper we will consider the existence of a periodic solution under a smallness condition on $\|f(t)\|^2_{W_2^{1,1}}$ of system (1.1) in $\mathbb{R}^N, N=2,3$, where the smallness assumption on $\int_0^T\left(\|f(t)\|^2_{L^{\frac{6}{5}}}+\|f(t)\|^4_{H^1}\right)dt$ may be redundant by employing some different energy estimates.

The rest of the paper is organized as follows. In Section 2, we will reformulate the problem, then introduce the function space of solutions and some preliminaries lemmas for later use. In Section 3, we will obtain the existence of periodic solutions for system (2.2) by uniform estimates and topological degree theory in a bounded domain. And the proof of the main result will be given in the last section.

Notations: Throughout this paper, for simplicity, we will omit the variables t,x of functions if it does not cause any confusion. C denotes a generic positive constant which may vary in different estimates. Moreover, we use H^s to denote the usual L^2-Sobolev spaces with normal $\|\cdot\|_{H^s}$ and $L^p, 1\le p\le\infty$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. Finally, denote the t-anisotropic Sobolev spaces as

$$W_p^{m,k}((0,T)\times\Omega^L)=\{u: D^\alpha u, D_t^\beta u\in L^p((0,T)\times\Omega^L),\ \text{for any } |\alpha|\le m, |\beta|\le k\},$$

with the norm

$$\|u\|_{W_p^{m,k}}=\sum_{|\alpha|\le m}\|D^\alpha u\|_{L^p}+\sum_{|\beta|\le k}\|D_t^\beta u\|_{L^p}.$$

And for $0<\alpha<1$, denote $C^{\alpha,\frac{\alpha}{2}}((0,T)\times\Omega^L)$ be the set of all functions u such that $|u|_{\alpha,\frac{\alpha}{2}}<\infty$, where

$$|u|_{\alpha,\frac{\alpha}{2}}=[u]_{\alpha,\frac{\alpha}{2}}+\|u\|_{L^\infty},$$

here $[\cdot]_{\alpha,\frac{\alpha}{2}}$ is the semi-norm defined by

$$[u]_{\alpha,\frac{\alpha}{2}}=\sup_{(x,t)\neq(y,s)}\frac{|u(x,t)-u(y,s)|}{(|x-y|^2+|t-s|)^{\frac{\alpha}{2}}}.$$

2. Preliminaries

Without loss of generality, we take the constant δ in system (1.1) to be 1. We rewrite problem (1.1) in a new form, by the change of variables $\varrho=\rho-\bar\rho, v=u, \phi=\Phi$, then we can obtain

$$\begin{cases}\varrho_t+\bar\rho\operatorname{div}v=-\operatorname{div}(\varrho v),\\ \quad(\bar\rho+\varrho)v_t-\mu\Delta v-(\mu+\lambda)\nabla\operatorname{div}v+P'(\bar\rho+\varrho)\nabla\varrho+(\bar\rho+\varrho)v-(\bar\rho+\varrho)\nabla\phi\\ =-(\bar\rho+\varrho)(v\cdot\nabla)v+(\bar\rho+\varrho)f,\\ \Delta\phi=\varrho.\end{cases}\tag{2.1}$$

Problems (1.1) and (2.1) are obviously equivalent. Thus, we will concentrate on system (2.1) in the following. Firstly, we concern the following regularized problem in a bounded domain

$$\begin{cases}\varrho_t+\bar\rho\operatorname{div}v-\epsilon\Delta\varrho=-\operatorname{div}(\varrho v),\\ \quad(\bar\rho+\varrho)v_t-\mu\Delta v-(\mu+\lambda)\nabla\operatorname{div}v+P'(\bar\rho+\varrho)\nabla\varrho+(\bar\rho+\varrho)v-(\bar\rho+\varrho)\nabla\phi\\ =-(\bar\rho+\varrho)(v\cdot\nabla)v+(\bar\rho+\varrho)f^L,\\ \Delta\phi=\varrho,\\ \int_{\Omega^L}\varrho dx=0,\end{cases}\tag{2.2}$$

where $\Omega^L=(-L,L)^N\subset\mathbb{R}^N, N=2,3,$ $f^L(x,t)$ is a time periodic function and an odd function on the space variable x with periodic boundary, satisfying

$$f^L\to f\ \text{in}\ W_2^{1,1}((0,T)\times\mathbb{R}^N),$$

and

$$\|f^L\|^2_{W_2^{1,1}((0,T)\times\Omega^L)}\le 2\|f\|^2_{W_2^{1,1}((0,T)\times\mathbb{R}^N)}.$$

Precisely, denote the solution space in bounded domain Ω^L and the whole space $\mathbb{R}^N, N=2,3$ by

$$\begin{aligned} X^L=\{(\rho,u,\nabla\Phi)(x,t):&(\rho,u,\nabla\Phi)\in L^\infty(0,T;H^2(\Omega^L));\ \ (\rho_t,u_t,\nabla\Phi_t)\in L^\infty(0,T;L^2(\Omega^L));\\ &\rho\in L^2(0,T;H^2(\Omega^L));\ \ (u,\nabla\Phi)\in L^2(0,T;H^3(\Omega^L));\\ &(\rho_t,u_t,\nabla\Phi_t)\in L^2(0,T;H^1(\Omega^L));\ \text{ and } (\rho,u,\nabla\Phi) \text{ satisfies } (a),(b),(c)\}, \end{aligned}$$

and

$$\begin{aligned} X=\{(\rho,u,\nabla\Phi)(x,t):&(\rho,u,\nabla\Phi)\in L^\infty(0,T;H^2(\mathbb{R}^N));\ \ (\rho_t,u_t,\nabla\Phi_t)\in L^\infty(0,T;L^2(\mathbb{R}^N));\\ &\rho\in L^2(0,T;H^2(\mathbb{R}^N));\ \ (u,\nabla\Phi)\in L^2(0,T;H^3(\mathbb{R}^N));\\ &(\rho_t,u_t,\nabla\Phi_t)\in L^2(0,T;H^1(\mathbb{R}^N));\ \text{ and } (\rho,u,\nabla\Phi) \text{ satisfies } (c)\}, \end{aligned}$$

with

(a) $(\rho,u,\nabla\Phi)$ is time periodic functions with periodic boundary condition;

(b) $\displaystyle\int_\Omega \rho(x,t)\,dx=0$;

(c) $\rho(x,t)=\rho(-x,t), u(x,t)=-u(-x,t), \nabla\Phi(x,t)=-\nabla\Phi(-x,t)$.

Set

$$X_a^L=\{(\rho,u,\nabla\Phi)\in X^L:\ |||(\rho,u,\nabla\Phi)|||<a\},$$

for some positive constant a and with the norm $|||\cdot|||$ defined as

$$\begin{aligned} |||(\rho,u,\nabla\Phi)|||^2=&\sup_{0\le t\le T}\left(\|(\rho,u,\nabla\Phi)(t)\|_{H^2}^2+\|(\rho,u,\nabla\Phi)_t(t)\|_{L^2}^2\right)\\ &+\int_0^T\left(\|\rho(t)\|_{H^2}^2+\|(u,\nabla\Phi)(t)\|_{H^3}^2+\|(\rho,u,\nabla\Phi)_t(t)\|_{H^1}^2\right)dt. \end{aligned}$$

Now, we have the following result for problem (2.2), the proof of this proposition will be given in the end of next section.

PROPOSITION 2.1. *Assume that $f^L(x,t)$ is a smooth function and $f^L(x,t)\in W_2^{1,1}((0,T)\times\Omega^L)$, with $f^L(-x,t)=-f^L(x,t)$. If*

$$\|f^L\|_{W_2^{1,1}}^2\le h^*,$$

for some small constant $h^*>0$, then the problem (2.2) admits a solution $(\varrho^L,v^L,\nabla\phi^L)\in X_{a_0}^L$, here a_0 is a small constant independent of L and ϵ.

Several elementary inequalities are needed later, cf. [7], here the coefficients independent of the domain play an important role in passing the limit of the approximate solutions in the last section.

LEMMA 2.1. *Assume that $\Omega\subset\mathbb{R}^N$ is a bounded domain, and $\partial\Omega$ is locally Lipschitz continuous. If $u|_{\partial\Omega}=0$ (or $\int_\Omega u\,dx=0$), then for any $1\le p<N, 1\le q\le p^*=\frac{Np}{N-p}$,*

$$\left(\int_\Omega |u|^q\,dx\right)^{1/q}\le C(N,p,q)|mes\Omega|^{\frac1q-\frac1{p^*}}\left(\int_\Omega|\nabla u|^p\,dx\right)^{1/p}.$$

In particular, if $q=p^*=\frac{Np}{N-p}$, *then*

$$\Big(\int_\Omega |u|^{p^*}\,dx\Big)^{1/p^*}\le C(N,p,q)\Big(\int_\Omega |\nabla u|^p\,dx\Big)^{1/p}.$$

LEMMA 2.2. *Assume that* $\Omega\subset\mathbb{R}^N$ *is a bounded domain, and* $\partial\Omega$ *is locally Lipschitz continuous. If* $u|_{\partial\Omega}=0$ *(or* $\int_\Omega u\,dx=0$*), then*

$$\|u\|_{L^3}\le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2},\qquad \text{for } N=3,$$

$$\|u\|_{L^4}\le C\|u\|_{L^2}^{1/4}\|\nabla u\|_{L^2}^{3/4},\quad \|u\|_{L^\infty}\le C\|\nabla u\|_{H^1},\qquad \text{for } N=3,$$

$$\|u\|_{L^4}\le C\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2},\quad \|u\|_{L^\infty}\le C\|u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2},\qquad \text{for } N=2,$$

where C *is independent of* Ω. *Moreover, the above inequalities also hold if* $u(x)\to 0$ *as* $|x|\to\infty$.

3. Existence in bounded domain

The task of this section is to give the proof of Proposition 2.1. The proof is based on some uniform estimates and the topological degree theory. For the sake of clarity, we divide this section into three subsections. To begin with, we introduce a completely continuous operator $\mathcal{F}$ to the problem (3.1).

3.1. Introduction of an operator $\mathcal{F}$. For any $\tau\in[0,1]$, we define an operator

$$\mathcal{F}: X_a^L\times[0,1]\to X^L,$$

$$((\rho,u,\nabla\Phi),\tau)\to(\varrho,v,\nabla\phi),$$

with a being suitably small, here $(\varrho,v,\nabla\phi)$ is the solution of the following system with periodic boundary

$$\begin{cases}\varrho_t+\bar\rho\,\mathrm{div}\,v-\epsilon\Delta\varrho=Q_1(\rho,u,\tau),\\ \quad(\bar\rho+\tau\rho)v_t-\mu\Delta v-(\mu+\lambda)\nabla\mathrm{div}\,v+\frac{P'(\bar\rho)}{\bar\rho}(\bar\rho+\tau\rho)\nabla\varrho+(\bar\rho+\tau\rho)v-(\bar\rho+\tau\rho)\nabla\phi\\ =Q_2(\rho,u,\tau)+\tau(\bar\rho+\tau\rho)f^L,\\ \Delta\phi=\varrho,\\ \int_{\Omega^L}\varrho\,dx=0,\end{cases}\tag{3.1}$$

where

$$Q_1(\rho,u,\tau)=-\tau\,\mathrm{div}(\rho u),$$
$$Q_2(\rho,u,\tau)=\Big(\frac{P'(\bar\rho)}{\bar\rho}(\bar\rho+\tau\rho)-P'(\bar\rho+\tau\rho)\Big)\nabla\rho-\tau(\bar\rho+\tau\rho)(u\cdot\nabla)u.$$

REMARK 3.1. To guarantee the uniqueness of solutions , we impose the condition $\int_{\Omega^L}\varrho\,dx=0$ here. Since $\frac{d}{dt}\int_{\Omega^L}\varrho\,dx=0$ implies that when $(\varrho,v,\nabla\phi)$ is a solution of system (3.1), then $(\varrho+c,v,\nabla\phi)$ is also a solution for any constant c.

In what follows, we will concentrate on the properties of the operator $\mathcal{F}$. To show $\mathcal{F}$ is well defined, we shall establish the following lemma first.

LEMMA 3.1. *Assume that a is sufficiently small, then for any $(\rho,u,\nabla\Phi)\in X_a^L, \tau\in[0,1]$, the problem* (3.1) *admits a unique time periodic solution $(\varrho,v,\nabla\phi)\in X^L$.*

Proof. By Lemma 2.2, we have

$$\|\rho\|_{L^\infty}\le C\|\rho\|_{H^2}\le Ca. \tag{3.2}$$

By the smallness of a, one can get

$$\frac{\bar\rho}{2}\le\bar\rho+\tau\rho\le 2\bar\rho, \tag{3.3}$$

that is,

$$\frac{1}{2\bar\rho}\le\frac{1}{\bar\rho+\tau\rho}\le\frac{2}{\bar\rho}. \tag{3.4}$$

Define

$$\mathbb{A}=\begin{pmatrix}\epsilon\Delta & -\bar\rho\,\mathrm{div}\\ -\frac{P'(\bar\rho)}{\bar\rho}\nabla+\nabla\Delta^{-1} & \frac{\mu}{\bar\rho+\tau\rho}\Delta+\frac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla\,\mathrm{div}-1\end{pmatrix},$$

and set $U=(\varrho,v)$, $W=(\rho,u)$, $Q(W)=(Q_1,Q_2)$, $F=(0,\tau f^L)$, then the system (3.1) takes the form

$$U_t=\mathbb{A}U+Q(W)+F.$$

Now, we begin with the following initial value problem of (3.1) in Ω^L, that is

$$\begin{cases}\varrho_t+\bar\rho\,\mathrm{div}\,v-\epsilon\Delta\varrho=0,\\ v_t-\frac{\mu}{\bar\rho+\tau\rho}\Delta v-\frac{\mu+\lambda}{\bar\rho+\tau\rho}\nabla\,\mathrm{div}\,v+\frac{P'(\bar\rho)}{\bar\rho}\nabla\varrho+v-\nabla\phi=0,\\ \Delta\phi=\varrho,\\ (\varrho,v)(x,0)=(\varrho_0,v_0)(x),\end{cases} \tag{3.5}$$

with periodic boundary condition and the initial data $\varrho_0(x)$ is an even function with $\int_{\Omega^L}\varrho_0\,dx=0$, v_0 is an odd function. Obviously, the solution (ϱ,v) has the same properties as the initial data (ϱ_0,v_0).

Multiplying $(3.5)_2$ by v and the integration over Ω^L yields

$$\begin{aligned}&\frac12\frac{d}{dt}\int_{\Omega^L}v^2\,dx+\int_{\Omega^L}\left(\frac{\mu}{\bar\rho+\tau\rho}|\nabla v|^2+\frac{\mu+\lambda}{\bar\rho+\tau\rho}|\mathrm{div}\,v|^2\right)dx+\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}v\nabla\varrho\,dx\\ &\quad+\int_{\Omega^L}v^2\,dx-\int_{\Omega^L}v\nabla\phi\,dx\\ &=\int_{\Omega^L}\frac{\tau\mu}{(\bar\rho+\tau\rho)^2}\nabla v\nabla\rho v\,dx+\int_{\Omega^L}\frac{\tau(\mu+\lambda)}{(\bar\rho+\tau\rho)^2}\mathrm{div}\,v\nabla\rho v\,dx\\ &=:J.\end{aligned}$$

For the last term at left-hand side of the above equality, we have from $(3.5)_1$ and $(3.5)_3$ that

$$
\begin{aligned}
-\int_{\Omega^L} v\nabla\phi\,dx &= \int_{\Omega^L} \phi\,\mathrm{div}\,v\,dx\\
&= \frac{1}{\bar\rho}\int_{\Omega^L}\phi(-\varrho_t+\epsilon\Delta\varrho)\,dx\\
&= \frac{1}{\bar\rho}\int_{\Omega^L}\phi(-\Delta\phi_t+\epsilon\Delta\Delta\phi)\,dx\\
&= \frac{1}{2\bar\rho}\frac{d}{dt}\int_{\Omega^L}|\nabla\phi|^2\,dx+\frac{\epsilon}{\bar\rho}\int_{\Omega^L}|\Delta\phi|^2\,dx.
\end{aligned}
$$

To estimate J, we have if $N=3$,

$$
\begin{aligned}
J &\le C\frac{\tau(2\mu+\lambda)}{(\bar\rho-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{L^3}\|\nabla v\|_{L^2}\|v\|_{L^6}\\
&\le C\frac{\tau(2\mu+\lambda)}{(\bar\rho-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{H^1}\|\nabla v\|_{L^2}^2.
\end{aligned}
$$

And if $N=2$,

$$
\begin{aligned}
J &\le C\frac{\tau(2\mu+\lambda)}{(\bar\rho-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{L^4}\|\nabla v\|_{L^2}\|v\|_{L^4}\\
&\le C\frac{\tau(2\mu+\lambda)}{(\bar\rho-\tau\|\rho\|_{L^\infty})^2}\|\nabla\rho\|_{H^1}\|\nabla v\|_{L^2}^{\frac{3}{2}}\|v\|_{L^2}^{\frac{1}{2}}\\
&\le C[\frac{\tau(2\mu+\lambda)}{(\bar\rho-\tau\|\rho\|_{L^\infty})^2}]^4\|\nabla\rho\|_{H^1}^4\|v\|_{L^2}^2+\frac{\mu}{8\bar\rho}\|\nabla v\|_{L^2}^2.
\end{aligned}
$$

In view of inequality (3.2), then for some suitably small constant a, we see that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}(v^2+\frac{1}{\bar\rho}|\nabla\phi|^2)\,dx+\int_{\Omega^L}\left(\frac{\mu}{8\bar\rho}|\nabla v|^2+\frac{\mu+\lambda}{2\bar\rho}|\mathrm{div}\,v|^2\right)dx+\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}v\nabla\varrho\,dx\\
&\quad+\frac{1}{2}\int_{\Omega^L}v^2\,dx+\frac{\epsilon}{\bar\rho}\int_{\Omega^L}|\Delta\phi|^2\,dx\le 0.
\end{aligned}
$$

Multiplying Equation $(3.5)_1$ by ϱ and integrating it to obtain

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\varrho^2\,dx+\epsilon\int_{\Omega^L}|\nabla\varrho|^2\,dx+\bar\rho\int_{\Omega^L}\varrho\,\mathrm{div}\,v\,dx=0.
$$

Summing up the above two inequalities, we derive

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho^2}\varrho^2+v^2+\frac{1}{\bar\rho}|\nabla\phi|^2\right)dx+\int_{\Omega^L}\left(\frac{\mu}{8\bar\rho}|\nabla v|^2+\frac{\mu+\lambda}{2\bar\rho}|\mathrm{div}\,v|^2\right)dx\\
&\quad+\frac{1}{2}\int_{\Omega^L}v^2\,dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho^2}\int_{\Omega^L}|\nabla\varrho|^2\,dx+\frac{\epsilon}{\bar\rho}\int_{\Omega^L}|\Delta\phi|^2\,dx\le 0. \qquad (3.6)
\end{aligned}
$$

On the other hand, multiplying Equation $(3.5)_2$ by $\mu\Delta v+(\mu+\lambda)\nabla\,\mathrm{div}\,v$, we deduce that

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}(\mu|\nabla v|^2+(\mu+\lambda)|\mathrm{div}\,v|^2)\,dx+\int_{\Omega^L}\frac{1}{\bar\rho+\tau\rho}(\mu\Delta v+(\mu+\lambda)\nabla\,\mathrm{div}\,v)^2\,dx
$$

$$+\mu\int_{\Omega^L}|\nabla v|^2\,dx+(\mu+\lambda)\int_{\Omega^L}|\mathrm{div}\,v|^2\,dx-(2\mu+\lambda)\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}\nabla\varrho\nabla\,\mathrm{div}\,v\,dx$$
$$+(2\mu+\lambda)\int_{\Omega^L}\nabla\phi\nabla\,\mathrm{div}\,v\,dx=0. \tag{3.7}$$

The last term at left-hand side of Equation (3.7) can be estimated by Equations $(3.5)_1$ and $(3.5)_3$ that

$$\begin{aligned}\int_{\Omega^L}\nabla\phi\nabla\,\mathrm{div}\,v\,dx&=-\int_{\Omega^L}\Delta\phi\,\mathrm{div}\,v\,dx\\&=\frac{1}{\bar\rho}\int_{\Omega^L}\Delta\phi(\varrho_t-\epsilon\Delta\varrho)\,dx\\&=\frac{1}{2\bar\rho}\frac{d}{dt}\int_{\Omega^L}|\Delta\phi|^2\,dx+\frac{\epsilon}{\bar\rho}\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx.\end{aligned} \tag{3.8}$$

Applying ∇ to Equation $(3.5)_1$ and multiplying the resultant identity by $\nabla\varrho$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\varrho|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\varrho|^2\,dx+\bar\rho\int_{\Omega^L}\nabla\varrho\nabla\,\mathrm{div}\,v\,dx=0. \tag{3.9}$$

From the estimates Equations (3.7)–(3.9), we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left((2\mu+\lambda)\frac{P'(\bar\rho)}{\bar\rho^2}|\nabla\varrho|^2+\mu|\nabla v|^2+(\mu+\lambda)|\mathrm{div}\,v|^2+\frac{2\mu+\lambda}{\bar\rho}|\Delta\phi|^2\right)dx\\&\quad+\int_{\Omega^L}\frac{1}{\bar\rho+\tau\rho}\left(\mu\Delta v+(\mu+\lambda)\nabla\,\mathrm{div}\,v\right)^2dx+\int_{\Omega^L}\left(\mu|\nabla v^2|+(\mu+\lambda)|\mathrm{div}\,v|^2\right)dx\\&\quad+\epsilon(2\mu+\lambda)\frac{P'(\bar\rho)}{\bar\rho^2}\int_{\Omega^L}|\Delta\varrho|^2\,dx+(2\mu+\lambda)\frac{\epsilon}{\bar\rho}\int_{\Omega^L}|\nabla\Delta\phi|^2=0.\end{aligned} \tag{3.10}$$

This together with inequality (3.6) and the Poincaré inequality imply

$$\|(\varrho,v)(x,t)\|_{H^1}\le\|(\varrho_0,v_0)\|_{H^1}e^{-C\epsilon t},$$

that is,

$$\|e^{t\mathbb{A}}U_0\|_{H^1}\le\|U_0\|_{H^1}e^{-C\epsilon t}.$$

Then by Duhamel's principle, the solution to the system (3.1) can be written in a mild form as

$$U(t)=\int_{-\infty}^{t}e^{(t-s)\mathbb{A}}\big(Q(W)(s)+F(s)\big)\,ds,$$

and it satisfies that

$$\begin{aligned}\|U(t)\|_{H^1}&\le\int_{-\infty}^{t}\|e^{(t-s)\mathbb{A}}\big(Q(W)(s)+F(s)\big)\|_{H^1}\,ds\\&\le\int_{-\infty}^{t}e^{-C\epsilon(t-s)}\|Q(W)(s)+F(s)\|_{H^1}\,ds\\&=\sum_{i=0}^{\infty}\int_{t-(i+1)T}^{t-iT}e^{-C\epsilon(t-s)}\|Q(W)(s)+F(s)\|_{H^1}\,ds\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{\infty} \int_0^T e^{-C\epsilon((i+1)T-s)} \|Q(W)(s+t)+F(s+t)\|_{H^1}\, ds \\
&\le \sum_{i=0}^{\infty} \left(\int_0^T e^{-2C\epsilon((i+1)T-s)}\, ds \right)^{\frac{1}{2}} \left(\int_0^T \|Q(W)(s)+F(s)\|_{H^1}^2\, ds \right)^{\frac{1}{2}} \\
&\le C \left(\int_0^T \|Q(W)(s)+F(s)\|_{H^1}^2\, ds \right)^{\frac{1}{2}},
\end{aligned}
$$

where we have used the time periodic property of W and F, also we have

$$
\begin{aligned}
U(t+T) &= \int_{-\infty}^{t+T} e^{(t+T-s)\mathbb{A}} \big(Q(W)(s)+F(s)\big)\, ds \\
&= \int_{-\infty}^{t+T} e^{(t-(s-T))\mathbb{A}} \big(Q(W)(s-T)+F(s-T)\big)\, ds \\
&= \int_{-\infty}^{t} e^{(t-s)\mathbb{A}} \big(Q(W)(s)+F(s)\big)\, ds = U(t).
\end{aligned}
$$

That is, $U(t) \in L^\infty(0,T;H^1(\Omega^L))$ is a time periodic solution of problem (3.1) with time period T.

Moreover, by the classical theory of parabolic and elliptic equations, we have that for any $(\rho,u,\nabla\Phi) \in X_a^L, \tau \in [0,1]$, the problem (3.1) admits a time periodic solution $(\varrho,v,\nabla\phi) \in X^L$. To prove the uniqueness, assume that there exist two solutions $U_1 = (\varrho_1,v_1,\nabla\phi_1), U_2 = (\varrho_2,v_2,\nabla\phi_2)$ for some $(\rho,u,\nabla\Phi) \in X_a^L, \tau \in [0,1]$, then we have

$$
(U_1 - U_2)_t = \mathbb{A}(U_1 - U_2).
$$

Similar to the first half of this proof, Let $\varrho = \varrho_1 - \varrho_2, v = v_1 - v_2, \nabla\phi = \nabla\phi_1 - \nabla\phi_2$, we have

$$
\begin{aligned}
&\int_0^T \int_{\Omega^L} \left(\frac{\mu}{8\bar{\rho}} |\nabla v|^2 + \frac{\mu+\lambda}{2\bar{\rho}} |\operatorname{div} v|^2 \right) dx\, dt + \epsilon \frac{P'(\bar{\rho})}{\bar{\rho}^2} \int_0^T \int_{\Omega^L} |\nabla \varrho|^2\, dx\, dt \\
&\quad + \frac{1}{2} \int_0^T \int_{\Omega^L} v^2\, dx\, dt + \frac{\epsilon}{\bar{\rho}} \int_0^T \int_{\Omega^L} |\Delta\phi|^2\, dx\, dt \le 0.
\end{aligned}
$$

By Poincaré inequality, $(\varrho,v,\nabla\phi) = (0,0,0)$ which implies the uniqueness. Finally, if$(\varrho(x,t),\ v(x,t),\nabla\phi(x,t))$ is the periodic solution of problem (3.1), then $(\varrho(-x,t),-v(-x,t),-\nabla\phi(-x,t))$ is also the solution of problem (3.1), by the uniqueness, we easily obtain $(\varrho(x,t),v(x,t),\nabla\phi(x,t)) = (\varrho(-x,t),\ -v(-x,t),-\nabla\phi(-x,t))$. This completes the proof of this Lemma. □

Now, in the following lemma, we see that the operator $\mathcal{F}$ is completely continuous. The proof is similar to Lemma 2.2 and Lemma 2.3 in [6].

Lemma 3.2. *Assume that a is sufficiently small, then the operator $\mathcal{F}$ is compact and continuous.*

3.2. Uniform estimates. In this subsection, we focus on giving several uniform estimates for $(\varrho,v,\nabla\phi)$ of the following system, which play a crucial role in the proof of

Proposition 2.1 and the main theorem.

$$\begin{cases} \varrho_t+\bar{\rho}\operatorname{div} v-\epsilon\Delta\varrho=-\tau\operatorname{div}(\varrho v), \\ \quad(\bar{\rho}+\tau\varrho)v_t-\mu\Delta v-(\mu+\lambda)\nabla\operatorname{div} v+P'(\bar{\rho}+\tau\varrho)\nabla\varrho+(\bar{\rho}+\tau\varrho)v-(\bar{\rho}+\tau\varrho)\nabla\phi \\ =-\tau(\bar{\rho}+\tau\varrho)(v\cdot\nabla)v+\tau(\bar{\rho}+\tau\varrho)f, \\ \Delta\phi=\varrho, \\ \int_{\Omega^L}\varrho\,dx=0, \end{cases} \tag{3.11}$$

where $\tau\in(0,1]$. Since when $\tau=0$, similar to the proof given in the section 3 of [7], it is easy to obtain that $\mathcal{F}((\rho,u,\nabla\Phi),0)=0$.

By elaborate calculation, the uniform estimates for $(\varrho,v,\nabla\phi)$ independent of L and ϵ are derived as follows.

Lemma 3.3. *Let $\tau\in(0,1]$, if $|\varrho|\le\frac{\bar{\rho}}{2}$, then the solution $(\varrho,v,\nabla\phi)\in X^L$ to the system* (3.11) *satisfies*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar{\rho})}{\bar{\rho}}\varrho^2+(\bar{\rho}+\tau\varrho)v^2+|\nabla\phi|^2\right)dx+\frac{1}{2}\int_{\Omega^L}(\bar{\rho}+\tau\varrho)v^2\,dx \\ &\qquad+\int_{\Omega^L}\left(\mu|\nabla v|^2+\frac{\mu+\lambda}{2}|\operatorname{div} v|^2\right)dx+\frac{\epsilon}{2}\frac{P'(\bar{\rho})}{\bar{\rho}}\int_{\Omega^L}|\nabla\varrho|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\phi|^2\,dx \\ &\le C\tau\|\varrho\|_{H^1}^2\|\nabla\varrho\|_{H^1}^2+C\tau\epsilon\|v\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|f^L\|_{L^2}^2, \end{aligned} \tag{3.12}$$

where C is a constant independent of L and ϵ.

Proof. Multiplying Equation $(3.11)_2$ by v and integrating it over Ω^L by parts, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}(\bar{\rho}+\tau\varrho)v^2\,dx+\int_{\Omega^L}\left(\mu|\nabla v|^2+(\mu+\lambda)|\operatorname{div} v|^2\right)dx+P'(\bar{\rho})\int_{\Omega^L}v\nabla\varrho\,dx \\ &\qquad+\int_{\Omega^L}(\bar{\rho}+\tau\varrho)v^2\,dx-\int_{\Omega^L}(\bar{\rho}+\tau\rho)v\nabla\phi\,dx \\ &=\int_{\Omega^L}(P'(\bar{\rho})-P'(\bar{\rho}+\tau\varrho))v\nabla\varrho\,dx+\frac{\tau\epsilon}{2}\int_{\Omega^L}\Delta\varrho v^2\,dx+\tau\int_{\Omega^L}(\bar{\rho}+\tau\varrho)f^Lv\,dx, \end{aligned} \tag{3.13}$$

where we have used the Equation $(3.11)_1$ and the periodic boundary conditions. While for the last term at left-hand side of Equation (3.13), by Equation $(3.11)_1$, the Poisson Equation $(3.11)_3$ and the integration by parts, we get

$$\begin{aligned} -\int_{\Omega^L}(\bar{\rho}+\tau\rho)v\nabla\phi\,dx&=\int_{\Omega^L}\phi\operatorname{div}[(\bar{\rho}+\tau\rho)v]\,dx \\ &=\int_{\Omega^L}\phi(-\varrho_t+\epsilon\Delta\varrho)\,dx \\ &=\int_{\Omega^L}\phi(-\Delta\phi_t+\epsilon\Delta\Delta\phi)\,dx \\ &=\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\phi|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\phi|^2\,dx. \end{aligned} \tag{3.14}$$

Similarly, by multiplying Equation $(3.11)_1$ with ϱ and integrating it over Ω^L by parts, one can get that

$$\frac{d}{dt}\int_{\Omega^L}\varrho^2\,dx+\epsilon\int_{\Omega^L}|\nabla\varrho|^2\,dx+\bar{\rho}\int_{\Omega^L}\varrho\operatorname{div} v\,dx=-\tau\int_{\Omega^L}\operatorname{div}(\varrho v)\varrho\,dx. \tag{3.15}$$

This estimate Equation (3.15) together with Equations (3.13) and (3.14) imply that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho}\varrho^2+(\bar\rho+\tau\varrho)v^2+|\nabla\phi|^2\right)dx+\int_{\Omega^L}(\bar\rho+\tau\varrho)v^2\,dx\\
&\quad+\int_{\Omega^L}\left(\mu|\nabla v|^2+(\mu+\lambda)|\operatorname{div}v|^2\right)dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}|\nabla\varrho|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\phi|^2\,dx\\
&=-\tau\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}\operatorname{div}(\varrho v)\varrho\,dx+\int_{\Omega^L}(P'(\bar\rho)-P'(\bar\rho+\tau\varrho))v\nabla\varrho\,dx-\frac{\tau\epsilon}{2}\int_{\Omega^L}\nabla\varrho\nabla v^2\,dx\\
&\quad+\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)f^Lv\,dx\\
&\le C\tau\|\varrho\|_{L^4}^2\|\operatorname{div}v\|_{L^2}+C\tau\|\varrho\|_{L^4}^2\|\nabla\varrho\|_{L^4}^2+C\tau\epsilon\|v\|_{L^4}\|\nabla v\|_{L^4}\|\nabla\varrho\|_{L^2}\\
&\quad+C\tau\|f^L\|_{L^2}^2+\frac{1}{2}\int_{\Omega^L}(\bar\rho+\tau\varrho)v^2\,dx\\
&\le C\tau\|\varrho\|_{H^1}^2\|\nabla\varrho\|_{H^1}^2+C\tau\epsilon\|v\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|f^L\|_{L^2}^2+\epsilon\frac{P'(\bar\rho)}{2\bar\rho}\|\nabla\varrho\|_{L^2}^2\\
&\quad+\frac{\mu+\lambda}{2}\|\operatorname{div}v\|_{L^2}^2+\frac{1}{2}\int_{\Omega^L}(\bar\rho+\tau\varrho)v^2\,dx,
\end{aligned}
$$

where we have used Lemma 2.2, Hölder, Young inequalities and the fact that $(P'(\bar\rho)-P'(\bar\rho+\tau\varrho))\sim\tau\varrho$. Hence, the estimate (3.12) follows from the above inequality immediately. This completes the proof of this lemma. ∎

LEMMA 3.4. *Under the same conditions in Lemma 3.3, we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho|^2+\bar\rho|\nabla v|^2+|\Delta\phi|^2\right)dx+\frac{\bar\rho}{2}\int_{\Omega^L}|\nabla v|^2\,dx+\frac{\mu}{2}\int_{\Omega^L}|\Delta v|^2\,dx\\
&\quad+\frac{\mu+\lambda}{2}\int_{\Omega^L}|\nabla\operatorname{div}v|^2\,dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}|\Delta\varrho|^2\,dx+\epsilon\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx\\
&\le C\tau\|\varrho\|_{H^2}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|\varrho\|_{H^2}^2\|v_t\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\nabla v\|_{L^2}^2\\
&\quad+C\tau\|\varrho\|_{H^2}^2\|\nabla\phi\|_{H^2}^2+C\tau\|f^L\|_{L^2}^2+\xi_1\|v\|_{L^2}^2,
\end{aligned}\tag{3.16}
$$

where C,ξ_1 are constants independent of L and ϵ. Moreover, ξ_1 can be chosen to be arbitrarily small.

Proof. Multiplying Equation $(3.11)_2$ by Δv and then integrating it over Ω^L by parts to obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\bar\rho|\nabla v|^2\,dx+\int_{\Omega^L}\left(\mu|\Delta v|^2+(\mu+\lambda)|\nabla\operatorname{div}v|^2+\bar\rho|\nabla v|^2\right)dx\\
&=P'(\bar\rho)\int_{\Omega^L}\nabla\varrho\Delta v\,dx+\tau\int_{\Omega^L}\varrho v_t\Delta v\,dx+\tau\int_{\Omega^L}\varrho v\Delta v\,dx-\int_{\Omega^L}(\bar\rho+\tau\varrho)\nabla\phi\Delta v\,dx\\
&\quad+\int_{\Omega^L}\Big(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\Big)\nabla\varrho\Delta v\,dx+\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)(v\cdot\nabla)v\Delta v\,dx\\
&\quad-\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)f^L\Delta v\,dx.
\end{aligned}\tag{3.17}
$$

For the Poisson term, by Equations $(3.11)_1$ and $(3.11)_3$, we find

$$
\int_{\Omega^L}(\bar\rho+\tau\rho)\nabla\phi\Delta v\,dx=-\bar\rho\int_{\Omega^L}\Delta\phi\operatorname{div}v\,dx+\tau\int_{\Omega^L}\varrho\nabla\phi\Delta v\,dx
$$

$$
\begin{aligned}
&=\int_{\Omega^L}\Delta\phi(\varrho_t-\epsilon\Delta\varrho+\tau\operatorname{div}(\varrho v))\,dx+\tau\int_{\Omega^L}\varrho\nabla\phi\Delta v\,dx\\
&=\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\Delta\phi|^2\,dx+\epsilon\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx+\tau\int_{\Omega^L}\Delta\phi\operatorname{div}(\varrho v)\,dx+\tau\int_{\Omega^L}\varrho\nabla\phi\Delta v\,dx. \quad (3.18)
\end{aligned}
$$

While for the first term at right-hand side of Equation (3.17), it follows from the fact $\Delta\operatorname{div}v=\operatorname{div}\Delta v$ that

$$
\int_{\Omega^L}\nabla\varrho\Delta v\,dx=-\int_{\Omega^L}\varrho\Delta\operatorname{div}v\,dx=\int_{\Omega^L}\nabla\varrho\nabla\operatorname{div}v\,dx.
$$

Then applying the operator ∇ to Equation $(3.11)_1$, and multiplying the resultant equation by $\nabla\varrho$ yield

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\varrho|^2\,dx+\bar\rho\int_{\Omega^L}\nabla\operatorname{div}v\nabla\varrho\,dx+\epsilon\int_{\Omega^L}|\Delta\varrho|^2\,dx=-\tau\int_{\Omega^L}\nabla\operatorname{div}(\varrho v)\nabla\varrho\,dx\\
&=-\tau\int_{\Omega^L}\left(\frac{1}{2}|\nabla\varrho|^2\operatorname{div}v+\nabla\varrho\nabla v\nabla\varrho+\varrho\nabla\varrho\nabla\operatorname{div}v\right)dx. \quad (3.19)
\end{aligned}
$$

In light of Equations (3.17)–(3.19), we see that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho|^2+\bar\rho|\nabla v|^2+|\Delta\phi|^2\right)dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}|\Delta\varrho|^2\,dx+\mu\int_{\Omega^L}|\Delta v|^2\,dx\\
&\quad+(\mu+\lambda)\int_{\Omega^L}|\nabla\operatorname{div}v|^2\,dx+\bar\rho\int_{\Omega^L}|\nabla v|^2\,dx+\epsilon\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx\\
&=-\frac{\tau P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}\left(\frac{1}{2}|\nabla\varrho|^2\operatorname{div}v+\nabla\varrho\nabla v\nabla\varrho+\varrho\nabla\varrho\nabla\operatorname{div}v\right)dx+\tau\int_{\Omega^L}\varrho v_t\Delta v\,dx\\
&\quad+\int_{\Omega^L}\Big(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\Big)\nabla\varrho\Delta v\,dx+\tau\int_{\Omega^L}\varrho v\Delta v\,dx+\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)(v\cdot\nabla)v\Delta v\,dx\\
&\quad-\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)f^L\Delta v\,dx-\tau\int_{\Omega^L}\Delta\phi\operatorname{div}(\varrho v)\,dx-\tau\int_{\Omega^L}\varrho\nabla\phi\Delta v\,dx\\
&\le C\tau\|\nabla\varrho\|_{L^4}^2\|\nabla v\|_{L^2}+C\tau\|\varrho\|_{L^4}\|\nabla\varrho\|_{L^4}\|\nabla\operatorname{div}v\|_{L^2}+C\tau\|\varrho\|_{H^2}\|v_t\|_{L^2}\|\Delta v\|_{L^2}\\
&\quad+C\tau\|\varrho\|_{L^4}\|\nabla\varrho\|_{L^4}\|\Delta v\|_{L^2}+C\tau\|\varrho\|_{L^4}\|v\|_{L^4}\|\Delta v\|_{L^2}+C\tau\|v\|_{H^2}\|\nabla v\|_{L^2}\|\Delta v\|_{L^2}\\
&\quad+C\tau\|\varrho\|_{L^4}\|\nabla v\|_{L^2}\|\Delta\phi\|_{L^4}+C\tau\|\nabla\varrho\|_{L^4}\|v\|_{L^2}\|\Delta\phi\|_{L^4}+C\tau\|\varrho\|_{L^4}\|\Delta v\|_{L^2}\|\nabla\phi\|_{L^4}\\
&\quad+C\tau\|f^L\|_{L^2}\|\Delta v\|_{L^2}\\
&\le C\tau\|\varrho\|_{H^2}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|\varrho\|_{H^2}^2\|v_t\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\nabla v\|_{L^2}^2\\
&\quad+C\tau\|\varrho\|_{H^1}^2\|\Delta\phi\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\phi\|_{H^1}^2+\xi_1\|v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|\nabla\phi\|_{H^1}^2\\
&\quad+C\tau\|f^L\|_{L^2}^2+\frac{\bar\rho}{2}\|\nabla v\|_{L^2}^2+\frac{\mu}{2}\|\Delta v\|_{L^2}^2+\frac{\mu+\lambda}{2}\|\nabla\operatorname{div}v\|_{L^2}^2.
\end{aligned}
$$

This gives the estimate (3.16) and completes the proof of this lemma. □

LEMMA 3.5. *Under the same conditions in Lemma 3.3, we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(D_0\frac{P'(\bar\rho)}{\bar\rho}\varrho_t^2+D_0(\bar\rho+\tau\varrho)v_t^2+\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\bar\rho|\nabla\operatorname{div}v|^2+D_0|\nabla\phi_t|^2\right)dx\\
&\quad+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx+\frac{D_0\mu}{4}\int_{\Omega^L}|\nabla v_t|^2\,dx+\frac{(\mu+\lambda)D_0}{2}\int_{\Omega^L}|\operatorname{div}v_t|^2\,dx\\
&\quad+\frac{D_0}{2}\int_{\Omega^L}(\bar\rho+\tau\varrho)v_t^2\,dx+\frac{(\mu+\lambda)}{2}\int_{\Omega^L}|\Delta\operatorname{div}v|^2\,dx+\frac{\mu}{2}\int_{\Omega^L}|\operatorname{curl}\Delta v|^2\,dx
\end{aligned}
$$

$$
\begin{aligned}
&+\bar{\rho}\int_{\Omega^L}|\nabla \operatorname{div} v|^2\,dx+\epsilon D_0\int_{\Omega^L}|\Delta\phi_t|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\Delta\phi|^2\\
&+\epsilon\frac{P'(\bar{\rho})}{\bar{\rho}}D_0\int_{\Omega^L}|\nabla\varrho_t|^2\,dx+\epsilon\frac{P'(\bar{\rho})}{\bar{\rho}}\int_{\Omega^L}|\nabla\Delta\varrho|^2\,dx\\
\le& C\tau\|\varrho_t\|_{H^1}^2\|\varrho\|_{H^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|v_t\|_{L^2}^2+C\tau\|\varrho_t\|_{L^2}^2\|v\|_{H^2}^2+C\tau\|v_t\|_{H^1}^2\|v\|_{H^2}^2\\
&+C\tau\|\varrho_t\|_{H^1}^2\|\nabla\phi_t\|_{H^1}^2+C\tau\|\varrho_t\|_{L^2}^2\|\nabla\phi\|_{H^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2\\
&+C\tau\|\varrho\|_{H^2}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\varrho\|_{H^2}^2\|v_t\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|\varrho\|_{H^2}^2\|\nabla\phi\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla^3 v\|_{L^2}^2\\
&+C\tau\|\varrho_t\|_{L^2}^4+C\tau\|\nabla\varrho\|_{H^1}^4+\xi_2\|v\|_{H^2}^2+C\|\nabla v\|_{L^2}^2+C\tau\|f_t^L\|_{L^2}^2+C\tau\|f^L\|_{H^1}^2. \quad (3.20)
\end{aligned}
$$

where C,D_0,ξ_2 are constants independent of L and ϵ. Moreover, D_0 can be chosen to be suitably large, and ξ_2 can be chosen to be arbitrarily small.

Proof. Applying ∂_t to Equation $(3.11)_1$, we have

$$
\varrho_{tt}+\bar{\rho}\operatorname{div} v_t-\epsilon\Delta\varrho_t=-\tau\varrho_t\operatorname{div} v-\tau\varrho\operatorname{div} v_t-\tau v_t\nabla\varrho-\tau v\nabla\varrho_t. \quad (3.21)
$$

Multiplying the above equation by ϱ_t, integrating it by parts over Ω^L to conclude that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\varrho_t^2\,dx+\epsilon\int_{\Omega^L}|\nabla\varrho_t|^2\,dx+\bar{\rho}\int_{\Omega^L}\varrho_t\operatorname{div} v_t\,dx\\
=&-\frac{\tau}{2}\int_{\Omega^L}\varrho_t^2\operatorname{div} v\,dx-\tau\int_{\Omega^L}\nabla\varrho\varrho_t v_t\,dx-\tau\int_{\Omega^L}\varrho\varrho_t\operatorname{div} v_t\,dx. \quad (3.22)
\end{aligned}
$$

Applying ∂_t to Equation $(3.11)_2$, and multiplying the resultant equation by v_t, then integrating it by parts over Ω^L, it holds that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}(\bar{\rho}+\tau\varrho)v_t^2\,dx+\int_{\Omega^L}\left(\mu|\nabla v_t|^2+(\mu+\lambda)|\operatorname{div} v_t|^2\right)dx+\int_{\Omega^L}(\bar{\rho}+\tau\varrho)v_t^2\,dx\\
&-\int_{\Omega^L}(\bar{\rho}+\tau\varrho)\nabla\phi_t v_t\,dx-P'(\bar{\rho})\int_{\Omega^L}\varrho_t\operatorname{div} v_t\,dx\\
=&\int_{\Omega^L}\Big(P'(\bar{\rho}+\tau\varrho)-P'(\bar{\rho})\Big)\varrho_t\operatorname{div} v_t\,dx-\frac{\tau}{2}\int_{\Omega^L}\varrho_t v_t^2\,dx-\tau\int_{\Omega^L}\varrho_t v v_t\,dx\\
&+\tau\int_{\Omega^L}\varrho_t\nabla\phi v_t\,dx-\tau^2\int_{\Omega^L}\varrho_t(v\cdot\nabla)v\cdot v_t\,dx-\tau\int_{\Omega^L}(\bar{\rho}+\tau\varrho)(v_t\cdot\nabla)v\cdot v_t\,dx\\
&-\tau\int_{\Omega^L}(\bar{\rho}+\tau\varrho)(v\cdot\nabla)v_t\cdot v_t\,dx+\tau\int_{\Omega^L}(\bar{\rho}+\tau\varrho)f_t^L v_t\,dx+\tau^2\int_{\Omega^L}\varrho_t f^L v_t\,dx. \quad (3.23)
\end{aligned}
$$

In view of Equation (3.21), we may bound the Poisson term at left-hand side of Equation (3.23) as

$$
\begin{aligned}
-\int_{\Omega^L}(\bar{\rho}+\tau\varrho)\nabla\phi_t v_t\,dx&=\int_{\Omega^L}\phi_t\operatorname{div}((\bar{\rho}+\tau\varrho)v_t)\,dx\\
&=\int_{\Omega^L}\phi_t(-\varrho_{tt}+\epsilon\Delta\varrho-\tau\operatorname{div}(\varrho_t v))\,dx\\
&=\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\phi_t|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\phi_t|^2\,dx+\tau\int_{\Omega^L}\varrho_t v\nabla\phi_t\,dx. \quad (3.24)
\end{aligned}
$$

Combing Equations (3.22)–(3.24), we obtain

$$
\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar{\rho})}{\bar{\rho}}\varrho_t^2+(\bar{\rho}+\tau\varrho)v_t^2+|\nabla\phi_t|^2\right)dx+\epsilon\int_{\Omega^L}|\Delta\phi_t|^2\,dx+\mu\int_{\Omega^L}|\nabla v_t|^2\,dx
$$

$$
\begin{aligned}
&+(\mu+\lambda)\int_{\Omega^L}|\operatorname{div}v_t|^2\,dx+\int_{\Omega^L}(\bar\rho+\tau\varrho)v_t^2\,dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}|\nabla\varrho_t|^2\,dx\\
=&-\tau\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}\left(\frac12\varrho_t^2\operatorname{div}v+\nabla\varrho\varrho_t v_t+\varrho\varrho_t\operatorname{div}v_t\right)dx-\frac{\tau}{2}\int_{\Omega^L}\varrho_t v_t^2\,dx-\tau\int_{\Omega^L}\varrho_t v v_t\,dx\\
&+\int_{\Omega^L}\Big(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\Big)\varrho_t\operatorname{div}v_t\,dx-\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)(v_t\cdot\nabla)v\cdot v_t\,dx\\
&-\tau^2\int_{\Omega^L}\varrho_t(v\cdot\nabla)v\cdot v_t\,dx-\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)(v\cdot\nabla)v_t\cdot v_t\,dx-\tau\int_{\Omega^L}\varrho_t v\nabla\phi_t\,dx\\
&+\tau\int_{\Omega^L}\varrho_t\nabla\phi v_t\,dx+\tau^2\int_{\Omega^L}\varrho_t f^L v_t\,dx+\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)f_t^L v_t\,dx\\
\le& C\tau\|\varrho_t\|_{L^2}^2\|\operatorname{div}v\|_{H^2}+C\tau\|\varrho_t\|_{L^4}^2\|\nabla\varrho\|_{L^4}^2+C\tau\|\varrho_t\|_{L^2}\|\varrho\|_{H^2}\|\operatorname{div}v_t\|_{L^2}\\
&+\left\{\begin{array}{ll} C\tau\|\varrho_t\|_{L^2}\|v_t\|_{L^4}^2 & (\text{if } N=2)\\ C\tau\|\varrho_t\|_{L^3}\|v_t\|_{L^2}\|v_t\|_{L^6} & (\text{if } N=3)\end{array}\right\}+C\tau\|\varrho_t\|_{L^2}\|v\|_{H^2}\|v_t\|_{L^2}\\
&+C\tau\|v_t\|_{L^2}\|\nabla v\|_{L^4}\|v_t\|_{L^4}+C\tau\|\varrho_t\|_{L^4}\|v\|_{H^2}\|\nabla v\|_{L^4}\|v_t\|_{L^2}+C\tau\|v\|_{L^4}\|\nabla v_t\|_{L^2}\|v_t\|_{L^4}\\
&+C\tau\|\varrho_t\|_{L^4}\|v\|_{L^2}\|\nabla\phi_t\|_{L^4}+C\tau\|\varrho_t\|_{L^2}\|\nabla\phi\|_{H^2}\|v_t\|_{L^2}+C\tau\|\varrho_t\|_{L^4}\|f^L\|_{L^4}\|v_t\|_{L^2}\\
&+C\tau\|f_t^L\|_{L^2}^2+\frac14\int_{\Omega^L}(\bar\rho+\tau\varrho)v_t^2\,dx\\
\le& C\tau\|\varrho_t\|_{L^2}^2\|\operatorname{div}v\|_{H^2}+C\tau\|\varrho_t\|_{H^1}^2\|\varrho\|_{H^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|v_t\|_{L^2}^2+C\tau\|\varrho_t\|_{L^2}^2\|v\|_{H^2}^2\\
&+C\tau\|v_t\|_{H^1}^2\|v\|_{H^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C_{\xi_3}\tau\|\varrho_t\|_{H^1}^2\|\nabla\phi_t\|_{H^1}^2\\
&+C\tau\|\varrho_t\|_{L^2}^2\|\nabla\phi\|_{H^2}^2+C\tau\|f^L\|_{H^1}^2+C\tau\|f_t^L\|_{L^2}^2+\frac{\mu}{2}\|\nabla v_t\|_{L^2}^2+\frac{(\mu+\lambda)}{2}\|\operatorname{div}v_t\|_{L^2}^2\\
&+\frac12\int_{\Omega^L}(\bar\rho+\tau\varrho)v_t^2\,dx+\xi_3\|v\|_{L^2}^2,
\end{aligned}
$$

where ξ_3, C_{ξ_3} are constants independent of L and ϵ, and ξ_3 can be chosen to be arbitrarily small. Thus the above estimate implies that

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho}\varrho_t^2+(\bar\rho+\tau\varrho)v_t^2+|\nabla\phi_t|^2\right)dx+\epsilon\int_{\Omega^L}|\Delta\phi_t|^2\,dx+\frac{\mu}{2}\int_{\Omega^L}|\nabla v_t|^2\,dx\\
&\quad+\frac{\mu+\lambda}{2}\int_{\Omega^L}|\operatorname{div}v_t|^2\,dx+\frac12\int_{\Omega^L}(\bar\rho+\tau\varrho)v_t^2\,dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}|\nabla\varrho_t|^2\,dx\\
&\le C\tau\|\varrho_t\|_{L^2}^2\|\operatorname{div}v\|_{H^2}+C\tau\|\varrho_t\|_{H^1}^2\|\varrho\|_{H^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|v_t\|_{L^2}^2+C\tau\|\varrho_t\|_{L^2}^2\|v\|_{H^2}^2\\
&\quad+C\tau\|v_t\|_{H^1}^2\|v\|_{H^2}^2+C\tau\|\varrho_t\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C_{\xi_3}\tau\|\varrho_t\|_{H^1}^2\|\nabla\phi_t\|_{H^1}^2\\
&\quad+C\tau\|\varrho_t\|_{L^2}^2\|\nabla\phi\|_{H^2}^2+C\tau\|f^L\|_{H^1}^2+C\tau\|f_t^L\|_{L^2}^2+\xi_3\|v\|_{L^2}^2.
\end{aligned}\tag{3.25}
$$

On the other hand, applying the operator Δ to Equation $(3.11)_1$, then multiplying it by $\Delta\varrho$, we can arrive at

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\Omega^L}|\Delta\varrho|^2\,dx+\epsilon\int_{\Omega^L}|\nabla\Delta\varrho|^2\,dx+\bar\rho\int_{\Omega^L}\Delta\operatorname{div}v\Delta\varrho\,dx=-\tau\int_{\Omega^L}\Delta\operatorname{div}(\varrho v)\Delta\varrho\,dx\\
=&-\tau\int_{\Omega^L}\left(\frac12|\Delta\varrho|^2\operatorname{div}v+\nabla\varrho\Delta v\Delta\varrho+2\nabla^2\varrho\nabla v\Delta\varrho+\varrho\Delta\operatorname{div}v\Delta\varrho+2\nabla\varrho\nabla\operatorname{div}v\Delta\varrho\right)dx.
\end{aligned}\tag{3.26}
$$

Then by multiplying Equation $(3.11)_2$ with $\nabla\Delta\operatorname{div}v$ and integrating it over Ω^L yields

$$
\frac{\bar\rho}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\operatorname{div}v|^2\,dx+(2\mu+\lambda)\int_{\Omega^L}|\Delta\operatorname{div}v|^2\,dx+\bar\rho\int_{\Omega^L}|\nabla\operatorname{div}v|^2\,dx
$$

$$
\begin{aligned}
&-P'(\bar\rho)\int_{\Omega^L}\Delta\varrho\Delta\operatorname{div}v\,dx+\bar\rho\int_{\Omega^L}\Delta\phi\Delta\operatorname{div}v\,dx\\
=&\tau\int_{\Omega^L}\varrho\operatorname{div}v_t\Delta\operatorname{div}v\,dx+\tau\int_{\Omega^L}\nabla\varrho v_t\Delta\operatorname{div}v\,dx+\tau\int_{\Omega^L}P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2\Delta\operatorname{div}v\,dx\\
&+\int_{\Omega^L}\Big(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\Big)\Delta\varrho\Delta\operatorname{div}v\,dx+\tau\int_{\Omega^L}\operatorname{div}(\varrho v)\Delta\operatorname{div}v\,dx\\
&+\tau\int_{\Omega^L}\operatorname{div}\left((\bar\rho+\tau\varrho)(v\cdot\nabla)v\right)\Delta\operatorname{div}v\,dx-\tau\int_{\Omega^L}\nabla\varrho\nabla\phi\Delta\operatorname{div}v\,dx\\
&-\tau\int_{\Omega^L}\varrho\Delta\phi\Delta\operatorname{div}v\,dx-\tau\int_{\Omega^L}\operatorname{div}\big((\bar\rho+\tau\varrho)f^L\big)\Delta\operatorname{div}v\,dx. \qquad (3.27)
\end{aligned}
$$

The Poisson term at left-hand side of Equation (3.27) may be estimated as

$$
\begin{aligned}
&\bar\rho\int_{\Omega^L}\Delta\phi\Delta\operatorname{div}v\,dx=\bar\rho\int_{\Omega^L}\Delta\Delta\phi\operatorname{div}v\,dx\\
=&\int_{\Omega^L}\Delta\Delta\phi(-\varrho_t+\epsilon\Delta\varrho-\tau\operatorname{div}(\varrho v))\,dx\\
=&\int_{\Omega^L}\Delta\Delta\phi(-\Delta\phi_t+\epsilon\Delta\Delta\phi-\tau\operatorname{div}(\varrho v))\,dx\\
=&\frac12\frac{d}{dt}\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\Delta\phi|^2\,dx-\tau\int_{\Omega^L}\Delta\varrho\operatorname{div}(\varrho v)\,dx. \qquad (3.28)
\end{aligned}
$$

Summing up the estimates (3.26)–(3.28) to deduce

$$
\begin{aligned}
&\frac12\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\bar\rho|\nabla\operatorname{div}v|^2+|\nabla\Delta\phi|^2\right)dx+\epsilon\int_{\Omega^L}|\Delta\Delta\phi|^2\,dx\\
&+(2\mu+\lambda)\int_{\Omega^L}|\Delta\operatorname{div}v|^2\,dx+\bar\rho\int_{\Omega^L}|\nabla\operatorname{div}v|^2\,dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}|\nabla\Delta\varrho|^2\,dx\\
=&-\frac{\tau P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}\left(\frac12|\Delta\varrho|^2\operatorname{div}v+\nabla\varrho\Delta v\Delta\varrho+2\nabla^2\varrho\nabla v\Delta\varrho+\varrho\Delta\operatorname{div}v\Delta\varrho\right)dx\\
&-2\frac{\tau P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}\nabla\varrho\nabla\operatorname{div}v\Delta\varrho\,dx+\tau\int_{\Omega^L}(\varrho\operatorname{div}v_t\Delta\operatorname{div}v+\nabla\varrho v_t\Delta\operatorname{div}v)\,dx\\
&+\tau\int_{\Omega^L}P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2\Delta\operatorname{div}v\,dx+\int_{\Omega^L}\Big(P'(\bar\rho+\tau\varrho)-P'(\bar\rho)\Big)\Delta\varrho\Delta\operatorname{div}v\,dx\\
&+\tau\int_{\Omega^L}\operatorname{div}(\varrho v)\Delta\operatorname{div}v\,dx+\tau\int_{\Omega^L}\operatorname{div}\left((\bar\rho+\tau\varrho)(v\cdot\nabla)v\right)\Delta\operatorname{div}v\,dx\\
&-\tau\int_{\Omega^L}\nabla\varrho\nabla\phi\Delta\operatorname{div}v\,dx-\tau\int_{\Omega^L}\varrho\Delta\phi\Delta\operatorname{div}v\,dx+\tau\int_{\Omega^L}\Delta\varrho\operatorname{div}(\varrho v)\,dx\\
&-\tau\int_{\Omega^L}\operatorname{div}\big((\bar\rho+\tau\varrho)f^L\big)\Delta\operatorname{div}v\,dx\\
\le&C\tau\|\Delta\varrho\|_{L^2}^2\|\nabla v\|_{H^2}+C\tau\|\nabla\varrho\|_{H^1}\|\nabla v\|_{H^2}\|\Delta\varrho\|_{L^2}+C\tau\|\varrho\|_{H^2}\|\Delta\operatorname{div}v\|_{L^2}\|\Delta\varrho\|_{L^2}\\
&+C\tau\|\varrho\|_{H^2}\|\operatorname{div}v_t\|_{L^2}\|\Delta\operatorname{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{L^4}\|v_t\|_{L^4}\|\Delta\operatorname{div}v\|_{L^2}\\
&+C\tau\|\varrho\|_{L^4}\|\operatorname{div}v\|_{L^4}\|\Delta\operatorname{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{L^4}\|v\|_{H^2}\|\nabla v\|_{L^4}\|\Delta\operatorname{div}v\|_{L^2}\\
&+C\tau\|\nabla\varrho\|_{L^4}^2\|\Delta\operatorname{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{L^2}\|v\|_{H^2}\|\Delta\operatorname{div}v\|_{L^2}+C\tau\|\nabla v\|_{L^4}^2\|\Delta\operatorname{div}v\|_{L^2}\\
&+C\tau\|\Delta v\|_{L^2}\|v\|_{H^2}\|\Delta\operatorname{div}v\|_{L^2}+C\tau\|\nabla\varrho\|_{L^4}\|\Delta\operatorname{div}v\|_{L^2}\|\nabla\phi\|_{L^4}\\
&+C\tau\|\varrho\|_{H^2}\|\Delta\operatorname{div}v\|_{L^2}\|\Delta\phi\|_{L^2}+C\tau\|\nabla\varrho\|_{L^4}\|v\|_{L^4}\|\Delta\varrho\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+C\tau\|\varrho\|_{H^2}\|\operatorname{div} v\|_{L^2}\|\Delta\varrho\|_{L^2}+C\tau(\|\nabla\varrho\|_{L^4}\|f^L\|_{L^4}+\|\operatorname{div} f^L\|_{L^2})\|\Delta\operatorname{div} v\|_{L^2}\\
\le & C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{H^2}+C\tau\|\varrho\|_{H^2}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\varrho\|_{H^2}^2\|\nabla v_t\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v_t\|_{H^1}^2\\
&+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\varrho\|_{H^2}^2\|\nabla\phi\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\operatorname{div} v\|_{L^2}^2+C\tau\|f^L\|_{H^1}^2+\xi_4\|v\|_{H^1}^2+\mu\|\Delta\operatorname{div} v\|_{L^2}^2,
\end{aligned}
$$

where ξ_4 is a constant independent of L and ϵ and can be chosen to be arbitrarily small. Hence, the above inequality implies

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\bar\rho|\nabla\operatorname{div} v|^2+|\nabla\Delta\phi|^2\right)dx+\bar\rho\int_{\Omega^L}|\nabla\operatorname{div} v|^2\,dx\\
&+(\mu+\lambda)\int_{\Omega^L}|\Delta\operatorname{div} v|^2\,dx+\epsilon\frac{P'(\bar\rho)}{\bar\rho}\int_{\Omega^L}|\nabla\Delta\varrho|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\Delta\phi|^2\,dx\\
\le & C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{H^2}+C\tau\|\varrho\|_{H^2}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\varrho\|_{H^2}^2\|\nabla v_t\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v_t\|_{H^1}^2\\
&+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\varrho\|_{H^2}^2\|\nabla\phi\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\operatorname{div} v\|_{L^2}^2+C\tau\|f^L\|_{H^1}^2+\xi_4\|v\|_{H^1}^2.
\end{aligned}
\tag{3.29}
$$

Notice carefully that

$$
\|\nabla\Delta v\|_{L^2}^2\le C(\|\operatorname{div}\Delta v\|_{L^2}^2+\|\operatorname{curl}\Delta v\|_{L^2}^2),
$$

so we may apply the operator curl to Equation $(3.11)_2$ to obtain

$$
\begin{aligned}
\operatorname{curl}((\bar\rho+\tau\varrho)v_t)-\mu\operatorname{curl}\Delta v+\tau\operatorname{curl}((\bar\rho+\tau\varrho)(v\cdot\nabla)v)+\operatorname{curl}((\bar\rho+\tau\varrho)v)&\\
-\operatorname{curl}((\bar\rho+\tau\varrho)\nabla\phi)=\tau\operatorname{curl}\left((\bar\rho+\tau\varrho)f^L\right).&
\end{aligned}
$$

Multiplying the above equation by $\operatorname{curl}\Delta v$, and integrating it over Ω^L by parts to get

$$
\begin{aligned}
\mu\int_{\Omega^L}|\operatorname{curl}\Delta v|^2\,dx\le & C\|\nabla v_t\|_{L^2}^2+C\|\nabla v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v_t\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{L^2}^2\|v\|_{H^2}^2\\
&+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\phi\|_{H^1}^2+C\tau\|f^L\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\operatorname{curl}\Delta v\|_{L^2}^2.
\end{aligned}
\tag{3.30}
$$

Finally, multiplying inequality (3.25) by a suitably large constant D_0, adding the resulting inequality with inequalities (3.29), (3.30) yield inequality (3.20) immediately. This completes the proof of this lemma. □

Lemma 3.6. *Under the same conditions in Lemma 3.3, we have*

$$
\int_{\Omega^L}\varrho_t^2\,dx+\epsilon\frac{d}{dt}\int_{\Omega^L}|\nabla\varrho|^2\,dx\le C\|\operatorname{div} v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^2}^2,
\tag{3.31}
$$

$$
\int_{\Omega^L}|\nabla\varrho_t|^2\,dx+\epsilon\frac{d}{dt}\int_{\Omega^L}|\Delta\varrho|^2\,dx\le C\|\nabla\operatorname{div} v\|_{L^2}^2+C\tau\|\varrho\|_{H^2}^2\|v\|_{H^2}^2,
\tag{3.32}
$$

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega^L}P'(\bar\rho+\tau\varrho)|\nabla\varrho|^2\,dx+\bar\rho\int_{\Omega^L}|\Delta\phi|^2\,dx+\epsilon\int_{\Omega^L}|\nabla\varrho|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\varrho^2\,dx\le C\|v_t\|_{L^2}^2\\
&+C\|\Delta v\|_{L^2}^2+C\|\nabla\operatorname{div} v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|\nabla\phi\|_{H^1}^2+C\tau\|v\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|f^L\|_{L^2}^2,
\end{aligned}
\tag{3.33}
$$

$$\frac{1}{2}\int_{\Omega^L} P'(\bar\rho+\tau\varrho)|\Delta\varrho|^2\,dx+\bar\rho\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\varrho|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\varrho|^2\,dx$$
$$\leq C\|\operatorname{div} v_t\|_{L^2}^2+C\|\Delta\operatorname{div} v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v_t\|_{H^1}^2+C\tau\|\varrho\|_{H^2}^2\|\nabla\phi\|_{H^1}^2$$
$$+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^4+C\tau\|f^L\|_{H^1}^2, \quad (3.34)$$

$$\frac{1}{2}\int_{\Omega^L}(\bar\rho+\tau\varrho)|\nabla\phi|^2\,dx+P'(\bar\rho)\int_{\Omega^L}|\Delta\phi|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\phi|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}|\nabla\phi|^2\,dx$$
$$\leq C\|v_t\|_{L^2}^2+C\|\Delta v\|_{L^2}^2+C\|\nabla\operatorname{div} v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|v\|_{H^1}^2\|\nabla v\|_{H^1}^2$$
$$+C\|f^L\|_{L^2}^2, \quad (3.35)$$

$$\int_{\Omega^L}|\nabla\phi_t|^2\,dx+\epsilon\frac{d}{dt}\int_{\Omega^L}|\Delta\phi|^2\,dx\leq C\|v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^1}^2, \quad (3.36)$$

where C is a constants independent of L and ϵ.

Proof. Firstly, we multiply Equation $(3.11)_1$ by ϱ_t and $\Delta\varrho_t$ respectively, then the integration over Ω^L yields inequalities (3.31) and (3.32) immediately.

On the other hand, multiply Equation $(3.11)_2$ by $\nabla\varrho$, then integrate it over Ω^L. By the Poisson equation $(3.11)_3$, we obtain

$$\int_{\Omega^L} P'(\bar\rho+\tau\varrho)|\nabla\varrho|^2\,dx+\bar\rho\int_{\Omega^L}|\Delta\phi|^2\,dx+\int_{\Omega^L}(\bar\rho+\tau\varrho)v\nabla\varrho\,dx$$
$$=\tau\int_{\Omega^L}\varrho\nabla\phi\nabla\varrho\,dx-\int_{\Omega^L}(\bar\rho+\tau\varrho)v_t\nabla\varrho\,dx+\mu\int_{\Omega^L}\Delta v\nabla\varrho\,dx+(\mu+\lambda)\int_{\Omega^L}\nabla\operatorname{div} v\nabla\varrho\,dx$$
$$-\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)(v\cdot\nabla)v\nabla\varrho\,dx+\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)f^L\nabla\varrho\,dx. \quad (3.37)$$

While for the last term at left-hand side of Equation (3.37), we have from Equation $(3.11)_1$ that

$$\int_{\Omega^L}(\bar\rho+\tau\varrho)v\nabla\varrho\,dx=-\int_{\Omega^L}\operatorname{div}((\bar\rho+\tau\varrho)v)\,\varrho\,dx$$
$$=\int_{\Omega^L}(\varrho_t-\epsilon\Delta\varrho)\varrho\,dx$$
$$=\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\varrho^2\,dx+\epsilon\int_{\Omega^L}|\nabla\varrho|^2\,dx. \quad (3.38)$$

Plugging estimate (3.38) into Equation (3.37) gives

$$\int_{\Omega^L} P'(\bar\rho+\tau\varrho)|\nabla\varrho|^2\,dx+\bar\rho\int_{\Omega^L}|\Delta\phi|^2\,dx+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\varrho^2\,dx+\epsilon\int_{\Omega^L}|\nabla\varrho|^2\,dx$$
$$\leq C\tau\|\varrho\|_{H^1}^2\|\nabla\phi\|_{H^1}^2+C\|v_t\|_{L^2}^2+C\|\Delta v\|_{L^2}^2+C\|\nabla\operatorname{div} v\|_{L^2}^2$$
$$+C\tau\|v\|_{H^1}^2\|\nabla v\|_{H^1}^2+C\tau\|f^L\|_{L^2}^2+\frac{1}{2}\int_{\Omega^L}P'(\bar\rho+\tau\varrho)|\nabla\varrho|^2\,dx,$$

which implies inequality (3.33).

In addition, applying the operator div to Equation $(3.11)_2$ and multiplying it by $\Delta\varrho$ yields

$$\int_{\Omega^L} P'(\bar\rho+\tau\varrho)|\Delta\varrho|^2\,dx+\bar\rho\int_{\Omega^L}|\nabla\Delta\phi|^2\,dx+\int_{\Omega^L}\operatorname{div}((\bar\rho+\tau\varrho)v)\,\Delta\varrho\,dx$$

$$
\begin{aligned}
&=\tau\int_{\Omega^L}\Delta\varrho\,\mathrm{div}(\varrho\nabla\phi)\,dx+\tau\int_{\Omega^L}P''(\bar\rho+\tau\varrho)|\nabla\varrho|^2\Delta\varrho\,dx-\tau\int_{\Omega^L}\nabla\varrho v_t\Delta\varrho\,dx\\
&\quad-\int_{\Omega^L}(\bar\rho+\tau\varrho)\,\mathrm{div}\,v_t\Delta\varrho\,dx+(2\mu+\lambda)\int_{\Omega^L}\mathrm{div}\,\Delta v\Delta\varrho\,dx\\
&\quad-\tau\int_{\Omega^L}\mathrm{div}\,((\bar\rho+\tau\varrho)(v\cdot\nabla)v)\,\Delta\varrho\,dx+\tau\int_{\Omega^L}\mathrm{div}\left((\bar\rho+\tau\varrho)f^L\right)\Delta\varrho\,dx\\
&\le C\tau\|\varrho\|_{H^2}^2\|\nabla\phi\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v_t\|_{H^1}^2+C\|\,\mathrm{div}\,v_t\|_{L^2}^2\\
&\quad+C\|\Delta\,\mathrm{div}\,v\|_{L^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2+C\tau\|f^L\|_{H^1}^2\\
&\quad+\frac12\int_{\Omega^L}P'(\bar\rho+\tau\varrho)|\Delta\varrho|^2\,dx,
\end{aligned}
$$

where we have used the fact that $\Delta\phi=\varrho$ and the periodic boundary condition. And the last term at left-hand side of the above estimate can be estimated similar as Equation (3.38), that is,

$$
\int_{\Omega^L}\mathrm{div}\,((\bar\rho+\tau\varrho)v)\,\Delta\varrho\,dx=\int_{\Omega^L}(-\varrho_t+\epsilon\Delta\varrho)\Delta\varrho\,dx=\frac12\frac{d}{dt}\int_{\Omega^L}|\nabla\varrho|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\varrho|^2\,dx.
$$

Thus, inequality (3.34) follows from the above two estimates immediately.

Now we turn to estimate $\nabla\phi$, by multiplying Equation $(3.11)_2$ with $\nabla\phi$, we have

$$
\begin{aligned}
&\int_{\Omega^L}(\bar\rho+\tau\varrho)|\nabla\phi|^2\,dx+P'(\bar\rho)\int_{\Omega^L}|\Delta\phi|^2\,dx-\int_{\Omega^L}(\bar\rho+\tau\varrho)v\nabla\phi\,dx\\
&=\int_{\Omega^L}(\bar\rho+\tau\varrho)v_t\nabla\phi\,dx+\int_{\Omega^L}(P'(\bar\rho+\tau\varrho)-P'(\bar\rho))\,\nabla\varrho\nabla\phi\,dx-\mu\int_{\Omega^L}\Delta v\nabla\phi\,dx\\
&\quad-(\mu+\lambda)\int_{\Omega^L}\nabla\,\mathrm{div}\,v\nabla\phi\,dx+\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)(v\cdot\nabla)v\nabla\phi\,dx-\tau\int_{\Omega^L}(\bar\rho+\tau\varrho)f^L\nabla\phi\,dx\\
&\le C\|v_t\|_{L^2}^2+C\|\Delta v\|_{L^2}^2+C\|\nabla\,\mathrm{div}\,v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|v\|_{H^1}^2\|\nabla v\|_{H^1}^2\\
&\quad+C\|f^L\|_{L^2}^2+\frac12\int_{\Omega^L}(\bar\rho+\tau\varrho)|\nabla\phi|^2\,dx.
\end{aligned}
$$

In a similar way, by Equation $(3.11)_1$ and the Poisson equation $(3.11)_3$, the last term at left-hand side of the above estimate is bounded as

$$
\begin{aligned}
-\int_{\Omega^L}(\bar\rho+\tau\varrho)v\nabla\phi\,dx&=\int_{\Omega^L}\mathrm{div}\,((\bar\rho+\tau\varrho)v)\,\phi\,dx\\
&=\int_{\Omega^L}(-\Delta\phi_t+\epsilon\Delta\Delta\phi)\phi\,dx\\
&=\frac12\frac{d}{dt}\int_{\Omega^L}|\nabla\phi|^2\,dx+\epsilon\int_{\Omega^L}|\Delta\phi|^2\,dx.
\end{aligned}
$$

Thus the above two estimates gives inequality (3.35) immediately.

Finally, multiplying Equation $(3.11)_1$ by ϕ_t, notice that $\Delta\phi=\varrho$, we see that

$$
\begin{aligned}
\int_{\Omega^L}|\nabla\phi_t|^2\,dx+\frac{\epsilon}{2}\frac{d}{dt}\int_{\Omega^L}|\Delta\phi|^2\,dx&=\bar\rho\int_{\Omega^L}\mathrm{div}\,v\phi_t\,dx+\tau\int_{\Omega^L}\mathrm{div}(\varrho v)\phi_t\,dx\\
&=-\bar\rho\int_{\Omega^L}v\nabla\phi_t\,dx-\tau\int_{\Omega^L}\varrho v\nabla\phi_t\,dx\\
&\le C\|v\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^1}^2+\frac12\int_{\Omega^L}|\nabla\phi_t|^2\,dx.
\end{aligned}
$$

Then it follows from the above inequality that the estimate (3.36) holds. This completes the proof of this lemma. ∎

LEMMA 3.7. *Under the same conditions in Lemma 3.3, we have*

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\mu|\Delta v|^2+(\mu+\lambda)|\nabla \operatorname{div} v|^2+\bar{\rho}|\nabla v|^2+\frac{D_1P'(\bar{\rho})}{\bar{\rho}}|\Delta\varrho|^2+D_1\bar{\rho}|\nabla\operatorname{div} v|^2\right)dx\\
&\quad+\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(D_1|\nabla\Delta\phi|^2+2P'(\bar{\rho})\Delta\varrho\operatorname{div} v\right)dx+\epsilon D_1\frac{d}{dt}\int_{\Omega^L}|\nabla\varrho|^2\,dx\\
&\quad+\frac{\bar{\rho}}{2}\int_{\Omega^L}|\nabla v_t|^2\,dx+\frac{D_1}{2}\int_{\Omega^L}\varrho_t^2\,dx+\frac{D_1(\mu+\lambda)}{2}\int_{\Omega^L}|\Delta\operatorname{div} v|^2\,dx\\
&\quad+D_1\bar{\rho}\int_{\Omega^L}|\nabla\operatorname{div} v|^2\,dx+D_1\epsilon\frac{P'(\bar{\rho})}{\bar{\rho}}\int_{\Omega^L}|\nabla\Delta\varrho|^2\,dx+D_1\epsilon\int_{\Omega^L}|\Delta\Delta\phi|^2\,dx\\
\le{}&C\|v\|_{H^1}^2+C\|\Delta\phi\|_{L^2}^2+C\tau\|v_t\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^2}^2+C\tau\|\varrho\|_{H^2}^2\|\nabla\phi\|_{H^1}^2\\
&\quad+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla v\|_{H^2}+C\tau\|\Delta\operatorname{div} v\|_{L^2}^2\|\nabla\varrho\|_{H^1}^2\\
&\quad+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2+C\tau\|\varrho\|_{H^2}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|v_t\|_{H^1}^2\|\varrho\|_{H^2}^2+C\tau\|f^L\|_{H^1}^2, \qquad (3.39)
\end{aligned}
$$

where D_1 and C are constants independent of L and ϵ, and D_1 can be chosen to be suitably large.

Proof. Multiplying Equation $(3.11)_2$ by Δv_t and integrating it by parts over Ω^L to have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\Omega^L}\left(\mu|\Delta v|^2+(\mu+\lambda)|\nabla\operatorname{div} v|^2+\bar{\rho}|\nabla v|^2+2P'(\bar{\rho})\Delta\varrho\operatorname{div} v\right)dx\\
&\quad+\int_{\Omega^L}(\bar{\rho}+\tau\varrho)|\nabla v_t|^2\,dx\\
={}&\frac{\tau}{2}\int_{\Omega^L}v_t^2\Delta\varrho\,dx-\int_{\Omega^L}\left(P'(\bar{\rho}+\tau\varrho)-P'(\bar{\rho})\right)\Delta\varrho\operatorname{div} v_t\,dx+P'(\bar{\rho})\int_{\Omega^L}\varrho_t\Delta\operatorname{div} v\,dx\\
&\quad-\tau\int_{\Omega^L}P''(\bar{\rho}+\tau\varrho)|\nabla\varrho|^2\operatorname{div} v_t\,dx-\tau\int_{\Omega^L}\nabla(\varrho v)\nabla v_t\,dx+\int_{\Omega^L}\nabla\left((\bar{\rho}+\tau\varrho)\nabla\phi\right)\nabla v_t\,dx\\
&\quad-\tau\int_{\Omega^L}\nabla\left((\bar{\rho}+\tau\varrho)(v\cdot\nabla)v\right)\nabla v_t\,dx+\tau\int_{\Omega^L}\nabla\left((\bar{\rho}+\tau\varrho)f^L\right)\nabla v_t\,dx\\
\le{}&\frac{\bar{\rho}}{4}\|\nabla v_t\|_{L^2}^2+C\tau\|v_t\|_{H^1}^2\|\Delta\varrho\|_{L^2}^2+C\tau\|\varrho\|_{H^2}^2\|\Delta\varrho\|_{L^2}^2+C\|\varrho_t\|_{L^2}\|\Delta\operatorname{div} v\|_{L^2}\\
&\quad+C\tau\|\varrho\|_{H^1}^2\|v\|_{H^2}^2+C\tau\|\nabla\varrho\|_{H^1}^2\|v\|_{H^2}^2\|\nabla v\|_{H^1}^2+C\tau\|v\|_{H^2}^2\|\Delta v\|_{L^2}^2\\
&\quad+C\tau\|\nabla\varrho\|_{H^1}^2\|\nabla\phi\|_{H^1}^2+C\|\Delta\phi\|_{L^2}^2+C\tau\|\nabla v_t\|_{L^2}^2\|\nabla\varrho\|_{H^1}^2+C\tau\|f^L\|_{H^1}^2,
\end{aligned}
$$

This together with inequalities (3.29) and (3.31), by choosing D_1 suitably large, yields the desired estimate (3.39). This completes the proof of this lemma. ∎

3.3. Proof of Proposition 2.1. We are now in a position to prove the existence of time periodic solutions of Proposition 2.1. The proof is a combination of the uniform estimates obtained in the above subsection and the topological degree theory.

Proof. **(Proof of Proposition 2.1.)** To prove the existence of a solution $(\varrho,v,\nabla\phi)\in X_{a_0}^L$ of system (2.2) is equivalent to solving the equation

$$U-\mathcal{F}(U,1)=0,\ U=(\varrho,v,\nabla\phi)\in X_{a_0}^L.$$

That is, by the topological degree theory, to show

$$\deg\Big(I-\mathcal{F}(\cdot,1),\hat{B}_{a_0}(0),0\Big)\neq 0, \tag{3.40}$$

where $\hat{B}_{a_0}(0)$ is a ball of radius a_0 centered at the origin in X^L. To do this, we may show that there exits $a_0>0$ such that

$$(I-\mathcal{F}(\cdot,\tau))(\partial\hat{B}_{a_0}(0))\neq 0, \text{ for any } \tau\in[0,1] \tag{3.41}$$

by the topological degree theory.

By the fact that $\|\varrho\|_{L^\infty}\leq C\|\varrho\|_{H^2}\leq\frac{\bar{\rho}}{2}$, if a_0 is small enough. We can choose D_2,D_3 suitably large, then consider $D_2D_3\times(3.12)+D_2D_3\times(3.16)+D_2\times(3.20)+(3.31)+(3.32)+(3.33)+(3.34)+\times(3.35)+(3.36)$, the integration from 0 to T yields

$$\begin{aligned}
&\frac{D_2D_3}{4}\int_0^T\int_{\Omega^L}\left(\mu|\nabla v|^2+(\mu+\lambda)|\operatorname{div}v|^2+\bar{\rho}v^2+(\mu+\lambda)|\nabla\operatorname{div}v|^2\right)dx\,dt\\
&\quad+\int_0^T\int_{\Omega^L}\left(\frac{\mu D_2D_3}{4}|\Delta v|^2+\frac{D_2D_3\bar{\rho}}{2}|\nabla v|^2+\frac{(\mu+\lambda)}{4}D_2|\Delta\operatorname{div}v|^2\right)dx\,dt\\
&\quad+\frac{D_2\mu}{2}\int_0^T\int_{\Omega^L}|\operatorname{curl}\Delta v|^2\,dx\,dt+\frac{D_0D_2}{4}\int_0^T\int_{\Omega^L}\left(\mu|\nabla v_t|^2+(\mu+\lambda)|\operatorname{div}v_t|^2\right)dx\,dt\\
&\quad+\frac{\bar{\rho}D_0D_2}{4}\int_0^T\int_{\Omega^L}v_t^2\,dx\,dt+\int_0^T\int_{\Omega^L}\left(\varrho_t^2+|\nabla\varrho_t|^2+\varrho^2+|\nabla\varrho|^2+|\Delta\varrho|^2\right)dx\,dt\\
&\quad+\int_0^T\int_{\Omega^L}\left(|\nabla\phi|^2+|\nabla\phi_t|^2\right)dx\,dt+\epsilon D_0D_2\frac{P'(\bar{\rho})}{\bar{\rho}}\int_0^T\int_{\Omega^L}|\nabla\varrho_t|^2\,dx\,dt\\
&\quad+\epsilon D_2\frac{P'(\bar{\rho})}{\bar{\rho}}\int_0^T\int_{\Omega^L}|\nabla\Delta\varrho|^2\,dx\,dt+\epsilon D_0D_2\int_0^T\int_{\Omega^L}|\Delta\phi_t|^2\,dx\,dt\\
&\leq C\tau\sup_{0\leq t\leq T}\|\varrho\|_{H^2}^2\int_0^T\left(\|\nabla\varrho\|_{H^1}^2+\|\varrho_t\|_{H^1}^2+\|\nabla\phi\|_{H^2}^2+\|v_t\|_{H^1}^2\right)dt\\
&\quad+C\tau\sup_{0\leq t\leq T}\|v\|_{H^2}^2\int_0^T\left(\|\nabla v\|_{H^1}^2+\|\varrho\|_{H^2}^2+\|v_t\|_{H^1}^2+\|\varrho_t\|_{L^2}^2\right)dt\\
&\quad+C\tau\sup_{0\leq t\leq T}\|\varrho_t\|_{L^2}^2\int_0^T\left(\|\nabla\phi\|_{H^2}^2+\|\varrho_t\|_{L^2}^2\right)dt+C\tau\sup_{0\leq t\leq T}\|\nabla\varrho\|_{H^1}^2\int_0^T\|\nabla^3v\|_{L^2}^2\,dt\\
&\quad+C\tau\sup_{0\leq t\leq T}\left(\|\nabla\phi_t\|_{H^1}^2+\|v_t\|_{L^2}^2\right)\int_0^T\|\varrho_t\|_{H^1}^2\,dt+C\tau\int_0^T\|f^L\|_{H^1}^2\,dt\\
&\quad+C\tau\sup_{0\leq t\leq T}\|v\|_{H^2}^4\int_0^T\left(\|\varrho_t\|_{H^1}^2+\|\nabla\varrho\|_{H^1}^2\right)dt+C\tau\int_0^T\|f_t^L\|_{L^2}^2\,dt\\
&\leq C_1\tau a_0^4+C_2\tau a_0^6+C_3\tau\|f^L\|_{W_2^{1,1}}^2.
\end{aligned} \tag{3.42}$$

Consequently, there exists a time $t'\in(0,T)$ such that

$$\begin{aligned}
&\int_{\Omega^L}\left(\varrho_t^2+|\nabla\varrho_t|^2+v_t^2+|\nabla v_t|^2+|\operatorname{div}v_t|^2+|\nabla\phi_t|^2+v^2+|\nabla v|^2+|\operatorname{div}v|^2\right)(x,t')\,dx\\
&\quad+\int_{\Omega^L}\left(|\Delta v|^2+|\nabla\operatorname{div}v|^2+|\Delta\operatorname{div}v|^2+|\operatorname{curl}\Delta v|^2+\varrho^2+|\nabla\varrho|^2+|\Delta\varrho|^2\right)(x,t')\,dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega^L}|\nabla\phi|^2(x,t')\,dx+\epsilon\int_{\Omega^L}\left(|\nabla\varrho_t|^2+|\nabla\Delta\varrho|^2+|\Delta\phi_t|^2\right)(x,t')\,dx\\
\le & C_1'\tau a_0^4+C_2'\tau a_0^6+C_3'\tau\|f^L\|^2_{W_2^{1,1}}.
\end{aligned}\tag{3.43}
$$

On the other hand, it follows from the fact

$$
2P'(\bar\rho)\int_{\Omega^L}\Delta\varrho\operatorname{div}v\,dx\le\frac{\bar\rho}{2}\int_{\Omega^L}|\nabla v|^2\,dx+\frac{D_1P'(\bar\rho)}{2\bar\rho}\int_{\Omega^L}|\Delta\varrho|^2\,dx,
$$

and inequalities (3.12), (3.16), (3.20), (3.39) that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega^L}\left(\frac{P'(\bar\rho)}{\bar\rho}\varrho^2+(\bar\rho+\tau\varrho)v^2+|\nabla\phi|^2+\frac{P'(\bar\rho)}{\bar\rho}|\nabla\varrho|^2+\bar\rho|\nabla v|^2+|\Delta\phi|^2\right)dx\\
&\quad+\frac{d}{dt}\int_{\Omega^L}\left(D_0\frac{P'(\bar\rho)}{\bar\rho}\varrho_t^2+D_0(\bar\rho+\tau\varrho)v_t^2+D_0|\nabla\phi_t|^2+\frac{P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2+\bar\rho|\nabla\operatorname{div}v|^2\right)dx\\
&\quad+\frac{d}{dt}\int_{\Omega^L}\left(|\nabla\Delta\phi|^2+\mu|\Delta v|^2+(\mu+\lambda)|\nabla\operatorname{div}v|^2+\bar\rho|\nabla v|^2+\frac{D_1P'(\bar\rho)}{\bar\rho}|\Delta\varrho|^2\right)dx\\
&\quad+\frac{d}{dt}\int_{\Omega^L}\left(D_1\bar\rho|\nabla\operatorname{div}v|^2+D_1|\nabla\Delta\phi|^2+2P'(\bar\rho)\Delta\varrho\operatorname{div}v+2\epsilon D_1|\nabla\varrho|^2\right)dx\\
\le & C\|v\|^2_{H^2}+C\|\Delta\phi\|^2_{L^2}+C\tau\|\nabla\varrho\|^2_{H^1}\|\nabla v\|_{H^2}+C\tau\|v_t\|^2_{H^1}\|\Delta\varrho\|^2_{L^2}+C\tau\|\varrho\|^2_{H^1}\|v\|^2_{H^2}\\
&\quad+C\tau\|\nabla\varrho\|^2_{H^1}\|v\|^2_{H^2}\|\nabla v\|^2_{H^1}+C\tau\|v_t\|^2_{H^1}\|\varrho\|^2_{H^2}+C\tau\|\varrho\|^2_{H^2}\|\nabla\phi\|^2_{H^2}+C\tau\|\varrho_t\|^4_{L^2}\\
&\quad+C\tau\|\varrho_t\|^2_{H^1}\|v_t\|^2_{L^2}+C\tau\|\varrho_t\|^2_{H^1}\|v\|^2_{H^2}\|\nabla v\|^2_{H^1}+C\tau\|v_t\|^2_{H^1}\|v\|^2_{H^2}+C\tau\|\varrho_t\|^2_{H^1}\|\varrho\|^2_{H^2}\\
&\quad+C\tau\|\varrho_t\|^2_{H^1}\|\nabla\phi_t\|^2_{H^1}+C\tau\|\varrho_t\|^2_{L^2}\|\nabla\phi\|^2_{H^2}+C\tau\|\nabla\varrho\|^2_{H^1}\|\nabla^3v\|^2_{L^2}+C\tau\|\varrho_t\|^2_{L^2}\|v\|^2_{H^2}\\
&\quad+C\tau\|\varrho\|^2_{H^2}\|\nabla\varrho\|^2_{H^1}+C\tau\|v\|^2_{H^2}\|\nabla v\|^2_{H^1}+C\tau\|f_t^L\|^2_{L^2}+C\tau\|f^L\|^2_{H^1}.
\end{aligned}\tag{3.44}
$$

Hence, By inequality (3.43), we integrate inequality (3.44) from t' to t for $t\in[t',t'+T]$ to obtain

$$
\begin{aligned}
&\sup_{0\le t\le T}\int_{\Omega^L}\left(\varrho^2+v^2+|\nabla\phi|^2+|\nabla\varrho|^2+|\nabla v|^2+|\Delta v|^2+|\nabla\operatorname{div}v|^2+|\Delta\varrho|^2\right)dx\\
&\quad+\sup_{0\le t\le T}\int_{\Omega^L}\left(\varrho_t^2+v_t^2+|\nabla\phi_t|^2\right)dx\\
\le&\int_{\Omega^L}\left(\varrho^2+v^2+|\nabla\phi|^2+|\nabla\varrho|^2+|\nabla v|^2+|\Delta v|^2+|\nabla\operatorname{div}v|^2+|\Delta\varrho|^2\right)(x,t')\,dx\\
&\quad+\int_{\Omega^L}\left(\varrho_t^2+v_t^2+|\nabla\phi_t|^2\right)(x,t')\,dx+C\int_0^T\|v\|^2_{H^2}\,dt+C\int_0^T\|\Delta\phi\|^2_{L^2}\,dt\\
&\quad+C\tau\int_0^T\|\nabla v\|^2_{H^2}\,dt+C\tau\sup_{0\le t\le T}\left(\|\nabla\phi_t\|^2_{H^1}+\|v_t\|^2_{L^2}\right)\int_0^T\|\varrho_t\|^2_{H^1}\,dt\\
&\quad+C\tau\sup_{0\le t\le T}\|\varrho\|^2_{H^2}\int_0^T\left(\|\nabla\varrho\|^2_{H^1}+\|\varrho_t\|^2_{H^1}+\|\nabla\phi\|^2_{H^2}+\|v_t\|^2_{H^1}\right)dt\\
&\quad+C\tau\sup_{0\le t\le T}\|v\|^2_{H^2}\int_0^T\left(\|\nabla v\|^2_{H^1}+\|\varrho\|^2_{H^1}+\|v_t\|^2_{H^1}+\|\varrho_t\|^2_{L^2}\right)dt\\
&\quad+C\tau\sup_{0\le t\le T}\|v\|^4_{H^2}\int_0^T\left(\|\varrho_t\|^2_{H^1}+\|\nabla\varrho\|^2_{H^1}\right)dt+C\tau\sup_{0\le t\le T}\|\nabla\varrho\|^2_{H^1}\int_0^T\|\nabla^3v\|^2_{L^2}\,dt\\
&\quad+C\tau\sup_{0\le t\le T}\|\varrho_t\|^2_{L^2}\int_0^T\left(\|\nabla\phi\|^2_{H^2}+\|\varrho_t\|^2_{L^2}\right)dt+C\tau\int_0^T(\|f_t^L\|^2_{L^2}+\|f^L\|^2_{H^1})\,dt
\end{aligned}
$$

$$\leq C_4\tau a_0^4+C_5\tau a_0^6+C_6\tau\|f^L\|^2_{W_2^{1,1}}. \tag{3.45}$$

This together with inequality (3.42) yields

$$\begin{aligned}&\sup_{0\leq t\leq T}\int_{\Omega^L}\left(\varrho^2+v^2+|\nabla\phi|^2+|\nabla\varrho|^2+|\nabla v|^2+|\Delta v|^2+|\nabla \operatorname{div} v|^2+|\Delta\varrho|^2\right)dx\\&\quad+\sup_{0\leq t\leq T}\int_{\Omega^L}\left(\varrho_t^2+v_t^2+|\nabla\phi_t|^2\right)dx+\int_0^T\int_{\Omega^L}\left(\varrho_t^2+|\nabla\varrho_t|^2+v_t^2\right)dx\,dt\\&\quad+\int_0^T\int_{\Omega^L}\left(|\nabla v_t|^2+|\operatorname{div} v_t|^2+|\nabla\phi_t|^2+\varrho^2+|\nabla\varrho|^2+|\Delta\varrho|^2+|\nabla\phi|^2+v^2\right)dx\,dt\\&\quad+\int_0^T\int_{\Omega^L}\left(|\nabla v|^2+|\operatorname{div} v|^2+|\Delta v|^2+|\nabla\operatorname{div} v|^2+|\Delta\operatorname{div} v|^2+|\operatorname{curl}\Delta v|^2\right)dx\,dt\\&\quad+\epsilon\int_0^T\int_{\Omega^L}\left(|\nabla\varrho_t|^2+|\nabla\Delta\varrho|^2+|\Delta\phi_t|^2\right)dx\,dt\\&\leq C_7\tau a_0^4+C_8\tau a_0^6+C_9\tau\|f^L\|^2_{W_2^{1,1}}.\end{aligned} \tag{3.46}$$

Thus, when a_0 and $\|f^L\|^2_{W_2^{1,1}}$ are suitably small, it holds that

$$\begin{aligned}&|||(\varrho,v,\nabla\phi)|||^2+\epsilon\int_0^T\int_{\Omega^L}\left(|\nabla\varrho_t|^2+|\nabla\Delta\varrho|^2+|\Delta\phi_t|^2\right)dx\,dt\\&\leq\hat{C}_1\tau a_0^4+\hat{C}_2\tau a_0^6+\hat{C}_3\tau\|f^L\|^2_{W_2^{1,1}}\\&\leq\frac{1}{2}a_0^2.\end{aligned} \tag{3.47}$$

That is, condition (3.41) holds. Since $\mathcal{F}(\cdot,0)=0$, then

$$\deg\Big(I-\mathcal{F}(\cdot,1),\hat{B}_{a_0}(0),0\Big)=\deg\Big(I-\mathcal{F}(\cdot,0),\hat{B}_{a_0}(0),0\Big)=\deg\Big(I,\hat{B}_{a_0}(0),0\Big)=1.$$

Therefore, we have proved condition (3.40), which implies the problem (2.2) admits a solution $(\varrho,v,\nabla\phi)\in X^L_{a_0}$. This completes the proof of Proposition 2.1. ∎

4. Existence in the whole space

Now, we are devoted to showing the existence of time periodic solutions stated in Theorem 1.1.

Proof. **(Proof of Theorem 1.1.)** First, we denote $(\varrho^L,v^L,\nabla\phi^L)$ be the solution of the regularized problem (2.2). By Sobolev imbedding theorem, we see that $(\varrho^L,v^L,\nabla\phi^L)\in C^{\alpha,\frac{\alpha}{2}}((0,T)\times\Omega^L)$, and

$$[\varrho^L,v^L,\nabla\phi^L]_{\alpha,\frac{\alpha}{2}}\leq Ca_0.$$

Now, let $\epsilon\to0$, and then let $L\to+\infty$, for any fixed $\ell>0$, there exists a subsequence $\{(\varrho_n,v_n,\nabla\phi_n)\}_{n=1}^\infty$ and $(\varrho,v,\nabla\phi)\in X^\ell_{a_0}$, such that

$$(\varrho_n,v_n,\nabla\phi_n)\to(\varrho,v,\nabla\phi)\quad\text{uniformly in }\Omega^\ell;$$

$$(\varrho_{nt},v_{nt},\nabla\phi_{nt})\rightharpoonup^*(\varrho_t,v_t,\nabla\phi_t)\quad\text{in }L^\infty((0,T);L^2(\Omega^\ell));$$

$$(\varrho_n,v_n,\nabla\phi_n)\rightharpoonup^*(\varrho,v,\nabla\phi)\quad\text{in }L^\infty((0,T);H^2(\Omega^\ell));$$

$$(\varrho_{nt},v_{nt},\nabla\phi_{nt})\rightharpoonup(\varrho_t,v_t,\nabla\phi_t)\quad \text{in } L^2((0,T);H^1(\Omega^\ell));$$

$$\varrho_n\rightharpoonup\varrho\quad \text{in } L^2((0,T);H^2(\Omega^\ell));$$

$$(v_n,\nabla\phi_n)\rightharpoonup(v,\nabla\phi)\quad \text{in } L^2((0,T);H^3(\Omega^\ell)).$$

On the other hand, integrating inequality (3.44) from t to $t+\zeta$ and then integrating the resulting inequality from 0 to T over t to obtain

$$\begin{aligned}&\int_0^T\int_{\Omega^L}\left(\varrho^2+v^2+|\nabla\phi|^2+\varrho_t^2+v_t^2+|\nabla\phi_t|^2+|\nabla\varrho|^2\right)(x,t+\zeta)\,dx\,dt\\&\quad+\int_0^T\int_{\Omega^L}\left(|\nabla v|^2+|\Delta\varrho|^2+|\Delta v|^2+|\nabla\operatorname{div}v|^2\right)(x,t+\zeta)\,dx\,dt\\&\quad-\int_0^T\int_{\Omega^L}\left(\varrho^2+v^2+|\nabla\phi|^2+\varrho_t^2+v_t^2+|\nabla\phi_t|^2+|\nabla\varrho|^2\right)(x,t)\,dx\,dt\\&\quad-\int_0^T\int_{\Omega^L}\left(|\nabla v|^2+|\Delta\varrho|^2+|\Delta v|^2+|\nabla\operatorname{div}v|^2\right)(x,t)\,dx\,dt\\&\le C\zeta,\end{aligned}$$

where ζ is a suitably small constant and C is a constant independent of L. Thus, we have

$$(\varrho_{nt},v_{nt},\nabla\phi_{nt})\to(\varrho_t,v_t,\nabla\phi_t)\quad \text{strongly in } L^2((0,T);L^2(\Omega^\ell));$$

$$\varrho_n\to\varrho\quad \text{strongly in } L^2((0,T);H^1(\Omega^\ell));$$

$$(v_n,\nabla\phi_n)\to(v,\nabla\phi)\quad \text{strongly in } L^2((0,T);H^2(\Omega^\ell)).$$

Now, choosing a sequence L_n with $L_n\to+\infty$ as $n\to\infty$, let $\{(\varrho_n^k,v_n^k,\nabla\phi_n^k)\}$ be the convergent sequence in Ω^{L_k} given in the above sense. Then, denoted $\{(\varrho_n^{k+1},v_n^{k+1},\nabla\phi_n^{k+1})\}$ be a subsequence of $\{(\varrho_n^k,v_n^k,\nabla\phi_n^k)\}$, which convergence in $\Omega^{L_{k+1}}$, $(k=1,2,\cdots,n,\cdots)$. Repeating the argument as follows:

$$\begin{array}{llll}(\varrho_1^1,v_1^1,\nabla\phi_1^1) & (\varrho_2^1,v_2^1,\nabla\phi_2^1) & \cdots & (\varrho_n^1,v_n^1,\nabla\phi_n^1)\text{ converges in }\Omega^{L_1}\\(\varrho_1^2,v_1^2,\nabla\phi_1^2) & (\varrho_2^2,v_2^2,\nabla\phi_2^2) & \cdots & (\varrho_n^2,v_n^2,\nabla\phi_n^2)\text{ converges in }\Omega^{L_2}\\ \vdots & \vdots & \ddots & \vdots\\(\varrho_1^n,v_1^n,\nabla\phi_1^n) & (\varrho_2^n,v_2^n,\nabla\phi_2^n) & \cdots & (\varrho_n^n,v_n^n,\nabla\phi_n^n)\text{ converges in }\Omega^{L_n}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\end{array}$$

Hence, we get a Cantor diagonal subsequence $\{(\varrho_n^n,v_n^n,\nabla\phi_n^n)\}$ which converges to $(\varrho,v,\nabla\phi)$ in Ω^L for any $L>0$. By the arbitrariness of $L>0$, we see that $(\varrho,v,\nabla\phi)\in X_{a_0}$ is the time periodic solution of system (2.1) in $\mathbb{R}^N,N=2,3$. This completes the proof of Theorem 1.1. ∎

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